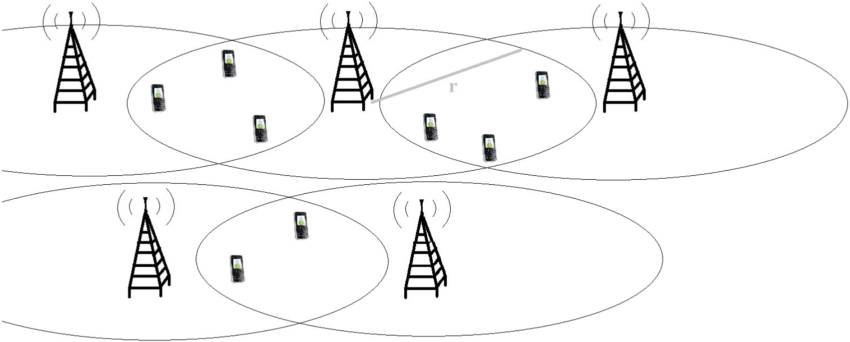
Covering codes

Élise Vandomme

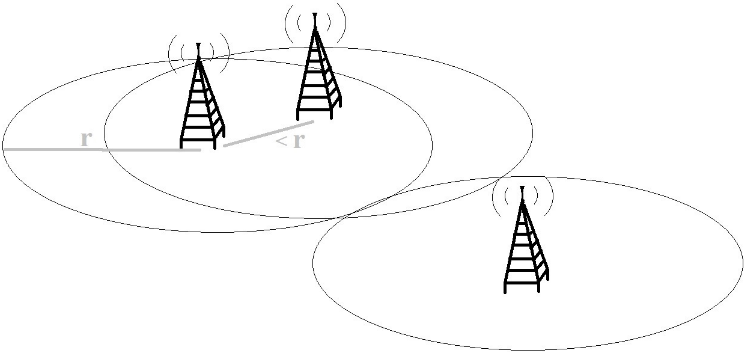
Combinatorics on Words and Tilings Workshop
Montréal – April 2017

Université du Québec à Montréal

Mobile network



Mobile network



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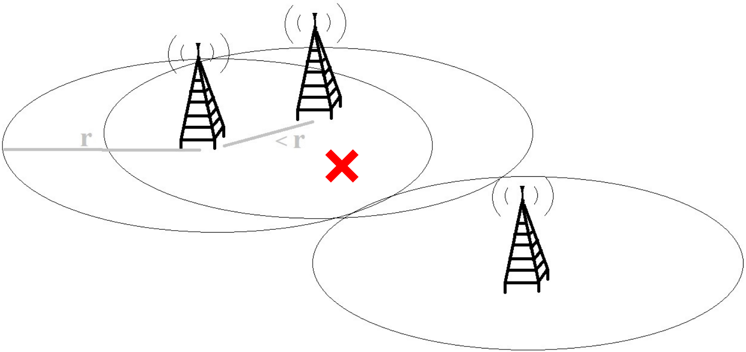

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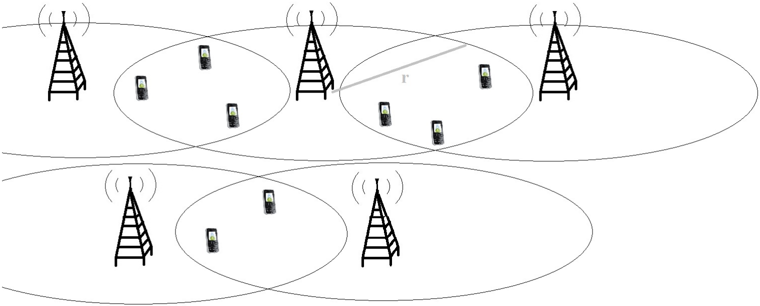

(r, a, b)-covering code with

- r: reach of the emitting stations
- a: number of emitting stations within reach of an emitting station
- b: number of emitting stations that reach of a phone

Translation in terms of graphs

A set $S \subseteq V$ is an (r, a, b)-covering code of $G = (V, E)$ if for any $u \in V$

$$\Big| \{ B_r(v) \mid u \in B_r(v),\ v \in S \} \Big| = \begin{cases} a & \text{if } u \in S \\ b & \text{if } u \notin S. \end{cases}$$

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Also known as isotropic coloring, perfect coloring.

