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# Average first-passage time of a quasi-Hamiltonian Mathieu oscillator with parametric and forcing excitations

H. Vanvinckenroye<sup>a,b</sup>, V. Denoël<sup>a,\*</sup><sup>a</sup> Structural Engineering Division, Faculty of Applied Sciences, University of Liège, Liège, Belgium<sup>b</sup> FRIA (F.R.S.-F.N.R.S), National Found for Scientific Research, Brussels, Belgium

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## ABSTRACT

A linear oscillator simultaneously subjected to stochastic forcing and parametric excitation is considered. The time required for this system to evolve from a low initial energy level until a higher energy state for the first time is a random variable. Its expectation satisfies the Pontryagin equation of the problem, which is solved with the asymptotic expansion method developed by Khasminskii. This allowed deriving closed-form expressions for the expected first passage time. A comprehensive parameter analysis of these solutions is performed. Beside identifying the important dimensionless groups governing the problem, it also highlights three important regimes which are called incubation, multiplicative and additive because of their specific features. Those three regimes are discussed with the parameters of the problem.

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## 1. Introduction

### 1.1. The considered governing equation

Many physical problems can be modeled by the forced and damped Mathieu equation

$$\ddot{x} + 2\xi\dot{x} + [1 + u]x = w \quad (1)$$

with  $x(t)$  the state variable as a function of time  $t$  and  $\xi$  a damping coefficient. This model involves parametric excitation, represented by  $u(t)$ , and external forcing  $w(t)$ . For instance a vertical motion of the support of a pendulum generates this kind of parametric excitation. It has been widely studied in its most simple case where the parametric excitation is harmonic, *viz.*  $u(t) = \lambda \cos \nu t$ , and, in some instances, including the possible large rotations of the pendulum, *i.e.* replacing  $x$  by  $\sin x$  in the above equation. This equation develops in various forms, from deterministic to stochastic, depending on the non-deterministic nature of the parameters of the problem and of the excitations. Notable works studying the transition from deterministic to stochastic conditions include the studies about the evolution of unstable regions for an excitation that continuously varies from periodic to stochastic [32]. Besides, Gitterman studies the stability and the period of the pendulum under deterministic and stochastic excitations of its support [13,12]. He observes that an increasing stochasticity of the excitation induces larger unstable ranges of parameters of lower intensity. In those works, it is shown that, assuming no

\* Corresponding author.

E-mail addresses: [hvanvinckenroye@ulg.ac.be](mailto:hvanvinckenroye@ulg.ac.be) (H. Vanvinckenroye), [v.denoel@ulg.ac.be](mailto:v.denoel@ulg.ac.be) (V. Denoël).

damping and depending on the frequency content and magnitude of the excitation, the pendulum describes oscillations or eventually complete rotations.

Narrow band and random phase excitations are investigated by Alevras and Yurchenko in [49,2] through a numerical path integration providing complex stability lobes in a configuration where the support is submitted to a vertical harmonic excitation of frequency equal to an integer multiple of the fundamental frequency. Higher energy lobes are observed for a frequency of the excitation close to the fundamental frequency of the pendulum or a fraction of it. Xu presents similar results in [46] for a harmonically excited pendulum by evidencing the basins of attraction in the phase plane. Mallick presents an analytical method providing an expression for the asymptotic probability distribution function of the energy in [24]. The current work mainly focuses on the stochastic version of the forced and parametric oscillator, without any deterministic component. Similar problems have been studied by other authors who developed analytical solutions of the deterministic problem including the different stable regions thanks to the harmonic balance method, the perturbation method and the critical velocity criterion [11,4,45].

In the following *quasi-Hamiltonian* oscillator, the concept of energy stored in the oscillator plays a central role. In fact, interest has been shown for the energy of a system that is governed by a Mathieu equation in the study of a pendulum submitted to wave excitations in [28,47,1] and finds a direct application in the extraction of energy from waves and heave, as also discussed earlier by [17,16]. In the same way, capsizing and rolling motions of ships under stochastic wave excitation can also be assimilated to similar oscillators and are studied by Moshchuk and Troesch in [27,39].

In order to understand and characterize different crane instabilities in gusty wind conditions, Voisin performed experimental analyses and determined the susceptibility of a tower crane to autorotation when it is left free to rotate in a given environment ([42,43]). This approach allows to experimentally assess field configurations against undesired crane autorotation which could potentially lead to dramatic failures. Following this model, the crane can be represented by a single-degree-of-freedom model composed of a rigid jib rotating around a fixed pivot with an angle  $\theta$ . The linearized governing equation of a jib with inertia  $I$  and damping coefficient  $C$  is characterized by the following stochastic Mathieu equation

$$I\ddot{\theta} + C\dot{\theta} - M_w(\theta, \dot{\theta}, u(t), w(t)) = 0 \quad (2)$$

in which the wind torque  $M_w$  depends on the two horizontal components  $u(t)$  and  $w(t)$  of the wind field and typically shows the features of a parametric excitation [41].

As another example, the deflection of a horizontal cable subjected to an axial oscillation of one anchorage is described by the modified Mathieu equation [7]

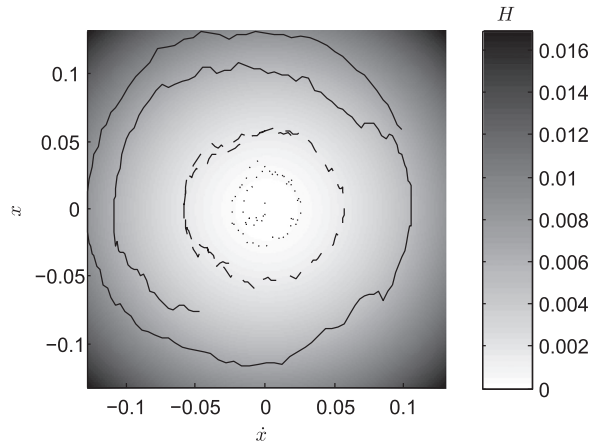
$$\ddot{\alpha} + 2\xi\dot{\alpha} + \left(1 + \frac{\lambda^2}{\pi^2} + x_b(t)\right)\alpha + \nu_2\alpha^2 + \nu_3\alpha^3 = 0. \quad (3)$$

with  $\alpha(t)$  the deflection at mid-span in the first mode,  $\xi$  the damping ratio,  $\lambda^2$  the Irvine parameter and  $\nu_1$  and  $\nu_2$  two parameters related to the cable elasticity. The motion of the cable support  $x_b(t)$  is assumed to be deterministic in many cases, but could also be represented by a stochastic process as a result of the buffeting action of wind on a bridge deck, for example [5].

Using the appropriate non-dimensionalization and discarding the nonlinear governing components, the governing equations of these problems, as well as a large number of other applications, can be cast under the format of Eq. (1) where  $u(t)$  and  $w(t)$  are stochastic processes. This is the general equation studied in this paper.

## 1.2. First passage time of the parametric and forced oscillators

As a heritage of deterministic dynamics, the externally and parametrically excited oscillator is usually studied with the scope to determine the stability zones, the amplitude of the limit cycle oscillations or steady state solutions. These objectives lose interest in undamped or even very slightly damped oscillators, since a steady-state configuration takes too long to develop. It is possible to demonstrate [35] that, for the considered governing equation, any large value of the generalized coordinate  $x$ , as large as desired, is encountered with probability one in the undamped case. This mathematical fact therefore suggests to reconsider the practical questions related to boat capsizing, large cable vibration or crane autorotations, from a different standpoint since any large amplitude could be reached by waiting sufficiently long enough (in case dissipation is neglected). We therefore decide to investigate the time required for the system to reach a certain displacement or amplitude, given an initial condition, or to reach a given energy barrier departing from a lower initial energy level. This is known as a *first passage* problem. The objective of such a problem is to determine the time required for a system to evolve from a known initial condition and to reach a given state. In a stochastic context, this time is a random variable. It is characterized by its probability density function which can be obtained by Monte Carlo simulations [9,29] or by resolution of the Chapman-Kolmogorov equation, via use of numerical [22] or semi-analytical methods such as a numerical path integration [48,21], approximation of the solution by a Galerkin scheme [37,36], a Poisson distribution based assumption [3]. These numerical approaches are important because there are very few problems in which the distribution of the first passage time can be established in closed-form [44,40]. This work seeks to provide an analytical understanding of this problem and therefore focuses on the average value of the first passage time instead of the complete distribution.



