A Discontinuous Galerkin Formulation of Kirchhoff-Love Shells: From Linear Elasticity to Finite Deformations

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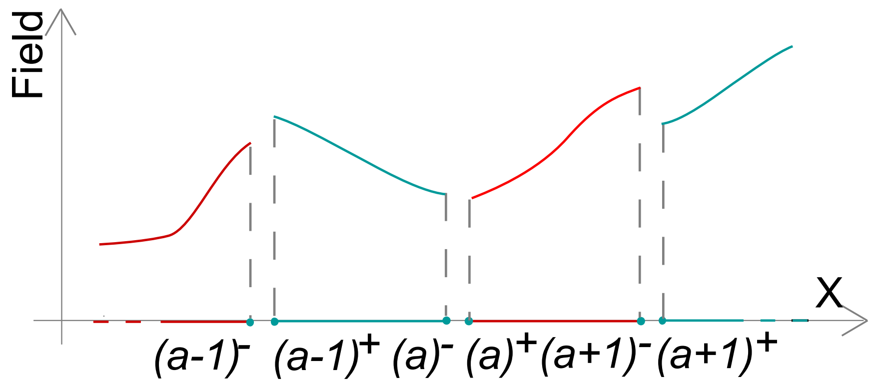
Topics

- Discontinuous Galerkin Methods (DG)
- Kirchhoff-Love Shell Kinematics
- Linear Shells
- Non-Linear Shells
- Conclusions & Perspectives

Discontinuous Galerkin Methods

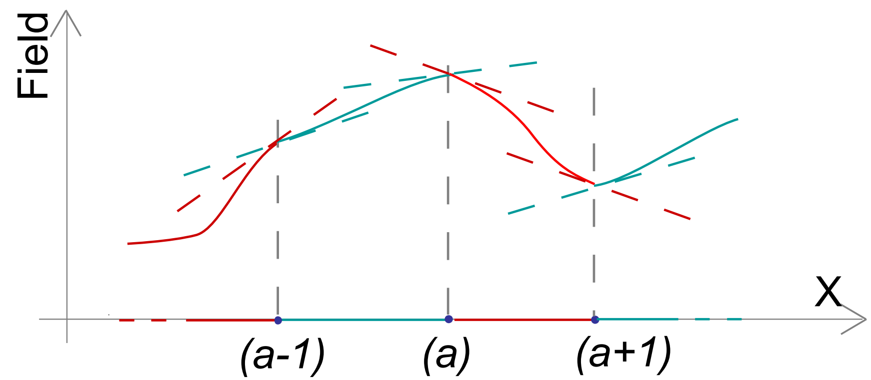
- Main idea
  - Finite-element discretization
  - Same **discontinuous** polynomial approximations for the
    - **Test** functions φ_h and
    - **Trial** functions $\delta\varphi$



  - Definition of operators on the interface trace:
    - **Jump** operator: $[\![\bullet]\!] = \bullet^+ - \bullet^-$
    - **Mean** operator: $\langle\bullet\rangle = \dfrac{\bullet^+ + \bullet^-}{2}$

Discontinuous Galerkin Methods

- Continuous field / discontinuous derivative
  - No new nodes
  - Weak enforcement of C^1 continuity
  - Displacement formulations of high-order differential equations
  - Usual shape functions in 3D (no new requirement)
  - Applications to
    - Beams, plates [Engel et al., CMAME 2002; Hansbo & Larson, CALCOLO 2002; Wells & Dung, CMAME 2007]
    - Linear shells [Noels & Radovitzky, CMAME, 2008]
    - Damage & Strain Gradient [Wells et al., CMAME 2004; Molari, CMAME 2006; Bala-Chandran et al. 2008]

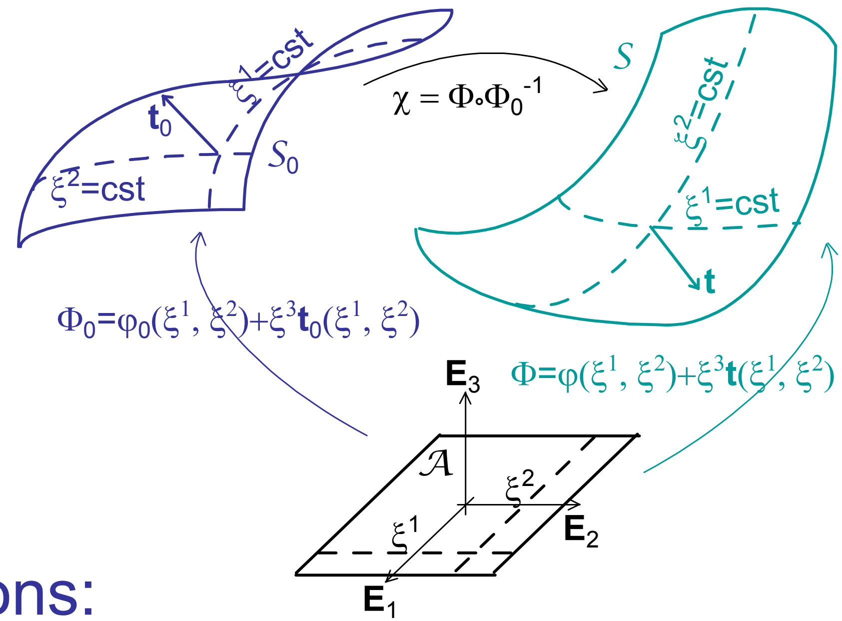


Kirchhoff-Love Shell Kinematics

- Deformation mapping

  $\mathbf{F} = \nabla\boldsymbol{\Phi} \circ [\nabla\boldsymbol{\Phi}_0]^{-1}$ with

  $\nabla\boldsymbol{\Phi} = \boldsymbol{g}_i \otimes \boldsymbol{E}^i$ & $\boldsymbol{g}_i = \nabla\boldsymbol{\Phi}\boldsymbol{E}_i = \dfrac{\partial\boldsymbol{\Phi}}{\partial\xi^i}$



- Shearing is neglected

$$\boldsymbol{t} = \frac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|} \Rightarrow \begin{cases} \boldsymbol{t}_{,\alpha} = \lambda^{\mu}_{\alpha}\boldsymbol{\varphi}_{,\mu} \\ \bar{j} = \|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\| \end{cases}$$

- Resultant equilibrium equations:

$$\frac{1}{\bar{j}}\left(\bar{j}\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \boldsymbol{n}^{\mathcal{A}} = 0 \quad \& \quad \frac{1}{\bar{j}}\left(\bar{j}\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha} - \boldsymbol{l} + \lambda\boldsymbol{t} + \tilde{\boldsymbol{m}}^{\mathcal{A}} = 0$$