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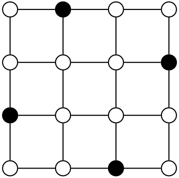
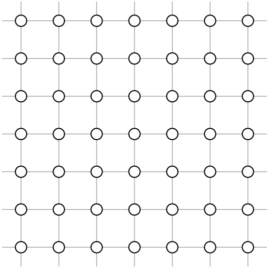
Also known as isotropic coloring, perfect coloring.
If $a = 1 = b$, they are called r-perfect code. [Biggs 1973]



$r = 1$

Finding an r-prefect code is NP-complete. [Kratochvíl 1988]

The infinite grid $\mathbb{Z}^2$



- Vertices : $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$
- Edge between (x_1, x_2) and (y_1, y_2) if $|x_1 - y_1| + |x_2 - y_2| = 1$

Manhattan distance d:

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

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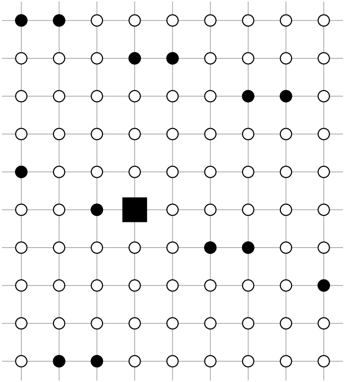

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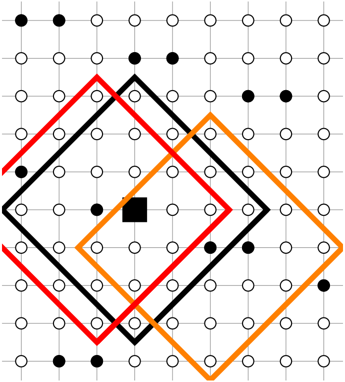
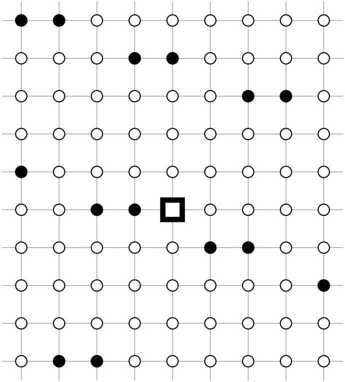


$a = 3$

The infinite grid $\mathbb{Z}^2$



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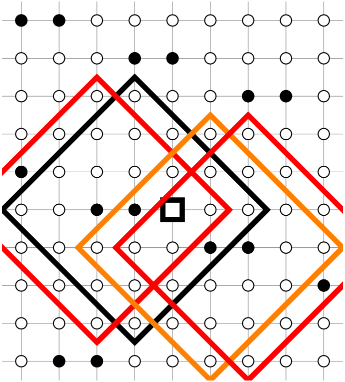

$a = 3$ and $b = 4$

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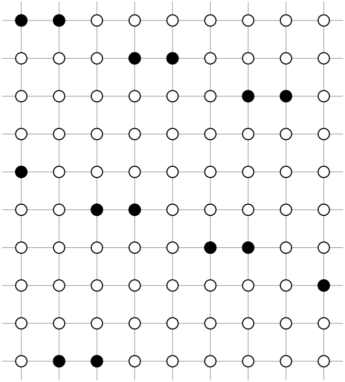

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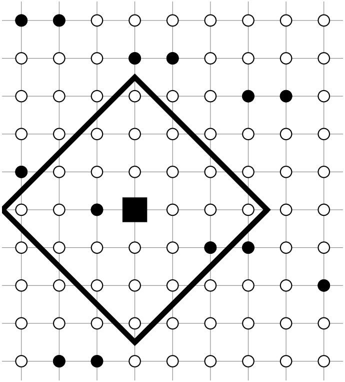

