MODERATE DEFLECTION ANALYSIS OF SHELLS OF REVOLUTION, USING CYLINDRICAL COMPONENTS OF THE DISPLACEMENTS

J.F. DEBONGNIE
Professor
ECOLE MOHAMMADIA D'INGENIEURS
RABAT - MOROCCO

Summary

This paper is devoted to the analysis of shells of revolution, using cylindrical components of the displacement. This procedure, coupled with a Fourier expansion by respect of the azimuth permits the generation of finite elements in which rigid body motions are represented exactly, even in the case of doubly curved shells.

1. INTRODUCTION

A particular feature of rotational solids is the fact that nonaxisymmetric motions may be treated by a Fourier expansion. In this approach, the geometry is described by only giving the half-meridional section $\theta = 0$. Considering a shell, the trace of its middle surface in the half-plane $\theta = 0$ is a curve which may be parametrized by a single variable, say ξ (see fig. 1). In practical applications, however, this procedure is advantageous only if the different azimuthal harmonics may be treated separarely, i.e. if they are decoupled in energy. Under certain conditions on the constitutive laws, which are precised in /1/ and verified in most cases, this fundamental property is true for linear elastostatics and linear elastodynamics. It is also verified for linear buckling analysis, provided the initial stresses are axisymmetric. In more general problems including nonlinear effects, all Fourier modes may be coupled and a tridimensional analysis using general shell elements, seems easier to perform.

As a consequence, there is - at least from a numericist view point - no real need to develop rotational shell theories whose generality exceed the frame of moderate deflections. By these words, reference is made to the assumption that no displacement gradient is greater in magnitude than $\Gamma^{\frac{1}{2}}$, Γ being the order of magnitude of the strains. This simplification, also known as "moderate rotations theory"/10/ (Koiter refers to it as "small finite deflections" /9/), may be considered as a first approximation of geometrical nonlinearities and constitutes a sufficient basis for buckling analysis.

As is well known, an exact representation of rigid body motions is of considerable importance in finite element applications. When the displacements are decomposed in the classical curvilinear basis of the shell, this condition cannot be fulfilled unless the shell is conical. Consequently, doubly curved shells need another treatment.

The oldest solution, which was pioneered by GRAFTON and STROME /5/ and subsequently followed by many authors /11,6/, consits in approximating the actual geometry by conical frustra. This procedure saves the representation of rigid body motions, but at the price of a crude geometrical approximation which, in particular, may be responsible for spurious bending stresses under pressure loads.

More accurate, but still of approximate nature, is the quasi-conical shell theory developed by the author/3/ from the concept of fictitious displacements. In this theory, the actual shell is viewed as a geometrical perturbation from a set of conical frustra. Since the system of displacement components is determined by the reference cones, rigid body motions may be represented exactly by the use of parametric elements.

Pursuing in this way of disconnection of the displacement basis from the geometry, the ultimate step consists in the use of cylindrical components. This is in fact no new idea, a finite element based on this concept having been developed in 1975 by DELPAK /4,6/. Delpak's analysis, however, is restricted to linear elasticity and axisymmetrical deformation. Fundamentally, it makes use of the classical expressions of strains in terms of locally directed displacements, a transformation into cylindrical components being performed a posteriori.

In this paper, a more general approach is developed, in which the strains are directly expressed in terms of cylindrical components of the displacements. In a first step, transverse shear effects are taken into account, and geometrical nonlinearities are included. The second step is the intoduction of the moderate deflections hypothesis. At this stage, Kirchhoff-Love conditions are intoduced. Concerning the old problem to determine whether Kirchhoff-Love assumptions have to be used or not, author's point of view is that it is a question of opportunity. On one hand, Kirchhoff-Love assumptions result in a simpler theory, working even for very thin shells for which the theory including transverse shear effects may exhibit numerical degeneracies. On the other hand, in situations where the shell has to be connected with volume elements, e.g. in the case of a shell filled with a solid propellant, this connection is easier when shear effects are taken into account, due to an exact correspondance of generalized displacements. Both approaches have thus their own field of application and no universal answer does exist to the question whether one is better than the other.

As any finite element analysis implies the connection of the elements at their common interfaces, conformity conditions have to be settled. This is done in section 9, for both approaches, it is to say with or without transverse shear effects.

