On a conjecture about regularity and ℓ-abelian complexity

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Bridges between Automatic Sequences, Algebra and Number Theory
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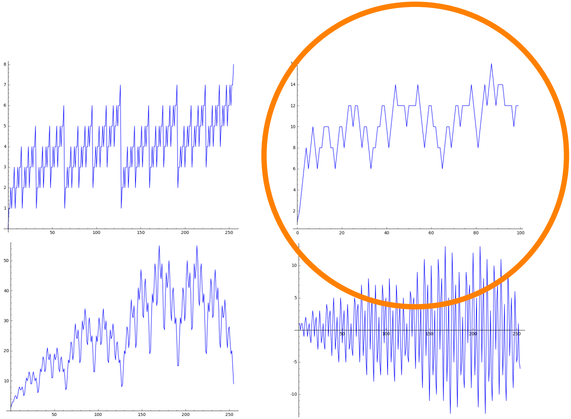
Yesterday...

k-regular sequences are much more chaotic...



Automatic sequences

Equivalence between

- $\mathbf{w} = (w_i)_{i \geq 0}$ is a k-automatic word
- $\mathbf{w} = \tau(\varphi^\omega(a))$ with φ k-uniform, τ 1-uniform, $a \in A$
- w_i is the output of a DFAO when reading $(i)_k$ [Cobham 72]

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- the k-kernel of $\mathbf{w}$

$$\mathcal{K}_k(\mathbf{w}) = \{w(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e\}$$

is finite [Eilenberg 1974]
Example: 2-kernel of the Thue–Morse word

$$\mathbf{t} = \quad 01101001100101101001011001101001 \cdots$$

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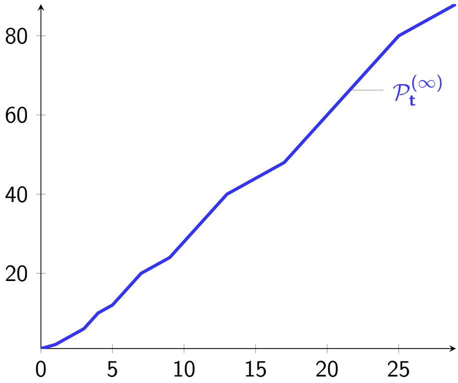
$$\mathcal{K}_2(\mathbf{t}) = \{\mathbf{t}, \bar{\mathbf{t}}\}$$

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$

Factor complexity $\mathcal{P}_{\mathbf{t}}^{(\infty)}$ [Brlek 1989, de Luca–Varricchio 1989]

$$\mathcal{P}_{\mathbf{t}}^{(\infty)}(n) = \begin{cases} 4n - 2\cdot 2^m - 4 & \text{if } 2\cdot 2^m < n \le 3\cdot 2^m \\ 2n + 4\cdot 2^m - 2 & \text{if } 3\cdot 2^m < n \le 4\cdot 2^m. \end{cases}$$



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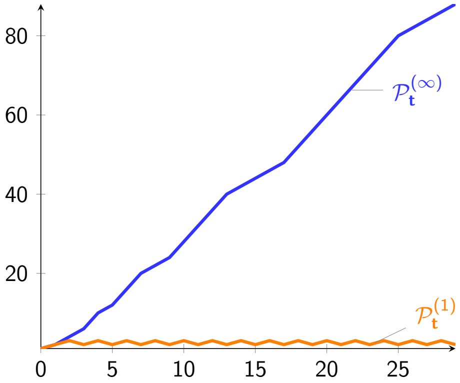

Abelian complexity $\mathcal{P}_{\mathbf{t}}^{(1)}$

$$\mathcal{P}_{\mathbf{t}}^{(1)}(2n) = 3 \text{ and } \mathcal{P}_{\mathbf{t}}^{(1)}(2n+1) = 2$$

ℓ-abelian complexity [Karhumäki–Saarela–Zamboni 2013]

Two words u, v are ℓ-abelian equivalent if

$$|u|_x = |v|_x \qquad \text{for any } x \text{ of length at most } \ell.$$

Example:

u	$\lvert u\rvert_0$	$\lvert u\rvert_1$	$\lvert u\rvert_{00}$	$\lvert u\rvert_{01}$	$\lvert u\rvert_{10}$	$\lvert u\rvert_{11}$
11010011	3	5	1	2	2	2
11101001	3	5	1	2	2	2

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Example: 2-abelian equivalent

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11101001	3	5	1	2	2	2	1

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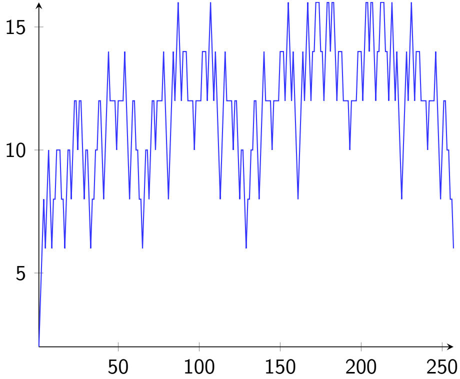
Number of factors of length n up to ℓ-abelian equivalence: $\mathcal{P}_{\mathbf{w}}^{(\ell)}(n)$

$$\mathcal{P}_{\mathbf{w}}^{(1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{w}}^{(\ell)}(n) \leq \mathcal{P}_{\mathbf{w}}^{(\ell+1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{w}}^{(\infty)}(n)$$

The ℓ-abelian complexity of a word $\mathbf{w}$ is the sequence $\mathcal{P}_{\mathbf{w}}^{(\ell)}(n)_{n\geq 0}$.