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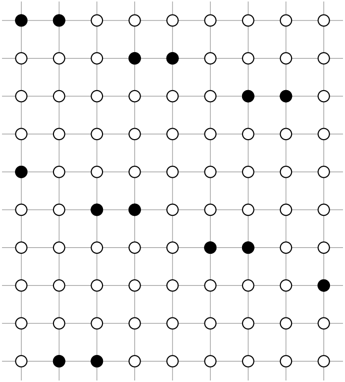

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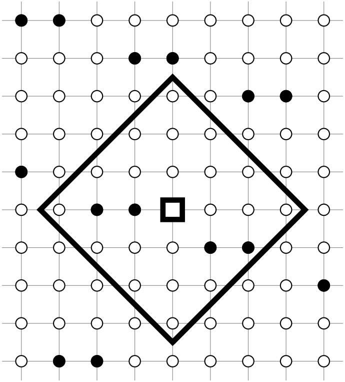

$a = 3$ and $b = 4$

Radius 1

Theorem (Axenovich 2003)

There exists a $(1, a, b)$-code in $\mathbb{Z}^2$ iff (a, b) is equal to one of:

$$\begin{matrix} (1,4), & (2,3), & (3,1), \\ (3,2), & (3,3), & (3,4), \\ (4,1), & (4,3), & (4,4), \end{matrix}$$

up to switching colors.

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$$(1, a, b)\text{-code} \quad \overset{\text{switching colors}}{\Longrightarrow} \quad (1, 5-b, 5-a)\text{-code}.$$

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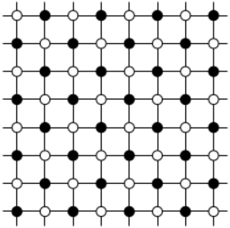
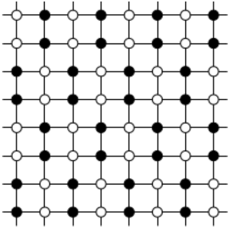
$$(1, a, b)\text{-code} \quad \overset{\text{switching colors}}{\Longrightarrow} \quad (1, 5-b, 5-a)\text{-code.}$$

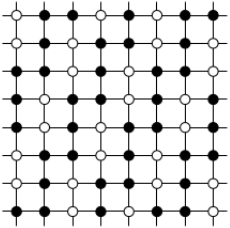
Unique up to isomorphism



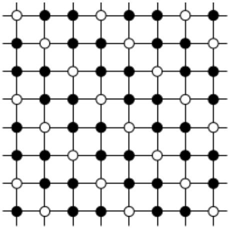
$(1, 4)$



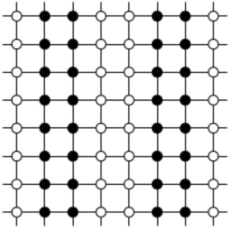
$(2, 3)$



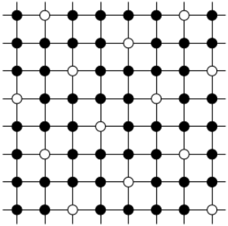
$(3, 3)$



$(3, 4)$

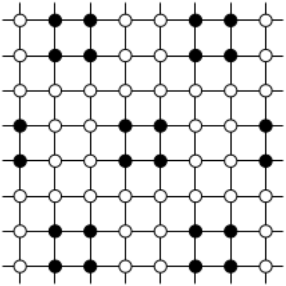


$(4, 1)$

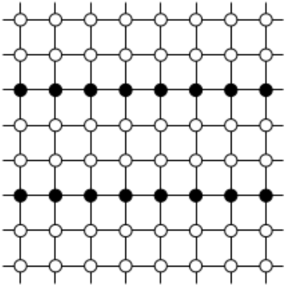


$(4, 4)$

Exactly two up to isomorphism



$(3,2)$



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Radius 1

Theorem (Puzynina 2004)

There exist non-periodic codes, but all of them can be obtained by periodic ones.

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Non-periodic $(1, a, b)$-code $\Longrightarrow (a, b) = (3, 2)$ or $(a, b) = (4, 3)$

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Example?

Radius 1 in higher dimension

Theorem (Dorbec, Gravier, Honkala, Mollard 2009)

Construction of periodic codes by extension of a $1D$-pattern.