  – in terms of resultant stresses:

$$\begin{cases} \boldsymbol{n}^{\alpha} = \dfrac{1}{\bar{j}}\displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\det\left(\nabla\boldsymbol{\Phi}\right) d\xi^3 \\ \tilde{\boldsymbol{m}}^{\alpha} = \dfrac{1}{\bar{j}}\displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \xi^3\boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\det\left(\nabla\boldsymbol{\Phi}\right) d\xi^3 \\ \boldsymbol{l} = \dfrac{1}{\bar{j}}\displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma}\boldsymbol{g}^{3}\det\left(\nabla\boldsymbol{\Phi}\right) d\xi^3 \end{cases}$$

  – and of resultant applied tension $\boldsymbol{n}^{\mathcal{A}}$ and torque $\tilde{\boldsymbol{m}}^{\mathcal{A}}$

Linear Shells

- Assumptions
  - Small displacements $\boldsymbol{\varphi}_{,\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \boldsymbol{u}_{,\alpha} \Longrightarrow \boldsymbol{t}(\boldsymbol{u}) = \boldsymbol{t}_0 + \boldsymbol{\Delta t}(\boldsymbol{u})$ (term in $\boldsymbol{u}_{,\alpha}$)
  - Test functions $\boldsymbol{u}_h$ and trial functions $\delta\boldsymbol{u}$ are C^0
  - Linear constitutive behavior
    - Resultant strain components

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{\varphi}_{,\beta} - \frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} = \frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{u}_{,\beta} + \frac{1}{2}\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta}$$

$$\begin{aligned}\rho_{\alpha\beta} &= \boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{t}_{,\beta} - \boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{t}_{0,\beta} \\ &= \boldsymbol{\varphi}_{0,\alpha\beta}\cdot\boldsymbol{t}_0\frac{e_{\mu\eta3}}{\bar{j}_0}\boldsymbol{u}_{,\mu}\cdot\left(\boldsymbol{\varphi}_{0,\eta}\wedge\boldsymbol{t}_0\right) + \frac{e_{\mu\eta3}}{\bar{j}_0}\boldsymbol{u}_{,\mu}\cdot\left(\boldsymbol{\varphi}_{0,\alpha\beta}\wedge\boldsymbol{\varphi}_{0,\eta}\right) - \boldsymbol{u}_{,\alpha\beta}\cdot\boldsymbol{t}_0\end{aligned}$$

High order term: $- \boldsymbol{u}_{,\alpha\beta}\cdot\boldsymbol{t}_0$

- Elastic constitutive behavior

$$\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{0,\beta} + \lambda^{\beta}_{\mu}\tilde{m}^{\alpha\mu}\boldsymbol{\varphi}_{0,\beta}$$

$$\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\alpha\beta}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0$$

$$\boldsymbol{l} = \lambda\boldsymbol{t}_0 + \lambda^{\alpha}_{\mu}\tilde{m}^{3\mu}\boldsymbol{\varphi}_{0,\alpha}$$

with

$$\tilde{n}^{\alpha\beta} = \frac{E\left(h_{\max}-h_{\min}\right)}{1-\nu^2}\mathcal{H}^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$$

$$\tilde{m}^{\alpha\beta} = \frac{E\left(h_{\max}-h_{\min}\right)^3}{12\left(1-\nu^2\right)}\mathcal{H}^{\alpha\beta\gamma\delta}\rho_{\gamma\delta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}$$

Linear Shells

- DG formulation of linear Kirchhoff-Love shell
  - Definition of a functional $I_h\left(\boldsymbol{u}_h, \varepsilon_{h\alpha\beta}, \rho_{h\alpha\beta}, \tilde{n}_h^{\alpha\beta}, \tilde{m}_h^{I\beta}, \lambda\right)$ accounting for discontinuities in Δt [Noels & Radovitzky, CMAME, 2008]

$$I_h\left(\boldsymbol{u}_h, \varepsilon_{h\alpha\beta}, \rho_{h\alpha\beta}, \tilde{n}_h^{\alpha\beta}, \tilde{m}_h^{I\beta}, \lambda\right) = \int_{\mathcal{A}_h}\left(\frac{1}{2}\varepsilon_{h\alpha\beta}\mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{h\gamma\delta} + \frac{1}{2}\rho_{h\alpha\beta}\mathcal{H}_m^{\alpha\beta\gamma\delta}\rho_{h\gamma\delta}\right)\bar{j}_0 d\mathcal{A} +$$

$$\int_{\mathcal{A}_h}\tilde{n}_h^{\alpha\beta}\left(\frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{u}_{h,\beta} + \frac{1}{2}\boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} - \varepsilon_{h\alpha\beta}\right)\bar{j}_0 d\mathcal{A} +$$

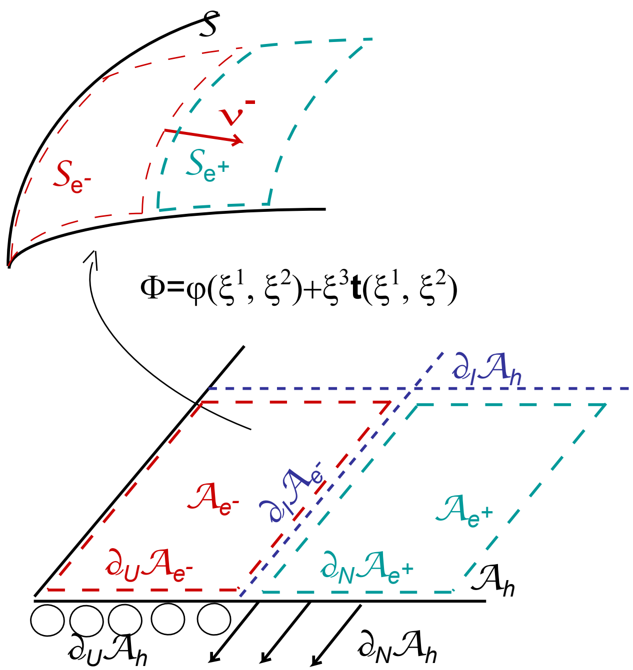
$$\int_{\mathcal{A}_h}\tilde{m}_h^{\alpha\beta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta} - \rho_{h\alpha\beta}\right)\bar{j}_0 d\mathcal{A} -$$

$$\int_{\mathcal{A}_h}\left(\boldsymbol{n}^{\mathcal{A}}\cdot\boldsymbol{u}_h + \tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\right)\bar{j}_0 d\mathcal{A} -$$

$$\int_{\partial_U\mathcal{A}_h}\left(\boldsymbol{u}_h - \bar{\boldsymbol{u}}\right)\cdot\left(\tilde{n}_h^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \lambda_{0\mu}^{\beta}\tilde{m}_h^{\alpha\mu}\boldsymbol{\varphi}_{0,\beta} + \lambda\boldsymbol{t}_0\right)\nu_\alpha\bar{j}_0 d\partial\mathcal{A} +$$

