Sub-Coulomb Transfer Reaction between Heavy Ions. Analytical Expressions.

J. CUGNON

Université de Liège, Physique Nucléaire Théorique
Institut de Physique au Sart Tilamn - B-4000 *Liège* 1, *Belgique*

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The particle transfer between heavy ions under the Coulomb barrier plays an interesting role in recent studies of nuclear as well as atomic physics. It is usually studied in the frame of the DWBA theory (1). The complexity of the formalism requires a numerical integration. Hence, the influence of some physical parameters on the energy dependence as well as the angular dependence of the cross-section or of the optimum Q-value is not clear. It is then desirable to obtain accurate approximations allowing to calculate the DWBA integral in a closed analytical form. The WKB approximation of the wave functions for the relative motion almost fits this desire. Recently (2), another method has been proposed, which lies on a suitable approximation of the relative motion wave function (which is described below) and on neglecting the radial variation of the form factor. The latter approximation is satisfactory for the charged-particle transfer case, where the effective domain of integration is very limited. Here, we go beyond this limitation and still get analytical form for the transition probability.

Our starting point will be the DWBA integral in the prior representation (1)

$$I = \int \chi_1^*(r) F(r) \chi_2(r)\, \mathrm{d}r\,, \tag{1}$$

where the χ_1 and χ_2 are the relative-motion wave functions in the entrance and exit channels respectively, and where r is the relative distance between the ions or between the cores, since at this stage, we neglect recoil effects. The quantity $F(r)$ is the form factor which is often written as

$$F(r) = N \frac{\exp[-qr/\hbar]}{r}\,. \tag{2}$$

(1) P. J. A. BUTTLE and L. J. B. GOLDFARB: *Nucl. Phys.*, **176** A, 299 (1971).
(2) J. CUGNON, D. GARDES and R. DA SILVEIRA: to be published (1977) and *European Conference on Nuclear Physics with Heavy Ions* (Caen, 1976), comm. no. 28.

In order to have an analytic expression for the integral (1), we invoke the fact that most of the contribution to the integral (1) comes from a relatively narrow domain of values of r near the classical turning points (1). Hence, we construct the functions χ_i, $i=1, 2$ by approximating the ion-ion potentials in channels 1 and 2 by constant gradient potentials in the region of interest, it is to say by approximating the potential energy curves by their tangents. We choose the tangential point as the one corresponding to the largest of the closest distances of approach in the two channels. We denote by R the associated relative distance. This choice is motivated by the classical situation, where the transfer mainly takes place at R. We neglect the variation of r^{-1} in the form factor and make it equal to R^{-1}, since its variation is not important on the effective domain of integration.

The transition probability is given by

$$\omega = \frac{2\pi N^2}{\hbar R^2}\left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a_1^*(p)\frac{q}{q^2+(p-p')^2}\, a_2(p')\,\mathrm{d}p\,\mathrm{d}p'\right|^2, \tag{3}$$

where the a_j's are the Fourier transforms of the χ's. For constant gradient potentials, the wave functions $a_j(p)$ are given, with the right normalization, in ref. (3)

$$\begin{cases} a_1(p) = \dfrac{1}{\sqrt{F_1}}\exp\left[\dfrac{i}{\hbar F_1}\left[(E-U_1)p-\dfrac{p^3}{6\mu_1}\right]\right], \\[2ex] a_2(p) = \dfrac{1}{\sqrt{2\pi\hbar F_2}}\exp\left[\dfrac{i}{\hbar F_2}\left[(E-U_2)p-\dfrac{p^3}{6\mu_2}\right]\right], \end{cases} \tag{4}$$

where U_1 and U_2 are the value of the potential-energy curves $\mathscr{V}_1(r)$ and $\mathscr{V}_2(r)$ at point R, and where F_1 and F_2 are minus their gradient at this point. We assume for the time being that the largest of the closest distances of approach corresponds to channel 2. In other words, $E=U_2$. Hence, eqs. (3)-(5) yield

$$\omega = \frac{N^2}{2\pi\hbar^2R^2F_1F_2}\left|\int_{-\infty}^{+\infty}\mathrm{d}p\,\exp\left[-\frac{i}{\hbar F_1}(E-U_1)p\right]\cdot\right. \\ \left.\cdot\int_{-\infty}^{+\infty}\mathrm{d}p'\,\frac{q}{q^2+(p'-p)^2}\exp\left[\frac{i}{6\hbar}\left(\frac{p^3}{F_1\mu_1}-\frac{p'^3}{F_2\mu_2}\right)\right]\right|^2. \tag{5}$$