2-abelian complexity of the Thue–Morse word



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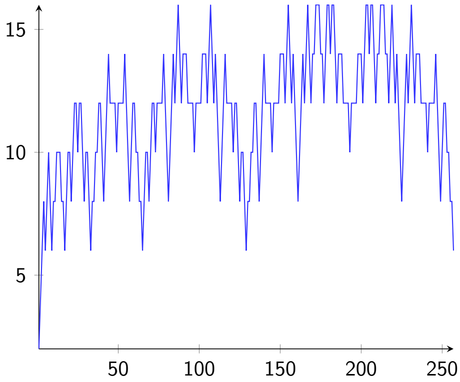

- Bounded? No [Berthé–Delecroix 2014, Karhumäki–Saarela–Zamboni 2014]

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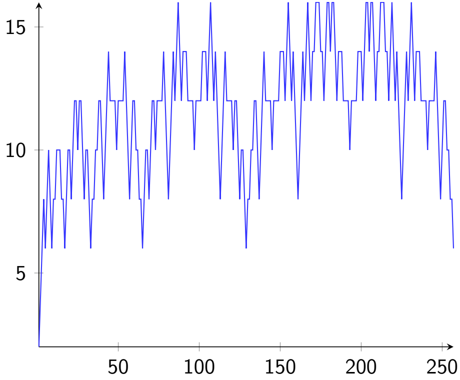

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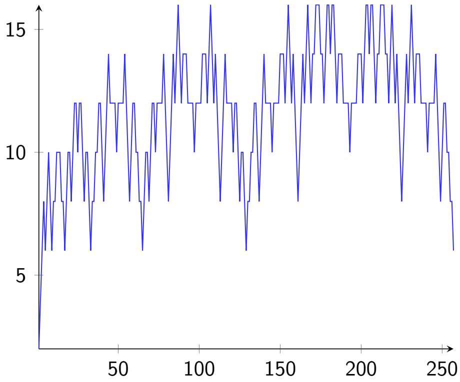

- Bounded? No [Berthé–Delecroix 2014, Karhumäki–Saarela–Zamboni 2014]
- Behavior? In $\log(n)$ [Karhumäki–Saarela–Zamboni 2014]
- Regular?

A definition of regularity [Allouche–Shallit 1992]

A sequence $\mathbf{s} = s(n)_{n \geq 0}$ is k-regular if the $\mathbb{Z}$-module generated by its k-kernel

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is finitely generated.

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Example: $s(n) =$ sum of digits in the representation in base 2 of n

$$\begin{aligned} &s(2n) = s(n) \text{ and } s(2n+1) = s(n) + 1 \\ &\Longrightarrow s(2^e n + r)_{n \geq 0} = s(n)_{n \geq 0} + s(r) \cdot 1_{n \geq 0} \\ &\Longrightarrow \mathbf{s} \text{ and } \mathbf{1} \text{ are generators} \\ &\Longrightarrow \mathbf{s} \text{ is 2-regular} \end{aligned}$$

Complexity and regularity

- The factor complexity of a k-automatic sequence is k-regular.
  [Carpi–D'Alonzo 2010, Charlier–Rampersad–Shallit 2012]
- The abelian complexity of
  - the Thue-Morse sequence
  - the paperfolding sequence [Madill–Rampersad 2013]
  - the period-doubling sequence [Karhumäki–Saarela–Zamboni 2014]
  - the 2-block coding of Thue-Morse sequence
    [Parreau–Rigo–Rowland–V. 2015]
  - the 2-block coding of period-doubling sequence
    [Parreau–Rigo–Rowland–V. 2015]
  - the Rudin-Shapiro sequence [Lü–Chen–Wen–Wu 2016]

  are 2-regular.
- The 2-abelian complexity of
  - the Thue-Morse sequence [Greinecker 2015, Parreau–Rigo–Rowland–V. 2015]
  - the period-doubling word [Parreau–Rigo–Rowland–V. 2015]

  are 2-regular.
- The ℓ-abelian complexity of the Cantor sequence is 3-regular for all $\ell \geq 1$ [Chen–Lü–Wu 2017]

How to prove regularity?

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We need to compute $\mathcal{P}_{\mathbf{t}}^{(\ell)}(n)$ for large n!