In $\mathbb{Z}^6$

A	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\lvert N[u] \cap A \rvert$	2	3	2	1	0	0	0	0	0	0	0	0	0	0	1
$w_1 = 0$															
$-w_1 = 0$															
$w_2 = 2$															
$-w_2 = -2$															
$w_3 = 4$															
$-w_3 = -4$															
$w_4 = 5$															
$-w_4 = -5$															
$w_5 = 7$															
$-w_5 = -7$															

In $\mathbb{Z}^6$

A	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$\lvert N[u] \cap A \rvert$	2	3	2	1	0	0	0	0	0	0	0	0	0	0	1
$w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$-w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$w_2 = 2$	○	○	●	●	●	○	○	○	○	○	○	○	○	○	○
$-w_2 = -2$	●	○	○	○	○	○	○	○	○	○	○	○	○	●	●
$w_3 = 4$	○	○	○	○	●	●	●	○	○	○	○	○	○	○	○
$-w_3 = -4$	○	○	○	○	○	○	○	○	○	○	○	●	●	●	○
$w_4 = 5$	○	○	○	○	○	●	●	●	○	○	○	○	○	○	○
$-w_4 = -5$	○	○	○	○	○	○	○	○	○	○	●	●	●	○	○
$w_5 = 7$	○	○	○	○	○	○	○	●	●	●	○	○	○	○	○
$-w_5 = -7$	○	○	○	○	○	○	○	○	●	●	●	○	○	○	○

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$w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$-w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$w_2 = 2$	○	○	●	●	●	○	○	○	○	○	○	○	○	○	○
$-w_2 = -2$	●	○	○	○	○	○	○	○	○	○	○	○	○	●	●
$w_3 = 4$	○	○	○	○	●	●	●	○	○	○	○	○	○	○	○
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$-w_4 = -5$	○	○	○	○	○	○	○	○	○	○	●	●	●	○	○
$w_5 = 7$	○	○	○	○	○	○	○	●	●	●	○	○	○	○	○
$-w_5 = -7$	○	○	○	○	○	○	○	○	●	●	●	○	○	○	○
	5	5	5	2	2	2	2	2	2	2	2	2	2	2	2

In $\mathbb{Z}^6$

A	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$\lvert N[u] \cap A \rvert$	2	3	2	1	0	0	0	0	0	0	0	0	0	0	1
$w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$-w_1 = 0$	●	●	●	○	○	○	○	○	○	○	○	○	○	○	○
$w_2 = 2$	○	○	●	●	●	○	○	○	○	○	○	○	○	○	○
$-w_2 = -2$	●	○	○	○	○	○	○	○	○	○	○	○	○	●	●
$w_3 = 4$	○	○	○	○	●	●	●	○	○	○	○	○	○	○	○
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$w_4 = 5$	○	○	○	○	○	●	●	●	○	○	○	○	○	○	○
$-w_4 = -5$	○	○	○	○	○	○	○	○	○	○	●	●	●	○	○
$w_5 = 7$	○	○	○	○	○	○	○	●	●	●	○	○	○	○	○
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	5	5	5	2	2	2	2	2	2	2	2	2	2	2	2

$C = \{(x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid x_1 - x_2 w_1 - \cdots - x_6 w_5 \in A\}$ is a $(1, 5, 2)$-code

Radius $r \geq 2$

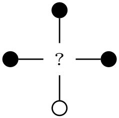
Theorem (Puzynina 2008)

All (r, a, b)-codes with $r \geq 2$ are periodic.

