148

International Journal of Modern Physics A, Vol. 11, No. 30 (1996) 5303–5325

STRONG COULOMB COUPLING IN THE TODOROV EQUATION

M. BAWIN and J. CUGNON
Université de Liège, Institut de Physique B5,
Sart Tilman, B-4000 Liège 1, Belgium

H. SAZDJIAN
Division de Physique Théorique, Institut de Physique Nucléaire,*
Université Paris XI, F-91406 Orsay Cedex, France

Received 1 February 1996

A positronium-like system with strong Coulomb coupling, considered in its pseudoscalar sector, is studied in the framework of relativistic quantum constraint dynamics with the Todorov choice for the potential. Case's method of self-adjoint extension of singular potentials, which avoids explicit introduction of regularization cut-offs, is adopted. It is found that, as the coupling constant α increases, the bound state spectrum undergoes an abrupt change at the critical value $\alpha = \alpha_c = 1/2$. For $\alpha > \alpha_c$, the mass spectrum displays, in addition to the existing states for $\alpha < \alpha_c$, a new set of an infinite number of bound states concentrated in a narrow band starting at mass $W = 0$; all the states have indefinitely oscillating wave functions near the origin. In the limit $\alpha \to \alpha_c$ from above, the oscillations disappear and the narrow band of low-lying states shrinks to a single massless state with a mass gap with the rest of the spectrum. This state has the required properties to represent a Goldstone boson and to signal spontaneous breakdown of chiral symmetry.

1. Introduction

The question of the possible existence of an ultraviolet stable fixed point in QED was investigated long ago by Gell-Mann and Low[1] and developed later by several authors.[2–4] If such a point were to exist, then the electron mass would be entirely dynamical in origin,[2–4] with a possible spontaneous breakdown of chiral symmetry.[5] Although perturbation theory calculations do not seem to point to the existence of such a solution, quenched lattice QED calculations displayed the existence of a phase transition at the critical value $\alpha_c \sim 0.3$ of the coupling constant α, with the occurrence of a spontaneous breakdown of chiral symmetry.[6] These observations were also confirmed with unquenched lattice calculations,[7] with $\alpha_c \sim 0.4$, but a vanishing of the Callan–Symanzik function β was not found there and the question of the validity of QED as a nontrivial consistent theory in the continuum limit was raised.

*Unité de Recherche des Universités Paris 11 et Paris 6 associée au CNRS.
PACS numbers: 03.65.Pm, 11.10.St, 12.20.-m, 11.30.Rd.

On the other hand, it is possible, in the continuum theory, to analyze a partial, but simpler, aspect of the phase transition problem, namely that of the bound state of strongly coupled positronium. If the ground state of the corresponding spectrum, for some value of α, were massless, then this would be the signal of a possible spontaneous breakdown of chiral symmetry and the prelude of a phase transition in QED. This result would not be, however, sufficient by itself to ensure the vanishing of the Callan–Symanzik β function, which should be shown by independent calculations; only in this case could the consistency of the whole procedure be guaranteed.

The above considerations have motivated us to further study the problem of strong Coulomb coupling[8,9] in the framework of relativistic quantum constraint dynamics (RQCD).[10,11] Let us emphasize, however, that the equations of RQCD in this paper do not include vacuum polarization and radiative correction diagrams, so that their consequences would be highly conjectural when applied to positronium in strongly coupled QED. One may however hope that these effects would bring quantitative changes only, leaving the qualitative aspects of the critical properties unchanged.

RQCD provides a manifestly covariant three-dimensional description of the internal motion of two-body systems and can be shown to be equivalent to a three-dimensional reduction of the Bethe–Salpeter equation.[12] When the approximation of local potentials is made, the corresponding wave equations can be analyzed rather easily and in many cases analytic solutions can be obtained. It was found that the Todorov form of the electromagnetic two-body potential, first introduced in the quasipotential approach,[13] leads to the existence of a critical value of the coupling constant, with $\alpha_c = 1/2$. For $\alpha > \alpha_c$, the potential becomes too singular and needs some regularization, provided by a cut-off radius r_0 in the Coulomb interaction. In the regularized theory, the bound state mass spectrum displays, for $\alpha \gtrsim 1/2$, a rapid fall of the ground state mass to values close to zero, thus indicating a drastic modification of the qualitative features of the bound system and presumably of the theory itself. The above procedure requires, however, a numerical treatment of the equations and makes it difficult to ascertain the existence of solutions for vanishing values of r_0.

The presence of a cut-off radius in the Coulomb interaction needs, in general, an adequate interpretation. As long as one studies the behavior of charged particles in supercritical Coulomb fields of heavy nuclei, the finite size of the latter naturally regularizes the Coulomb interaction at short distances. If, on the other hand, it is the strong interaction of pointlike particles that is considered, then the meaning of r_0 remains unclear.

A similar problem also occurs with the Bethe–Salpeter equation in the ladder approximation. It has been shown[14] that, when the coupling constant α is larger than a critical value α_c ($\sim \pi/3$ in the Landau gauge and $\sim \pi/4$ in the Feynman gauge), the theory undergoes spontaneous breakdown of chiral symmetry. However, for $\alpha \gtrsim \alpha_c$, the treatment and resolution of the equation necessitate the use of an ultraviolet cut-off Λ. While the introduction of the latter can naturally be justified in QCD as being an approximate way of parametrizing the asymptotic freedom of the theory,[15] it has not received a simple interpretation in QED. In this respect, Miransky *et al.*[16] suggested that, for $\alpha > \alpha_c$, QED undergoes an additional charge renormalization that absorbs the infinities of the pointlike limit; the renormalized charge remains equal to α_c and then might be identified with the ultraviolet fixed point of QED. It was also pointed out in this connection[17,18] that, for $\alpha = \alpha_c$, because of the new renormalization of α, the fermion composite operator $\bar{\psi}\psi$ acquires the dimension 2 instead of 3, and thus allows for the presence of renormalizable four-fermion interactions.

In quantum mechanics, there exists an alternative method for dealing with singular interactions of pointlike particles, without the need of introducing cut-offs: this is the self-adjoint extension method, discussed a long time ago for singular potentials of the type $1/r^n$ ($n \geq 2$) by Case.[19] In his classic paper, Case showed that all self-adjoint extensions of the Klein–Gordon–Coulomb (or Dirac–Coulomb) problem can be parametrized by a single constant B when the interaction becomes singular.

While Case's method is only of academic interest for problems concerning supercritical Coulomb fields of heavy nuclei, the size of the latter providing a natural short distance cut-off, it reveals its full power in the present problem of strong Coulomb interaction of pointlike particles. The self-adjoint extension parameter B can be interpreted as parametrizing the short distance behavior of the interaction and its choice amounts to fixing the energy of one of the bound states (the values of the masses of the constituent particles of the bound states and of the coupling constant α being already fixed) and calculating the other bound state energies with respect to this one, without making explicit cut-offs appear.

In the case of δ function interactions in two and three space dimensions, it was shown[20] that the self-adjoint extension method provides the renormalized version of the theory, when the cut-off of the regularized theory is removed and a corresponding renormalization of the coupling constant is performed. It is then natural to expect from the same method of approach, applied now to the $1/r^2$ singularity, to also provide the finite renormalized version of the theory, provided one of the bound state energies is fixed.

Motivated by these results, we have investigated with Case's method of self-adjoint extension the problem of strong Coulomb coupling in positronium-like systems. The relativistic wave equations of constraint theory, which, from now on, will be referred to as the Todorov equation, lead, for the relative motion in ${}^1S_0$ states to a final three-dimensional equation which is very similar in form to the Klein–Gordon (KG) equation,[9] and therefore Case's method can be readily applied to it.

Our main results are the following: We find that the system undergoes a first-order chiral phase transition at the critical value $\alpha = \alpha_c = 1/2$. While the ground state mass for $\alpha < 1/2$ can be continued to the domain $\alpha > 1/2$ and remains different from zero, a new set of an infinite number of states, concentrated in a finite domain of mass with accumulation at the value zero, appears, the zero mass state representing the new ground state of the system. This result occurs for any fixed value of Case's constant B. All the states have indefinitely oscillating wave functions near the origin. While tachyonic solutions formally exist, they are ruled out from the spectrum by the self-adjointness condition, and therefore the zero mass state remains the physical ground state of the spectrum.