The second integrand is peaked at $p'\simeq p$. Hence, we expand the argument of the exponential in power series of $\xi=p'-p$ and stop it at the first order. We obtain, after integration on ξ,

$$\omega = \frac{N^2}{2\pi^3\hbar^2F_1F_2R^2}\left|\int_{-\infty}^{+\infty}\mathrm{d}p\,\exp\left[i(\alpha p+\beta p^3)-\gamma p^2\right]\right|^2, \tag{6}$$

(3) L. D. LANDAU and E. M. LIFSHITZ: *Quantum Mechanics* (Oxford, 1965), chapt. III.

where we have put

$$(7) \qquad \alpha = -\frac{1}{\hbar F_1}(E - U_1)\,, \qquad \beta = \frac{1}{6\hbar}\left(\frac{1}{F_1\mu_1} - \frac{1}{F_2\mu_2}\right), \qquad \gamma = \frac{\varkappa}{2F_2\mu_2}\,.$$

The function in the integral is analytic in the whole complex plane $p = x + iy$. In order to transform the integral in eq. (6), that we call $I(\alpha, \beta, \gamma)$, we integrate along the contours shown in fig. 1. It can be seen that the integration along the lateral parts

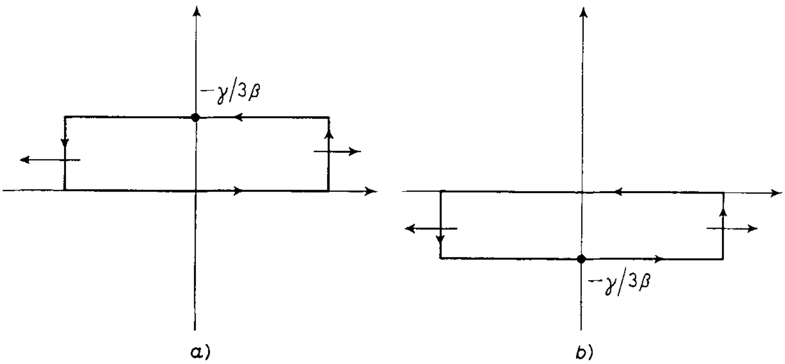

Fig. 1. – Contours of integration used for the evaluation of integral (6) according to the sign of β: *a*) $\beta > 0$, *b*) $\beta < 0$.

of the contour is vanishing when these parts are pushed towards infinity. A little algebra leads to the following results

$$(8) \qquad I(\alpha, \beta, \gamma) = \exp\left[\frac{1}{27\beta^2}(9\alpha\beta\gamma + 2\gamma^3)\right]\int_{-\infty}^{+\infty} \mathrm{d}p \exp\left[\frac{ip}{3\beta}(\gamma^2 + 3\alpha\beta) + i\beta p^3\right].$$

The remaining integral is a representation of the Airy function (see [4]). Hence

$$(9) \qquad I(\alpha, \beta, \gamma) = \frac{2\pi}{(3\beta)^{\frac{1}{3}}}\exp\left[\frac{\gamma}{27\beta^2}[3(3\alpha\beta + \gamma^2) - \gamma^2]\right] Ai\,\frac{3\alpha\beta + \gamma^2}{3\beta(3\beta)^{\frac{1}{3}}}\,.$$

It can be of interest to mention the following asymptotic forms

$$(10) \qquad I(\alpha, \beta, \gamma) = \left[\frac{\pi}{(3\alpha\beta + \gamma^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \exp\left[\frac{\gamma}{27\beta^2}[3(3\alpha\beta + \gamma^2) - \gamma^2]\right]\cdot$$

$$\cdot \exp\left[-\frac{2}{27\beta^2}(3\alpha\beta + \gamma^2)^{\frac{3}{2}}\right],$$

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[4] M. ABRAMOWITZ and I. A. STEGUN: *Handbook of Mathematical Functions* (New York), chapt. 10.

which is valid if $3\alpha\beta + \gamma^2 \gg \frac{5}{3}(\beta^4)^{\frac{1}{3}}$, and

$$(11)\qquad I(\alpha,\beta,\gamma) = 2\left[\frac{\pi}{(-3\alpha\beta-\gamma^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}\exp\left[\frac{\gamma}{27\beta^2}\left[3(3\alpha\beta+\gamma^2)-\gamma^2\right]\right]\cdot \cdot\cos\left(\frac{2}{27\beta^2}(-\gamma^2-3\alpha\beta)^{\frac{3}{2}}-\frac{\pi}{4}\right),$$

which holds for $\gamma^2 + 3\alpha\beta < 0$ and $|3\alpha\beta + \gamma^2| \gg \frac{5}{3}(\beta^4)^{\frac{1}{3}}$.