Naive idea

- Construct the first N letters of $\mathbf{t}$ with N large enough
- If the value of $\mathcal{P}_{\mathbf{t}}^{(\ell)}(n)$ is unchanged for several values of N, then we can suppose that the detected value of $\mathcal{P}_{\mathbf{t}}^{(\ell)}(n)$ is correct.

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$\rightarrow$ Impossible to compute $\mathcal{P}_{\mathbf{t}}^{(\ell)}(n)$ for large n

Proposition

Two words u, v (of length at least $\ell - 1$) are ℓ-abelian equivalent if and only if

(a) $|u|_x = |v|_x$ for any x of length ℓ;

(b) $pref_{\ell-1}(u) = pref_{\ell-1}(v)$.

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For $\ell = 2$, we associate a vector in $\mathbb{N}^{10}$ to each word $u = u_1 u_2 \cdots u_{n-1} u_n$,:

$$\Psi_2(u) = \begin{pmatrix} |u_1|_0 \\ |u_1|_1 \\ |u|_{00} \\ |u|_{01} \\ |u|_{10} \\ |u|_{11} \\ |u_{n-1}u_n|_{00} \\ |u_{n-1}u_n|_{01} \\ |u_{n-1}u_n|_{10} \\ |u_{n-1}u_n|_{11} \end{pmatrix} \qquad \Psi_2(11101) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Two words u and v are 2-abelian equivalent if and only if

(a) $[\Psi_2(u)]_{2+i} = [\Psi_2(v)]_{2+i}$ for $i \in \{1, \ldots, 2^2\}$,

(b) $[\Psi_2(u)]_i = [\Psi_2(v)]_i$ for $i \in \{1, 2\}$.

In this case, we write $\Psi_2(u) \sim \Psi_2(v)$.

	001	010	011	100	101	110
$\lvert u_1 \rvert_0$	1	1	1	0	0	0
$\lvert u_1 \rvert_1$	0	0	0	1	1	1
$\lvert u \rvert_{00}$	1	0	0	1	0	0
$\lvert u \rvert_{01}$	1	1	1	0	1	0
$\lvert u \rvert_{10}$	0	1	0	1	1	1
$\lvert u \rvert_{11}$	0	0	1	0	0	1
$\lvert u_{n-1} u_n \rvert_{00}$	0	0	0	1	0	0
$\lvert u_{n-1} u_n \rvert_{01}$	1	0	0	0	1	0
$\lvert u_{n-1} u_n \rvert_{10}$	0	1	0	0	0	1
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From a factor of length n to a factor of length $2n - 1$

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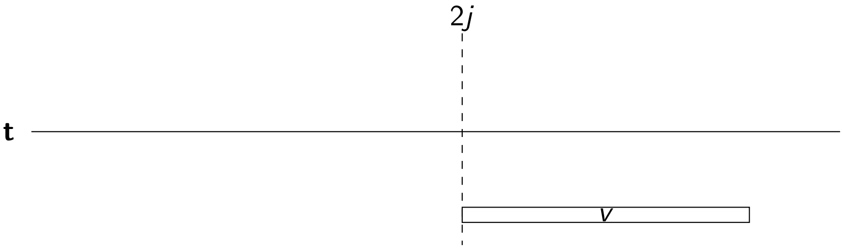

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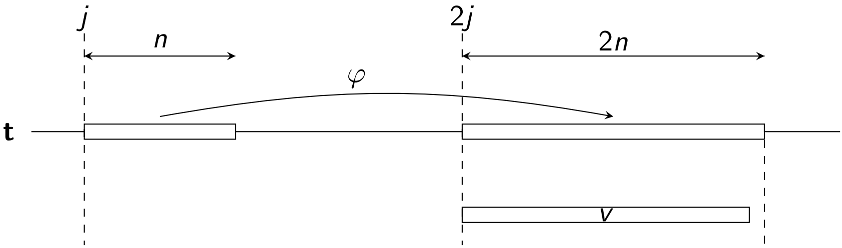

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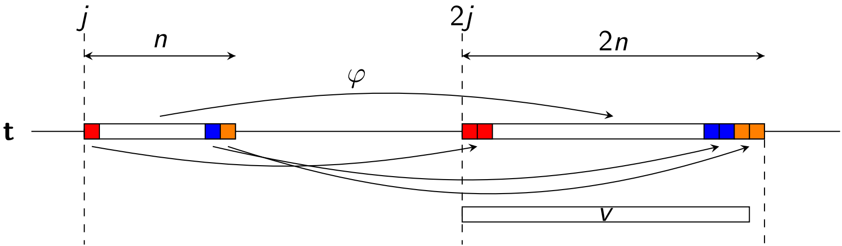

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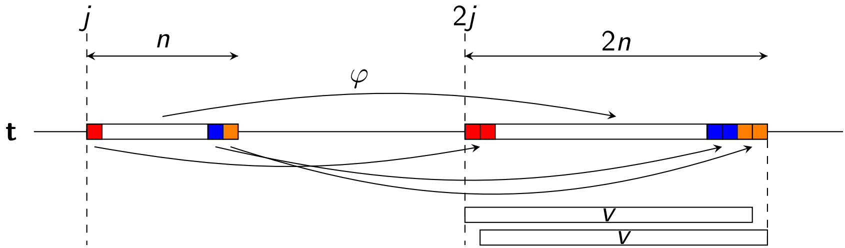