**Fig. 1.** Fragments of the trajectory of the system in the phase plane and contours of the Hamiltonian. Numerical values:  $\xi = 0.01$ ,  $S_u = 10^{-2}$  and  $S_w = 10^{-4}$ .

Since the problem at hand is particularly interesting when the damping and the intensities of the excitations are small – otherwise the steady regime develops fast enough – the considered oscillator actually happens to be a *quasi-Hamiltonian* system for which the total internal energy  $H(t)$  evolves on a slow time scale [24]. The energy balance of the governing equation, obtained by time integration of the power fluxes, yields

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \int (\xi \dot{x}^2 + u x \dot{x}) dt = \int w \dot{x} dt. \tag{4}$$

This indeed shows that the total internal energy (referred to as *the Hamiltonian*, by extension, in the sequel) defined by

$$H = \frac{x^2}{2} + \frac{\dot{x}^2}{2}, \tag{5}$$

composed of potential and kinetic energies, is slowly varying, since  $\dot{H} = \text{ord}(\epsilon)$  if  $\{\xi, u, w\} = \text{ord}(\epsilon)$ . This is also illustrated in Fig. 1 which shows, in the phase-plane  $(x, \dot{x})$ , three fragments of one realization of the system, considered to be subjected to two  $\delta$ -correlated white noises of intensities  $S_u = 10^{-2}$  and  $S_w = 10^{-4}$ , while the damping ratio is set to  $\xi = 0.01$ . It is seen that the trajectories are nearly tangent to the ellipses of constant energy, which indicates that the Hamiltonian varies by only a small quantity over one period of revolution of the unperturbed dynamical system. These observations support the quasi-Hamiltonian nature of the system, which is, in this case, a consequence of the smallness of  $\xi$ ,  $S_u$  and  $S_w$ .

The first passage problem of quasi-Hamiltonian systems has been studied in the literature in the undamped configuration ( $\xi = 0$ ) and without external forcing term ( $w=0$ ). The stochastic parametric excitation  $u(t)$  has first been considered to be a  $\delta$ -correlated process [38] and more recently an Ornstein-Uhlenbeck process [31]. In the latter case, Potapov [31] also proposes an estimation of the problem stability based on its Liapunov exponents, an approach that is also followed in [23] for quasi non-integrable Hamiltonian systems under Gaussian and Poisson white noises through the averaged Itô equation.

Still considering the undamped configuration ( $\xi = 0$ ) and without external forcing ( $w=0$ ), the stochastic differential equation governing the Hamiltonian reads

$$dH = k_1 H dt + k_2 H dB, \tag{6}$$

with  $k_1 = S_u/2$  and  $k_2 = \sqrt{S_u}/2$  two parameters depending on the spectral form of the parametric excitation  $u$ , and  $dB$  the increment of a  $\delta$ - correlated Brownian noise  $B(t)$  [33,6]. This equation can be solved explicitly [33]:

$$H(t) = H_0 \exp \left[ k_1 B_t + \left( k_1 - \frac{k_2^2}{2} \right) t \right]. \tag{7}$$

The first passage time of the energy level  $H_c$ , starting from a lower initial energy  $H_0$ , follows an inverse Gaussian distribution with parameters  $\frac{\ln(H_c/H_0)}{k_1 - k_2^2/2}$  and  $\frac{\ln(H_c/H_0)^2}{k_2^2}$ , so that its mean first passage time can be expressed as [33]

$$\mathbb{E} \left[ \inf \{ t | H(t) \geq H_c \} \right] := U(H_0) = \frac{4}{S_u} \ln \frac{H_c}{H_0}. \tag{8}$$

It is rather rare that the stochastic differential equation of a (more complex but realistic) problem can take a simple explicit solution as (7). The complete probability density function of the first passage time is therefore seldom available. However, the mean first passage time of a stochastic system through the boundary  $\partial \mathcal{D}$  of a given domain  $\mathcal{D}$ , to which the system belongs at initial state, is ruled by the Pontryagin equation, see Section 2.

Many solutions of more or less complex problems in terms of mean first passage time can be found in the literature [6,35,25,26,22]. Among them the first passage time for a stochastic fractional derivative system with power-law restoring force [22] shows the typical range of difficulties that can be tackled today. An important contribution in the field concerns the works of Khasminskii [20] which consist in the formal development of an asymptotic analysis of the Pontryagin equation. This is precisely the approach that is followed in this work.

Based on this review of the literature, we have identified the first passage time of an undamped system subjected to both parametric and external forcing as a novel contribution. This problem is solved in closed form and the solutions are thoroughly discussed in Section 2. Then, two variants are presented. The first one develops and discusses the second order terms and the boundary layer contribution of Khasminskii's theory while the second variant considers the leading order solution for the slightly damped (parametrically and externally forced) oscillator.

## 2. The undamped, externally and parametrically forced oscillator

The problem considered in this Section is

$$\ddot{x} + (1 + u)x = w \tag{9}$$

where  $u(t)$  and  $w(t)$  are Brownian  $\delta$ -correlated white noises of small intensities  $S_u$  and  $S_w$ . Formally this problem is represented in the state-space  $\mathbf{x} = (x, \dot{x}) = (q, p)$  by its Itô formulation for Markov times, i.e. for each  $t > t_0$ , by

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)dt + \mathbf{b}(\mathbf{x}, t)d\mathbf{B}, \tag{10}$$

where  $\mathbf{x} = \begin{bmatrix} q \\ p \end{bmatrix}$ ,  $\mathbf{f} = \begin{bmatrix} p \\ -q \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 0 & 0 \\ -q & 1 \end{bmatrix}$  and where  $\mathbf{B} = \begin{bmatrix} B_u \\ B_w \end{bmatrix}$  is the vector of Brownian motions characterized by the power spectrum matrix

$$\mathbf{S} = \begin{bmatrix} S_u & S_{uw} \\ S_{uw} & S_w \end{bmatrix} = \varepsilon \boldsymbol{\nu} = \varepsilon \begin{bmatrix} \nu_{uu} & \nu_{uw} \\ \nu_{uw} & \nu_{ww} \end{bmatrix}, \tag{11}$$

where  $\varepsilon \ll 1$  and  $\boldsymbol{\nu}$  is an order-one matrix. It is interesting to notice that (10) is a perturbation of a conservative system which evolves along closed trajectories of constant total internal energy  $H$ . The period of revolution of a complete orbit of the unperturbed system ( $\varepsilon = 0$ , so that  $u = w = 0$ ),

$$T = 2 \int_{q_1}^{q_2} \frac{dq}{\dot{q}} = 2 \int_{-\sqrt{2H}}^{\sqrt{2H}} \frac{1}{\sqrt{2H - q^2}} dq = 2\pi, \tag{12}$$

is independent of the considered energy level  $H$ .