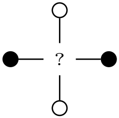
Radius $r \geq 2$

Axenovich divides (r, a, b)-codes into

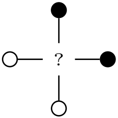
- Type A: $\exists$ a vertex such that



or



- Type B: $\forall$ vertex, we have



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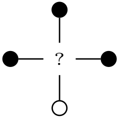

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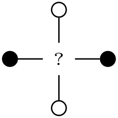
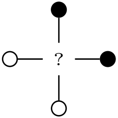


  Type A $\implies |a - b| \leq 4$

- Type B: $\forall$ vertex, we have



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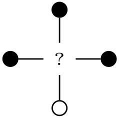

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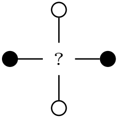

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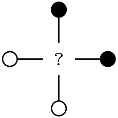
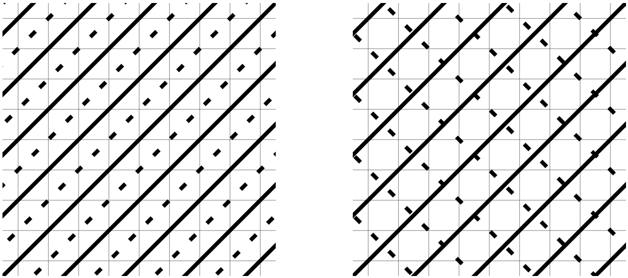


  Type B $\Longrightarrow$ which values of a and b?

Theorem (Axenovich 2003)

If c is an (r, a, b)-covering code of $\mathbb{Z}^2$ and $|a - b| > 4$, then c is a $\mathbf{p}$-periodic diagonal colouring for some $\mathbf{p} = (p, 0)$.

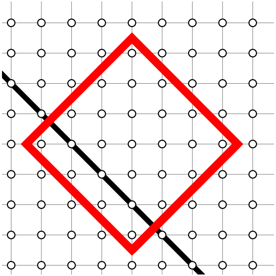


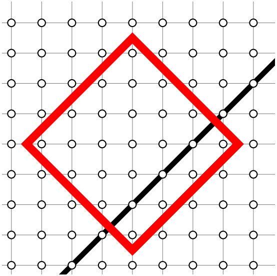
diagonal coloring

Consequence

We can assume that
φ is an (r, a, b)-code with $r \geq 2$ and $|a - b| > 4$
$\Longrightarrow \exists p \in \mathbb{Z}$ such that

- $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (1, 1)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2$,
- $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2$.





Projection and Folding

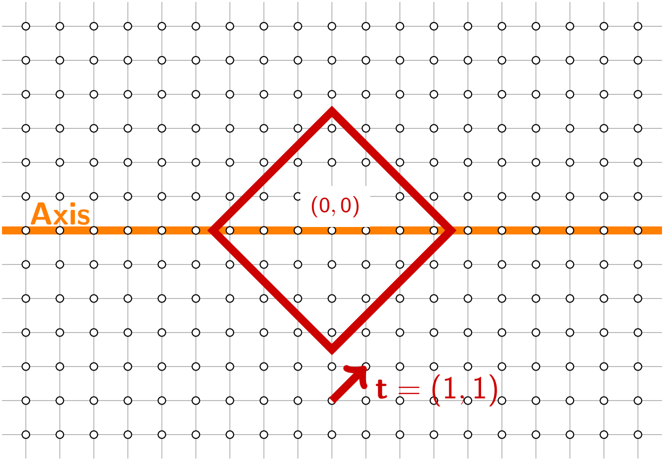
Hypotheses :

- $\varphi : \mathbb{Z}^2 \to \{\bullet, \circ\}$
- $t, p \in \mathbb{N}$
- $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (t, 1)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2,$
- $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2.$

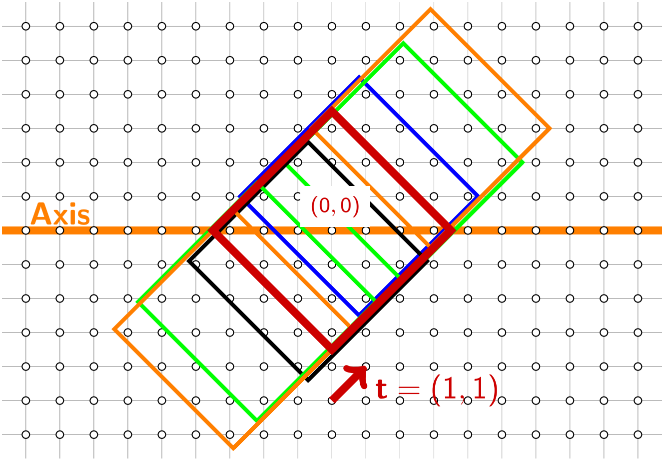
Goal :

Identify vertices of a given ball playing the "same role".