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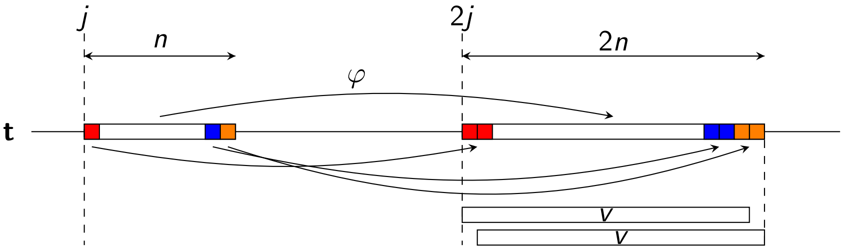
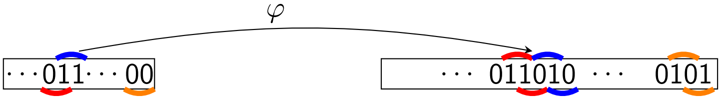
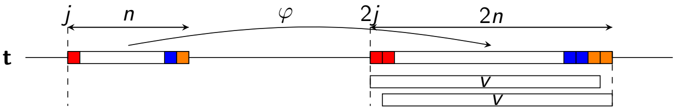


We know precisely what is happening





odd length factor at even position ($p = 0, r = 1$)

$$M^{(0,1)} = \left(\begin{array}{cc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

odd length factor at odd position ($p = 1, r = 1$)

$$M^{(1,1)} = \left(\begin{array}{cc|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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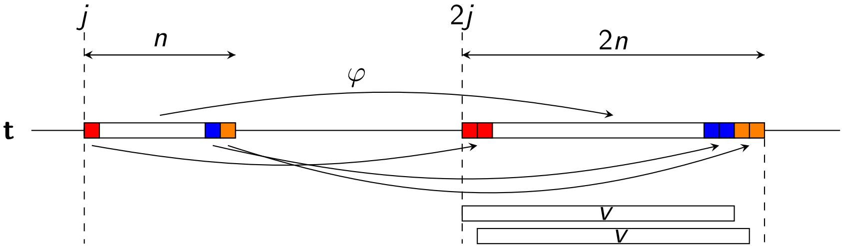
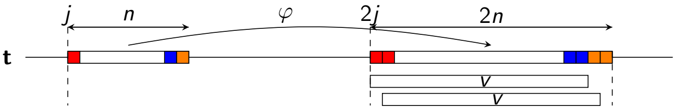
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Computation for even length factors

From a factor of length n to a factor of length $2n-2$





even length factor at even position $(p = 0, r = 0)$

$$M^{(0,0)} = \left(\begin{array}{cc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 \\
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even length factor at odd position $(p = 1, r = 0)$

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\end{array}\right)$$

Generalization for $\ell \geq 3$

$$\Psi_\ell(u) = (\underbrace{|pref_{\ell-1}(u)|_{a_{i_1}\dots a_{i_{\ell-1}}}, i_j \in \{1,\dots,|A|\},}_{\text{size } |A|^{\ell-1}}$$

$$\underbrace{|u|_{a_{i_1}\dots a_{i_\ell}}, i_j \in \{1,\dots,|A|\},}_{\text{size } |A|^{\ell}}$$

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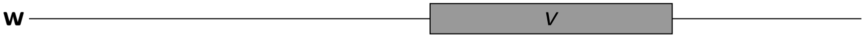
(a) $[\Psi_\ell(u)]_{|A|^{\ell-1}+i} = [\Psi_\ell(v)]_{|A|^{\ell-1}+i}$ for $i \in \{1,\dots,|A|^\ell\}$;

(b) $[\Psi_\ell(u)]_i = [\Psi_\ell(v)]_i$ for $i \in \{1,\dots,|A|^{\ell-1}\}$.

In this case, we note $\Psi_\ell(u) \sim \Psi_\ell(v)$.

Idea

Let φ be a k-uniform morphism and $\mathbf{w} = \varphi(\mathbf{w})$.



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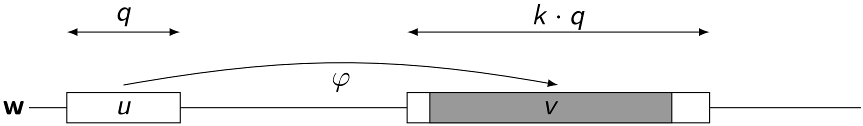

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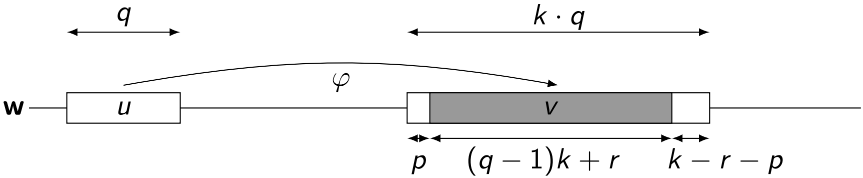

with $q \geq 1$, $p \in \{0, \ldots, k-1\}$ and $r \in \{2-k, \ldots, -1, 0, 1\}$.