In the limit $\alpha \to \alpha_c$ from above, the short-distance oscillations disappear from the wave functions, and the states accumulated around the zero mass solution shrink to a single massless state with a definite mass gap with the rest of the spectrum. It turns out that the latter state has the required properties to represent a Goldstone boson and hence to signal a spontaneous breakdown of chiral symmetry. The fact that for $\alpha > \alpha_c$ a sensible theory, with finite and nonvanishing couplings to the observable currents, can be defined only for $\alpha = \alpha_c + 0$ strongly suggests the possible identification of α_c with an ultraviolet stable fixed point of the theory, with the distinction between two phases, governed by $\alpha_c - 0$ and $\alpha_c + 0$, respectively, and joined to each other by a first-order chiral phase transition.

The paper is organized as follows: Section 2 discusses Case's method for Klein–Gordon particles in an external Coulomb field and is included to make the paper self-contained. Section 3 is devoted to the study, in the framework of RQCD, of the strongly coupled positronium spectrum in its pseudoscalar sector. Section 4 deals with the question of an eventual appearance of tachyonic states in the spectrum of states. In Sec. 5, the limit $\alpha \to \alpha_c + 0$ is considered and the presence of a Goldstone boson established. The summary and discussion of results follow in Sec. 6. In the appendix, some results of RQCD are summarized.

2. Case's Method for the Klein–Gordon Equation

For the sake of completeness and for easy comparison with RQCD, we discuss here Case's method[19] for a Klein–Gordon particle of charge e, mass M and energy E in a Coulomb potential $V(r) = -\alpha Z/r$. We have α=1/137 and $\hbar = c = 1$ in our system of units. The radial KG equation for s-states is

$$u'' + \left[E^2 - M^2 + \frac{2E\alpha Z}{r} + \frac{\alpha^2 Z^2}{r^2}\right] u = 0\,. \tag{1}$$

From Eq. (2.1) one derives the orthogonality condition for two solutions u_1 and u_2 of energy E_1 and E_2 ($E_1 \neq E_2$):

$$\begin{aligned}(u_1, u_2)_{\mathrm{KG}} &= \frac{1}{2}\int_0^\infty \left(E_1 + E_2 + \frac{2\alpha Z}{r}\right) u_1 u_2 \, dr \\ &= \frac{1}{2}(u_2' u_1 - u_1' u_2)\Big|_0^\infty = 0\,.\end{aligned} \tag{2}$$

Note further that the KG norm is given by

$$(u, u)_{\mathrm{KG}} = \int_0^\infty \left(E + \frac{\alpha Z}{r}\right) u^2 \, dr\,. \tag{3}$$

For $\alpha Z > 1/2$, the square integrable solutions of Eq. (2.1) (vanishing at $r = \infty$) are given by

$$u(E, r) = cW_{k,\mu}(\rho)\,, \tag{4}$$

where c is a normalization constant, $W_{k,\mu}$ is the Whittaker function[8] and where

$$\rho = 2(M^2 - E^2)^{\frac{1}{2}} r\,, \tag{5}$$

$$k = \frac{E\alpha Z}{(M^2 - E^2)^{\frac{1}{2}}}\,, \tag{6}$$

$$\mu = i\lambda\,, \quad \lambda = \left(\alpha^2 Z^2 - \frac{1}{4}\right)^{\frac{1}{2}}. \tag{7}$$

Using the formula:[21]

$$\lim_{\rho\to 0} W_{k,\mu}(\rho) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - k)}\,\rho^{\frac{1}{2}+\mu} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - k)}\,\rho^{\frac{1}{2}-\mu}\,, \tag{8}$$

one finds:

$$\lim_{\rho\to 0} u \sim \rho^{\frac{1}{2}} \cos(\beta + \lambda \ln \rho)\,, \tag{9}$$

with

$$\beta = \arg \frac{\Gamma(-2i\lambda)}{\Gamma(\frac{1}{2} - i\lambda - k)}\,. \tag{10}$$

It is easy to verify that the general behavior of the solution of Eq. (2.1) close to $r = 0$ is given by:[19]

$$\lim_{r\to 0} u \sim r^{\frac{1}{2}} \cos\left(\lambda \ln(Mr) + B\right), \tag{11}$$

where B is an arbitrary constant. Comparison of Eqs. (2.9) and (2.11) yields the relation:

$$\arg\Gamma(1 - 2i\lambda) + \frac{\pi}{2} - \arg\Gamma\left(\frac{1}{2} - i\lambda - k\right)$$

$$+ \lambda \ln 2 + \lambda \ln \frac{(M^2 - E^2)^{\frac{1}{2}}}{M} = B + n\pi\,, \tag{12}$$

where n is an arbitrary integer. Furthermore, it can be shown[19] that keeping the same value of B for all states guarantees the orthogonality condition (2.2). Therefore, choosing the value of B provides a self-adjoint extension of the KG equation and allows us to obtain the corresponding spectrum by solving Eq. (2.12) for E. Notice that this equation is invariant under adding to B any multiple of π and, therefore, it is sufficient to consider the values of B in the interval $[0, \pi]$.

It is of interest to investigate the small λ behavior of (2.12). Using the formula:[22]

$$\arg\Gamma(x + iy) = y\psi(x) + \sum_{n=0}^{\infty}\left(\frac{y}{x+n} - \arctan\frac{y}{x+n}\right), \tag{13}$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,, \tag{14}$$

one finds from Eq. (2.12):

$$-\,2\lambda\psi(1) + \frac{\pi}{2} + \lambda\psi\left(\frac{1}{2} - k\right) + \lambda \ln 2\sqrt{1 - \frac{E^2}{M^2}} + \mathcal{O}(\lambda^2) = B\,. \tag{15}$$

(Without loss of generality, we can drop the $n\pi$ term for small values of λ.) For this equation to be satisfied (for $B \neq \frac{\pi}{2}$), as $\lambda \to 0$, it is necessary that

$$\lim_{\lambda\to 0} \lambda\psi\left(\frac{1}{2} - k\right) = B - \frac{\pi}{2}\,, \tag{16}$$

or that

$$\lim_{\lambda\to 0} k = p + \frac{1}{2}\,, \tag{17}$$

where p is any nonnegative integer. Using Eq. (2.6), it is easy to see that one recovers the usual ($\ell = 0$) spectrum of the KG equation for αZ tending to 1/2 from below. This means that the spectrum is continuous through $\alpha Z = \frac{1}{2}$ for any value of $B \neq \frac{\pi}{2}$. However, the slope of the energy curves (in λ or αZ) is not continuous. For small values of λ, one can show that the eigenvalues correspond to the following behavior:

$$k(\lambda) \simeq p + \frac{1}{2} + \frac{\lambda}{B - \pi/2}\,. \tag{18}$$

On the other hand, for $B = \frac{\pi}{2}$, another state with no correspondence with the spectrum for $\alpha Z < \frac{1}{2}$ appears. Its energy is given, from Eq. (2.15), by the equation

$$\psi\left(\frac{1}{2} - k\right) + \ln 2\sqrt{1 - \frac{E^2}{M^2}} - 2\psi(1) = 0\,, \tag{19}$$

and has the value $E/M \simeq -0.049$.

We now turn to the question of a possible instability of the system, described by the KG equation (2.1) with a pointlike attractive Coulomb interaction, with respect to the spontaneous pair creation. This would correspond to the existence of a critical value of $Z = Z_c$ such that $E(Z_c) = -M$, or, according to Eq. (2.6), $k = -\infty$. To see whether such a solution of Eq. (2.12) exists for small values of λ, it is advantageous to transform the third term of Eq. (2.12) by using the following formula:[23]

$$\begin{aligned}\operatorname{Im} \ln \Gamma(x + iy) &= \arg \Gamma(x + iy) \\ &= \left(x - \frac{1}{2}\right) \arctan \frac{y}{x} + y\left\{ \ln x + \frac{1}{2} \ln \left[1 + \left(\frac{y}{x}\right)^2\right] - y \right. \\ &\quad \left. - \frac{y}{12(x^2 + y^2)} + \frac{1}{360} b - \frac{1}{1260}(ba_1 + ab_1) - \cdots \right\}\,,\end{aligned} \tag{20}$$

with

$$a = \frac{x}{x^2+y^2}a_1 - \frac{y}{x^2+y^2}b_1\,, \qquad b = \frac{y}{x^2+y^2}a_1 + \frac{x}{x^2+y^2}b_1\,,$$
$$a_1 = \frac{x^2-y^2}{(x^2+y^2)^2}\,, \qquad b_1 = \frac{2xy}{(x^2+y^2)^2}\,. \tag{21}$$