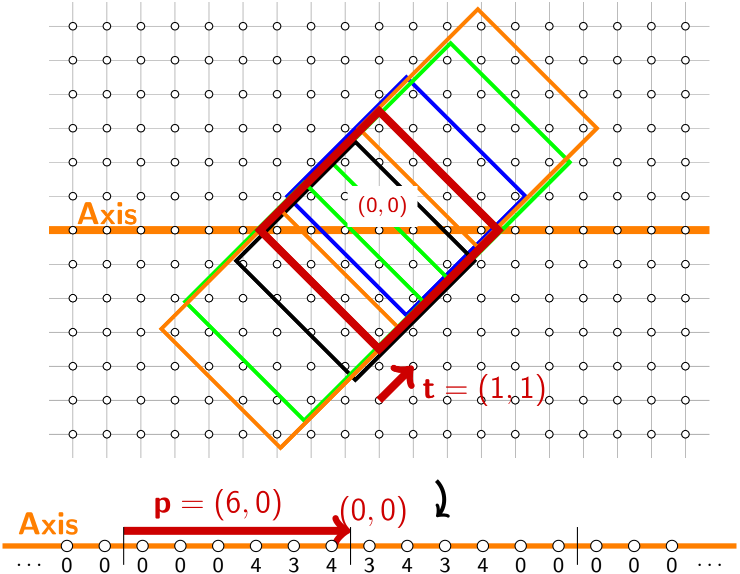
Projection and folding



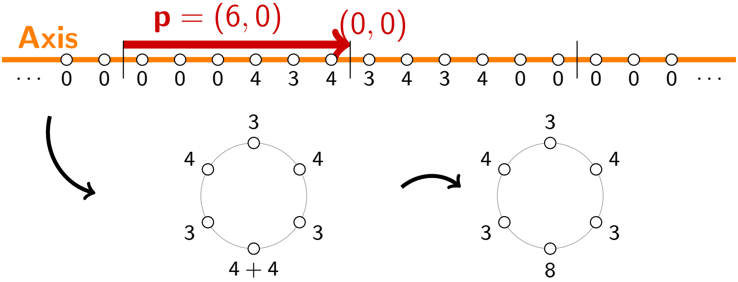
Projection and folding



Projection and folding



Projection and folding



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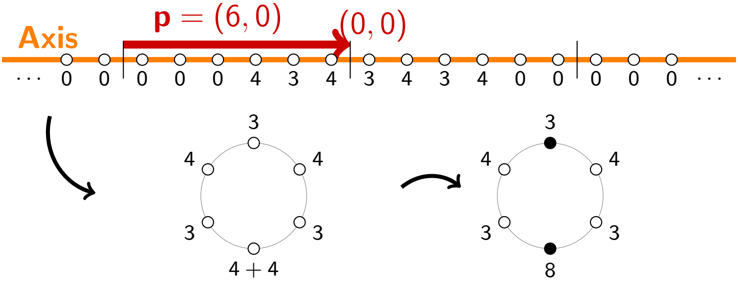

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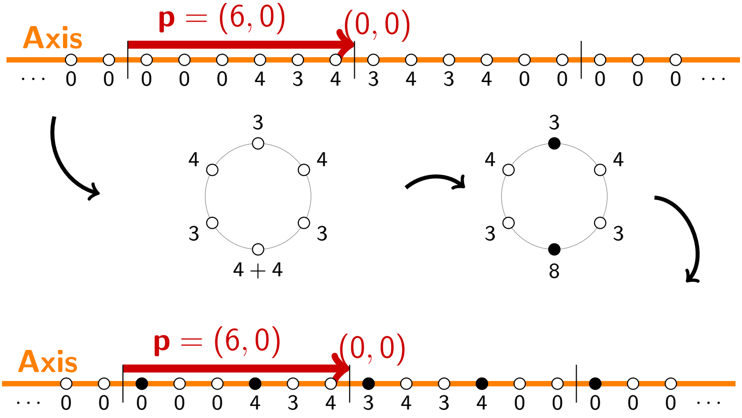