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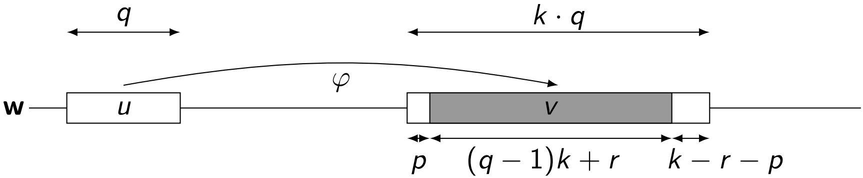


with $q \geq 1$, $p \in \{0, \ldots, k-1\}$ and $r \in \{2-k, \ldots, -1, 0, 1\}$.
Then

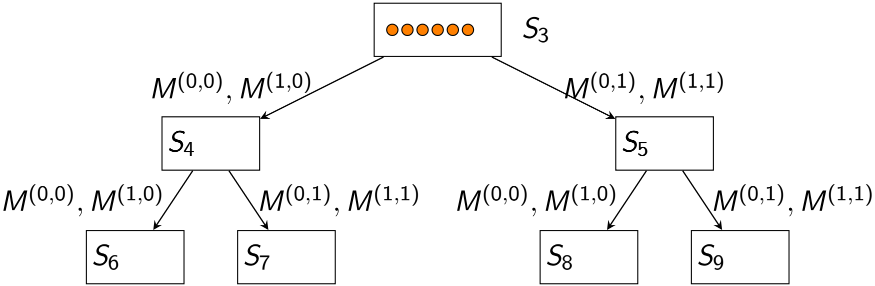
$$\Psi_\ell(v) = \left(\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline C & B_2 & D \\ \hline 0 & 0 & B_3 \end{array}\right) \Psi_\ell(u)$$

From matrices to the 2-abelian complexity of $\mathbf{t}$

$$S_3 = \{\mathbf{v} \in \mathbb{N}^{10} \mid \exists u \in A^3 : \mathbf{v} = \Psi_2(u) \text{ and } u \text{ is a factor of } \mathbf{t}\}$$

$$S_4 = \{M^{(0,0)}\mathbf{v},\ M^{(1,0)}\mathbf{v} \mid \mathbf{v} \in S_3\}/_\sim$$

$$S_5 = \{M^{(0,1)}\mathbf{v},\ M^{(1,1)}\mathbf{v} \mid \mathbf{v} \in S_3\}/_\sim$$



$$\mathcal{P}^2_{\mathbf{t}}(n) = \#S_n$$

How to prove regularity?

Find and prove relations for the sequences of the 2-kernel

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- Find? `Mathematica` experiments $\qquad x_{2^e+r} = \mathcal{P}_{\mathbf{t}}^{(2)}(2^e n + r)$

$$
\begin{aligned}
x_5 &= x_3\\
x_9 &= x_3\\
x_{12} &= -x_6 + x_7 + x_{11}\\
x_{13} &= x_7\\
x_{16} &= x_8\\
x_{17} &= x_3\\
x_{18} &= x_{10}\\
x_{20} &= -x_{10} + x_{11} + x_{19}\\
x_{21} &= x_{11}\\
x_{22} &= -x_3 - 2x_6 + x_7 + 3x_{10} + x_{11} - x_{19}\\
x_{23} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19}\\
x_{24} &= -x_3 + x_7 + x_{10}\\
x_{25} &= x_7\\
x_{26} &= -x_3 + x_7 + x_{10}\\
x_{27} &= -2x_3 + x_7 + 3x_{10} - x_{19}\\
x_{28} &= -2x_3 + x_7 + 3x_{10} - x_{14} + x_{15} - x_{19}\\
x_{29} &= x_{15}\\
x_{30} &= -x_3 + 3x_6 - x_7 - x_{10} - x_{11} + x_{15} + x_{19}\\
x_{31} &= -3x_3 + 6x_6 - 2x_{11} - 3x_{14} + 2x_{15} + x_{19}\\
x_{32} &= x_8\\
x_{33} &= x_3\\
x_{34} &= x_{10}\\
x_{35} &= x_{11}\\
x_{36} &= -x_{10} + x_{11} + x_{19}\\
x_{37} &= x_{19}\\
x_{38} &= -x_3 + x_{10} + x_{19}
\end{aligned}
$$