In the $\gamma \to 0$ limit, eq. (12) reduces to the result obtained in [2]

$$(12)\qquad I(\alpha,\beta,0) = \frac{2\pi}{(3\beta)^{\frac{1}{3}}}\,Ai\,\frac{\alpha}{(3\beta)^{\frac{1}{3}}}\,.$$

Because of the behaviour of the Airy function, $\alpha \simeq 0$ plays, in the $\gamma = 0$ case, the role of an effective threshold for the reaction. In other words, the reaction yield is drastically reduced for positive values of $\alpha/(3\beta)^{\frac{1}{3}}$. According to eq. (7), it is clear that the effective threshold corresponds to a crossing point of the energy curves in the entrance and in the exit channel. The variation of the form factor ($\gamma \neq 0$) changes this effective threshold.

The method we have sketched here can be extended to $l \neq 0$ transfer case. Furthermore, it is possible to use it together with semi-classical approximations to derive formulae for the differential cross-section and for the optimum Q-value as a function of angle and energy. This will be shown in detail in a subsequent paper [5] as well as the comparison with the experiment and with other methods [6-8] in the same spirit. In the following, we just give a few typical remarks concerning the neutral-particle transfer case.

Let $Z_1 Z_2$ be the product of the charges and let us consider an exothermic reaction $Q > 0$. In that case, $F_1 = F_2 = F = Z_1 Z_2 e^2 R^{-1}$, and the distance R is given by $R = (Z_1 Z_2)e^2 E^{-1}$, where E is the incident energy. The coefficients α, β, γ (see eq. (10)) are now

$$(13)\qquad \alpha = \frac{Q}{\hbar F}\,,\qquad \beta = \frac{1}{6\hbar F}\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right),\qquad \gamma = \frac{\varkappa}{2F\mu_1}\,.$$

We define the optimum Q-value as the Q-value for which $|I|^2$ (or ω) is maximum, or, in other words, for which $\partial|I|^2/\partial\alpha = 0$. Looking at eq. (9), and considering that β is always small for neutron transfer, one can see that large values of γ can shift the maximum of $|I|^2$ significantly from the maximum of the Airy function towards the positive value of $3\alpha\beta + \gamma^2$. Approximating the Airy function as in eq. (13), we find that $Q_{\text{opt}} \simeq 0$. This corresponds to the classical result. However, if γ is small, the effect of the quantum nature of the relative motion can be important. Indeed, in that case, the maximum of ω^2 lies in the vicinity of the maximum of the Airy function. Approximating the latter by a Gaussian, we find

$$(14)\qquad Q_{\text{opt}} = -\frac{\hbar^2\varkappa^2}{2\mu_1}\frac{\mu_2}{\mu_2-\mu_1} + \frac{\hbar\varkappa}{1.12}(2\hbar F)^{\frac{1}{3}}\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)^{-\frac{1}{3}} - 1.018\left(\frac{\hbar^2F^2}{2}\right)^{\frac{1}{3}}\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)^{\frac{1}{3}}.$$

(5) J. CUGNON: to be published.
(6) Z. E. SWITKOWSKI, R. M. WIELAND and A. WINTHER: *Phys. Rev. Lett.*, **33**, 840 (1974).
(7) TER-MARTIROSYAN: *Žurn. Èksp. Teor. Fiz.*, **29**, 713 (1955).
(8) W. E. FRAHN: *Nucl. Phys.*, **272** A, 413 (1976).

All these three terms are due to quantum effects. The last term arises from the difference of relative motion between the entrance and the exit channels. That is the only term given in ref. (2). The first two terms originate from the form factor. The first one looks like the expression proposed in ref. (6). However, it contains the influence of the sign $\mu_2 - \mu_1$, which changes according as the particle is transferred from the heavy or from the light ion. This aspect is absent from the formula provided in ref. (6) in the neutron transfer case.

Another interesting point is the energy dependence of ω at low energy, a question which has an interest of the astrophysical point of view. The relevant values of Q are the positive ones. Otherwise, ω is drastically reduced. Since β is small, $\gamma^2 + 3\alpha\beta$ is expected to be positive and large enough in order that expression (10) is valid. We expand in power series of $3\alpha\beta/\gamma^2$, and get, to the second order,

$$\omega \sim \exp\left[-\eta \frac{Q^2}{E^2}\frac{k}{\varkappa}\right], \tag{15}$$

where η is the Sommerfeld parameter. This expression is slightly different from the one proposed in ref. (6), but it also predicts that the transfer reaction yield can be larger than the compound reaction yield at low energies, provided that the Q-value is not very different from the optimum value.

In conclusion, we have worked out analytical expressions of the reaction yield for heavy ions induced transfer under the Coulomb barrier. They have been obtained under acceptable simplifying assumptions. We have discussed some aspects of the neutron transfer case. We think that our formulae could be useful for qualitative discussions of several aspects of the transfer reaction. Among these, let us mention the Q-dependence of the cross-section.