2. GEOMETRICAL DESCRIPTION

As mentioned in the introduction, the trace of the middle surface of a rotational shell in the meridian half-plane $\theta = 0$ is a curve which may be parametrized by a single variable ξ . The shape of the shell depends upon the form of the applications $\xi \mapsto R(\xi)$ and $\xi \mapsto Z(\xi)$. In what follows, it will be assumed that ξ is <u>normalized</u> in the sense that

$$R^2_{,\xi} + Z^2_{,\xi} = 1 \quad . \tag{1}$$

In practical applications, the functions $R(\xi)$ and $Z(\xi)$ are approximated by polynomials of some degree k whose coefficients are determined with the aid of nodes, in the same manner as in parametric finite elements. This being done, the position vector $\underset{\sim}{s}(\xi, \theta, x^3)$ of any point of the shell may be written in the following form

$$\underset{\sim}{s} = R\, \underset{\sim}{e}_R + Z\, \underset{\sim}{e}_Z + x^3 \left(-Z_{,\xi}\, \underset{\sim}{e}_R + R_{,\xi}\, \underset{\sim}{e}_Z \right) \tag{2}$$

In this expression, and throughout the present paper, $\underset{\sim}{e}_R$, $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_Z$ are the three vectors forming the normed basis of cylindrical coordinates.

3. DISPLACEMENTS

For the displacement field, a linear structure along the normal to the middle-surface will be considered, namely

$$\underset{\sim}{v} = u\, \underset{\sim}{e}_R + v\, \underset{\sim}{e}_\theta + w\, \underset{\sim}{e}_Z + x^3 (\alpha\, \underset{\sim}{e}_R + \beta\, \underset{\sim}{e}_\theta + \gamma\, \underset{\sim}{e}_Z) \quad , \tag{3}$$

where u, v, w, α , β , γ are functions of ξ and θ . The finite element model is constructed as follows. In a first step, the displacements are expanded in Fourier series

$$
\begin{aligned}
u &= \sum_{n=0} \sum_{m=0}^{1} u_{nm}(\xi) \cos(n\theta + m\,\frac{\pi}{2}) \\
v &= \sum_{n=0} \sum_{m=0}^{1} v_{nm}(\xi) \sin(n\theta + m\,\frac{\pi}{2}) \\
w &= \sum_{n=0} \sum_{m=0}^{1} w_{nm}(\xi) \cos(n\theta + m\,\frac{\pi}{2}) \qquad\qquad (4) \\
\alpha &= \sum_{n=0} \sum_{m=0}^{1} \alpha_{nm}(\xi) \cos(n\theta + m\,\frac{\pi}{2}) \\
\beta &= \sum_{n=0} \sum_{m=0}^{1} \beta_{nm}(\xi) \sin(n\theta + m\,\frac{\pi}{2})
\end{aligned}
$$

(system continues)

$$\gamma = \sum_{n=0} \sum_{m=0}^{1} \gamma_{nm}(\xi) \cos(n\theta + m\frac{\pi}{2})$$

Admitting on the moduli some restrictions which are ordinarily verified /1/, this decomposition ensures that harmonics corresponding to different values of the couple (n,m) are energetically decoupled. The second step consists in a polynomial approximation by finite elements of each harmonic in terms of ξ. The rigid body displacements are contained in this model provided u, v, w are at least polynomials of degree k and α, β, γ at least polynomials of degree (k - 1) in terms of ξ.

4. STRAIN EXPRESSIONS FOR LARGE DISPLACEMENTS

In the strained configuration, the position vector $\underset{\sim}{s}$ of a point of the shell is transformed in s' by the following rule

$$\begin{aligned} s' = {} & (R + u)\, \underset{\sim}{e}_R + v\, \underset{\sim}{e}_\theta + (Z + w)\, \underset{\sim}{e}_Z \\ & + x^3((-Z_{,\xi} + \alpha)\, \underset{\sim}{e}_R + \beta\, \underset{\sim}{e}_\theta + (R_{,\xi} + \gamma)\, \underset{\sim}{e}_Z) \end{aligned} \tag{5}$$

Throughout the text, primes will be used to distinguish quantities attached to the strained configuration. The base vectors $\underset{\sim}{g}_\xi$, $\underset{\sim}{g}_\theta$, $\underset{\sim}{g}_3$ of the unstrained shell are

$$\left\{ \begin{aligned} \underset{\sim}{g}_\xi &= \underset{\sim}{s}_{,\xi} = R_{,\xi}\, \underset{\sim}{e}_R + Z_{,\xi}\, \underset{\sim}{e}_Z + x^3(-Z_{,\xi\xi}\, \underset{\sim}{e}_R + R_{,\xi\xi}\, \underset{\sim}{e}_Z) \\ \underset{\sim}{g}_\theta &= \underset{\sim}{s}_{,\theta} = R\, \underset{\sim}{e}_\theta + x^3(-Z_{,\xi}\, \underset{\sim}{e}_\theta) \\ \underset{\sim}{g}_3 &= \underset{\sim}{s}_{,3} = -Z_{,\xi}\, \underset{\sim}{e}_R + R_{,\xi}\, \underset{\sim}{e}_Z \end{aligned} \right. \tag{6}$$