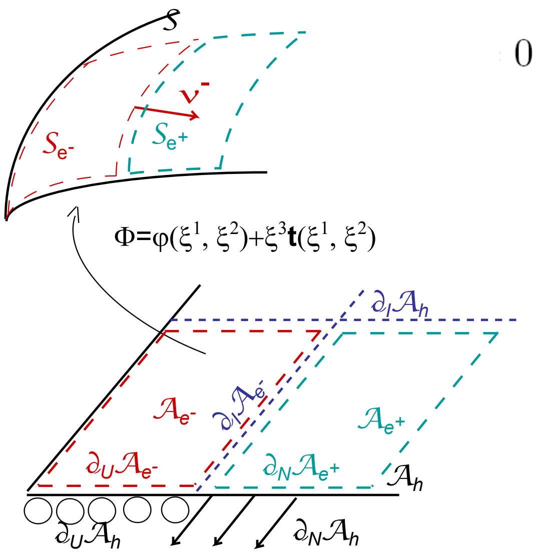
$$\int_{\partial_T\mathcal{A}_h\cup\partial_I\mathcal{A}_h}\left[\!\left[\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\right]\!\right]\cdot\left\langle\bar{j}_0\tilde{m}_h^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \bar{j}_0\tilde{m}_h^{3\alpha}\boldsymbol{t}_0\right\rangle\bar{\nu}_\alpha d\partial\mathcal{A} -$$

$$\int_{\partial_N\mathcal{A}_h}\bar{\boldsymbol{n}}\cdot\boldsymbol{u}_h\bar{j}_0 d\partial\mathcal{A} - \int_{\partial_M\mathcal{A}}\bar{\boldsymbol{m}}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)\bar{j}_0 d\partial\mathcal{A}$$



Linear Shells

- DG formulation of linear Kirchhoff-Love shell
  - Minimization of the functional ⟹ new weak form
  - Introduction of the stabilization parameter β



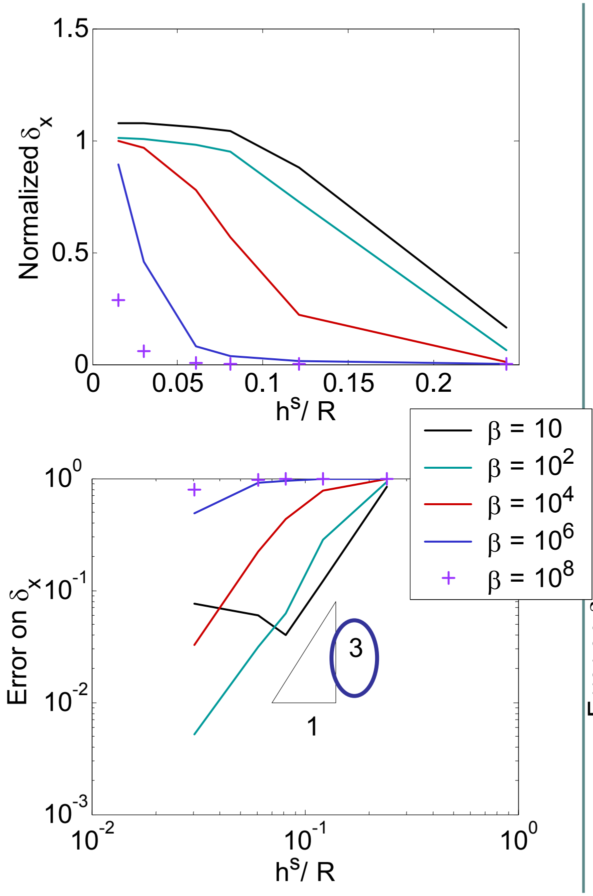
$$
\begin{aligned}
0 = & \int_{\mathcal{A}_h} \left(\frac{1}{2}\boldsymbol{\varphi}_{0,\gamma}\cdot\boldsymbol{u}_{h,\delta} + \frac{1}{2}\boldsymbol{u}_{h,\gamma}\cdot\boldsymbol{\varphi}_{0,\delta}\right)\mathcal{H}_n^{\alpha\beta\gamma\delta}\left(\frac{1}{2}\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{u}_{,\beta} + \frac{1}{2}\boldsymbol{\varphi}_{0,\beta}\cdot\delta\boldsymbol{u}_{,\alpha}\right)\bar{j}_0 d\mathcal{A} + \\
& \int_{\mathcal{A}_h} \left(\boldsymbol{\varphi}_{0,\gamma}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\delta} + \boldsymbol{u}_{h,\gamma}\cdot\boldsymbol{t}_{0,\delta}\right)\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{\Delta t}\left(\boldsymbol{u}\right)_{,\beta} + \delta\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0 d\mathcal{A} + \\
& \int_{\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} [\![\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)]\!]\cdot\left\langle\boldsymbol{\varphi}_{0,\gamma}\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\delta\boldsymbol{\Delta t}\left(\boldsymbol{u}\right)_{,\beta} + \delta\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0\right\rangle\nu_\delta^- d\partial\mathcal{A} + \\
& \int_{\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} [\![\delta\boldsymbol{\Delta t}\left(\boldsymbol{u}\right)]\!]\cdot\left\langle\boldsymbol{\varphi}_{0,\gamma}\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\boldsymbol{\varphi}_{0,\alpha}\cdot\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)_{,\beta} + \boldsymbol{u}_{h,\alpha}\cdot\boldsymbol{t}_{0,\beta}\right)\bar{j}_0\right\rangle\nu_\delta^- d\partial\mathcal{A} + \\
& \int_{s\in\partial_I\mathcal{A}_h\cup\partial\mathcal{A}_h} \frac{\beta}{h^s}[\![\delta\boldsymbol{\Delta t}]\!]\cdot\boldsymbol{\varphi}_{0,\gamma}\nu_\delta^-\left\langle\mathcal{H}_m^{\alpha\beta\gamma\delta}\bar{j}_0\right\rangle[\![\boldsymbol{\Delta t}\left(\boldsymbol{u}_h\right)]\!]\cdot\boldsymbol{\varphi}_{0,\alpha}\nu_\beta^- d\partial\mathcal{A}
\end{aligned}
$$

  - Properties for polynomial approximation of order k
    - Consistent, stable for $\beta > C^k$
    - Convergence rate: k-1 in the e-norm, k+1 in the L^2-norm

