A Duality for Boolean Contact Algebras

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1 Introduction

The well-known de Vries duality, established by H. de Vries in 1962, states that the category of compact Hausdorff spaces is dually equivalent to that of de Vries algebras [4]. The notion of Boolean *contact* algebra (BCA) was developed independently in the context of region-based theory of space. Düntsch and Winter established in [5] a representation theorem for BCAs, showing that every BCA is isomorphic to a dense subalgebra of the regular closed sets of a T_1 weakly regular space. It appears that BCAs are a direct generalization of de Vries algebras, and that the representation theorem for complete BCAs generalizes de Vries duality for objects. During a conference, Vakarelov raised the question of dualizing morphisms. We answer this question using concepts similar to those of modal logic's neighborhood semantics.

2 de Vries Duality and the Representation Theorem

A *de Vries algebra* (DVA), is a complete Boolean algebra B endowed with a binary relation $\prec$ satisfying the following axioms:

DV1 $0 \prec 0$;

DV2 $a \prec b \Rightarrow a \leq b$;

DV3 $a \leq b \prec c \Rightarrow a \prec c$;

DV4 $a \prec b, c \prec d \Rightarrow a \wedge c \prec b \wedge d$;

DV5 $a \prec b \Rightarrow -b \prec -a$;

DV6 $a \prec b \neq 0 \Rightarrow \exists c \neq 0$ such that $a \prec c \prec b$,

where $-a$ denotes the Boolean complement of a. A filter x of B is a *round filter* if for each $b \in x$ there is some $a \in x$ such that $a \prec b$; maximal round filters are called *ends*. Then the set $\mathcal{E}(B)$ of all ends, equipped with the topology having the sets $r_B(a) = \{x \in \mathcal{E}(B) : x \ni a\}$ as a basis, is a compact Hausdorff space. This leads to a dual equivalence between the category of compact Hausdorff spaces and the category of de Vries algebras with suitable morphisms [4].

Boolean contact algebras were studied independently as a formalization of Whiteheadean vision of space. A *Boolean contact algebra* (BCA) is a Boolean algebra B endowed with a binary relation $\mathcal{C}$ satisfying the following axioms:

C1 $a \mathcal{C} b \Rightarrow a \neq 0$;

C2 $a \neq 0 \Rightarrow a \mathcal{C} a$;

C3 $a \mathcal{C} b \Rightarrow b \mathcal{C} a$;

C4 $a \mathcal{C} b, b \leq c \Rightarrow a \mathcal{C} c$;

C5 $a \mathcal{C} (b \vee c) \Rightarrow a \mathcal{C} b$ or $a \mathcal{C} c$;

C6 $a \not\leq b \Rightarrow \exists c \in B$ such that $a \mathcal{C} c$ and $c \perp b$,

where $\perp$ denotes the complement of the relation $\mathcal{C}$.

The relations $\prec$ and $\perp$ are linked by $a \prec b \Leftrightarrow a \perp -b$. The axioms DV1-DV5 are then equivalent to C1-C5. However, the axiom C6 is weaker than DV6.

Due to the lack of the axiom DV6, round filters and ends do not work anymore. Those have to be replaced by the notions of clan and cluster. A non-empty subset Γ of B is a *clan* if its complement is an ideal and if $a, b \in \Gamma \Rightarrow a \mathcal{C} b$. A maximal clan is called a *cluster*. The set $\mathrm{clust}(B)$ of clusters of B is then equipped with the topology having the sets $\eta_B(a) = \{\Gamma \in \mathrm{clust}(B) : \Gamma \ni a\}$ as a basis for closed sets. This topological space appears to be T_1 and weakly regular.

The representation theorem for BCAs, due to Düntsch and Winter [5], states that η_B is a dense embedding from B to the algebra $\mathrm{RC}(\mathrm{clust}(B))$ of regular closed sets of $\mathrm{clust}(B)$ endowed with the contact relation $F \mathcal{C} G \Leftrightarrow F \cap G \neq \emptyset$.

3 A Duality for Morphisms

Let $\beta : B \to B'$ be a map between two complete BCAs satisfying

CM1 $\beta(a \vee b) = \beta(a) \vee \beta(b)$;

CM2 $\beta(1) = 1$;

CM3 $a \perp b \Rightarrow \beta(a) \perp \beta(b)$.

In the presence of the axiom DV6, it is not difficult to define a dual morphism between $\mathrm{clust}(B')$ and $\mathrm{clust}(B)$, as the inverse image of any cluster is a clan, which is contained in a unique cluster. If β additionally satisfies

CM4 $\beta(a) = \bigwedge\{\beta(b) : a \perp -b\}$,

one easily recovers de Vries duality. However, in general, a clan may be contained in several clusters. We then define a morphism from $\mathrm{clust}(B')$ to $\mathrm{clust}(B)$ to be a map N from $\mathrm{clust}(B')$ to $\mathrm{clan}(\mathrm{RC}(\mathrm{clust}(B)))$, defined as follows

$$N(\Gamma') = \{F \in \mathrm{RC}(\mathrm{clust}(B)) : \beta(\eta_B^{-1}(F)) \in \Gamma'\}.$$

While this definition may seem unnatural, it is quite similar to the accessibility relation in neighborhood semantics.

This leads to two dualities: one involving the category of BCAs with their natural morphisms (satisfying CM1-CM3) and another one extending de Vries duality.

References

[1] Guram Bezhanishvili, Nick Bezhanishvili, and John Harding. Modal compact Hausdorff spaces. *J. Log. Comput.*, 25(1):1–35, 2015.

[2] Guram Bezhanishvili, Nick Bezhanishvili, Sumit Sourabh, and Yde Venema. Irreducible Equivalence Relations, Gleason Spaces, and de Vries Duality. *Appl. Categ. Struct.*, pages 1–21, 2016.

[3] Guram Bezhanishvili and John Harding. Proximity frames and regularization. *Appl. Categ. Struct.*, 22(1):43–78, 2014.

[4] Hendrik de Vries. Compact spaces and compactifications, an algebraic approach. Van Gorcum, Assen, 1962.

[5] Ivo Düntsch and Michael Winter. A representation theorem for Boolean contact algebras. *Theor. Comput. Sci.*, 347(3):498–512, 2005.

[6] Sabine Koppelberg, Ivo Düntsch, and Michael Winter. Remarks on contact relations on Boolean algebras. *Algebra Univers.*, 68(3-4):353–366, 2012.

[7] Dimiter Vakarelov. Dynamic mereotopology II: Axiomatizing some whiteheadean type space-time logics. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence S. Moss, editors, *Advances in Modal Logic*, pages 538–558. College Publications, 2012.

[8] Dimiter Vakarelov, Georgi Dimov, Ivo Düntsch, and Brandon Bennett. A proximity approach to some region-based theories of space. *J. Appl. Non-Class. Log.*, 12(3-4):527–559, 2002.