The corresponding metric tensor has thus the following components

$$\left\{ \begin{aligned} g_{\xi\xi} &= 1 + 2x^3(-R_{,\xi} Z_{,\xi\xi} + Z_{,\xi} R_{,\xi\xi}) + (x^3)^2(R^2_{,\xi\xi} + Z^2_{,\xi\xi}) \\ g_{\theta\theta} &= R^2 - 2x^3 R\, Z_{,\xi} + (x^3)^2\, Z^2_{,\xi} \\ g_{33} &= 1 \\ g_{\xi\theta} &= g_{\xi 3} = g_{\theta 3} = 0 \end{aligned} \right. \tag{7}$$

The Green strain tensor $\underset{\approx}{\gamma}$ is defined by

$$\gamma_{ij} = \tfrac{1}{2}(g'_{ij} - g_{ij}) \tag{8}$$

To obtain a two-dimensional shell theory, an assumption is necessary on the pinch γ_{33}. The simplest one consists to impose that normals to the middle surface are not stretched during the deformation process, i.e.

$$g'_{33} = (-Z_{,\xi} + \alpha)^2 + \beta^2 + (R_{,\xi} + \gamma)^2 = 1 \qquad (9)$$

or, equivalently,

$$\gamma_{33} = -Z_{,\xi}\alpha + R_{,\xi}\gamma + \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = 0 \,. \qquad (10)$$

Using Greek letters for indices ξ and θ , one has the general relation

$$\gamma_{\lambda\mu} = \overset{o}{\gamma}_{\lambda\mu} - x^3 \rho_{\lambda\mu} + (x^3)^2 \psi_{\lambda\mu} \qquad (11)$$

where $\overset{o}{\gamma}_{\lambda\mu}$ are the membrane strains and $\rho_{\lambda\mu}$ the changes of curvature. (It is for convenience that this name is used, though it is abusive when shear effects are present). From general order of magnitude evaluations for shells, based on compatibility equations, following a step pioneered by CHIEN /7/ for thin shells, it may be proved /8/ that the term $(x^3)^2 \psi_{\lambda\mu}$ is negligibly small. It will therefore not be computed. After any calculations, the membrane strains and changes of curvature are found as follows

$$\left\{\begin{aligned}
\overset{o}{\gamma}_{\xi\xi} &= R_{,\xi}\, u_{,\xi} + Z_{,\xi}\, w_{,\xi} + \tfrac{1}{2}\,(u^2_{,\xi} + v^2_{,\xi} + w^2_{,\xi}) \\
\overset{o}{\gamma}_{\theta\theta} &= R(v_{,\theta} + u) + \tfrac{1}{2}(u_{,\theta} - v)^2 + \tfrac{1}{2}(v_{,\theta} + u)^2 + \tfrac{1}{2}w^2_{,\theta} \\
2\,\overset{o}{\gamma}_{\xi\theta} &= R_{,\xi}\,(u_{,\theta} - v) + R\, v_{,\xi} + u_{,\xi}\,(u_{,\theta} - v) + v_{,\xi}\,(v_{,\theta} + u) \\
&\quad + w_{,\xi}\, w_{,\theta} \\
-\rho_{\xi\xi} &= R_{,\xi}\,\alpha_{,\xi} + Z_{,\xi}\,\gamma_{,\xi} - Z_{,\xi\xi}\, u_{,\xi} + R_{,\xi\xi}\, w_{,\xi} \\
&\quad + u_{,\xi}\,\alpha_{,\xi} + v_{,\xi}\,\beta_{,\xi} + w_{,\xi}\,\gamma_{,\xi} \\
-\rho_{\theta\theta} &= R(\beta_{,\theta} + \alpha) - Z_{,\xi}\,(v_{,\theta} + u) + (u_{,\theta} - v)(\alpha_{,\theta} - \beta) \\
&\quad + (v_{,\theta} + u)(\beta_{,\theta} + \alpha) + w_{,\theta}\,\gamma_{,\theta} \\
-2\,\rho_{\xi\theta} &= R_{,\xi}\,(\alpha_{,\theta} - \beta) + Z_{,\xi}\,\gamma_{,\theta} + R\,\beta_{,\xi} - Z_{,\xi}\, v_{,\xi} \\
&\quad - Z_{,\xi\xi}\,(u_{,\theta} - v) + R_{,\xi\xi}\, w_{,\theta} + u_{,\xi}\,(\alpha_{,\theta} - \beta) \\
&\quad + v_{,\xi}\,(\beta_{,\theta} + \alpha) + w_{,\xi}\,\gamma_{,\theta} + \alpha_{,\xi}\,(u_{,\theta} - v) \\
&\quad + \beta_{,\xi}\,(v_{,\theta} + u) + \gamma_{,\xi}\, w_{,\theta} \,.
\end{aligned}\right. \qquad (12)$$