One finds, using Eq. (2.12) for large values of $|k|$ and Euler's constant $\gamma = -\psi(1) = 0.57721\ldots$, that the critical value of Z_c is given by

$$2\lambda_c\gamma + \lambda_c \ln(2\alpha Z_c) + \mathcal{O}(\lambda_c^2) = B - \frac{\pi}{2}\,, \tag{22}$$

where $\lambda_c = (\alpha^2 Z_c^2 - 1/4)^{1/2}$. Note the cancellation between the two terms containing $\ln(M^2 - E^2)$ in Eq. (2.12). From Eq. (2.22), we see that a critical value of $\alpha Z = \alpha Z_c > 1/2$ can only occur for $B \neq \frac{\pi}{2}$. Also note that λ_c is exactly 0 if $B = \frac{\pi}{2}$, and will be larger and larger if B is increasing. There is no solution of Eq. (2.22) (at least for small values of λ_c) for $B \lesssim \frac{\pi}{2}$, as the left-hand side should be positive. The above comments are illustrated in Figs. 1 and 2 which show respectively how the energy of the lower bound states vary with λ for $B = 1.56 < \frac{\pi}{2}$ and $B = 1.58 > \frac{\pi}{2}$ respectively. One can see that no instability will occur (i.e. no state with $E = -M$) for $\lambda < 2$ in the first case ($B < \frac{\pi}{2}$), while, for $B > \frac{\pi}{2}$, there is a state with energy $E = -M$ for $\lambda \simeq 0$. Furthermore, the solution given by Eq. (2.19) corresponds (for small λ) to the boundary case $B = \pi/2$ between the two domains $B < \pi/2$ and $B > \pi/2$.

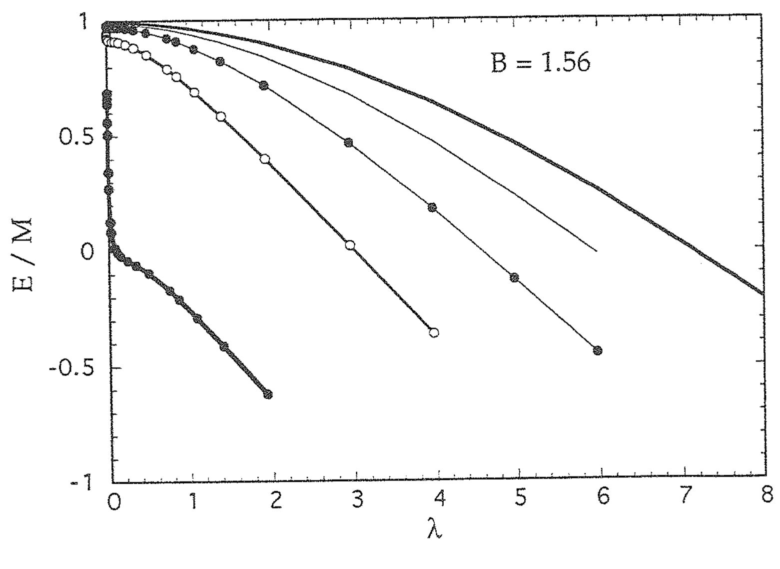

Fig. 1. $\ell = 0$ spectrum generated by Case's method for solving the KG equation (2.1) in the strong coupling case ($\alpha Z11/2$), with Case's constant $B = 1.56$. The variable λ is defined by Eq. (2.7).

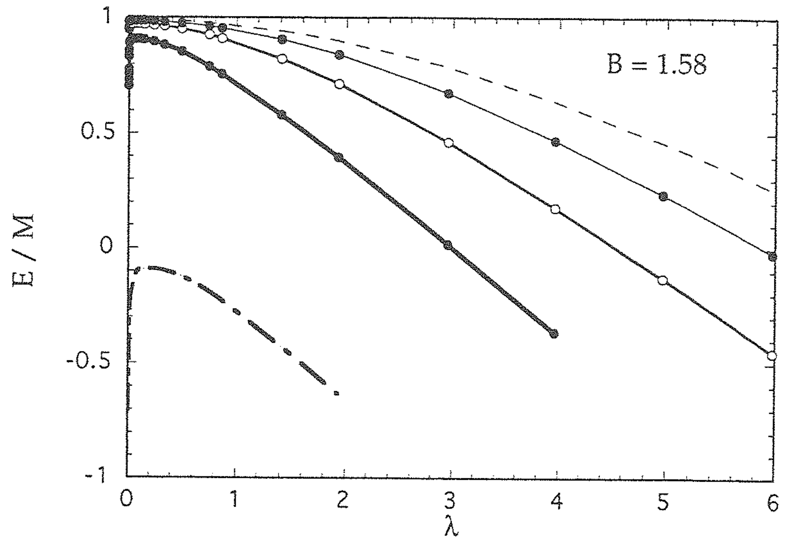

Fig. 2. Same as Fig. 1, with $B = 1.58$. The eigenvalues for all states, except the lowest one, are continuously continuing the eigenvalues for $\alpha Z < 1/2$.

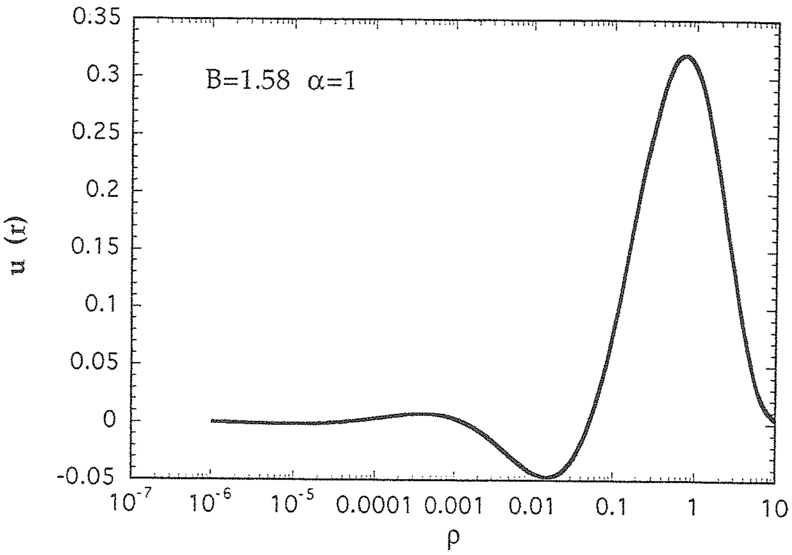

Fig. 3. Wave function of the lowest state generated by Case's method for solving the KG equation (2.1), with $\alpha Z = 1$ and Case's constant $B = 1.58$, corresponding to the dot–dashed curve in Fig. 2.

For the KG equation, the sign of the KG norm unambiguously distinguishes between particle and possible antiparticle bound states. We have calculated numerically the norm of the states of $E \simeq -M$ wave functions and established that they are indeed particle states. We further show in Fig. 3 the radial wave function corresponding to the lowest state for $\alpha Z = 1$ and $B = 1.58$ (the dot-dashed curve in Fig. 2). It, of course, displays an infinite number of nodes consistent with the behavior (2.9), but the outermost of these nodes occurs at a distance representing a rather small fraction (here, it is of the order of one tenth) of the "radius" of the state. Globally, the wave function has a "nodeless" structure, typical of a ground state wave function.

We thus find that the stability property with respect to pair creation of a Coulomb source with vanishing radius and charge $Z > 137/2$ crucially depends on the choice of the self-adjoint extension characterized by the constant B. In principle, this constant should be determined by comparison with experimental data. However, the case of physical interest corresponds here to a finite radius R of the source and has been extensively discussed in the literature.[24] We shall not pursue further the investigation of this academic problem. Let us simply note that the Coulomb field of a charge with radius R satisfying $MR \ll 1$ is supercritical for $\alpha Z > \alpha Z_c = 1/2$,[8] so that the choice $B > \pi/2$ would be mandatory to reproduce finite radii results.

3. The Todorov Equation

We now consider the problem of two particles of equal mass m and opposite charges with spin one-half, in mutual electromagnetic interaction. Within RQCD the $^1S_0$ states of the system with the Todorov choice of the interaction[13] are described in the c.m. frame by the radial equation[8,9] (see also the appendix):

$$\left[-\frac{d^2}{dr^2} + m_W^2 - \left(\varepsilon_W - A(r)\right)^2\right]\varphi = 0\,, \tag{23}$$

with

$$A(r) = -\frac{\alpha}{r}\,, \tag{24}$$

$$\varepsilon_W = \frac{W^2 - 2m^2}{2W}\,, \tag{25}$$

$$m_W = \frac{m^2}{W}\,, \tag{26}$$

W being the c.m. energy of the two-body system. Equation (3.1) correctly describes the physical positronium $^1S_0$ energy levels to order α^4,[10,11] with α being the fine structure constant. Our interest is in the solution of Eq. (3.1) for large arbitrary values of α. From now on, we shall consider α as a free parameter.