$$
\begin{aligned}
x_{39} &= -x_3 + x_{11} + x_{19}\\
x_{40} &= -x_3 + x_{10} + x_{11}\\
x_{41} &= x_{11}\\
x_{42} &= -x_3 + x_{10} + x_{11}\\
x_{43} &= -2x_3 + 3x_{10}\\
x_{44} &= -2x_3 - x_6 + x_7 + 3x_{10}\\
x_{45} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19}\\
x_{46} &= -2x_3 - 3x_6 + 2x_7 + 5x_{10} + x_{11} - 2x_{19}\\
x_{47} &= -2x_3 + x_7 + 3x_{10} - x_{19}\\
x_{48} &= -x_3 + x_7 + x_{10}\\
x_{49} &= x_7\\
x_{50} &= -x_3 + x_7 + x_{10}\\
x_{51} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19}\\
x_{52} &= -2x_3 - 3x_6 + 2x_7 + 5x_{10} + x_{11} - 2x_{19}\\
x_{53} &= -2x_3 + x_7 + 3x_{10} - x_{19}\\
x_{54} &= -4x_3 + 3x_6 + x_7 + 3x_{10} - x_{11} - 2x_{14} + x_{15}\\
x_{55} &= -4x_3 + 3x_6 + x_7 + 3x_{10} - x_{11} - 3x_{14} + 2x_{15}\\
x_{56} &= -x_3 + x_{10} + x_{15}\\
x_{57} &= x_{15}\\
x_{58} &= -x_3 + x_{10} + x_{15}\\
x_{59} &= -2x_3 + 3x_6 - x_7 - x_{11} + x_{15} + x_{19}\\
x_{60} &= -4x_3 + 6x_6 + x_{10} - 2x_{11} - 3x_{14} + 2x_{15} + x_{19}\\
x_{61} &= -3x_3 + 6x_6 - 2x_{11} - 3x_{14} + 2x_{15} + x_{19}\\
x_{62} &= -x_3 + 3x_6 - x_7 - x_{10} - x_{11} + x_{15} + x_{19}\\
x_{63} &= x_{15}
\end{aligned}
$$

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https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m

Regularity via relations

If the relations hold, then any sequence $\mathbf{x}_n$ for $n \geq 32$ is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{19}$.

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Example: $\mathbf{x}_{154} = \mathcal{P}_{\mathbf{t}}^{(2)}(128n + 26)_{n \geq 0}$

Using $\mathbf{x}_{58} = -\mathbf{x}_3 + \mathbf{x}_{10} + \mathbf{x}_{15}$,

$$\begin{aligned}\mathcal{P}_{\mathbf{t}}^{(2)}(128n + 26) &= \mathcal{P}_{\mathbf{t}}^{(2)}(32(4n) + 26)\\ &= -\mathcal{P}_{\mathbf{t}}^{(2)}(2(4n) + 1) + \mathcal{P}_{\mathbf{t}}^{(2)}(8(4n) + 2) + \mathcal{P}_{\mathbf{t}}^{(2)}(8(4n) + 7)\\ &= -\mathcal{P}_{\mathbf{t}}^{(2)}(8n + 1) + \mathcal{P}_{\mathbf{t}}^{(2)}(32n + 2) + \mathcal{P}_{\mathbf{t}}^{(2)}(32n + 7).\end{aligned}$$

So

$$\mathbf{x}_{154} = -\mathbf{x}_9 + \mathbf{x}_{34} + \mathbf{x}_{39} = -2\mathbf{x}_3 + \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{19}.$$

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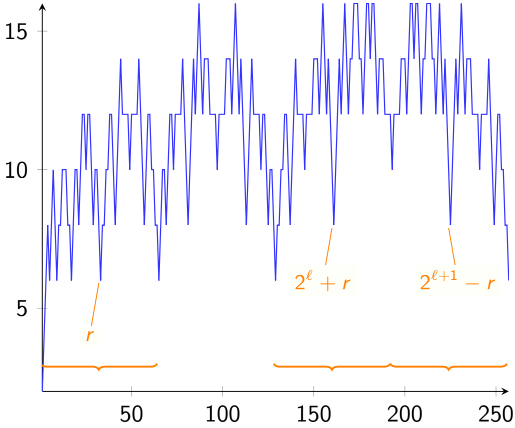
Theorem (Greinecker 2015)

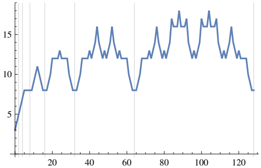
The relations hold and the 2-abelian complexity of **t** is 2-regular.