Let us now turn to transverse shear strains. They are of the general form

$$\gamma_{\lambda 3} = \overset{o}{\gamma}_{\lambda 3} + x^3 \rho_{\lambda 3} ,$$

with

$$\rho_{\xi 3} = (-Z_{,\xi} + \alpha)(-Z_{,\xi\xi} + \alpha_{,\xi}) + \beta\, \beta_{,\xi} + (R_{,\xi} + \gamma)(R_{,\xi\xi} + \gamma_{,\xi})$$

$$= g'_{33,\xi}$$

and, similarly,

$$\rho_{\theta 3} = (-Z_{,\xi} + \alpha)\,\alpha_{,\theta} + \beta\, \beta_{,\theta} + (R_{,\xi} + \gamma)\,\gamma_{,\theta} = g'_{33,\theta}$$

Thus, due to condition (9), both strains are constant by respect of x^3. Their explicit expressions are

$$2\,\gamma_{\xi 3} = R_{,\xi}\alpha + Z_{,\xi}\gamma + R_{,\xi}\, w_{,\xi} - Z_{,\xi}\, u_{,\xi} + \alpha u_{,\xi} + \beta\, v_{,\xi} + \gamma\, w_{,\xi}$$

$$2\,\gamma_{\theta 3} = R\beta + R_{,\xi}\, w_{,\theta} - Z_{,\xi}(u_{,\theta} - v) + \alpha\,(u_{,\theta} - v) + \beta(v_{,\theta} + u) + \gamma\, w_{,\theta} \qquad (13)$$

Up to now, <u>tensorial</u> components of the strains and changes of curvature have been used. <u>Physical</u> components are prefered as design variables. To make the distinction, physical components will be noted ε for the strains and K for the changes of curvature. The correspondance is as follows

$$\begin{cases} \varepsilon_{\xi} = \overset{o}{\gamma}_{\xi\xi} , & \varepsilon_{\theta} = \dfrac{1}{R^2}\overset{o}{\gamma}_{\theta\theta} , & \varepsilon_{\xi\theta} = \dfrac{1}{R}\overset{o}{\gamma}_{\xi\theta} \\ K_{\xi} = \rho_{\xi\xi} , & K_{\theta} = \dfrac{1}{R^2}\rho_{\theta\theta} , & K_{\xi\theta} = \dfrac{1}{R}\rho_{\xi\theta} \\ \varepsilon_{\xi 3} = \gamma_{\xi 3} , & \varepsilon_{\theta 3} = \dfrac{1}{R}\gamma_{\theta 3} & \end{cases} \qquad (14)$$

5. CONSTITUTIVE LAWS

By similar arguments as those that KOITER /12/ used in the frame of the Kirchhoff-Love hypothesis, the following expression may be used for the energy by unit of middle surface area when shear effects are present /8/

$$V = \tfrac{1}{2}\, \overset{o}{C}{}^{\alpha\beta\lambda\mu}\, t\, \overset{o}{\gamma}_{\alpha\beta}\, \overset{o}{\gamma}_{\lambda\mu} + \tfrac{1}{2}\, \overset{o}{C}{}^{\alpha\beta\lambda\mu}\, \frac{t^3}{12}\, \rho_{\alpha\beta}\, \rho_{\lambda\mu} + \tfrac{1}{2}\, \overset{o}{G}{}^{\alpha\beta}\, t\, \gamma_{\alpha 3}\gamma_{\beta 3} \qquad (15)$$

This expression implies a relative error which does not exceed

$$O\left(\frac{t^2}{\mathcal{R} L}\right) \quad , \quad O\left(\frac{t^2}{\mathcal{R}^2}\right) \quad , \quad O(\Gamma) \; ,$$

whichever of these quantities may be critical. Here, $\mathcal{R}$ is the order of magnitude of the radii of curvature, L is the "wave length" of the deformation, and t is the thickness of the shell. $\overset{\circ}{C}{}^{\alpha\beta\lambda\mu}$ and $\overset{\circ}{G}{}^{\alpha\beta}$ are the moduli at $x^3 = 0$. The following stress resultants may be deduced by derivation

$$\begin{aligned} &\underline{\text{Membrane resultants}} \quad n^{\alpha\beta} = \overset{\circ}{C}{}^{\alpha\beta\lambda\mu}\, t\, \overset{\circ}{\gamma}_{\lambda\mu} \\ &\underline{\text{Moments}} \quad m^{\alpha\beta} = \overset{\circ}{C}{}^{\alpha\beta\lambda\mu}\, (t^3/12)\, \rho_{\lambda\mu} \\ &\underline{\text{Shear resultants}} \quad q^{\alpha} = \overset{\circ}{G}{}^{\alpha\beta}\, \gamma_{\beta 3} \end{aligned} \tag{16}$$

These are of course tensorial components. The physical ones will be noted by capital letters and are related to the preceeding ones by the following relations

$$\left\{ \begin{aligned} N_{\xi} &= n^{\xi\xi} \;, & N_{\theta} &= R^2\, n^{\theta\theta} \;, & N_{\xi\theta} &= R\, n^{\xi\theta} \\ M_{\xi} &= m^{\xi\xi} \;, & M_{\theta} &= R^2\, m^{\theta\theta} \;, & M_{\xi\theta} &= R\, m^{\xi\theta} \\ Q_{\xi} &= q^{\xi} \;, & Q_{\theta} &= R\, q^{\theta} & & \end{aligned} \right. \tag{17}$$