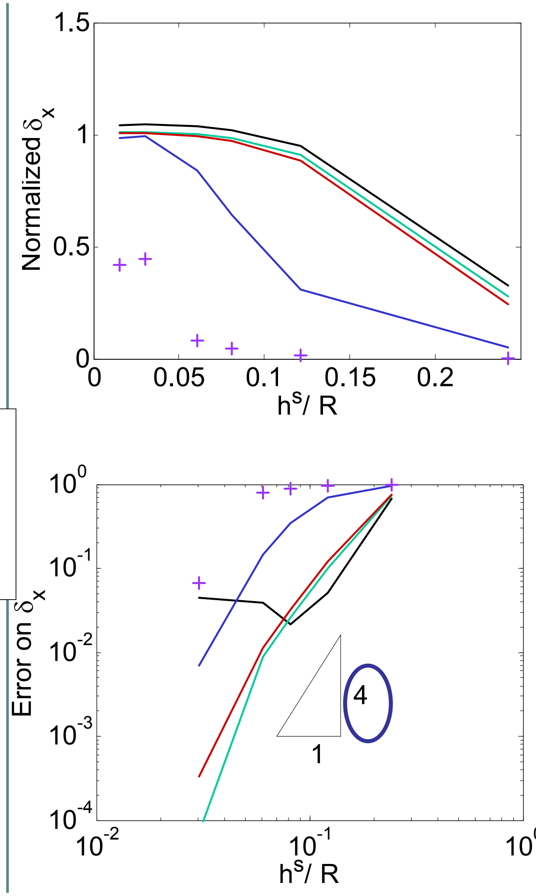
Linear Shells

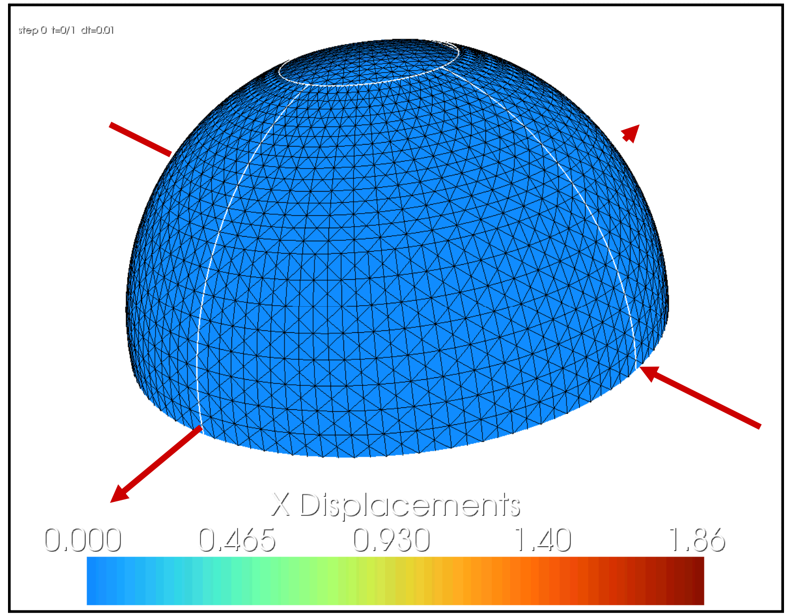
- Pinched open-hemisphere

8-node bi-quad



16-node bi-cubic





Double curvature

- Instability if $\beta \leq 10$
- Locking if $\beta > 1000$ (quad.) and if $\beta > 100000$ (cubic)
- Convergence in $\mathbf{L}^2$ norm: $k+1$

Non-linear Shells

- Material behavior
  - Through the thickness integration by Simpson's rule
  - At each Simpson point
    - Internal energy $W(\mathbf{C}=\mathbf{F}^T\mathbf{F})$ with

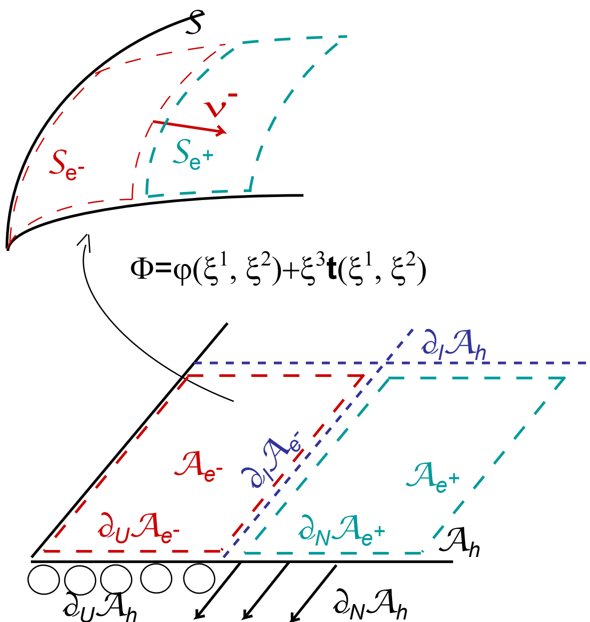
$$\begin{cases} \mathbf{C} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \, \boldsymbol{g}_0^i \otimes \boldsymbol{g}_0^j = \mathrm{g}_{ij} \, \boldsymbol{g}_0^i \otimes \boldsymbol{g}_0^j \\ \boldsymbol{\sigma} = \sigma^{ij} \, \boldsymbol{g}_i \otimes \boldsymbol{g}_j = 2 \dfrac{\det\left(\boldsymbol{\nabla\Phi}_0\right)}{\det\left(\boldsymbol{\nabla\Phi}\right)} \dfrac{\partial W}{\partial \mathrm{g}_{ij}} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \end{cases}$$

    - Iteration on the thickness ratio $\lambda_h = \dfrac{h_{\max} - h_{\min}}{h_{\max 0} - h_{\min 0}}$ in order to reach the plane stress assumption $\sigma^{33}=0$
  - Simpson's rule leads to the resultant stresses:

$$\begin{cases} \boldsymbol{n}^\alpha = \dfrac{1}{\bar{j}} \displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma} \boldsymbol{g}^\alpha \det\left(\boldsymbol{\nabla\Phi}\right) d\xi^3 \\ \tilde{\boldsymbol{m}}^\alpha = \dfrac{1}{\bar{j}} \displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \xi^3 \boldsymbol{\sigma} \boldsymbol{g}^\alpha \det\left(\boldsymbol{\nabla\Phi}\right) d\xi^3 \\ \boldsymbol{l} = \dfrac{1}{\bar{j}} \displaystyle\int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma} \boldsymbol{g}^3 \det\left(\boldsymbol{\nabla\Phi}\right) d\xi^3 \end{cases}$$