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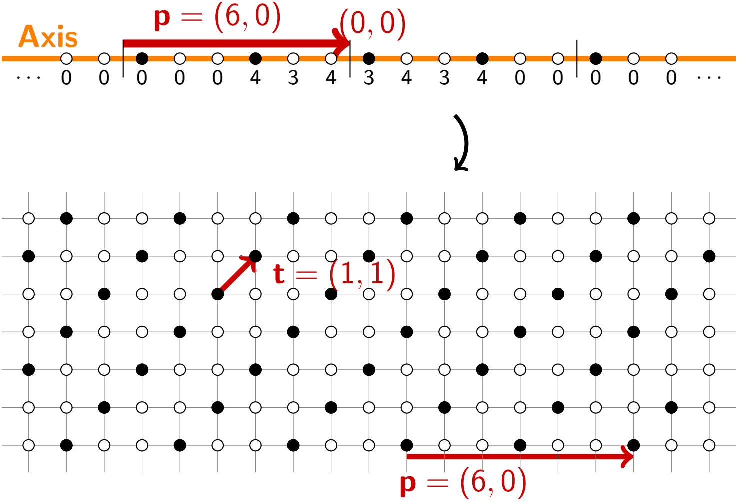
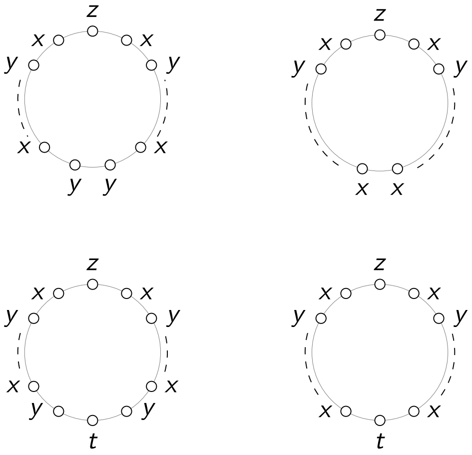


$\exists$ a $(3, 11, 7)$-covering code of $\mathbb{Z}^2$

Constant 2-labellings

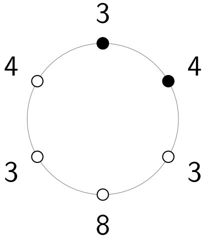
We only have to study particular colorings in 4 types of cycles!



Constant 2-labellings

A coloring is a constant 2-labelling of a weighted cycle $\mathcal{C}_p$ if for all rotations of the coloring

- v black $\implies \sum_{u \text{ black}} w(u) = \alpha$ constant
- v white $\implies \sum_{u \text{ black}} w(u) = \beta$ constant



v black $\qquad \sum_{u \text{ black}} w(u) = 7$

v white

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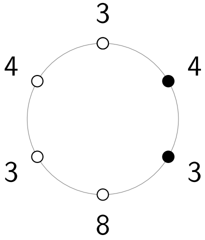

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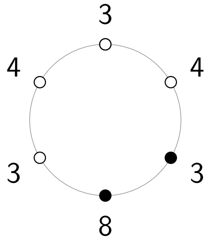

v black	$\sum_{u \text{ black}} w(u) = 7$
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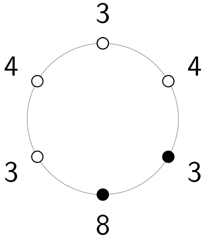

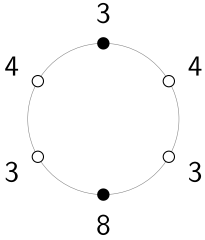

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✗

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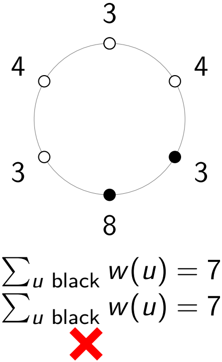