A more general approach

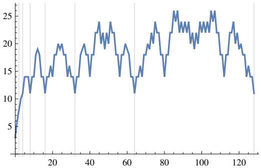


- Symmetry of the form $\mathcal{P}_{\mathbf{t}}^{(2)}(2^{\ell+1} - r) = \mathcal{P}_{\mathbf{t}}^{(2)}(2^\ell + r)$
- Some relation between $\mathcal{P}_{\mathbf{t}}^{(2)}(2^\ell + r)$ and $\mathcal{P}_{\mathbf{t}}^{(2)}(r)$

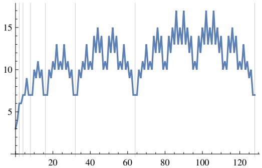
It is the case for lots of 2-abelian complexity functions



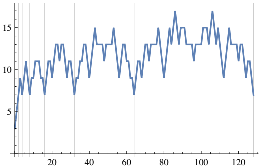
$0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 01$



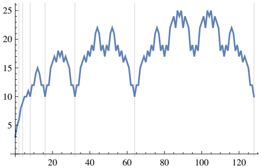
$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 01$



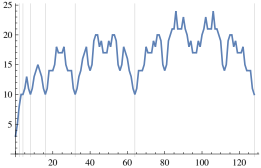
$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 11$



$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 21$



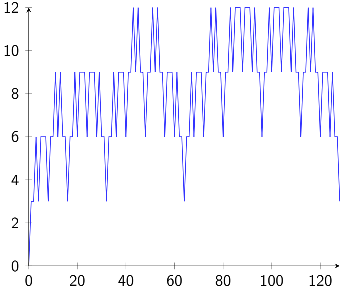
$0 \mapsto 01,\ 1 \mapsto 20,\ 2 \mapsto 01$



$0 \mapsto 01,\ 1 \mapsto 20,\ 2 \mapsto 10$

On a simpler function

Abelian complexity of the fixed point of $0 \mapsto 12,\ 1 \mapsto 12,\ 2 \mapsto 00$



- Recurrence: $\mathcal{P}^{(1)}_{\mathbf{x}}(2^\ell + r) = \mathcal{P}^{(1)}_{\mathbf{x}}(r) + 3$
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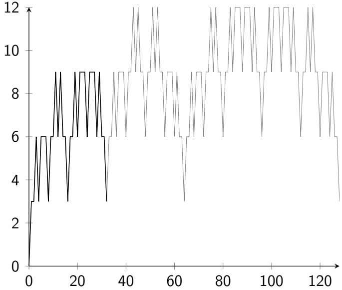

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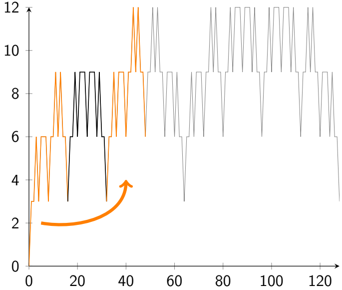

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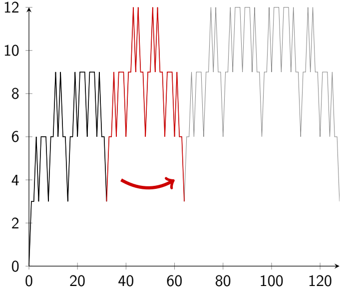

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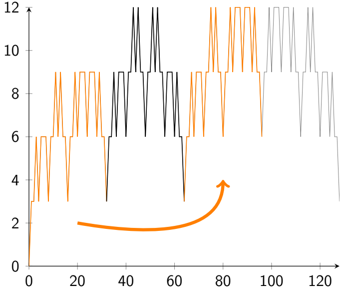

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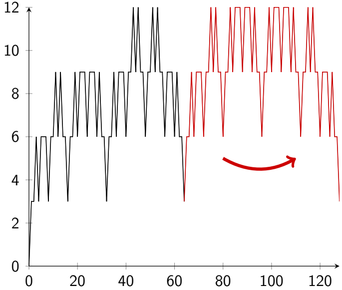

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Symmetry and recurrence relations

Do these nice symmetry and recurrence relations imply regularity?

- These relations use the most significant digits
- The kernel is made with the least significant digits

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Theorem (Parreau–Rigo–Rowland–V. 2015)

If $s(n)_{n\geq 0}$ satisfies

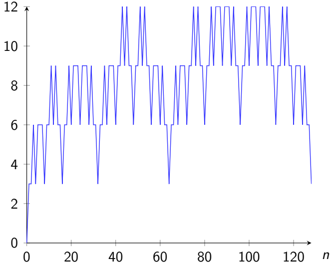
$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1} \end{cases}$$

then $s(n)_{n\geq 0}$ is 2-regular.

Consequences of the relations and the regularity

Using the recurrence and reflection relations, we immediately have that:

- it is not bounded,
- it is equal to $c\ell/2$ in $2^\ell + 2^{\ell-2} + 2^{\ell-4} + ... + 2^2 + 1$,
- it is constant and minimal in 2^ℓ.



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But how to prove the recurrence and reflection relations?

For abelian complexity of the fixed point of $0 \to 12, 1 \to 12, 2 \to 00$

$$\mathbf{x} = 120012121200120012001212120012121200\cdots$$

- Consider
$$\Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0$$
- It is closely related to the abelian complexity since 1 and 2 alternate.
- Prove the recurrence and reflection relations for Δ_0
$$\Delta_0(2^\ell + r) = \begin{cases} \Delta_0(r) + 2 & \text{if } r \leq 2^{\ell-1} \\ \Delta_0(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1} \end{cases}$$

But how to prove the recurrence and reflection relations?