Non-linear Shells

- Discontinuous Galerkin formulation
  - New weak form obtained from the momentum equations
  - Integration by parts on each element $\mathcal{A}^e$ but $\delta \boldsymbol{t}$ is discontinuous



$$0 = \int_{\mathcal{A}_e} \left(\bar{j}\boldsymbol{n}^{\alpha}(\boldsymbol{\varphi}_h)\right)_{,\alpha} \cdot \delta\boldsymbol{\varphi} d\mathcal{A} + \int_{\mathcal{A}_e} \boldsymbol{n}^{\mathcal{A}} \cdot \delta\boldsymbol{\varphi}\bar{j} d\mathcal{A} + \int_{\mathcal{A}_e} \left[\left(\bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h)\right)_{,\alpha} - \bar{j}\boldsymbol{l}\right] \cdot \delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_e} \tilde{\boldsymbol{m}}^{\mathcal{A}} \cdot \delta\boldsymbol{t}\lambda_h \bar{j} d\mathcal{A}$$

$$-\sum_e \int_{\bar{\mathcal{A}}_e} \bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h) \cdot (\delta\boldsymbol{t}\lambda_h)_{,\alpha} d\mathcal{A} + \sum_e \int_{\partial\mathcal{A}_e} \bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h) \cdot \delta\boldsymbol{t}\lambda_h \nu_{\alpha} d\mathcal{A}$$

$$\int_{\mathcal{A}_h} \bar{j}\boldsymbol{n}^{\alpha}(\boldsymbol{\varphi}_h) \cdot \delta\boldsymbol{\varphi}_{,\alpha} d\mathcal{A} + \int_{\mathcal{A}_h} \bar{j}\boldsymbol{l} \cdot \delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h} \bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h) \cdot (\delta\boldsymbol{t}\lambda_h)_{,\alpha} d\mathcal{A} + \int_{\partial_I\mathcal{A}_h \cup \partial_T\mathcal{A}_h} [\![\delta\boldsymbol{t} \cdot \bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}]\!] \nu_{\alpha}^{-} d\partial\mathcal{A} = \int_{\partial_N\mathcal{A}_h} \bar{j}\bar{\boldsymbol{n}} \cdot \delta\boldsymbol{\varphi} d\mathcal{A} + \int_{\partial_M\mathcal{A}_h} \bar{j}\bar{\tilde{\boldsymbol{m}}} \cdot \delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h} \boldsymbol{n}^{\mathcal{A}} \cdot \delta\boldsymbol{\varphi}\bar{j} d\mathcal{A} + \int_{\mathcal{A}_h} \tilde{\boldsymbol{m}}^{\mathcal{A}} \cdot \delta\boldsymbol{t}\lambda_h \bar{j} d\mathcal{A}$$

Non-linear Shells

- Interface terms rewritten as the sum of 3 terms
  - Introduction of the numerical flux ***h***

$$\int_{\partial_I \mathcal{A}_h} [\![\bar{j}\tilde{\boldsymbol{m}}^{\alpha}(\boldsymbol{\varphi}_h)\cdot\delta\boldsymbol{t}\lambda_h]\!]\,\nu_{\alpha}^{-}\,d\mathcal{A} \rightarrow \int_{\partial_I \mathcal{A}_h} [\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{h}\left(\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)^{+},\,\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)^{-},\,\nu_{\alpha}^{-}\right)d\mathcal{A}$$

    - Has to be consistent: $\boldsymbol{h}\left(\lambda_h\bar{j}\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}},\,\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}},\,\nu_{\alpha}\right) = \lambda_h\bar{j}\tilde{\boldsymbol{m}}^{\alpha}_{\text{exact}}\nu_{\alpha}^{-}$
    - One possible choice: $\boldsymbol{h}\left(\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)^{+},\,\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)^{-},\,\nu_{\alpha}^{-}\right) = \nu_{\alpha}^{-}\left\langle\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right\rangle$
  - Weak enforcement of the compatibility

$$\int_{\partial_I \mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\left\langle\delta\left(\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right)\right\rangle\nu_{\alpha}^{-}\,d\partial\mathcal{A}$$

$$\int_{\partial_I \mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\left\langle\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\delta\boldsymbol{\varphi}_{,\gamma}\cdot\boldsymbol{t}_{,\delta}+\boldsymbol{\varphi}_{,\gamma}\cdot\delta\boldsymbol{t}_{,\delta}\right)\boldsymbol{\varphi}_{,\beta}+\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\cdot\boldsymbol{\varphi}_{,\beta}\;\;\delta\boldsymbol{\varphi}_{,\beta}\right\rangle\nu_{\alpha}^{-}\,d\partial\mathcal{A}$$

  - Stabilization controlled by parameter β, for all mesh sizes h^s

$$\int_{\partial_I \mathcal{A}_h\cup\partial_T \mathcal{A}_h} [\![\boldsymbol{t}(\boldsymbol{\varphi}_h)]\!]\cdot\boldsymbol{\varphi}_{,\beta}\left\langle\frac{\beta\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}}{h^s}\right\rangle[\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{\varphi}_{,\gamma}\nu_{\alpha}^{-}\nu_{\delta}^{-}\,d\partial\mathcal{A}$$

Non-linear Shells

- New weak formulation

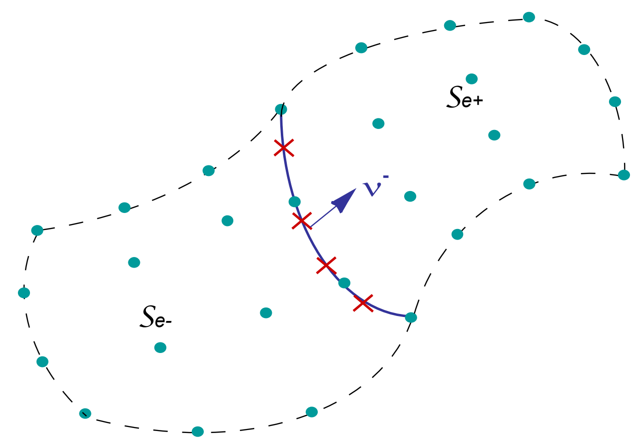
$$\int_{\mathcal{A}_h} \bar{j}\boldsymbol{n}^{\alpha}\left(\boldsymbol{\varphi}_h\right)\cdot\delta\boldsymbol{\varphi}_{,\alpha}d\mathcal{A} + \int_{\mathcal{A}_h} \bar{j}\tilde{\boldsymbol{m}}^{\alpha}\left(\boldsymbol{\varphi}_h\right)\cdot\left(\delta\boldsymbol{t}\lambda_h\right)_{,\alpha}d\mathcal{A} + \int_{\mathcal{A}_h}\bar{j}\boldsymbol{l}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} +$$