Let  $\mathcal{D}$  be a closed domain in the phase plane and an initial condition  $\mathbf{x}_0 \in \mathcal{D}$ . The average first passage time  $U(\mathbf{x}_0)$  for the trajectories of the dynamical system to reach the boundary  $\partial\mathcal{D}$  is given by the Pontryagin equation [30]

$$\mathcal{L}\{U(\mathbf{x}_0)\} = -1, \quad \text{for } \mathbf{x}_0 \in \mathcal{D} \tag{13}$$

with the boundary conditions

$$U(\mathbf{x}_0) = 0, \quad \forall \mathbf{x}_0 \in \partial\mathcal{D} \quad \text{and} \quad |U(\mathbf{0})| < \infty, \tag{14}$$

which translates that the mean first passage time is equal to zero for trajectories starting on the boundary and that the average time required to reach the boundary starting from  $\mathbf{x}_0 = 0$ , is finite. For the sake of simplicity in the notations, the subscript "0" is omitted in the following developments. This convention prevails for both the initial position and velocity,  $q_0$  and  $p_0$  which are the components of  $\mathbf{x}_0$  and, later, for the Hamiltonian  $H_0$ . In (13),  $\mathcal{L}\{\cdot\}$  is the Backward-Kolmogorov operator associated with the governing equation, see e.g. [35], given by

$$\mathcal{L}\{\cdot\} = \frac{1}{2} \text{Tr} \left\{ \left[ \frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{x}} (\cdot) \right] \boldsymbol{\sigma} \right\} + \mathbf{f}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}}, \tag{15}$$

where

$$\boldsymbol{\sigma} = \varepsilon \mathbf{b}(\mathbf{x}, t) \boldsymbol{\nu} \mathbf{b}^T(\mathbf{x}, t) = \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & q^2 \nu_{uu} + \nu_{ww} - 2q \nu_{uw} \end{bmatrix} \tag{16}$$

is a drift matrix of order  $\varepsilon$  and  $\mathbf{x} = (q, p)$  is the state vector.

The higher order derivative in  $\mathcal{L}\{\cdot\}$  is therefore multiplied by a small coefficient, of order  $\varepsilon$ , which is responsible for a well-known boundary layer in perturbation methods [15]. With this in mind and following Khasminskii's approach [20], an asymptotic expansion method is developed in order to establish closed-form expressions for the solution of the Pontryagin equation. This not only avoids the numerical solution of (13) but also provides a much better understanding of the features

of the problem, as explicit expressions for the mean first passage time are obtained, under the sole hypothesis that  $\varepsilon \ll 1$ , *i.e.* that the dimensionless intensities of the external and parametric excitations are small. Following this approach, the operator is decomposed into two operators, each one acting at its own scale in  $\varepsilon$ , as

$$\mathcal{L}\{\cdot\} = \mathcal{L}_1\{\cdot\} + \varepsilon \mathcal{L}_2\{\cdot\}, \quad (17)$$

where, after replacement of the derivatives in  $\mathbf{x}$ ,

$$\mathcal{L}_1 = p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p}; \quad \mathcal{L}_2 = \frac{1}{2} (q^2 \nu_u + \nu_w - 2q\nu_{uw}) \frac{\partial^2}{\partial p^2}. \quad (18)$$

Following the matched asymptotic expansion solution applied in [25,26] to the capsizing of boats in random seas (with external forcing only, though), a composite solution to (13) is provided as the sum of the outer and inner solutions

$$U(p, q) = U_n(p, q) + G_n(p, q), \quad (19)$$

where  $U_n$  is the outer solution obtained with the regular ansatz

$$U_n(p, q) = \frac{1}{\varepsilon} u_0(p, q) + u_1(p, q) + \dots + \varepsilon^{n-1} u_n(p, q) \quad (20)$$

and  $G_n$  stands for the inner solution in the boundary layer, in the neighborhood of  $\partial\mathcal{D}$ . The error of the approximate solution has the same order as the first neglected term, *i.e.*  $\varepsilon^n$ . In order not to include the overlap between the inner and outer solutions that anyway needs to be discarded in the composite solution [15], the outer and inner problems are respectively solved with the following righthand sides,  $\mathcal{L}\{U_n\} = -1$  and  $\mathcal{L}\{G_n\} = 0$ . While the efforts presented in this Section concentrate on the establishment of  $u_0$ , the outer solution will be developed up to the second term  $u_1$  in Section 3. Collecting terms of likewise powers of  $\varepsilon$  in (13) yields:

$$\text{ord}(\varepsilon^{-1}): \quad \mathcal{L}_1 u_0 = 0 \quad (21)$$

$$\text{ord}(\varepsilon^0): \quad \mathcal{L}_1 u_1 + \mathcal{L}_2 u_0 = -1 \quad (22)$$

$$\text{ord}(\varepsilon^1): \quad \mathcal{L}_1 u_2 + \mathcal{L}_2 u_1 = 0 \quad (23)$$

The leading order solution  $U_0$  is actually nothing but the result of the stochastic averaging method [34], which roughly assumes that the Hamiltonian is constant along one period of motion. The higher order terms provided by the asymptotic expansion extend the validity of the developments to moderate values of the small parameter, *i.e.*  $\varepsilon \lesssim 1$ . However, for problems slightly more complex than (9), the second and higher correction terms take awkward expressions which cuts down the advantages of Khasminskii's asymptotic expansion over the more usual stochastic averaging approach. This is discussed in Section 3.

Operator  $\mathcal{L}_1$  represents the derivative along the direction of the conservative system, *i.e.* along the orbits of constant Hamiltonian  $H$ . As a result, the leading order Eq. (21) means that  $u_0$  is constant along each orbit of constant energy. It is consequently a function of the Hamiltonian  $H$  only. Averaging (22) along a period  $T$  of the orbit, provides the information to determine  $u_0(H)$ . Indeed, as the orbits are closed, averaging  $\mathcal{L}_1 u_1$  along each of these trajectories gives zero and Eq. (22) becomes  $\langle \mathcal{L}_2 u_0 \rangle = -1$ , or

$$\frac{1}{2} \left[ \langle q^2 \nu_u + \nu_w - 2q\nu_{uw} \rangle \frac{du_0}{dH} + \langle p^2 (q^2 \nu_u + \nu_w - 2q\nu_{uw}) \rangle \frac{d^2 u_0}{dH^2} \right] = -1, \quad (24)$$

where the following relations have been used for the partial derivatives

$$\frac{\partial u_0}{\partial p} = p \frac{du_0}{dH}; \quad \frac{\partial^2 u_0}{\partial p^2} = \frac{du_0}{dH} + p^2 \frac{d^2 u_0}{dH^2} \quad (25)$$

since  $u_0 \equiv u_0(H)$  is a function of the initial Hamiltonian only and the operator  $\langle \cdot \rangle$  represents the average over one period  $T = 2\pi$  of the unperturbed motion,

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdot dt. \quad (26)$$

The averaging of Eq. (24) is derived, term by term, by changing the variables  $q$  and  $p$  into the energy-phase variables  $k$  and  $\theta$  with

$$p = 2k \cos \theta; \quad q = 2k \sin \theta \quad (27)$$

so that the Hamiltonian is now given by  $H = 2k^2$ .

Finally, at order  $\varepsilon^0$ , the averaged Pontryagin equation reads

$$\left(\frac{H}{2\nu_u} + \frac{1}{2\nu_w}\right)\frac{du_0}{dH} + \left(\frac{H^2}{4\nu_u} + \frac{H}{2\nu_w}\right)\frac{d^2u_0}{dH^2} = -1 \tag{28}$$

where  $u_0$ , the leading order solution in the outer domain, is a function of the Hamiltonian  $H$  ( $\equiv H_0$ ) in the initial configuration. The governing equation and consequently its solution are independent of the cross spectral density  $S_{uw}$ . The general solution of (28) is of the form

$$u_0(H) = -\frac{8\nu_w + \nu_u C_1}{2\nu_u\nu_w} \ln(H\nu_u + 2\nu_w) + C_1 \frac{\ln H}{2\nu_w} + C_2, \tag{29}$$