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- Deduce the recurrence and reflection relations for $\mathcal{P}_{\mathbf{x}}^{(1)}$

$$\mathcal{P}_{\mathbf{x}}^{(1)}(2^\ell + r) = \begin{cases} \mathcal{P}_{\mathbf{x}}^{(1)}(r) + 3 & \text{if } r \leq 2^{\ell-1} \\ \mathcal{P}_{\mathbf{x}}^{(1)}(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1} \end{cases}$$

Consequence

The abelian complexity of the fixed point of $0 \to 12, 1 \to 12, 2 \to 00$

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is 2-regular.

- It is the 2-block coding of the period-doubling word

$$\mathbf{p} = 0100010101000100 0100\cdots$$

- The abelian complexity of $\mathbf{x}$ is closely related to the 2-abelian complexity of $\mathbf{p}$

$$\mathcal{P}^{(2)}_{\mathbf{p}}(n+1) = \mathcal{P}^{(1)}_{\mathbf{x}}(n) \quad \text{if } n \text{ is odd}$$

Consequence

The abelian complexity of the fixed point of $0 \to 12, 1 \to 12, 2 \to 00$

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- It is the 2-block coding of the period-doubling word

$$\mathbf{p} = 0100010101000100 0100\cdots$$

- The abelian complexity of $\mathbf{x}$ is closely related to the 2-abelian complexity of $\mathbf{p}$

$$\mathcal{P}_{\mathbf{p}}^{(2)}(n+1) = \mathcal{P}_{\mathbf{x}}^{(1)}(n) \quad \text{if } n \text{ is odd}$$

Theorem (Parreau–Rigo–Rowland–V. 2015)

The 2-abelian complexity of the period-doubling word is regular.

Back to the 2-abelian complexity of Thue-Morse

- Consider the 2-block coding of Thue-Morse

$$132120132012132120121320\cdots$$

fixed point of $0 \to 12, 1 \to 13, 2 \to 20, 3 \to 21$.

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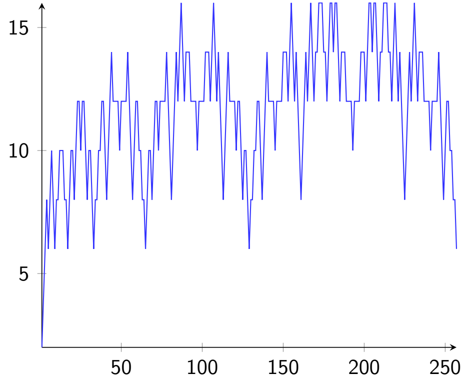
- Deduce the abelian complexity of the 2-block coding is 2-regular.

Theorem (Parreau–Rigo–Rowland–V. 2015)

The 2-abelian complexity of the Thue–Morse word satisfies a "slightly more complicated" recurrence and symmetry relation. It is 2-regular.



Summary

- The factor complexity of a k-automatic sequence is k-regular.
  [Carpi–D'Alonzo 2010, Charlier–Rampersad–Shallit 2012]
- The abelian complexity of
  - the Thue-Morse sequence
  - the paperfolding sequence [Madill–Rampersad 2013]
  - the period-doubling sequence [Karhumäki–Saarela–Zamboni 2014]
  - the 2-block coding of Thue-Morse sequence [P.–R.–R.–V. 2015]
  - the 2-block coding of period-doubling sequence [P.–R.–R.–V. 2015]
  - the Rudin-Shapiro sequence [Lü–Chen–Wen–Wu 2016]

  are 2-regular.
- The 2-abelian complexity of
  - the Thue-Morse sequence [Greinecker 2015, P.–R.–R.–V. 2015]
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Conjecture

The ℓ-abelian complexity of a k-automatic sequence is always k-regular.

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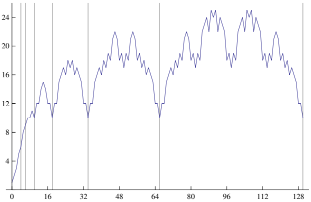
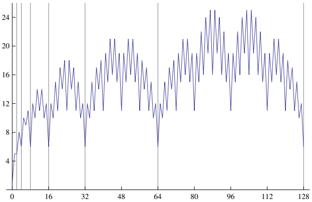
Perspectives

It seems that lots of (ℓ-)abelian complexity functions satisfy similar recurrence.





For the 3-abelian complexity of period-doubling word **p**, the abelian complexity of the 3-block coding **z** of **p** seems to satisfy:

$$\mathcal{P}_{\mathbf{z}}^{(1)}(2^\ell + r) = \begin{cases} \mathcal{P}_{\mathbf{z}}^{(1)}(r) + 5 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ even} \\ \mathcal{P}_{\mathbf{z}}^{(1)}(r) + 7 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ odd} \\ \mathcal{P}_{\mathbf{z}}^{(1)}(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}. \end{cases}$$

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- Same thing holds for the period-doubling word **p**
- Link between reflection symmetry and closed under "reversal and coding"?

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Submission deadline

May 5