$$\int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h} [\![\boldsymbol{t}\left(\boldsymbol{\varphi}_h\right)]\!]\cdot\left\langle\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}\left(\delta\boldsymbol{\varphi}_{,\gamma}\cdot\boldsymbol{t}_{,\delta}+\boldsymbol{\varphi}_{,\gamma}\cdot\delta\boldsymbol{t}_{,\delta}\right)\boldsymbol{\varphi}_{,\beta}+\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\cdot\boldsymbol{\varphi}_{,\beta}\ \ \delta\boldsymbol{\varphi}_{,\beta}\right\rangle\nu_{\alpha}^{-}d\partial\mathcal{A}$$

$$\int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h}[\![\delta\boldsymbol{t}]\!]\cdot\left\langle\bar{j}\lambda_h\tilde{\boldsymbol{m}}^{\alpha}\right\rangle\nu_{\alpha}^{-}d\partial\mathcal{A} + \int_{\partial_I\mathcal{A}_h\cup\partial_T\mathcal{A}_h}[\![\boldsymbol{t}\left(\boldsymbol{\varphi}_h\right)]\!]\cdot\boldsymbol{\varphi}_{,\beta}\left\langle\frac{\beta\bar{j}_0\mathcal{H}_m^{\alpha\beta\gamma\delta}}{h^s}\right\rangle[\![\delta\boldsymbol{t}]\!]\cdot\boldsymbol{\varphi}_{,\gamma}\nu_{\alpha}^{-}\nu_{\delta}^{-}d\partial\mathcal{A} =$$

$$\int_{\partial_N\mathcal{A}_h}\bar{j}\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{\varphi}d\mathcal{A} + \int_{\partial_M\mathcal{A}_h}\bar{j}\bar{\tilde{\boldsymbol{m}}}\cdot\delta\boldsymbol{t}\lambda_h d\mathcal{A} + \int_{\mathcal{A}_h}\boldsymbol{n}^{\mathcal{A}}\cdot\delta\boldsymbol{\varphi}\bar{j}d\mathcal{A} + \int_{\mathcal{A}_h}\tilde{\boldsymbol{m}}^{\mathcal{A}}\cdot\delta\boldsymbol{t}\lambda_h\bar{j}d\mathcal{A}$$

- Implementation
  - Shell elements
    - Membrane and bending responses
    - 2x2 (4x4) Gauss points for bi-quadratic (bi-cubic) quadrangles
  - Interface element
    - 3 interface contributions
    - 2 (4) Gauss points for quadratic (cubic) meshes
    - Contributions of neighboring shells evaluated at these points



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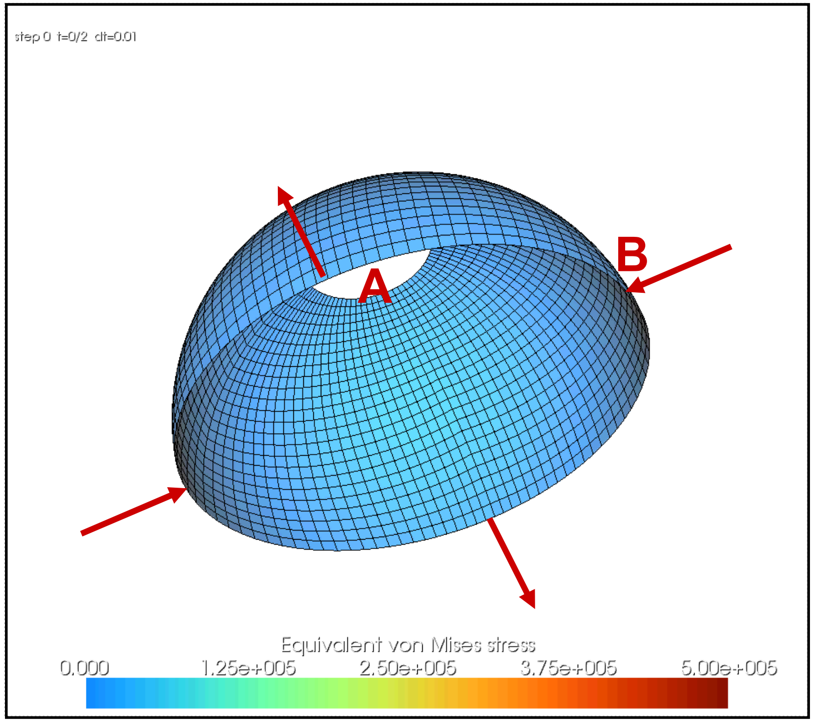
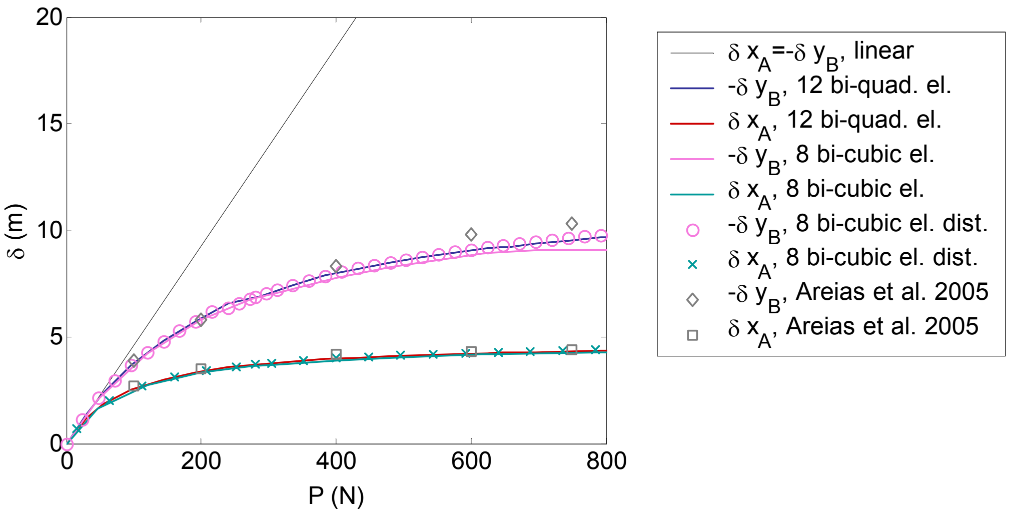
Non-linear Shells

- Pinched open hemisphere
  - Properties:
    - 18° hole
    - Thickness 0.04 m; Radius 10 m
    - Young 68.25 MPa; Poisson 0.3
  - Comparison of DG method
    - Quadratic, cubic & distorted el.