where constants of integration  $C_1$  and  $C_2$  need to be determined to satisfy the boundary conditions (14). Notice that the first boundary condition can also be satisfied in the outer solution which implies that there is no inner (boundary layer) solution at order  $\varepsilon^{-1}$ . Although the second condition straightforwardly yields  $C_1 = 0$ , the first boundary condition requires a little more attention. Indeed, the boundary condition is *a priori* written on the boundary of the domain of  $\mathcal{D}$  of any arbitrary shape in the state space  $(x, \dot{x})$ . However, because  $u_0(H)$  is a function of  $H$  only, there is no way to satisfy the boundary condition for any  $\mathbf{x}_0 \in \partial\mathcal{D}$ , otherwise than to have  $\mathcal{D}$  being a disk in the phase space. Because of this, we actually decide to restrict our study to the determination of the first passage time through a domain shaped like a circle, *i.e.* through the contours of equation  $H = H_c$ . In other words, the considered problem can be expressed as the first passage time through a state of a specifically chosen energy  $H_c$ , while starting from a smaller initial energy  $H_0$ . If the domain  $\mathcal{D}$  was not shaped like a disk, an underestimation of the first passage time might be obtained by replacing the domain by the inscribing circle as the rotation is fast compared to the energy. Accounting for the boundary condition  $u_0(H_c) = 0$ , we finally obtain

$$U_0(H_0) = \frac{u_0}{\varepsilon} = \frac{4}{S_u} \ln\left(\frac{H_c S_u + 2S_w}{H_0 S_u + 2S_w}\right) \tag{30}$$

where the subscript “0” has been re-introduced to indicate this corresponds to trajectories starting with an initial energy equal to  $H_0$ . As the Itô formulation (10) is only valid for positive times, this solution is only valid for a target energy higher than the initial energy. Although the analytical form (30) takes a simple expression, it seems that this result has never been derived before; it is a novel contribution of this work.

The average first passage time therefore takes a logarithmic form where the intensity of the parametric excitation is multiplied by the energy and appears in both the logarithm and the multiplying factor. Expression (30) presents two limit cases:

- when there is no parametric excitation, *i.e.*  $S_w = 0$ , the general solution degenerates into  $U_0(H_0) = 2\frac{\Delta H}{S_w}$  which indicates that the first passage time scales linearly with  $\Delta H = H_c - H_0$ , the difference between the target energy barrier and the initial energy in the system. With the terminology introduced below, this corresponds to an incubation regime, no matter the intensity of the forcing term;
- when there is no forcing term, *i.e.*  $S_w = 0$ , the general solution degenerates into  $U_0(H_0) = \frac{4}{S_u} \ln \frac{H_c}{H_0} = \frac{4}{S_u} \ln\left(1 + \frac{\Delta H}{H_0}\right)$ , which corresponds to the solution developed in [33], see (8). In this case, the first passage time scales with the ratio  $H_c/H_0$  on a logarithmic scale. Furthermore, a non-zero initial energy is required for the oscillator to exit its initial configuration. Then, for any  $H_0 > 0$ , the oscillator can reach any energy barrier in a finite time, on average. With the terminology introduced below, this corresponds to a multiplicative regime, no matter the intensity of the parametric excitation.

These two limiting cases reflect that the parametric excitation  $w(t)$  and the external forcing  $u(t)$  show themselves differently in the problem. Notice that the linear and logarithmic scalings that are obtained here agree with the well-known responses of undamped linear oscillators under deterministic excitations. On one hand, the envelope of the response under external forcing grows unbounded and linearly in case of harmonic excitation tuned to the natural frequency of the oscillator ( $w = \sin t$ ). On the other hand, in the unstable configuration of the undamped Mathieu equation, the response envelope grows exponentially fast for a harmonic parametric excitation tuned to twice the natural frequency of the oscillator ( $u = \sin 2t$ ).

The interplay between the two competing sources of excitation, *i.e.* does the first passage time scale on a log or linear scale with  $H_c$ ?, depends on  $S_u, S_w, H_0$  and  $H_c$ , as well as the relative smallness of some dimensionless groups made of these four parameters. To investigate this question, we rewrite (30) as

$$U_0(H_0) = \frac{4}{S_u} \ln\left(1 + \frac{\Delta H S_u}{H_0 S_u + 2S_w}\right) \tag{31}$$

with  $\Delta H = H_c - H_0$ .

**Incubation regime.** This mathematical formulation naturally reveals one particular regime, when the argument of the natural logarithm is close to one, i.e.

$$\frac{\Delta H S_u}{H_0 S_u + 2S_w} \ll 1 \quad \rightarrow \quad \Delta H \ll H_0 + \frac{2S_w}{S_u}. \quad (32)$$

It is always possible to choose, for given  $H_0$ ,  $S_w$  and  $S_u$ , small enough values of the target energy level satisfying condition (32). Indeed, the system needs to pass first by these states of energy slightly larger than the initial one before reaching, eventually, states of much higher energy. These states belong to an *incubation regime* during which the general solution (31) may be linearized. Recalling that  $\ln(1 + \epsilon) = \epsilon + \text{ord}(\epsilon^2)$  for  $\epsilon \ll 1$ , the mean first passage time in this incubation regime reads,

$$U_0 = \frac{4}{S_u} \left[ \frac{\Delta H S_u}{H_0 S_u + 2S_w} + \text{ord} \left( \frac{\Delta H S_u}{H_0 S_u + 2S_w} \right)^2 \right] \simeq \frac{4\Delta H}{H_0 S_u + 2S_w} \quad (33)$$

which indicates that the mean first passage time is proportional to the increase in the Hamiltonian  $\Delta H = H_c - H_0$ . For very small values of  $\Delta H$ , the first passage time decreases and might reach the order of the period of the oscillator. In this case, the asymptotic method induces large errors on the result which is not reliable anymore.

This approximation ceases to be valid when (32) is not fulfilled anymore, i.e. for  $\Delta H \simeq H_0 + \frac{2S_w}{S_u}$ , which corresponds to an energy barrier  $H_c = H_0 + \Delta H$  that is at least twice as large as the initial energy level. Although this is *a priori* prohibited because condition (32) is not satisfied, substituting  $H_0 + \frac{2S_w}{S_u}$  for  $\Delta H$  in (33) would yield  $U \simeq \frac{4}{S_u}$ . Because (33) is monotonic, the duration of the incubation regime, is therefore one order of magnitude smaller than  $\frac{4}{S_u}$ . In order to give a simple definition, we could for instance and arbitrarily define an *incubation time* by

$$U_{0,\text{incub}} := \frac{1}{2S_u}, \quad (34)$$

which corresponds to  $\epsilon = \frac{1}{8}$ , a number that is assumed to be small compared to 1. From a practical point of view, this means that the first passage time might be estimated with (33) and that the resulting estimation is valid, provided it is shorter than  $U_{0,\text{incub}}$ . As a consequence of our arbitrary definition choice for  $U_{0,\text{incub}}$ , the error on the estimated first passage time is, in the worst case,  $1 - \ln\left(1 + \frac{1}{8}\right) / \left(\frac{1}{8}\right) = 5.8\%$ .

Now, Eq. (31) can be rewritten

$$U_0(H_0) = \frac{4}{S_u} \ln \left( 1 + \frac{\Delta H^*}{H_0^* + 1} \right) = 8U_{0,\text{incub}} \ln \left( 1 + \frac{\Delta H^*}{H_0^* + 1} \right) \quad (35)$$

where the dimensionless groups  $H_0^*$  and  $\Delta H^*$  are defined by

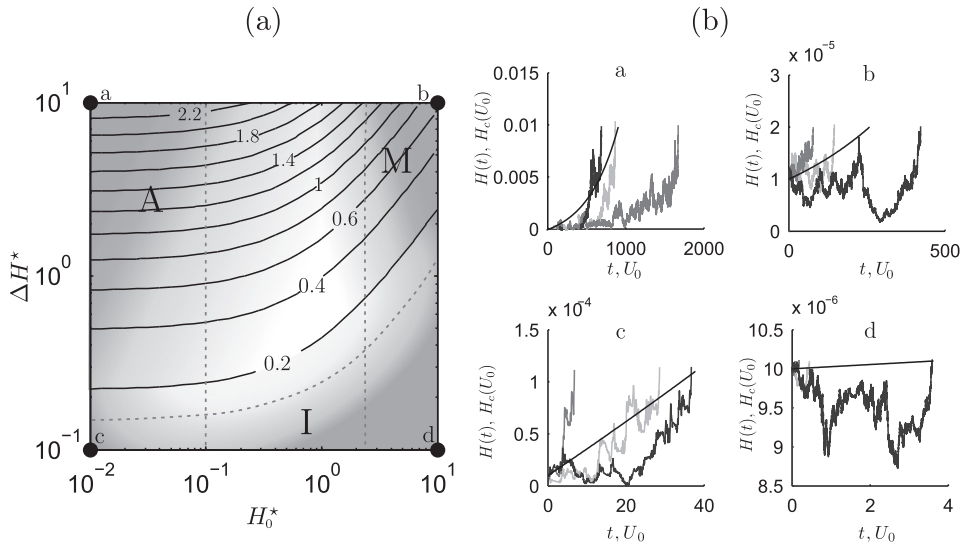
$$H_0^* = \frac{H_0 S_u}{2S_w}; \quad \Delta H^* = \frac{\Delta H S_u}{2S_w}. \quad (36)$$