As Eq. (3.1) is very similar to the Klein–Gordon equation (2.1), Case's method can be readily applied to it to obtain its solutions in a similar way. From Eq. (3.1) we find, for two square integrable solutions φ_1 and φ_2 of energy W_1 and W_2 ($W_1 \neq W_2$), the orthogonality condition:

$$(\varphi_1, \varphi_2) = \frac{1}{2}(\varphi_2{}'\varphi_1 - \varphi_1{}'\varphi_2)\Big|_0^\infty = 0\,, \tag{27}$$

where the scalar product is now defined by:

$$(\varphi_1, \varphi_2) = \int_0^\infty \left[\frac{W_1 + W_2}{4} - A(r)\left(1 + \frac{2m^2}{W_1 W_2}\right)\right]\varphi_1\varphi_2\, dr\,. \tag{28}$$

The conserved norm N corresponding to Eq. (3.6) is given by:

$$N = \int_0^\infty \left(\frac{W}{2} - A(r)\left(1 + \frac{2m^2}{W^2}\right)\right)\varphi^2\, dr\,. \tag{29}$$

For $\alpha > 1/2$, the solutions of Eq. (3.1), vanishing at $r = \infty$, are given by:

$$\varphi(r) = cW_{\tilde{k},\mu}(\rho)\,, \tag{30}$$

with

$$\tilde{k} = \frac{\alpha\varepsilon_W}{(m_W^2 - \varepsilon_W^2)^{1/2}} = \frac{\alpha(W^2 - 2m^2)}{W(4m^2 - W^2)^{1/2}}\,, \tag{31}$$

$$\mu = i\lambda\,, \qquad \lambda = \left(\alpha^2 - \frac{1}{4}\right)^{1/2}\,, \tag{32}$$

$$\rho = 2Kr\,, \tag{33}$$

$$K = (m_W^2 - \varepsilon_W^2)^{1/2} = \frac{1}{2}(4m^2 - W^2)^{1/2}\,. \tag{34}$$

As $r \to 0$, the wave function $\varphi(r)$ exhibits the behavior described in Eqs. (2.8), (2.9) and (2.11) (with k replaced by $\tilde{k}$). Choosing the same value of B for all the wave functions guarantees the validity of the orthogonality condition (3.5) for $W_1 \neq W_2$. The corresponding energy spectrum is then given by a relation similar to Eq. (2.12):

$$\arg\Gamma(1 - 2i\lambda) + \frac{\pi}{2} - \arg\Gamma\left(\frac{1}{2} - i\lambda - \tilde{k}\right) + \lambda\ln\left(\frac{2K}{m}\right) = B + n\pi\,. \tag{35}$$

It is easy to see, using Eqs. (2.13) and (2.14), that, for small values of λ, there are solutions of Eq. (3.13) corresponding to

$$\lim_{\lambda\to 0}\tilde{k} = p + \frac{1}{2}\,, \tag{36}$$

where p is any nonnegative integer, meaning that the $\alpha < \frac{1}{2}$ spectrum[8] is extended continuously through $\alpha = \frac{1}{2}$.

On the other hand, Eq. (3.13) possesses a new set of solutions with energies located in the positive vicinity of zero. To exhibit them, we consider small values of λ and large negative values of $\tilde{k}$, corresponding to small and positive values of W [see Eqs. (3.9) and (3.12)]. Using Eqs. (2.20) and (2.21), we can rewrite Eq. (3.13) in this case as

$$-2\lambda + \lambda \ln\left(\frac{1}{2} - \tilde{k}\right) + \lambda \ln\left(\frac{2K}{m}\right) + \mathcal{O}(\lambda^2) = B - \frac{\pi}{2} + n\pi\,, \tag{37}$$

or, using Eqs. (3.9) and (3.12), as

$$-2\lambda - \lambda \ln\left(\frac{W}{m}\alpha\right) + \lambda \ln 2 + \mathcal{O}(\lambda^2) = B - \frac{\pi}{2} + n\pi\,. \tag{38}$$

There are thus solutions of vanishing W as $\lambda \to 0$:

$$-\lambda \ln\left(\frac{W}{m}\alpha\right) \to B - \frac{\pi}{2} + n\pi\,, \tag{39}$$

i.e.

$$\frac{W}{m} \simeq \alpha^{-1} e^{-(B+n\pi-\pi/2)/\lambda}\,, \tag{40}$$

where $n = 0, 1, 2, \ldots$ for $B > \pi/2$ and $n = 1, 2, 3, \ldots$ for $B \leq \pi/2$ (the reason being that the right-hand side of Eq. (3.17) should be positive for small values of W).

The absence of these new types of solution in the Klein–Gordon equation case is due essentially to the fact that in Eq. (2.12) the term $\ln((M^2 - E^2)^{1/2}/M)$ cancels, in the limit $E \to -M$, the contribution coming from the large negative values of k; no such cancellation exists in the present case, for the equivalent parameter K [Eq. (3.12)] remains finite and different from zero in the limit $W \to 0$.

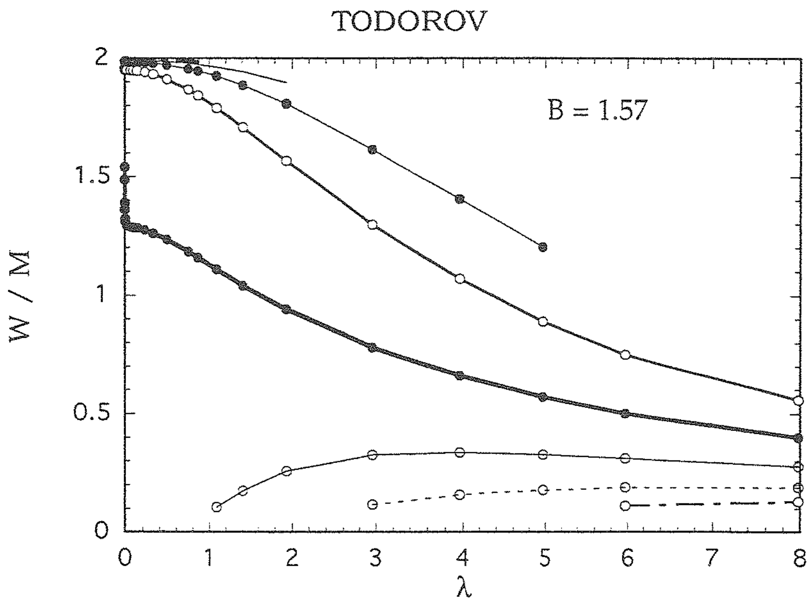

Fig. 4. $\ell = 0$ spectrum generated by Case's method for solving the RQCD equation (3.1) in the strong coupling case (α11/2) and for Case's constant $B = 1.57$. The λ parameter is defined by Eq. (3.10). All curves above and including the heavy bold curve are connected continuously to the spectrum obtained for α tending to 1/2 from below. As indicated in the text, all the other curves are merging to the origin of the axes as $\lambda \to 0$ (only the explicitly calculated parts are shown).

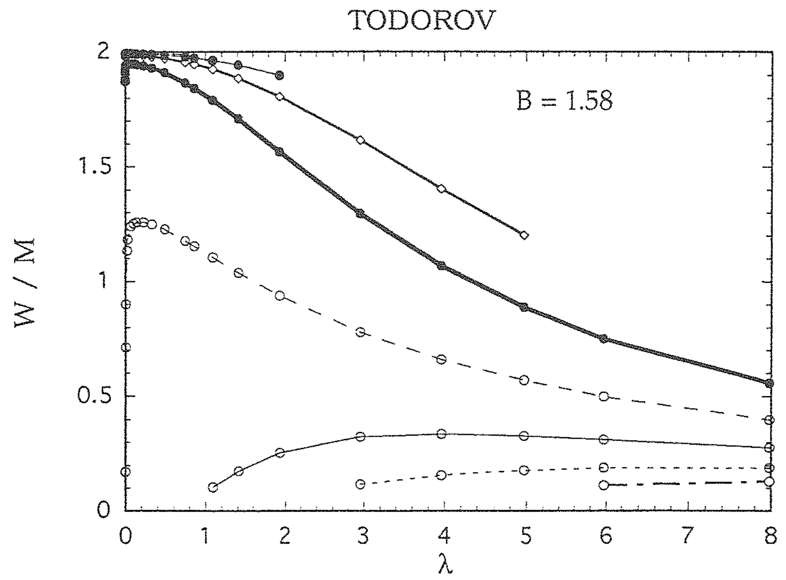

Fig. 5. Similar results as in Fig. 4, but for $B = 1.58$. The same conventions have been used.