6. MODERATE DEFLECTIONS

The moderate deflection approximation is characterized by the fact that all displacement gradients are at most $O(\Gamma^{\frac{1}{2}})$, Γ being the order of magnitude of the strains. In the present case, this hypothesis may be explicited as follows

$$u_{,\xi} \;;\; v_{,\xi} \;;\; \frac{1}{R}(u_{,\theta} - v) \;;\; \frac{1}{R}(v_{,\theta} + u) \;;\; \frac{1}{R} w_{,\theta} \;;\; \alpha \;;\; \beta \;;\; \gamma \;\leqslant O(\Gamma^{\frac{1}{2}}). \tag{18}$$

In each strain and change of curvature, relative errors $O(\Gamma^{\frac{1}{2}})$ are admitted. Let us examine the simplifications which proceed from this hypothesis.

6.1 - Membrane strains

One has

$$R_{,\xi}\, u_{,\xi} + Z_{,\xi}\, w_{,\xi} = \varepsilon_{\xi} - \tfrac{1}{2}\,(u^2_{,\xi} + v^2_{,\xi} + w^2_{,\xi}) = \varepsilon_{\xi} - O(\Gamma) = O(\Gamma) \tag{19}$$

Similarly,

$$\frac{1}{R}\,(v_{,\theta} + u) = O(\Gamma) \tag{20}$$

and

$$(R_{,\xi}/R)(u_{,\theta} - v) + v_{,\xi} + (Z_{,\xi}/R)w_{,\theta} = O(\Gamma) \tag{21}$$

These results implie directly

$$\varepsilon_\theta = \frac{1}{R}(v_{,\theta} + u) + \frac{1}{2R^2}((u_{,\theta} - v)^2 + w^2_{,\theta}) + O(\Gamma^2). \tag{22}$$

Concerning ε_ξ, note that

$$\tfrac{1}{2}(u^2_{,\xi} + w^2_{,\xi}) = \tfrac{1}{2}(R_{,\xi}u_{,\xi} + Z_{,\xi}w_{,\xi})^2 + \tfrac{1}{2}(Z_{,\xi}u_{,\xi} - R_{,\xi}w_{,\xi})^2,$$

so that, from (19),

$$\varepsilon_\xi = R_{,\xi}u_{,\xi} + Z_{,\xi}w_{,\xi} + \tfrac{1}{2}(Z_{,\xi}u_{,\xi} - R_{,\xi}w_{,\xi})^2 + \tfrac{1}{2}v^2_{,\xi} + O(\Gamma^2) \tag{23}$$

In the expression of $2\varepsilon_{\xi\theta}$ appears the term

$$v_{,\xi}\,\frac{1}{R}(v_{,\theta} + u) = O(\Gamma^{3/2}),$$

which may be neglected. Now,

$$\begin{aligned} u_{,\xi}(u_{,\theta} - v) + w_{,\xi}w_{,\theta} &= \\ &= (R_{,\xi}u_{,\xi} + Z_{,\xi}w_{,\xi})(R_{,\xi}(u_{,\theta} - v) + Z_{,\xi}w_{,\theta}) \\ &\quad + (Z_{,\xi}u_{,\xi} - R_{,\xi}w_{,\xi})(Z_{,\xi}(u_{,\theta} - v) - R_{,\xi}w_{,\theta}) \end{aligned}$$

and, owing to relation (19), the first product of the right hand side is $O(R\,\Gamma^{3/2})$, so that, finally,

$$\begin{aligned} 2\varepsilon_{\xi\theta} &= (R_{,\xi}/R)(u_{,\theta} - v) + v_{,\xi} + (Z_{,\xi}/R)w_{,\theta} \\ &\quad + \frac{1}{R}(Z_{,\xi}u_{,\xi} - R_{,\xi}w_{,\xi})(Z_{,\xi}(u_{,\theta} - v) - R_{,\xi}w_{,\theta}) \\ &\quad + O(\Gamma^{3/2}) \end{aligned} \tag{24}$$

6.2 - Changes of curvature

The linear terms of the changes of curvature are of the following orders of magnitude

$$\Gamma^{\frac{1}{2}}/L \quad , \quad \Gamma^{\frac{1}{2}}/\mathcal{R} \quad .$$