**Multiplicative regime.** If  $\Delta H^* \ll H_0^* + 1$ , we recover condition (32), the logarithm may be linearized, the first passage time is proportional to  $\Delta H^*$  and is much smaller than the incubation time. Otherwise, if  $\Delta H^* \gtrsim H_0^* + 1$ , the logarithm cannot be linearized, and the expected first passage time is of the order of the incubation time or more. Two other limiting cases are interesting.

Either  $H_0^* \gg 1$  and the expected first passage time required to go from a relatively large initial energy level to an even larger energy level tends to  $4/S_u \ln(1 + \Delta H^*/H_0^*) = 4/S_u \ln(H_c/H_0)$ . It therefore depends on by how much the initial energy is multiplied to obtain the target energy level. This regime is therefore called the *Multiplicative regime*. In this conditions, the first passage time is independent of the forcing excitation intensity  $S_w$ . In the overlap between the multiplicative and the incubation regimes the linearized solution reads  $U_0 = \frac{4}{S_u} \frac{\Delta H}{H_0}$ .

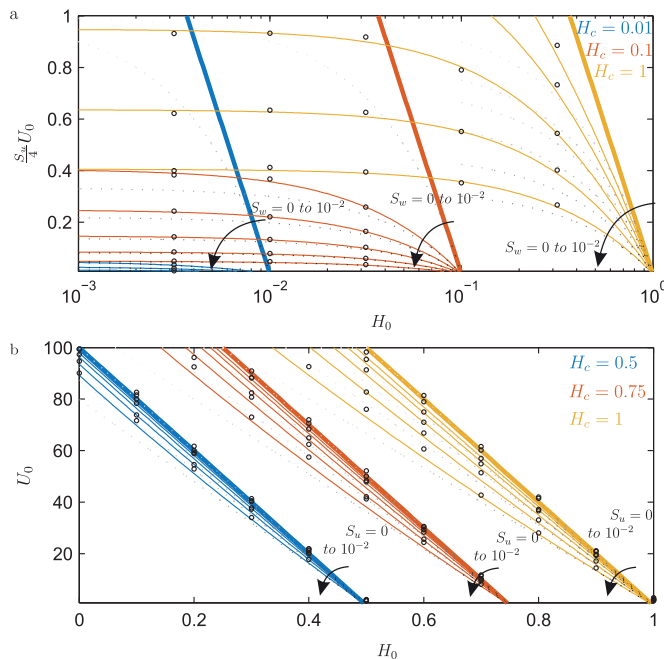
**Additive regime.** Alternatively,  $H_0^* \ll 1$  and the (large) first passage tends to  $4/S_u \ln(1 + \Delta H^*)$ . In this latter case, no matter the smallness of the initial energy  $H_0$  in the system, provided it is much smaller than  $2S_w/S_u$ , it does not influence the expected first passage time. In this regime, the expected first passage time only depends on the increase in energy  $\Delta H^*$ . This regime is therefore called the *Additive regime*. In the overlap between the additive and the incubation regimes the linearized solution reads  $U_0 = \frac{2}{S_w} \Delta H$ , which recovers the limit case  $S_u = 0$ .

Fig. 2 (a) presents the ratio  $\frac{U_0 S_u}{4} = \frac{U_0}{8U_{0,\text{incub}}}$  as a function of  $H^*$  and  $\Delta H^*$  and identifies the three regimes (incubation, additive



**Fig. 2.** (a) Dimensionless first passage time  $\frac{U_0 S_u}{4} = \frac{U_0}{8U_{0,incub}}$  as a function of  $H_0^*$  and  $\Delta H^*$  and identification of the Incubation (I), Multiplicative (M) and Additive (A) regimes (b) Simulations and first passage time with  $S_u = 10^{-3}$  and  $H_0 = 10^{-5}$  for the four corners of the left diagram.

and multiplicative). For different values of  $H^*$  and  $\Delta H^*$ , Fig. 2 (b) shows four realizations of  $H(t)$  as well as the expected first passage time  $H_c(U_0)$ . Subplots a, b, c and d correspond to the four corners of Fig. 2 (a), i.e.  $H_0^* = 10^{-2}$  or  $10^1$  and  $\Delta H^* = 10^{-1}$  or  $10^1$ . The parametric excitation is fixed at  $S_u = 10^{-3}$ , which corresponds to an incubation time  $U_{0,incub} = 500$  and simply means that  $H_c(U_0)$  increases more or less linearly for expected first passage times shorter than 500. Whether or not the expected first passage time versus the target energy level curve  $U_0(H_c)$  exhibits a nonlinear profile depends on the target energy level. In case a, corresponding to the additive regime, the curve  $U_0(H_c)$  is nonlinear as the first passage time is longer than the incubation time and the expected first passage time is governed by the energy increase  $\Delta H^*$ . This is confirmed in the upper left corner of Fig. 2 (a) which presents horizontal asymptotes in the additive regime. Similarly, case b representing the multiplicative regime presents a nonlinear  $U_0(H_c)$  curve and, in this regime, the first passage time is governed by the ratio  $H_c/H_0$ . This is visible by the unit slope of the iso-curves in Fig. 2 (b). Finally, cases c and d corresponding to the incubation regime with a first passage time



**Fig. 3.** (a) First passage times for different values of  $S_w$  and  $S_u = 10^{-2}$  (b) and for different values of  $S_u$  and  $S_w = 10^{-2}$ .



much smaller than 500 present a linear increase of the expected first passage time with the target energy.

The bottom left corner and the upper right corner in Fig. 2 represent the two limiting cases where the loading is either forcing or parametric, respectively. The additive regime therefore appears as a novelty of the combination of these two types of excitation.

Fig. 3 presents the first passage time of a slightly perturbed system ( $\epsilon$  is of order  $10^{-3}$ ) for different target energy barriers  $H_c$  and different excitations (forced excitation in (a) and parametric excitation in (b)). The logarithmic evolution when  $S_w = 0$  and the linear evolution when  $S_u = 0$  are respectively observable in bold in each Figure. The small circles represent Monte Carlo simulations. As expected, the first order solution  $U_0$  provides accurate results for this small value of  $\epsilon$ . The linearized solution (33) is represented in dotted line and fits well the exact solution for values of  $U_0$  that are much smaller than the incubation time, i.e. on the upper Figure  $\frac{S_u}{4}U_0 \ll 1$ . In both graphs, increasing the parametric or forced excitation decreases the first passage time.

As a first variant, in the next Section, the second order solution is developed in order to improve the accuracy of the solution for larger values of  $\epsilon$ , i.e. for larger excitation intensities. As a second variant, the last Section will present the leading order solution of the damped oscillator.