Equation (3.18) shows that, for a fixed value of B and sufficiently small λ, there is an infinite number of states concentrated in a narrow band of energy and having the zero energy as an accumulation point. Clearly, the zero energy state is the new ground state of the bound state spectrum. The energy spectrum given by Eq. (3.13) is plotted in Figs. 4 and 5 and illustrates the above comments again for values of $B < \pi/2$ (Fig. 4) and $B > \pi/2$ (Fig. 5). Only a few states corresponding to Eq. (3.18) are shown.

4. The Fate of Tachyons

To complete the analysis of the energy spectrum, we discuss in this section the question of the status of tachyonic states. The presence of such states in the spectrum of physical states would invalidate the conclusions of Sec. 3, since the zero energy state would no longer represent the ground state of the spectrum, the latter then displaying rather the characteristics of an unstable vacuum.

Tachyons are characterized by negative values of W^2, i.e. by imaginary values of W. Inspection of Eqs. (3.9) and (3.12) shows that in this case $\tilde{k}$ becomes imaginary, while K remains real.

Equation (3.1) still has the Whittaker functions $W_{\tilde{k},\mu}$ as normalizable solutions. We define

$$\tilde{k} = i\eta\,, \tag{41}$$

and examine the behavior of the wave function near the origin; it is given by Eq. (2.8):

$$W_{\tilde{k},\mu} \simeq \frac{\Gamma(-2i\lambda)}{\Gamma(\frac{1}{2} - i\lambda - i\eta)} \rho^{\frac{1}{2}+i\lambda} + \frac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2} + i\lambda - i\eta)} \rho^{\frac{1}{2}-i\lambda} . \tag{42}$$

In order to display the main qualitative features of this type of solution, we consider, in the following, large values of $|\eta|$, while keeping λ small. Then, the functions $\Gamma(\frac{1}{2} \pm i\lambda - i\eta)$ can be expanded in terms of $i\lambda$:

$$\Gamma\left(\frac{1}{2} \pm i\lambda - i\eta\right) \simeq \Gamma\left(\frac{1}{2} - i\eta\right) e^{\pm(i\lambda \ln|\eta| + \epsilon(\eta)\lambda\pi/2)} , \tag{43}$$

[$\epsilon(\eta)$ is the sign of η,] leading in turn to the following behavior of $W_{\tilde{k},\mu}$:

$$W_{\tilde{k},\mu}(\rho) \simeq \frac{1}{\Gamma(\frac{1}{2} - i\eta)} \{\Gamma(-2i\lambda) e^{i\lambda \ln|\eta| + \epsilon(\eta)\lambda\pi/2} \rho^{\frac{1}{2}+i\lambda} + \Gamma(2i\lambda) e^{-i\lambda \ln|\eta| - \epsilon(\eta)\lambda\pi/2} \rho^{\frac{1}{2}-i\lambda}\} . \tag{44}$$

The wave function φ has the same formal behavior as in Eqs. (2.9) or (2.11), with β defined as

$$\beta = \arg\Gamma(-2i\lambda) + \lambda \ln|\eta| - i\epsilon(\eta)\lambda\frac{\pi}{2} . \tag{45}$$

Notice that β, and hence B, is complex.

In order to study the orthogonality conditions for these states, we observe that when tachyons exist, there is a doubling of states: for each "eigenvalue" $i\eta$, the value $-i\eta$ is also a solution. The rules of constructing scalar products for such states have been studied in the two papers of Ref. 25. For a state with complex "eigenvalue" $i\eta$ one also considers the state with the complex conjugate "eigenvalue" $-i\eta$, called the associated vector. While the norm of a state with complex eigenvalue is zero, its scalar product with its associated vector is nonzero.[25] Therefore, the following rule should be adopted for the choice of admissible states. Define admissible states as those corresponding to a definite sign of η in the "eigenvalues" $i\eta$. Then, the corresponding adjoint states in the scalar product are the associated vectors (with "eigenvalues" $-i\eta$); these appear there complex-conjugated.

The analysis of Case can then be repeated for the tachyonic states. Once the same complex value of B is chosen for the tachyonic solutions, then these will satisfy among themselves the orthogonality conditions (3.5) and might constitute admissible states. [Notice that associated vectors cannot be considered as admissible states, for, according to Eq. (4.5), they would not have the same value of B; they appear only as the adjoints of admissible states.] However, the orthogonality condition (3.5) fails when considered between a tachyonic state and a "normal" state (with $W^2 > 0$ and $W > 0$), because the coefficient B cannot be chosen as the same for both types of solution (it must be complex for the former and real for the latter). This is the reason why tachyons are rejected from the Hilbert space of physical states.

Therefore, we end up with the conclusion that the physical spectrum is free of tachyons and its ground state is the zero energy state.

5. The Limit $\alpha \to \frac{1}{2} + \epsilon$ and Spontaneous Chiral Symmetry Breaking

The occurrence of a zero mass ground state, with the quantum numbers of a pseudoscalar boson, in the bound state spectrum is suggestive of spontaneous chiral symmetry breaking. However, the oscillatory nature of the wave functions near the origin makes it difficult to define their couplings to the axial vector current. Furthermore, the accumulation of an infinite number of states around the zero mass state [Eq. (3.18)] does not allow the disentanglement of the ground state from the rest of the states of its neighborhood.

Within these circumstances, the limiting value $\alpha = \frac{1}{2} + \epsilon$ ($\epsilon = +0$) plays a particular role for a physical interpretation of the theory. In this limit, the oscillating behaviors of the type (2.9) or (2.11) disappear from the wave functions and, according to Eq. (3.18), all the low mass states shrink to a single state with zero mass. A definite mass gap appears between the zero mass ground state and the other massive states of the spectrum.

To study in more detail the properties of the corresponding wave functions, it is preferable to reanalyze Eq. (3.1) for the particular value $\alpha = \frac{1}{2}+\epsilon$. The normalizable solutions are the Whittaker functions $W_{\tilde{k},0}$, which behave near the origin as

$$W_{\tilde{k},0}(\rho) \simeq -\frac{(2\rho)^{1/2}}{\Gamma\left(-\tilde{k}+\frac{1}{2}\right)}\left(\ln 2\rho + \psi\left(-\tilde{k}+\frac{1}{2}\right) - 2\psi(1)\right), \tag{46}$$

the various parameters and variables being already defined in Eqs. (3.9)–(3.12), while the general behavior of the solutions of Eq. (3.1) is

$$\varphi \sim r^{1/2}\big(a\ln(mr) + b\big), \tag{47}$$

where a and b are constants.

The orthogonality condition (3.5) requires from the admissible normalizable solutions to satisfy the condition

$$\frac{b}{a} = A, \tag{48}$$

A being the same (arbitrary) constant for all the solutions. Equation (5.1) then yields the eigenvalue equation:

$$\ln\left(\frac{2K}{m}\right) + \psi\left(-\tilde{k}+\frac{1}{2}\right) - 2\psi(1) = A. \tag{49}$$

In order that the corresponding solutions be the limits of those found for $\alpha > 1/2$ [Eqs. (3.14) and (3.18)], it is necessary that the constant A equal $+\infty$. One therefore finds again the solutions (3.14), as well as the additional single solution $W = 0$, corresponding to $\tilde{k} \to -\infty$.

The infinite value of A means that for the solutions of the type (3.14) the logarithmic piece in Eq. (5.2) is absent and the corresponding wave functions behave as $r^{1/2}$ near the origin. This is not true for the solution corresponding to $W = 0$.

for the limit $\tilde{k} \to -\infty$ cannot be straightforwardly taken in the Whittaker function $W_{\tilde{k},0}$. To analyze more accurately the properties of the corresponding wave function, we consider small values of W ($W = W_0 \simeq 0$), make in Eq. (3.1) the change of variable

$$y = \frac{m^2 r}{W_0}, \tag{50}$$

and keep only leading terms, it being understood that the limit $W_0 \to 0$ should be taken at the end of calculations of physical quantities. Equation (3.1) becomes at leading order in W_0

$$\frac{d^2}{dy^2}\varphi - \frac{1}{y}\varphi + \frac{1}{4y^2}\varphi = 0, \tag{51}$$

the solution of which is

$$\varphi_0 = c_0 m y^{\frac{1}{2}} e^{-2y^{1/2}} \Psi\left(\frac{1}{2}, 1, 4y^{1/2}\right), \tag{52}$$

where c_0 is a dimensionless normalization constant and Ψ (also denoted by U in the literature) is the confluent hypergeometric function which behaves at infinity with a power law;[21] its dominant behavior near the origin is given by

$$\lim_{x\to 0} \Psi\left(\frac{1}{2}, 1, x\right) = -\frac{1}{\Gamma(\frac{1}{2})}\left(\ln x + \psi\left(\frac{1}{2}\right) - 2\psi(1)\right). \tag{53}$$

The solution (5.7) is a nodeless function and represents the ground state of the spectrum. [The function $\Psi(a, c; x)$ does not have positive zeros for a and c real and either $a > 0$ or $a - c + 1 > 0$. Also, it can be checked that the formal tachyonic solutions, found in Sec. 4, are absent in the present case.] It has a distribution-like behavior, due to the fact that it is defined by the limiting procedure $W_0 \to 0$: taking the limit $W_0 \to 0$, while keeping r fixed, shows that the wave function is actually peaked at the origin (recall that the complete wave function is φ/r).