The nonlinear terms are of order Γ/L and may therefore be neglected as negligibly small in comparison of the linear ones. This results in the following simplified expressions

$$\begin{cases} -K_{\xi} = R_{,\xi}\,\alpha_{,\xi} + Z_{,\xi}\,\gamma_{,\xi} - Z_{,\xi\xi}\,u_{,\xi} + R_{,\xi\xi}\,w_{,\xi} \\ -K_{\theta} = (1/R)(\beta_{,\theta} + \alpha) - (Z_{,\xi}/R^2)(v_{,\theta} + u) \\ -2K_{\xi\theta} = (R_{,\xi}/R)(\alpha_{,\theta} - \beta) + \beta_{,\xi} + (Z_{,\xi}/R)\,\gamma_{,\theta} \\ \qquad -(Z_{,\xi}/R)\,v_{,\xi} - (Z_{,\xi\xi}/R)(u_{,\theta} - v) + (R_{,\xi\xi}/R)\,w_{,\theta} \end{cases} \quad (25)$$

6.3 - Transverse shear strains

Here, the nonlinear terms are visibly of order Γ and in principle, no simplification may be done. This conclusion has to be qualified . In many applications, the transverse shear terms of the energy play the role of a penalty functional for the Kirchhoff-Love conditions /6/. As will be seen in the following section, nonlinear terms may be neglected in these condition, at the same error level as the rest of the theory. In this view, it is thus perfectly consistent to omit nonlinear terms of the transverse shear strains in most applications. They will consequently be written in square brackets in the explicit expressions which follow

$$2\,\varepsilon_{\xi 3} = R_{,\xi}\alpha + Z_{,\xi}\gamma + R_{,\xi}\,w_{,\xi} + Z_{,\xi}\,u_{,\xi}$$

$$\left[+\alpha\, u_{,\xi} + \beta\, v_{,\xi} + \gamma\, w_{,\xi} \right]$$

$$2\,\varepsilon_{\theta 3} = \beta + (R_{,\xi}/R)\,w_{,\theta} - (Z_{,\xi}/R)(u_{,\theta} - v)$$

$$\left[+\alpha\,(u_{,\theta} - v) + \beta\,(v_{,\theta} + u) + \gamma\, w_{,\theta} \right]$$

6.4 - Pinch strain

The group $(-Z_{,\xi}\alpha + R_{,\xi}\gamma)$ is *a priori* of order $\Gamma^{\frac{1}{2}}$, and the exact condition $\gamma_{33} = 0$ may be written

$$-Z_{,\xi}\alpha + R_{,\xi}\gamma = -\tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = O(\Gamma).$$

Admitting a relative error $O(\Gamma^{\frac{1}{2}})$, this condition may be replaced by its linearized form

$$-Z_{,\xi}\alpha + R_{,\xi}\gamma = 0 \qquad (26)$$

7. THE KIRCHHOFF-LOVE CONDITION IN THE FRAME OF MODERATE DEFLECTIONS

The well-known Kirchhoff-Love condition consists to neglect transverse shear strains, i.e. to impose

$$\varepsilon_{\xi 3} = \varepsilon_{\theta 3} = 0 \tag{27}$$

From the vanishing of $\varepsilon_{\theta 3}$, it is found

$$\beta = -(R_{,\xi}/R)\, w_{,\theta} + (Z_{,\xi}/R)(u_{,\theta} - v) + 0(\Gamma) \tag{28}$$

The simultaneous vanishing of $\varepsilon_{\xi 3}$ and ε_{33} leads to the following system

$$\begin{cases} R_{,\xi}\,\alpha + Z_{,\xi}\,\gamma = -R_{,\xi}\, w_{,\xi} + Z_{,\xi}\, u_{,\xi} + 0(\Gamma) & (29) \\ -Z_{,\xi}\,\alpha + R_{,\xi}\,\gamma = 0(\Gamma) & (30) \end{cases}$$

Owing to the general error level $0(\Gamma^{\frac{1}{2}})$ of the theory, quantities $0(\Gamma)$ appearing in equations (28) to (30) are negligibly small as compared to the displacement gradients composing the left sides of these equations. Now, by virtue of the normalization condition (1), α and γ may be explicited from (27) and (28). This gives

$$\begin{cases} \alpha = -R^2_{,\xi}\, w_{,\xi} + R_{,\xi}\, Z_{,\xi}\, u_{,\xi} & (31) \\ \gamma = -R_{,\xi}\, Z_{,\xi}\, w_{,\xi} + Z^2_{,\xi}\, u_{,\xi} & (32) \end{cases}$$

Results (28), (31), (32) could of course be introduced directly in the changes of curvature. Such a direct step, however, would be laborious, and a more effective step is as follows. Firstly, from (29),

$$-K_{,\xi} = R_{,\xi}\,\alpha_{,\xi} + Z_{,\xi}\,\gamma_{,\xi} + R_{,\xi\xi}\, w_{,\xi} - Z_{,\xi\xi}\, u_{,\xi} =$$

$$= -R_{,\xi\xi}\,\alpha - Z_{,\xi\xi}\,\gamma - R_{,\xi}\, w_{,\xi\xi} + Z_{,\xi}\, u_{,\xi\xi}$$