### 3. Variant 1: Second order and boundary layer solutions

First order developments have shown that the averaging is more natural in the energy-phase space. In fact, the governing Eq. (9) has been solved in the  $(q,p)$  space but the same solution would have been obtained using the variables  $k$  and  $\theta$  from the very beginning, i.e. solving

$$d\tilde{\mathbf{x}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t)dt + \tilde{\mathbf{b}}(\tilde{\mathbf{x}}, t)d\mathbf{B}, \tag{37}$$

where  $\tilde{\mathbf{x}} = \begin{bmatrix} k \\ \theta \end{bmatrix}$ ,  $\tilde{\mathbf{f}} = \begin{bmatrix} \delta_1 \\ 1 + \delta_2 \end{bmatrix}$  and  $\tilde{\mathbf{b}} = \begin{bmatrix} -k \cos \theta \sin \theta \frac{\cos \theta}{2} \\ \sin^2 \theta - \frac{\sin \theta}{2k} \end{bmatrix}$  and where the Wong-Zakai correction terms [35,14,10] are given by

$$\delta = \sum_{j=1}^2 \sum_{k=1}^2 \frac{1}{2} \sigma_{ij} \frac{\partial^2 \tilde{\mathbf{x}}}{\partial x_i \partial x_j} = \epsilon \left( \nu_w - 4k\nu_{uw} \sin \theta + 4k^2\nu_u \sin^2 \theta \right) \begin{bmatrix} \frac{\sin^2 \theta}{8k} \\ \frac{\cos \theta \sin \theta}{4k^2} \end{bmatrix}. \tag{38}$$

The corresponding drift matrix  $\tilde{\sigma} = \tilde{\mathbf{b}}(\tilde{\mathbf{x}}, t)\nu\tilde{\mathbf{b}}^T(\tilde{\mathbf{x}}, t)$  is still of order  $\epsilon$ . The invariant operators  $\mathcal{L}_1$  and  $\mathcal{L}_2$  become

$$\mathcal{L}_1\{\cdot\} = \frac{\partial}{\partial \theta}; \quad \epsilon \mathcal{L}_2\{\cdot\} = \delta_1 \frac{\partial}{\partial k} + \delta_2 \frac{\partial}{\partial \theta} + \frac{1}{2} \tilde{\sigma}_{11} \frac{\partial^2}{\partial k^2} + \frac{1}{2} \tilde{\sigma}_{22} \frac{\partial^2}{\partial \theta^2} + \tilde{\sigma}_{12} \frac{\partial^2}{\partial k \partial \theta}. \tag{39}$$

Again, from here, the subscript “0” is dropped. It will be reintroduced in (49). This formulation clearly indicates that the operator  $\mathcal{L}_1$  represents the variation along the orbits of constant energy  $H$  or  $k$ , as announced and used in the previous Section.

Similarly to what was done to determine  $u_0$ , the second order term  $u_1$  is obtained by solving (22), with  $u_0$  being pushed to the righthand side since it is now known,

$$\mathcal{L}_1\{u_1\} = \frac{\partial u_1}{\partial \theta} = -1 - \mathcal{L}_2\{u_0\}. \tag{40}$$

The solution of this equation takes the form

$$u_1(k, \theta) = u_{11}(k, \theta) + u_{12}(k), \tag{41}$$

where

$$u_{11}(k, \theta) = \frac{k^2 S_u \sin 2\theta}{2(k^2 S_u + S_w)^2} \left[ k^2 S_u (\cos 2\theta - 2) - 3S_w + \frac{8}{3k} \frac{\sin^2 \theta}{\sin 2\theta} S_{uw} \left( 2k^2 (2 \cos \theta + \cos 2\theta) - 6 \frac{S_w}{S_u} \right) \right]. \tag{42}$$

is the general solution of the differential equation in  $\theta$  while  $u_{12}(k)$  plays the role of a constant of integration, independent of  $\theta$ . The cross correlation  $S_{uw}$  influences the second order solution while it did not affect the first order one and, depending on the initial conditions, can be the governing term of  $u_{11}$  as the energy  $k$  appears in the denominator of  $S_{uw}$ . More details and complete derivation of expressions for  $u_{11}$  and  $u_{12}$  are given in Appendix A. The constant of integration is determined at the next order, with Eq. (23) averaged over one period of the unperturbed oscillator

$$\langle \mathcal{L}_1\{u_2\} \rangle + \langle \mathcal{L}_2\{u_{11}\} \rangle + \langle \mathcal{L}_2\{u_{12}\} \rangle = 0 \tag{43}$$

whose solution reads, after some developments (see Appendix A),

$$u_{12}(k) = \frac{4k\nu_{uw}}{3} \frac{(k^2\nu_u + 3\nu_w)}{(k^2\nu_u + \nu_w)^2} + \frac{C_1}{\nu_w} \ln\left(\frac{k^2}{k^2\nu_u + \nu_w}\right) + C_2 \tag{44}$$

and where constant  $C_1$  is shown to vanish. The constant  $C_2$  is determined by matching with the boundary layer solution in order to respect the first boundary condition. Details are reported in Appendix A for the sake of brevity.

Unlike for the general form (29) of  $u_0$ , the boundary conditions (14) cannot be satisfied with the general form of solution for  $u_1$ . A boundary layer solution therefore develops in the neighborhood of  $\partial\mathcal{D}$ , which is here restricted to a circle, in order to satisfy the second boundary condition. It is obtained by following the standard steps for the derivation of an asymptotic boundary solution [15,18,19,8]. Therefore, the stretched coordinate  $\zeta = \frac{H-H_c}{\sqrt{\varepsilon}}$  is introduced in order to focus on the small region in the neighborhood of  $H = H_c$ . Then, the equation to be solved  $\mathcal{L}\{G_n\} = 0$  is considered at the different orders in  $\sqrt{\varepsilon}$  and a regular ansatz in the stretched coordinate system

$$G_n(\zeta, \theta) = g_1(\zeta, \theta) + \sqrt{\varepsilon}g_2(\zeta, \theta) + \dots + \varepsilon^{\frac{n-1}{2}}g_n(\zeta, \theta) \tag{45}$$

is substituted into the governing equation. The leading order equation reads

$$\frac{\partial g_1}{\partial \theta} + 4H_c\tilde{\sigma}_{11}(H_c, \theta)\frac{\partial^2 g_1}{\partial \zeta^2} = 0 \tag{46}$$

so that the leading order solution  $g_1(\zeta, \theta)$  satisfying the boundary condition  $g_1(0, \theta) = -u_1(H_c, \theta)$  and the matching condition  $g_1(\zeta, \theta) \rightarrow 0$  when  $\zeta \rightarrow -\infty$  is

$$g_1(\zeta, \theta) = \sum_{n=1}^{\infty} a_n e^{\sqrt{\frac{nG_1}{2}}\zeta} \sin\left(n\alpha(\theta) - \sqrt{\frac{nG_1}{2}}\zeta\right) \tag{47}$$

with

$$G_1 = \frac{2\pi}{\int_0^T 4H_c\tilde{\sigma}_{11}(H_c, s)ds}, \quad \alpha(\theta) = 2\pi \frac{\int_0^\theta 4H_c\tilde{\sigma}_{11}(H_c, s)ds}{\int_0^T 4H_c\tilde{\sigma}_{11}(H_c, s)ds}$$

and where coefficients  $a_n$  are shown to be given by

$$a_n = -\frac{1}{\pi} \int_0^{2\pi} u_1(H_c, \theta(\alpha)) \sin(n\alpha) d\alpha. \tag{48}$$

Up to second order, the mean first passage time is finally given by

$$U(p_0, q_0) = \frac{1}{\varepsilon}u_0[H(p_0, q_0)] + u_1[k(p_0, q_0), \theta(p_0, q_0)] + g_1\left[\frac{H(p_0, q_0) - H_c}{\sqrt{\varepsilon}}, \theta(p_0, q_0)\right], \tag{49}$$