To compute the normalization constant of this wave function (as well as of the others), it is necessary to reconstruct the whole sixteen-component spinor wave function and to use its relationship with the Bethe–Salpeter wave function, which ultimately fixes the normalization coefficients. Some details of these calculations can be found in the appendix. One finds for the norm N of Eq. (3.17) the expression:

$$N = \frac{W^2}{8\pi}, \tag{54}$$

where W is the mass of the bound state.

As to the massless state, after using the change of variable (5.5) in Eq. (3.7), we find that the dominant contribution for small W comes from the third term in the integral. This leads to the following behavior of the normalization constant c_0 of the wave function $\varphi_0(y)$ [Eq. (5.7)] for small W:

$$c_0 \sim W_0^2/m^2. \tag{55}$$

We now turn to the calculation of the coupling constants of the bound states to the axial vector current. These are defined as

$$\langle 0|j^R_{\mu 5}(0)|P\rangle = P_\mu F\,, \tag{56}$$

where P is the four-momentum of the pseudoscalar state and $j^R_{\mu 5}$ is the renormalized axial vector current. In general, in the absence of anomalies, the axial vector current undergoes only a finite multiplicative renormalization by radiative corrections.[26] This feature is, however, the result of compensating contributions from propagator and vertex renormalizations (with factors Z_2 and Z_A^{-1}, respectively). More explicitly, one has

$$j_{\mu 5} = Z_2 Z_A^{-1} j^R_{\mu 5}\,, \tag{57}$$

where $j_{\mu 5}$ is the unrenormalized current, with $Z_2 Z_A^{-1}$ finite.

From operator product expansion[27] and renormalization group analysis[28] one finds:

$$\langle 0|j^R_{05}|P\rangle = -Z_A \text{ Tr } \gamma_0\gamma_5\phi_{\text{BS}}(x)\big|_{x\to 0}\,, \tag{58}$$

where ϕ_{BS} is the Bethe–Salpeter wave function. Using the relationship of the constraint theory wave function ψ with the Bethe–Salpeter wave function ϕ_{BS} [Eqs. (A.6)–(A.8)] one obtains (in the c.m. frame):

$$\langle 0|j^R_{05}|P\rangle = -Z_A \text{ Tr } \gamma_0\gamma_5\left(1+\frac{2\alpha}{Wr}\right)^{-1/2}\psi(r)\bigg|_{r\to 0}\,, \tag{59}$$

$$WF = 2Z_A\left(1+\frac{2\alpha}{Wr}\right)^{-1/2}\text{Tr } \psi_4(r)\bigg|_{r\to 0} = Z_A\frac{8m}{Wr}\varphi(r)\bigg|_{r\to 0}\,. \tag{60}$$

The renormalization constant Z_A should render finite the physical coupling constants F. Let Λ be the ultraviolet cut-off (in momentum space) of the four-dimensional theory and let r_0 be the corresponding short-distance cut-off (in x-space) of the three-dimensional theory. We shall admit the weak relation

$$\lim_{\Lambda\to\infty} r_0(\Lambda) = 0\,, \tag{61}$$

and shall transpose several known qualitative results of the four-dimensional theory into the three-dimensional one. When $r_0 \neq 0$, the electron has a bare mass $m_0(r_0)$ that vanishes with r_0 with some power ν:[2,4]

$$\lim_{r_0\to 0} m_0(r_0) \sim m(mr_0)^\nu\,,\quad \nu > 0\,. \tag{62}$$

For $m_0 \neq 0$, the Ward identities of the axial vector current imply that the Goldstone boson acquires a mass W_0, which behaves in terms of m_0 as[5,15,29]

$$\lim_{r_0\to 0} W_0^2(r_0) \sim m_0(r_0)m\,. \tag{63}$$

We shall assume that $\nu < 2$; because of Eq. (5.18), it is only in this case (including eventually the limiting case $\nu = 2$) that the wave function (5.7) can be consistently defined. As a matter of rough comparison, the analog of ν in the four-dimensional theory, calculated at the two loop level, is equal to $3\alpha/(2\pi)+(3/4)(\alpha/(2\pi))^2 \simeq 0.25$ for $\alpha = 1/2$.[2,4]

We designate by W_1 and F_1 the mass and coupling constant of a massive state of the bound state spectrum and by W_0 and F_0 the similar quantities of the ground state (the Goldstone boson) when $r_0 \neq 0$. Taking into account the behaviors of the corresponding wave functions near the origin [$\varphi_1 \sim mc_1(mr)^{1/2}$, $\varphi_0 \sim mc_0(m^2r/W_0)^{1/2}\ln(m^2r_0/W_0)$], Eq. (5.15) leads for the two cases to the following equations:

$$F_1 = Z_A\frac{8m^3c_1}{W_1^2(mr_0)^{1/2}}, \tag{64}$$

$$F_0 = Z_A\frac{8m^3c_0\ln(m^2r_0/W_0)}{W_0^2(W_0r_0)^{1/2}}. \tag{65}$$

In Eq. (5.19), c_1 is a nonvanishing normalization constant in the limit $r_0 \to 0$, while the behavior of c_0 in Eq. (5.20) is given by Eq. (5.10). In order to maintain finite the value of F_0 in the limit $r_0 \to 0$, we must have:

$$Z_A \sim (W_0r_0)^{1/2}/\ln(mr_0)^{-1} \sim (mr_0)^{\frac{1}{2}+\frac{\nu}{4}}/\ln(mr_0)^{-1}. \tag{66}$$

Replacing Z_A in Eq. (5.19) yields $F_1 = 0$. Therefore we obtain, when $r_0 \to 0$, the following behaviors of the coupling constants:

$$F_0 \neq 0, \qquad F_1 = 0. \tag{67}$$

These are precisely the complementary conditions for having spontaneous breakdown of chiral symmetry: only the Goldstone boson couples to the axial vector current.

The behavior of Z_A when $r_0 \to 0$, given by Eq. (5.21), is in qualitative agreement with its behavior in the four-dimensional theory: in the Feynman gauge, Z_2, and hence Z_A, vanishes when the ultraviolet cut-off Λ goes to infinity.[4]

We can also calculate the matrix elements of the divergence operator $\partial^\mu j_{\mu5}$; actually, this should only lead to a check of the covariance property of the formalism. We find:

$$\begin{aligned}\langle 0|\partial^\mu j_{\mu5}^R|P\rangle &= -iW^2F = -Z_A 2im\left(1+\frac{2\alpha}{Wr}\right)^{-1}\mathrm{Tr}\ \gamma_5\psi(r)\Big|_{r\to0}\\ &= -Z_A 4im\left(1+\frac{2\alpha}{Wr}\right)^{-1}\mathrm{Tr}\ \psi_3(r)\Big|_{r\to0} = -Z_A 8im\frac{\varphi}{r}\Big|_{r\to0},\end{aligned} \tag{68}$$

which yields back Eq. (5.15). [In obtaining the above results, the wave equations of ψ have been used; similar calculations can be found (see the second paper of

Ref. 11). According to Eqs. (5.22), for all states of the bound state spectrum, the matrix elements (5.23) vanish, thus ensuring axial vector current conservation.

We end up with the conclusion that, in the limit $\alpha = \frac{1}{2} + \epsilon$, the theory displays the features of spontaneous chiral symmetry breaking: (i) the presence of a zero mass ground state in the spectrum, with a mass gap with the rest of the bound states; (ii) a nonvanishing coupling of the Goldstone boson to the axial vector current with a decoupling of the massive states from the latter.