Noting that, from (~~29~~), 31 and 32

$$R_{,\xi\xi}\,\alpha + Z_{,\xi\xi}\,\gamma = (R_{,\xi}\, R_{,\xi\xi} + Z_{,\xi}\, Z_{,\xi\xi})(-R_{,\xi}\, w_{,\xi} + Z_{,\xi}\, u_{,\xi})$$

where appears the group

$$R_{,\xi}\, R_{,\xi\xi} + Z_{,\xi}\, Z_{,\xi\xi} = \tfrac{1}{2}(R^2_{,\xi} + Z^2_{,\xi})_{,\xi} = 0\,,$$

the following result is obtained

$$K_{\xi} = R_{,\xi}\, w_{,\xi\xi} - Z_{,\xi}\, u_{,\xi\xi} \quad . \tag{33}$$

Concerning K_{θ} , one first computes

$$(1/R)\ \beta_{,\theta} = -(R_{,\xi}/R^2)\ w_{,\theta\theta} + (Z_{,\xi}/R^2)(u_{,\theta\theta} - v_{,\theta})$$

and

$$(\alpha/R) = -(R^2_{,\xi}/R)\ w_{,\xi} + (R_{,\xi}\, Z_{,\xi}/R)\ u_{,\xi} \quad ,$$

from which

$$K_{\theta} = (R_{,\xi}/R^2) w_{,\theta\theta} - (Z_{,\xi}/R^2) u_{,\theta\theta} + (R_{,\xi}/R)(R_{,\xi}\, w_{,\xi} - Z_{,\xi}\, u_{,\xi})$$
$$+ (Z_{,\xi}/R^2)\ u \quad . \tag{34}$$

The computation of $K_{\xi\theta}$ may be started by deducing from (29)

$$R_{,\xi}\, \alpha_{,\theta} + Z_{,\xi}\, \gamma_{,\theta} = -R_{,\xi}\, w_{,\xi\theta} + Z_{,\xi}\, u_{,\xi\theta} \quad .$$

A differentiation of the following equivalent form of (28)

$$R\,\beta + R_{,\xi}\, w_{,\theta} - Z_{,\xi}\,(u_{,\theta} - v) = 0$$

with respect to ξ leads

$$R\,\beta_{,\xi} + R_{,\xi\xi}\, w_{,\theta} - Z_{,\xi\xi}\,(u_{,\theta} - v) = -R_{,\xi}\,\beta - R_{,\xi}\, w_{,\xi\theta}$$
$$+ Z_{,\xi}\,(u_{,\xi\theta} - v_{,\xi}) \quad ,$$

from which

$$-2\, K_{\xi\theta} = (1/R)(-2\, R_{,\xi}\, w_{,\xi\theta} + 2\, Z_{,\xi}\, u_{,\xi\theta} - 2\, R_{,\xi}\,\beta - 2\, Z_{,\xi}\, v_{,\xi})$$

and, finally,

$$K_{\xi\theta} = (R_{,\xi}/R) w_{,\xi\theta} - (Z_{,\xi}/R) u_{,\xi\theta}$$
$$-(R_{,\xi}/R^2)(R_{,\xi}\, w_{,\theta} - Z_{,\xi}\,(u_{,\theta} - v)) + (Z_{,\xi}/R) v_{,\xi} \tag{35}$$

8. ON THE CONFORMITY CONDITIONS FOR FINITE ELEMENTS BASED UPON THE PRESENT THEORY

Two adjacent elements are connected on a nodal line, whose trace on the meridian half-plane is reduced to a single point. The question now arises, to know what are the variables to be connected on this nodal line.

When transvers shear effects are taken into account, all components of the displacement and the rotation have to be transmitted. However, an interrelation exists between α and γ . In fact, considering the vector

$$\underset{\sim}{\omega} = \alpha \underset{\sim}{e}_R + \beta \underset{\sim}{e}_\theta + \gamma \underset{\sim}{e}_Z \quad , \tag{36}$$

the condition

$$\varepsilon_{33} = - Z_{,\xi}\alpha + R_{,\xi}\gamma = 0$$

may be interpreted as the vanishing of the normal component ω_3 of $\underset{\sim}{\omega}$. As a consequence, only β and the tangential component

$$\omega_t = R_{,\xi}\alpha + Z_{,\xi}\gamma \tag{37}$$

have to be connected. Let us mention here that the condition ε_{33} = may be obtained in practice by a penalty method /6/.

In the frame of the Kirchhoff-Love conditions, it follows from (28) that β is automatically transmitted whenever the displacements are connected on a nodal circle. Consequently, the only rotation that has to be connected in this case is ω_t. Its expression, taking account of (29) and (30), is

$$\omega_t = - R_{,\xi}\, w_{,\xi} + Z_{,\xi}\, u_{,\xi} \tag{38}$$

9. CONCLUSIONS

A general theory of rotational shells using cylindrical components of the displacements has been developed. This approach permits an exact representation of rigid body motions by finite elements using a Fourier decomposition. Shear effects have been considered, but the Kirchhoff-Love hypothesis has also be exploited. Finally, conformity conditions have been examined in both cases. It is interesting to note that the present theory is not significantly more complex than the conical shell theory and rather simpler than the quasi-conical shell theory. This fact, added to its exactness, make it very attractive for applications.