    and literature

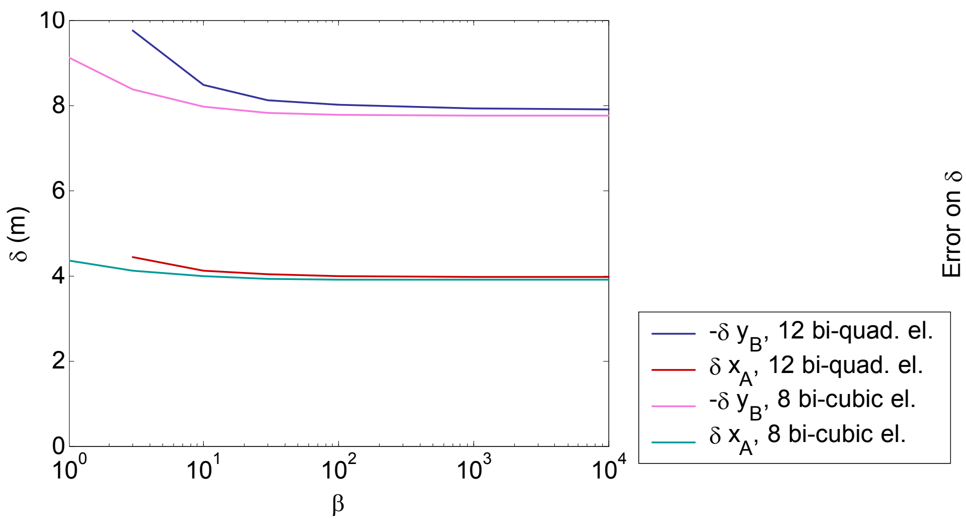




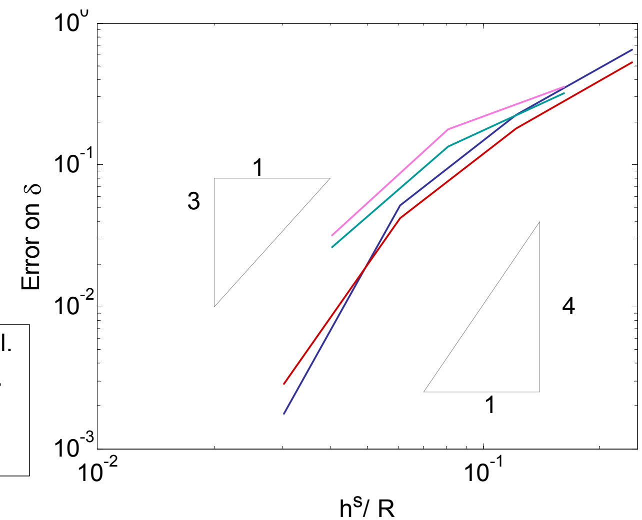
Non-linear Shells

- Pinched open hemisphere

Influence of the stabilization parameter



Influence of the mesh size

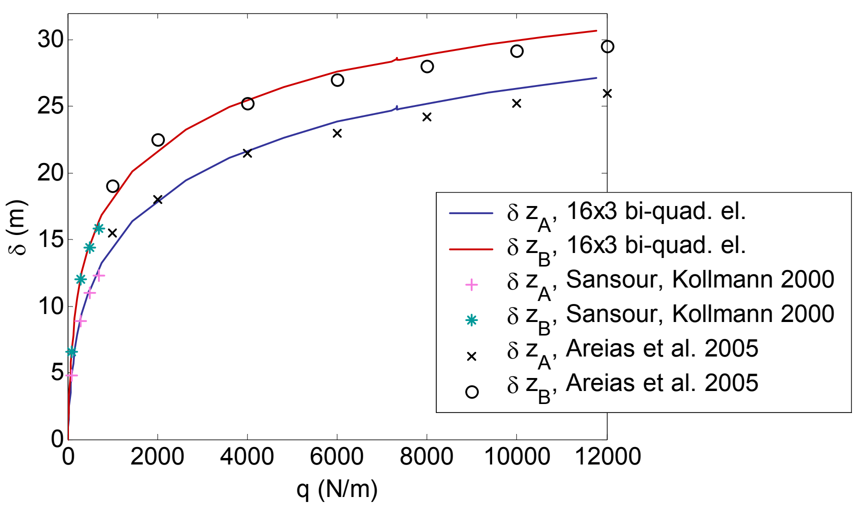


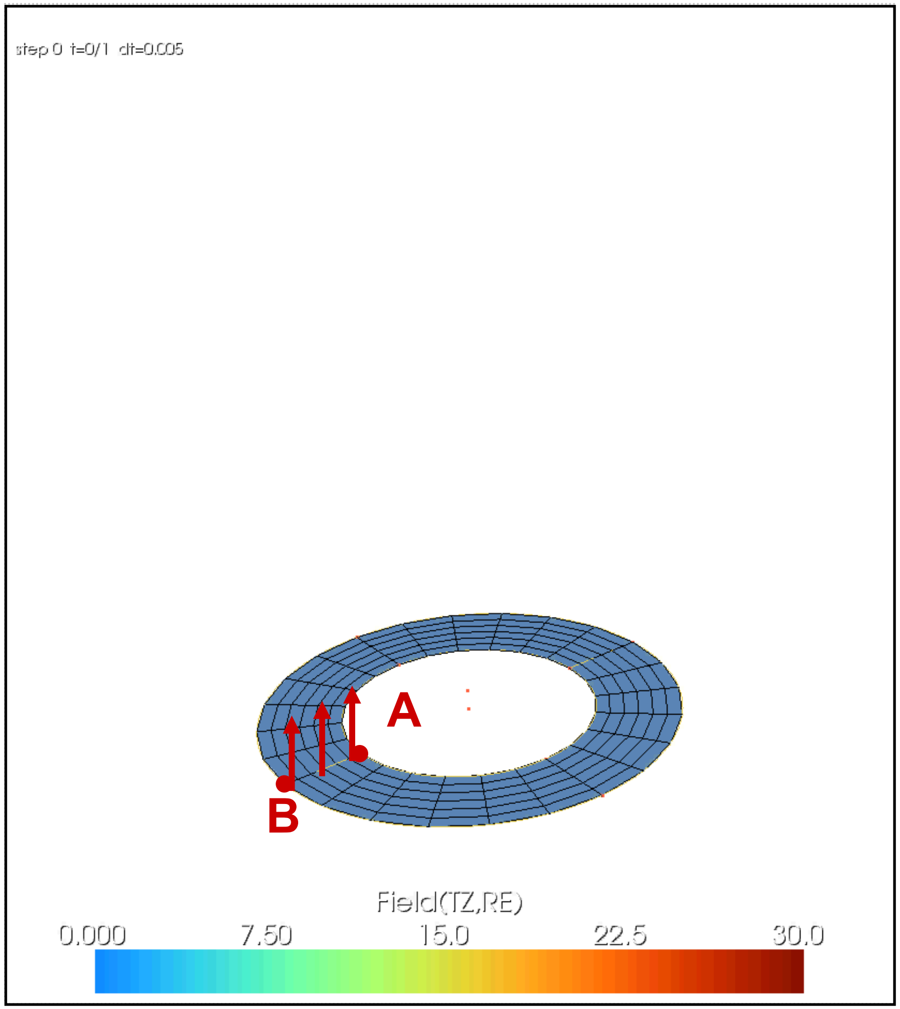
  – Stability if $\beta > 10$
  – Order of convergence in the $\mathbf{L}^2$-norm in k+1