For completeness, let us also describe the situation that results from the limit $\alpha = \frac{1}{2} - \epsilon$. For $\alpha < \frac{1}{2}$, the positronium spectrum has the usual structure,[8] the same as for small α, without massless bound states, and the limit $\alpha \to \frac{1}{2}$ from below does not introduce any qualitative changes, the ground state remaining massive. In this case the renormalization constant Z_A is determined from the finiteness of F_1 [Eq. (5.19)]. The latter should be different from zero; otherwise, the axial vector current would be conserved and, in the absence of a Goldstone boson, chiral symmetry would be realized through its normal mode, implying a parity doubling of degenerate states; this is not realized in the bound state spectrum; furthermore, in QED, with only electrons as massive fermions, this also is not possible. We therefore conclude that for $\alpha = \frac{1}{2} - \epsilon$, the axial vector current is not conserved and chiral symmetry is explicitly broken by the electron mass.

6. Summary and Discussion

We applied Case's method of self-adjoint extension of singular potentials, to the study of strong Coulomb coupling in the pseudoscalar sector, in the framework of the Todorov equation. We found that, as the coupling constant α increases, the bound state spectrum undergoes, at the critical value $\alpha = \alpha_c = 1/2$, an abrupt qualitative change. For $\alpha > \alpha_c$, the mass spectrum displays, in addition to the existing states for $\alpha < \alpha_c$, a new set of an infinite number of bound states, concentrated in a narrow band starting at mass $W = 0$. The bound states have indefinitely oscillating wave functions near the origin.

In the limit $\alpha \to \alpha_c$ from above, the short-distance oscillations disappear and the states accumulated around the zero mass state, shrink to a single massless state, representing the ground state of the spectrum, with a definite mass gap with the rest of the states. This state has the required properties to represent a Goldstone boson and hence it signals a transition to a new phase where chiral symmetry is spontaneously broken. It is tempting to relate this feature to the expected possibility from the existence of an ultraviolet stable fixed point in QED and is therefore suggestive of an identification of the critical value α_c with the Gell-Mann–Low eigenvalue α_0. The fact that the two boundary values $\alpha_c - \epsilon$ and $\alpha_c + \epsilon$ correspond to different phases, the former governing a phase where chiral symmetry is broken by the electron mass term and the latter governing a phase where chiral symmetry is spontaneously broken, necessitates the introduction of a similar distinction for α_0, with boundary values $\alpha_0 - \epsilon$ and $\alpha_0 + \epsilon$, with the search for the relevant domains in the theory.

In the model potential we were considering, the contribution of the one-photon exchange diagram, besides kinematic factors, is represented in the three-dimensional theory by α/r, which means that no distinction was made between the large- and short-distance behaviors of the effective charge (the physical coupling constant α being, in general, determined from the large distance behavior of the photon propagator). In order to be able to determine the domains of each of the above phases with respect to the values of the physical coupling constant, we define the effective charge in four-dimensional momentum space[1–3] as $\alpha_{\text{eff}}(-q^2/m^2) \equiv \alpha d(-q^2/m^2, \alpha)$; d is the Lorentz invariant part of the transverse part of the photon propagator multiplied by q^2; α is the physical coupling constant, measured at large distances: $\alpha_{\text{eff}}(0) = \alpha$. The asymptotic value of α_{eff} is α_0 $[\alpha_{\text{eff}}(\infty) = \alpha_0]$, at which value the Gell-Mann–Low function ψ vanishes. Because of the positivity of the photon two-point spectral function, one has in general the inequality $\alpha_{\text{eff}}(-q^2/m^2) < \alpha_0$ for $q^2 < 0$,[3] which implies in particular that $\alpha < \alpha_0$.

Another particular value of α is provided by the zero of the Callan–Symanzik function β, which we denote by α_1, with $\alpha_1 < \alpha_0$, satisfying $\alpha_1 d^{as}(1, \alpha_1) = \alpha_0$, where d^{as} is the asymptotic part of d.[3] If radiative corrections are estimated to be of the order of α/π in general, then, for $\alpha \sim \alpha_0 = 1/2$, they are of the order of 20% and α_1 should be of the order of 0.4. It was shown by Adler[3] that, according to the ways of summing diagrams, either α_1 or α_0 are essential singularities for the corresponding defining functions (β or ψ). The value α_1 appears then as a natural separation point between two subdomains in the domain of variation $0 < \alpha < \alpha_0$. The following scheme might provide a possible description of the conditions of occurrences of each of the phases mentioned above.

When $0 < \alpha < \alpha_1$, the asymptotic behavior of the photon propagator is governed by $\alpha_0 - \epsilon$ and we are in the phase where chiral symmetry is broken by the electron mass. In this phase, the renormalized vertex function $m\bar{\Gamma}^5$ (defined in Ref. 5), corresponding to the divergence of the axial vector current, is different from zero and the axial vector current is not conserved (in the absence of anomalies). The structure of the two-body spectrum is the same as for small values of α, with a massive ground state.

When α jumps from $\alpha_1 - \epsilon$ to $\alpha_1 + \epsilon$, this induces, through the relationship between the β and ψ functions, a similar jump of the bare coupling constant from $\alpha_0 - \epsilon$ to $\alpha_0 + \epsilon$ and we enter the phase where chiral symmetry is spontaneously broken. In this phase $m\bar{\Gamma}^5$ is identically zero and the axial vector current is conserved. The two-body spectrum has now, in addition to the existing states for $\alpha < \alpha_1$, a massless ground state.

When $\alpha > \alpha_0$, because of the positivity condition already mentioned, the unitarity of the theory breaks down.

Our conclusions also join those obtained by Miransky *et al.*[16] from the Bethe–Salpeter equation, who conjectured that the critical value α_c could be identified with the Gell-Mann–Low eigenvalue. The difference in the numerical values of α_c

found in the two approaches ($\alpha_c = 1/2$ here and $\alpha_c = \pi/4$ in the ladder approximation of the Bethe–Salpeter equation in the Feynman gauge) is presumably related to the different approximations used in the kernels of the bound state wave equations. The Todorov potential, used in the present approach, takes into account multi-photon exchange diagrams and correctly reproduces the physical positronium and muonium spectra to order α^4;[10,11] this is not the case for the ladder approximation of the Bethe–Salpeter equation in covariant gauges. Let us nevertheless stress once again that these conclusions remain conjectural when applied to positronium in strongly coupled QED, as we have ignored so far vacuum polarization and radiative correction diagrams.

In the course of the present analysis, the effects of anomalies were ignored; these are known to modify the Ward identities of the axial vector current.[30] However, in QED, it turns out that these effects disappear at zero momentum transfer[31] and hence they do not seem to be able to give a mass to the Goldstone boson, when the latter exists. It is only in non-Abelian gauge theories that nonperturbative effects, like those of instantons, succeed, through the anomalous Ward identities, in providing the Goldstone boson with a mass.[32]

Finally, a comment on the structure of the Goldstone boson is in order. This state, in the present mechanism of chiral symmetry breaking, is not of the same nature as that of the massive states. In particular, it does not result from the continuous decrease of the mass of a massive state down to zero, when the physical coupling constant increases, but rather appears abruptly as a new type of solution when the physical coupling constant exceeds a critical value.

Acknowledgments

The work of M. B. was supported by the National Fund for Scientific Research, Belgium.

Appendix

The wave equations of constraint theory for a fermion–antifermion system can be written in the form[11]

$$(\gamma_1 \cdot p_1 - m_1)\tilde{\Psi} = (-\gamma_2 \cdot p_2 + m_2)\tilde{V}\tilde{\Psi}\,, \tag{A.1a}$$

$$(-\gamma_2 \cdot p_2 - m_2)\tilde{\Psi} = (\gamma_1 \cdot p_1 + m_1)\tilde{V}\tilde{\Psi}\,, \tag{A.1b}$$

where $\tilde{\Psi}$ is a sixteen-component spinor wave function of rank two and is represented as a 4×4 matrix; the Dirac matrices γ_2 act on $\tilde{\Psi}$ from the right. The compatibility condition of the two equations (A.1) allow one to eliminate the relative time variable and to define an internal three-dimensional wave function.

After using the parametrization

$$\tilde{V} = \tanh V \tag{A.2}$$

and making the change of function

$$\tilde{\Psi} = (\cosh V)\Psi\,, \tag{A.3}$$

the norm of the internal three-dimensional wave function, denoted by ψ, becomes (in the c.m. frame):

$$\int d^3\mathbf{x}\ \mathrm{Tr}\left\{\psi^\dagger\left[1 + 4\gamma_{10}\gamma_{20}P_0^2\frac{\partial V}{\partial P^2}\right]\psi\right\} = 2P_0\,, \tag{A.4}$$

where P_μ is the total four-momentum of the system.