REFERENCES

/1/ J.F. DEBONGNIE - "Modélisation de Problèmes Hydroélastiques par Eléments Finis"
Ph. D. Thesis, University of Liège, 1978

/2/ J.F. DEBONGNIE, E. CARNOY - "Comparaison de différents modèles d'éléments finis plans pour l'analyse linéaire des coques"
Proc. Int. Cong. on "Méthodes Numériques dans les Sciences de l'Ingénieur", GAMNI, nov. 1978; DUNOD, Paris, 1979

/3/ J.F. DEBONGNIE - "Physical Interpretation and Generalization of Marguerre's Shallow Shell Theory"
Int. J. Engng. Sc. , vol 17, pp. 387 - 399, 1979

/4/ R. DELPAK - "Role of the Curved Parametric Element in Linear Analysis of Rotational Shells"
Ph. D. Thesis, Dept. of Civil Engng. and Building, The Polytechnic of Wales, 1975

/5/ P.E. GRAFTON, D.R. STROME - "Analysis of axi-symmetrical shells by the direct stiffness method"
J.A.I.A.A, 1, pp. 2342-2347, 1963

/6/ O.C. ZIENKIEWICZ - "The Finite Element Method"
3d. edition, Mc Graw Hill, 1977

/7/ W.Z. CHIEN - "The intrinsic theory of thin shell and plates"
University of Cal., Publ. in Math., new series, 2, p.103, 1944

/8/ J.F. DEBONGNIE - "Etude critique de quelques formulations de la theorie des coques en vue de l'application aux éléments finis"
Unpublished.

/9/ W.T. KOITER - "On the nonlinear theory of thin elastic shells"
Reprt Nr. 310 of the Laboratory of Engineering Mechanics, Dept. of Mech. Engng., Technological University, Delft.

/10/ E. CARNOY - "Mixed finite elements based upon Marguerre theory for the study of geometrical nonlinear behavior of thin shells"
Comp. Meth. Appl. Mech. Engng. 29, pp. 121-146, 1981

/11/ J.H. PERCY, T.H.H. PIAN, S.KLEIN, D.R. NAVARATNA - "Application of matrix displacement method to linear elastic analysis of shells of revolution"
J.A.I.A.A., vol. 3, Nr. 11, pp. 2138-2145, 1965

/12/ W.T. KOITER - "A consistent first approximation in the general

theory of thin elastic shells". Part 1, "Foundations and linear theory"
Lab. voor toegepaste mechanica der Technische Hogeschool, Delft,
August 5, 1959.

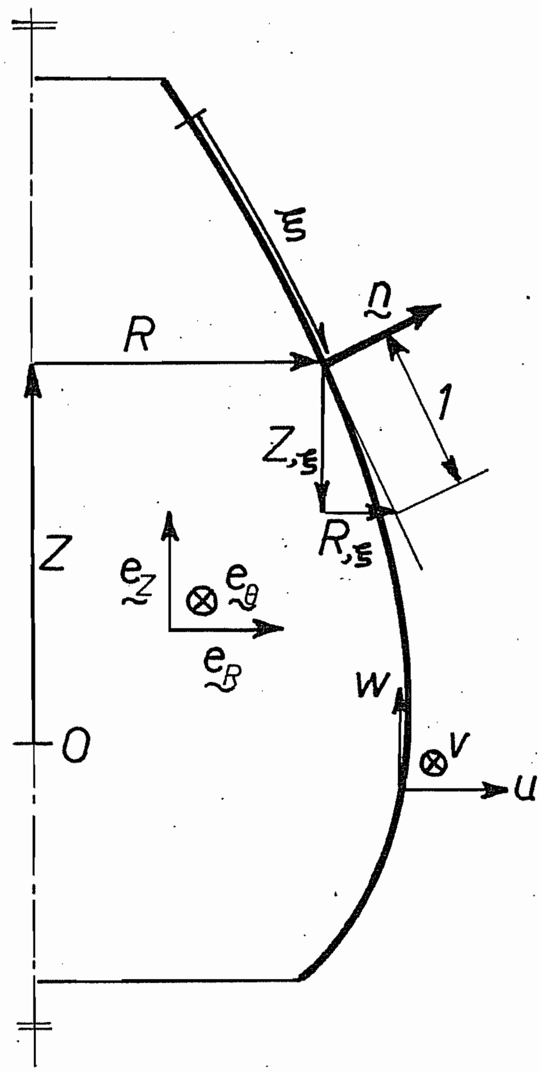

Fig. 1