Non-linear Shells

- Plate ring
  - Properties:
    - Radii 6 -10 m
    - Thickness 0.03 m
    - Young 12 GPa; Poisson 0
  - Comparison of DG method
    - Quadratic elements

    and literature

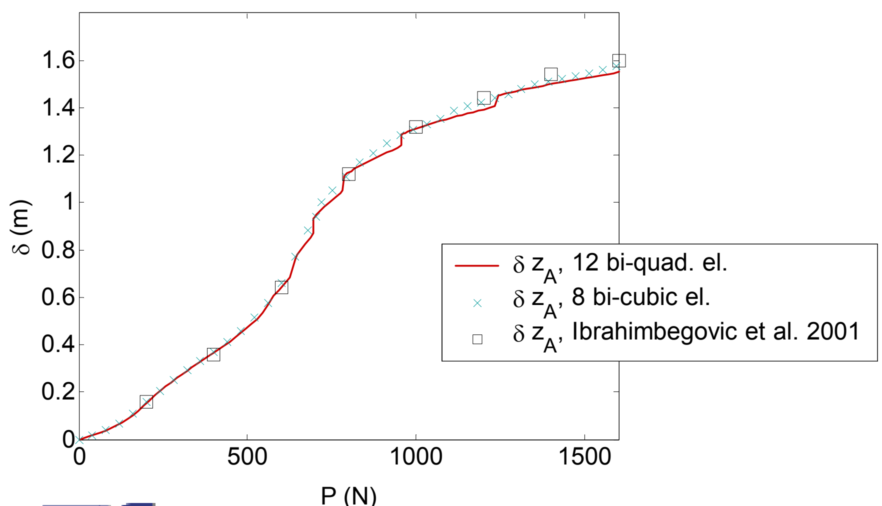


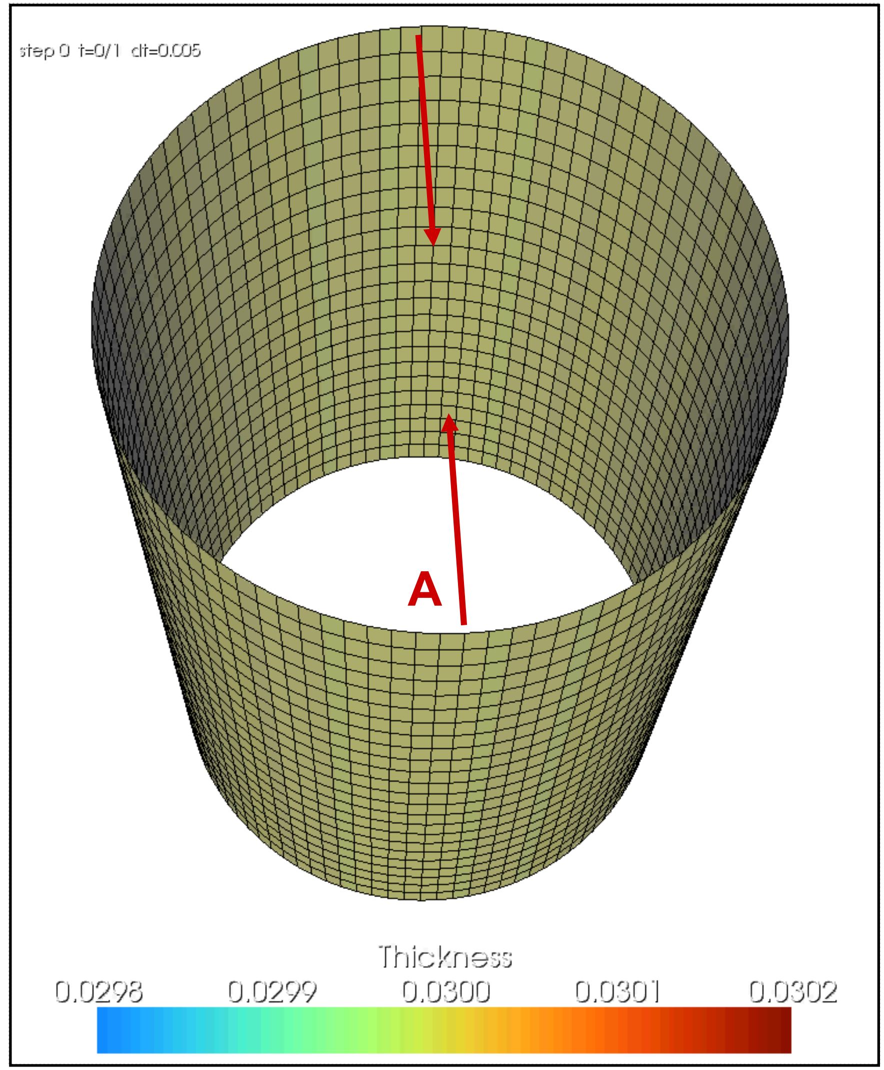


Non-linear Shells

- Clamped cylinder
  - Properties:
    - Radius 1.016 m; Length 3.048 m; Thickness 0.03 m
    - Young 20.685 MPa; Poisson 0.3
  - Comparison of DG method
    - Quadratic & cubic elements

    and literature





Conclusions & Perspectives

- Development of a discontinuous Galerkin framework for non-linear Kirchhoff-Love shells
  - Displacement formulation (no additional degree of freedom)
    - Strong enforcement of C^0 continuity
    - Weak enforcement of C^1 continuity
  - Quadratic elements:
    - Method is stable if $\beta \geq 10$
    - Reduced integration
  - Cubic elements:
    - Method is stable if $\beta \geq 10$
    - Full Gauss integration
  - Convergence rate:
    - k-1 in the energy norm
    - k+1 in the L2-norm
- Perspectives: plasticity, dynamics …