In this representation the Todorov potential[13] takes the form (in the Feynman gauge):

$$V = \gamma_1\cdot\gamma_2\frac{1}{4}\ln\left(1+\frac{2\alpha}{Wr}\right)\,, \qquad W = \sqrt{P^2}\,. \tag{A.5}$$

Equations (A.1) can be solved by first decomposing ψ (the internal part of Ψ) on the basis of the matrices 1, γ_0, γ_5 and $\gamma_0\gamma_5$:

$$\psi = \psi_1 + \gamma_0\psi_2 + \gamma_5\psi_3 + \gamma_0\gamma_5\psi_4\,, \tag{A.6}$$

with ψ_i $(i = 1,\ldots,4)$ considered as 2×2 matrices in the spin subspace. The relationships of these components with the wave function φ used throughout the text are (for the equal mass case and the quantum numbers $s=0$, $\ell=0$, $j=0$):

$$\begin{aligned} \psi_1 &= \frac{2}{W}(\mathbf{s}_1-\mathbf{s}_2)\cdot\mathbf{p}\frac{\varphi}{r}\,, & \psi_2 &= 0\,,\\ \psi_3 &= \left(1+\frac{2\alpha}{Wr}\right)\frac{\varphi}{r}\,, & \psi_4 &= \frac{2m}{W}\left(1+\frac{2\alpha}{Wr}\right)^{1/2}\frac{\varphi}{r}\,, \end{aligned} \tag{A.7}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are the spin operators of particles 1 and 2 in the 2×2 component subspace of the ψ_i's.

In perturbation theory, a relationship can be established by means of an iterative series, between Eqs. (A.1) and the Bethe–Salpeter equation.[12] In general, the potential $\tilde{V}$ is a three-dimensional nonlocal operator in x-space, but becomes a local function when appropriate approximations are used. In particular, when the nonlocal operator $(m^2+\mathbf{p}^2)^{-1/2}$ is replaced by a mean value like $(m^2+\langle\mathbf{p}^2\rangle)^{-1/2}$, $\tilde{V}$ becomes local in $\mathbf{x}$ (in the c.m. frame) and dependent on $(m^2+\langle\mathbf{p}^2\rangle)^{-1/2}$.

The Todorov potential (A.5) results, however, from a slightly different approximation: it is a function of $\mathbf{x}$ and $W/2$, rather than of $\mathbf{x}$ and $(m^2+\langle\mathbf{p}^2\rangle)^{-1/2}$. One should then replace the latter quantity by $W/2$. It turns out that this approximation provides even better results, since the Todorov potential reproduces the correct spectrum to order α^4 for positronium and muonium.[10,11] In this approximation, at zero relative time ($x^0=0$), the relationship between the Bethe–Salpeter wave function ϕ_{BS} and ψ takes the form:

$$\phi_{\rm BS}(x^0 = 0, \mathbf{x}) = e^{\gamma_{10}\gamma_{20}V}\psi(\mathbf{x})\,, \tag{A.8}$$

where V is given in Eq. (A.5). The normalization constant in the right-hand side of Eq. (A.4) takes account of this relationship.

References

1. M. Gell-Mann and F. E. Low, *Phys. Rev.* **95**, 1300 (1954).
2. K. Johnson, M. Baker and R. Willey, *Phys. Rev.* **136**, B1111 (1964); K. Johnson, R. Willey and M. Baker, *Phys. Rev.* **161**, 1699 (1967); M. Baker and K. Johnson, *Phys. Rev.* **183**, 1292 (1969); *Phys. Rev.* **D3**, 2516 (1971).
3. S. Adler, *Phys. Rev.* **D5**, 3091 (1972).
4. S. L. Adler and W. A. Bardeen, *Phys. Rev.* **D4**, 3045 (1971); **6**, 734(E) (1972).
5. H. Pagels, *Phys. Rev. Lett.* **28**, 1482 (1972); *Phys. Rev.* **D7**, 3689 (1973).
6. J. Bartholomew *et al.*, *Nucl. Phys.* **B230**, 222 (1984).
7. M. Göckeler *et al.*, *Nucl. Phys.* **B371**, 713 (1992).
8. M. Bawin and J. Cugnon, *Phys. Lett.* **B246**, 1 (1990).
9. M. Bawin, J. Cugnon and H. Sazdjian, *Int. J. Mod. Phys.* **A9**, 5711 (1994).
10. H. W. Crater and P. Van Alstine, *Ann. Phys. (N.Y.)* **148**, 57 (1983); *Phys. Rev.* **D36**, 3007 (1987); H. W. Crater, R. I. Becker, C. Y. Wong and P. Van Alstine, *Phys. Rev.* **D46**, 5117 (1992).
11. H. Sazdjian, *Phys. Rev.* **D33**, 3401 (1986); *ibid.* 3425; *J. Math. Phys.* **29**, 1620 (1988); J. Mourad and H. Sazdjian, *J. Math. Phys.* **35**, 6379 (1994).
12. H. Sazdjian, *J. Math. Phys.* **28**, 2618 (1987); in *Extended Objects and Bound Systems — Proc. Karuizawa Int. Symp.* eds. O. Hara, S. Ishida and S. Naka (World Scientific, Singapore, 1992), p. 117.
13. I. T. Todorov, *Phys. Rev.* **D3**, 2351 (1971); in *Properties of Fundamental Interactions*, eds. A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, Part C, p. 951.
14. T. Maskawa and H. Nakajima, *Prog. Theor. Phys.* **52**, 1326 (1974); **54**, 860 (1975); R. Fukuda and T. Kugo, *Nucl. Phys.* **B117**, 250 (1976).
15. K. Higashijima, *Prog. Theor. Phys. Suppl.* **104**, 1 (1991).
16. P. I. Fomin, V. P. Gusynin and V. A. Miransky, *Phys. Lett.* **B78**, 136 (1978); V. A. Miransky, *Phys. Lett.* **B91**, 421 (1980); V. A. Miransky and P. I. Fomin, *Sov. J. Part. Nucl.* **16**, 203 (1985).
17. C. N. Leung, S. T. Love and W. A. Bardeen, *Nucl. Phys.* **B273**, 649 (1986).
18. J. B. Kogut, E. Dagotto and A. Kocić, *Nucl. Phys.* **B317**, 253 (1989).
19. K. M. Case, *Phys. Rev.* **80**, 797 (1950).
20. R. Jackiw, in *M.A.B. Bég Memorial Volume*, eds. A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991), p. 25.
21. M. A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, Washington, 1965), pp. 504–505.
22. M. A. Abramowitz and I. A. Stegun, *ibidem*, p. 256.
23. A. A. Abramov, *Tables of* $\ln\Gamma(Z)$ *for Complex Argument* (Pergamon, New York, 1960), p. 3.
24. V. S. Popov, *Yad. Fiz.* **12**, 429 (1970) [*Sov. J. Nucl. Phys.* **12**, 429 (1971)]; J. Rafelski, L. P. Fulcher and A. Klein, *Phys. Rep.* **C38**, 229 (1978); Ya. B. Zel'dovich and V. S. Popov, *Usp. Fiz. Nauk.* **105**, 403 (1971) [*Sov. Phys. Usp.* **14**, 673 (1971)].
25. B. Schroer and J. A. Swieca, *Phys. Rev.* **D2**, 2938 (1970); V. A. Rizov, H. Sazdjian and I. T. Todorov, *Ann. Phys. (N.Y.)* **165**, 59 (1985).
26. G. Preparata and W. I. Weisberger, *Phys. Rev.* **175**, 1965 (1968), App. C.
27. K. Wilson, *Phys. Rev.* **179**, 1499 (1969).

28. H. D. Politzer, *Phys. Rep.* **C14**, 129 (1974); D. J. Gross, in *Methods in Field Theory — Les Houches Lectures 1975*, eds. R. Balian and J. Zinn-Justin (North-Holland and World Scientific, Amsterdam and Singapore, 1976), p. 141.
29. Y. Nambu and L. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961); **124**, 246 (1961); M. Gell-Mann, R. J. Oakes and B. Renner, *Phys. Rev.* **175**, 2195 (1968); J. Gasser and H. Leutwyler, *Phys. Rep.* **C87**, 77 (1982).
30. S. L. Adler, *Phys. Rev.* **177**, 2426 (1969).
31. S. L. Adler and W. A. Bardeen, *Phys. Rev.* **182**, 1517 (1969).
32. G. 't Hooft, *Phys. Rep.* **142**, 357 (1986).