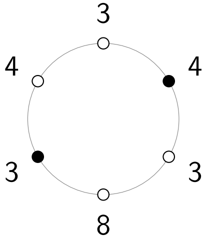

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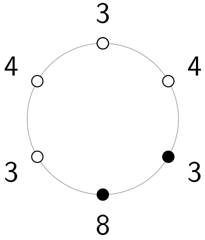

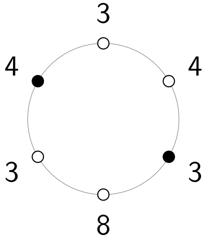

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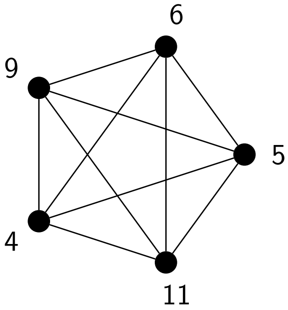
	✗	✓

Properties

Proposition

For any $G = (V, E)$, $v \in V$, $w : V \to \mathbb{R}$ and $A \subseteq Aut(G)$, a monochromatic coloring is a constant 2-labelling.



$$\alpha = \sum_{u \text{ black}} w(u) = \sum_{u \in V} w(u)$$

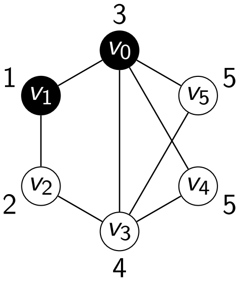
NB : β is not defined.

Properties

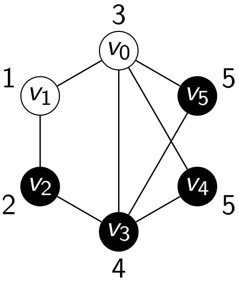
Proposition

For any $G = (V, E)$, $v \in V$, $w : V \to \mathbb{R}$ and $A \subseteq Aut(G)$, φ is a constant 2-labelling iff $\overline{\varphi}$ is a constant 2-labelling.

$A = Aut(G), v = v_3$



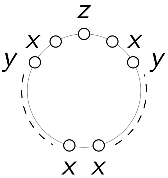
$\alpha = 6$
$\beta = 4$



$\overline{\alpha} = \sum_{u \in V} w(u) - \beta = 16$
$\overline{\beta} = \sum_{u \in V} w(u) - \alpha = 14$

Example of results

Lemma (Gravier, V.)



If c is a non-trivial constant 2-labelling of such cycle, then the number of vertices is a multiple of 3 and c is 3-periodic of pattern period $\bullet\bullet\circ$.

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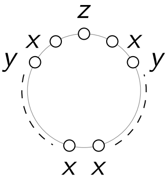

If c is a non-trivial constant 2-labelling of such cycle, then the number of vertices is a multiple of 3 and c is 3-periodic of pattern period $\bullet \bullet \circ$.

For $r \geq 2$ and $|a - b| > 4$, $\exists$ an (r, a, b)-code of $\mathbb{Z}^2$ iff $\exists$ a constant 2-labelling of some cycle $\mathcal{C}_p$ with adequate constants.

Characterization

Theorem (Gravier, V.)

Let $r, a, b \in \mathbb{N}$ be such that $|a-b| > 4$ and $r \geq 2$. For all (r, a, b)-codes of $\mathbb{Z}^2$, the values of a and b can be given explicitly.

If φ is an (r, a, b)-code with $|a-b| > 4$,

- φ is one of the periodic diagonal colorings given by Axenovich's theorem.
- We can apply the projection and folding method.
- Using constant 2-labellings, we have the possible values of a and b.

Perspectives

Many $(1, a, b)$-covering codes of $\mathbb{Z}^d$ are periodic.

[Dorbec, Gravier, Honkala, Mollard 2009]

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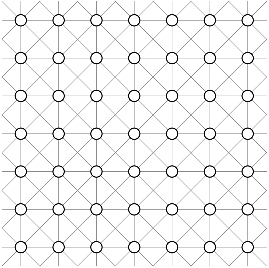
Same question for the King Lattice



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