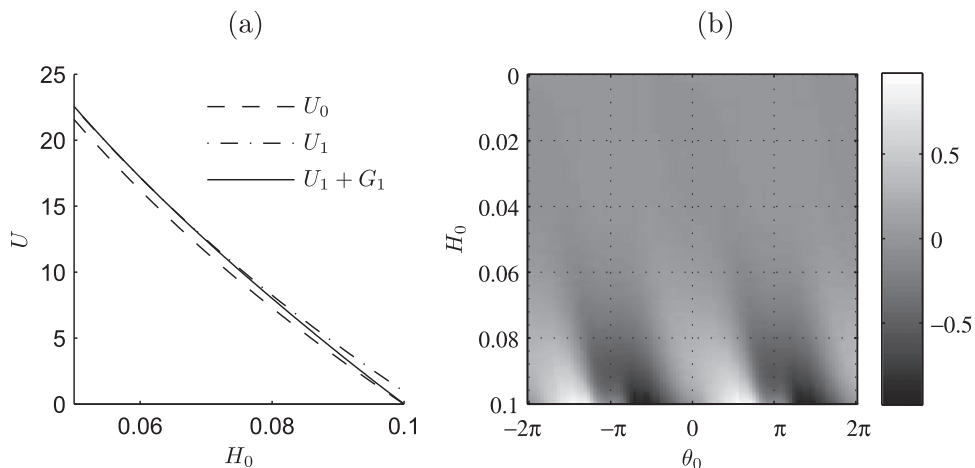
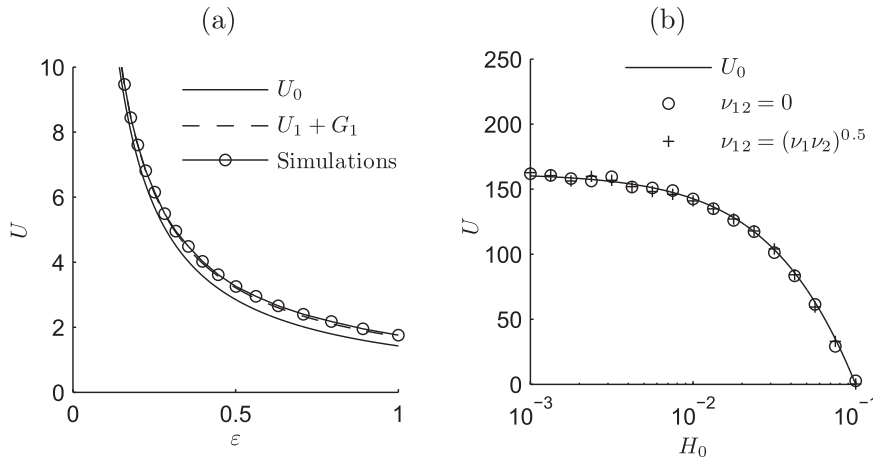


Fig. 4. (a) Comparison of the first-order, second-order and boundary layer terms with  $\theta = 3\pi/4$  (b) Evolution of the boundary layer solution  $g_1(b)$  with  $S_u = 0, S_w = 0.001, H_c = 0.1$ .



**Fig. 5.** (a) Comparison of the first and second order solution with simulations for different values of  $\varepsilon$  with  $\nu_u = \nu_w = 1$ ,  $H_0 = 0.01$ ,  $H_c = 1$  and  $\theta = 3\pi/4$  (b) Comparison of the solution for independent and fully correlated white noises with  $S_u = 0.1$ ,  $S_w = 0.01$  and  $H_c = 0.1$ .

with  $u_0(H_0)$ ,  $u_1(k, \theta)$  and  $g_1(\zeta, \theta)$  given by relations (30), (42) and (47). Notice the subscript “0” has been re-introduced in the last results in order to stress out that the first passage time depends on the initial energy level  $H_0$  and the initial position  $q_0$  and velocity  $p_0$ .

Fig. 4 (a) presents the first and second order solution  $U_0$  and  $U_1$ , together with the boundary layer solution  $g_1$ . The boundary condition  $U(H_c) = 0$  is respected for the leading order solution  $U_0$  and when the second order and the boundary layer are both taken into account. While the first order solution depends on the Hamiltonian only, the two second order solutions depend on both the Hamiltonian  $H_0$  and the phase  $\theta_0$ . Fig. 4 (b) presents the evolution of the boundary layer in the  $(H_0, \theta_0)$ -plane. The highest values are observed for  $H = H_c$ , on the boundary and  $g_1$  exponentially vanishes for decreasing initial energy levels.

Fig. 5 (a) presents the evolution of the first- and second-order solutions with the excitation intensities. The comparison with Monte Carlo simulations illustrates the accuracy of the asymptotic expansion as a function of the smallness of the parameter  $\varepsilon$ . As expected, the first-order solution matches almost perfectly for small intensities until values of  $\varepsilon$  of about 0.1 while the second-order solution is virtually perfect for values as large as  $\varepsilon \simeq 1$ . The precision of the method decreases with increasing values of  $\varepsilon$  but still provides very good results. Fig. 5 (b) compares the first-order solution  $U_0$  with simulations for fully correlated and independent white noises. As the excitation intensities are small ( $\varepsilon = 0.1$ ), the first-order solution is a good approximation and the first passage time is therefore independent of the correlation between the forced and parametric excitations. This is confirmed by the superposition of the Monte Carlo simulations (dots and crosses).

**4. Variant 2: Damped oscillator**

In order to limit the complexity of the following developments, only leading order terms of the first passage time are determined in the case of the damped system. Following the observations of Section 3, this means that the following results shall be used for small (and not moderate) values of  $\varepsilon$ . Also, in order to lighten the presentation, only main results are presented in the bulk of the text; details are presented in Appendix B.

The studied system is

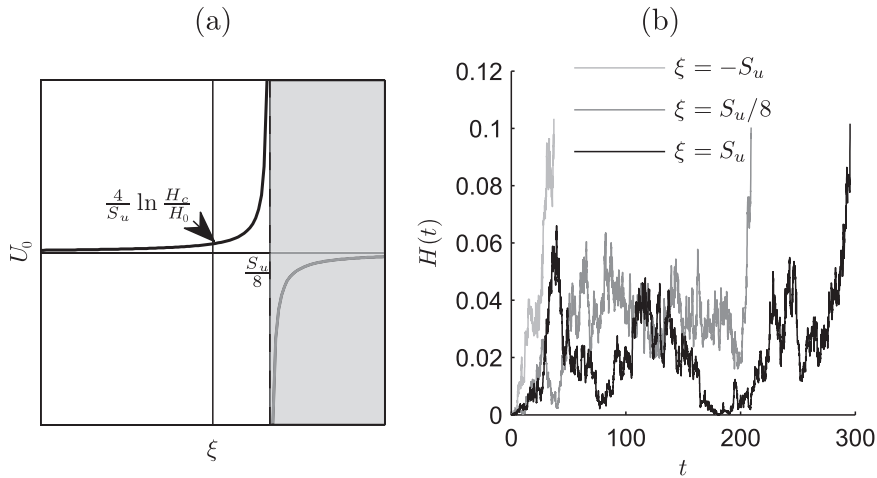
$$\ddot{x} + 2\xi\dot{x} + (1 + u)x = w, \tag{50}$$

with  $\xi$  the small damping ratio, of order  $\varepsilon$  at most, so that rewriting  $\xi = \varepsilon\nu_\xi$ , the re-scaled damping  $\nu_\xi$  is of order one at most. The first passage time is the solution of Eqs. (21) and (22) with the new Itô formulation of the differential equation. This leads to the same formulation for  $\mathcal{L}_1$  and an additional term in  $\mathcal{L}_2$ , see (B.3). Performing the same developments as before in the undamped case (which are reproduced in Appendix B), the expected first passage time finally reads

$$U_0(H_0) = \frac{4}{S_u(1-a)} \left[ \ln \left( 1 + \frac{\Delta H^*}{H_0^*} \right) + \frac{(1 + H_0^* + \Delta H^*)^a - (1 + H_0^*)^a}{a} - \int_{H_0^*}^{H_0^* + \Delta H^*} \frac{(1+t)^a}{t} dt \right]. \tag{51}$$

with  $a = \frac{8\nu_\xi\varepsilon}{\nu_u} = \frac{8\xi\varepsilon}{S_u}$ . This solution is only valid for positive first passage times. There are two interesting limiting cases:

- when there is no parametric excitation, i.e.  $S_u = 0$ , the general solution degenerates into



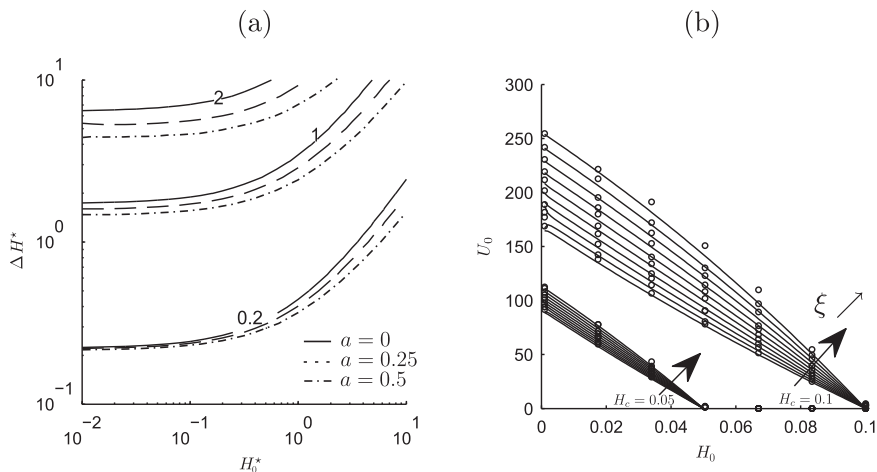
**Fig. 6.** (a) First passage time  $U_0$  as a function of the damping coefficient  $\xi$  for  $S_w = 0$  (b) Realisations of the Hamiltonian for  $H_0 = 10^{-5}$ ,  $S_u = 10^{-2}$ ,  $S_w = 10^{-3}$  and different values of  $\xi$ .

$$U_0(H_0) = -\frac{1}{2\xi} \ln \left( 1 + \frac{\Delta H^*}{H_0^*} \right) + \frac{\text{Ei}(aH_0^* + a\Delta H^*) - \text{Ei}(aH_0^*)}{2\xi}, \text{ with Ei}(\cdot) \text{ the Cauchy principal value integral defined by } \text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt.$$

The limit solution for  $\xi = 0$  is given by  $U_0(H_0) = 2 \frac{H_c - H_0}{S_w}$  which was already obtained in the undamped configuration. It is interesting though that the linearity of the solution, *i.e.* the expected first passage proportional to the increase in the Hamiltonian, regularly vanishes as damping is introduced into the problem.

- when there is no forcing term, *i.e.*  $S_w = 0$ , the solution degenerates into  $U_0(H_0) = \frac{4}{S_u(1-a)} \ln \left( 1 + \frac{\Delta H^*}{H_0^*} \right)$ . In this case, the damping does not modify the form of the first passage time, which still increases like the logarithm of the ratio  $H_c/H_0$ . This solution presents a positive first passage time for  $a < 1$ , which means that the energy of the system can increase, on average, if the damping ratio has an intensity below a certain threshold,  $\xi < S_u/8$ . For  $a \geq 1$ , the dissipation mechanism drives the dynamical system to lower energy levels, on average. The evaluated expected first passage time is negative. It has no meaning anymore since the Itô formulation on which the developments are based is no more valid. For a damping ratio equal to the critical threshold, the dissipated energy balances the injected energy and the first passage time is not defined, as illustrated in Fig. 6 (a). Fig. 6 (b) presents time series of the system for different values of  $\xi$  so that the first passage time is positive, not defined or invalid (negative, on average) for  $\xi = -S_u$ ,  $S_u/8$  or  $S_u$ .

Let us notice that the case where  $S_u = 0$  and  $S_w = 0$  is deterministic and provides in the undamped case a harmonic



**Fig. 7.** (a) Representation of  $\frac{U_0 S_u}{4} = \frac{U_0}{8U_{0,incub}}$  as a function of  $H^*$  and  $\Delta H^*$  for different damping ratios (b) First passage time  $U_0$  for different damping coefficients  $\xi = 0$  to 0.005 with  $H_c = 0.05$  and 0.1,  $S_u = 0$ ,  $S_w = 10^{-3}$ .

**Table 1**  
Summary of the analytical results and limit cases at first order.

(51)	$S_u = 0$	$S_w = 0$	$\xi = 0$
$S_u = 0$	$-\frac{1}{2\xi} \ln\left(\frac{H_c}{H_0}\right) + \frac{\text{Ei}(2\xi H_c/S_w) - \text{Ei}(2\xi H_0/S_w)}{2\xi}$	Deterministic	$2\frac{H_c - H_0}{S_w}$
$S_w = 0$	sym.	$\frac{4}{S_u - 8\xi} \ln\frac{H_c}{H_0}$ if $\xi < S_u/8$	$\frac{4}{S_u} \ln\frac{H_c}{H_0}$
$\xi = 0$	sym.	sym.	$\frac{4}{S_u} \ln\left(\frac{H_c S_u + 2S_w}{H_0 S_u + 2S_w}\right)$

motion of constant energy and in the damped case an exponentially decreasing energy. The system is governed by  $\ddot{x} + 2\xi\dot{x} + x = 0$ , whose solution is

$$q(t) = e^{-\xi t} \left( q_0 \cos \omega_d t + \frac{p_0 + \xi q_0}{\omega_d} \sin \omega_d t \right) \tag{52}$$

with  $\omega_d = \sqrt{1 - \xi^2}$ . The initial position and velocity are given by  $q_0$  and  $p_0$ . For small damping ratio, the Hamiltonian of the system evolves as  $H(t) = H_0 e^{-2\xi t}$  and agrees with  $U_0(H) = \frac{-1}{2\xi} \ln \frac{H_c}{H}$ , the limit solution of (51) for  $\{S_u, S_w\} \rightarrow 0$ . In this case the concept of expected first passage time has no physical meaning if the damping is positive.

By analogy with the undamped oscillator, the first passage time in the incubation regime, i.e. for  $\Delta H^* \ll H_0^* + 1$  is evaluated as the leading order term of the MacLaurin series expansion of (51) for  $\Delta H^*$ , which yields

$$U_0(H_0) = \underbrace{\frac{4}{S_u} \frac{\Delta H^*}{H_0^* + 1}}_{U_0(H_0)|_{a=0}} \underbrace{\frac{H_0^* + 1 - (H_0^* + 1)^a}{(1 - a)H_0^*}}_{f(a, H_0^*)} \tag{53}$$

Eq. (53) shows that the first passage time is still proportional to the dimensionless group  $\Delta H^*/(H_0^* + 1)$  with an additional factor ( $f(a, H_0^*) \geq 1$ ) modeling the increase of the first passage time with the damping ratio. Notice that  $f(a, H_0^*)$  is independent of the energy increase  $\Delta H^*$  and is asymptotic to  $1 + [1 - \ln(H_0^* + 1)]H_0^{*-1}a + \text{ord}(a^2)$  as  $a \rightarrow 0$ .

Fig. 7 (a) shows the reduced expected first passage time  $\frac{U_0 S_u}{4}$ , exactly as in Fig. 2 but with non zero damping ratios. It is seen that the damping ratio has little influence on the first passage time in the incubation regime, while it extends the expected first passage time in the additive regime. In the multiplicative regime (upper right corner), the damping ratio changes the slope of the curve of equal first passage time, which means that the first passage time is governed by a power of  $\Delta H^*/H_0^*$  smaller than unity. In all regimes, increasing the damping ratio increases the first passage time.

Fig. 7 (b) presents the first passage time as a function of the initial energy  $H_0$  for different values of  $\xi$  and  $H_c$ . The small circles represent Monte Carlo simulations while the full line is the analytical solution. As expected, the first passage time increases with the damping.

Table 1 presents a summary of the analytical results and their limits.

### 5. Conclusion

Based on an asymptotic approach, this work derives and discusses an analytical solution of the average first passage time of a quasi-Hamiltonian oscillator simultaneously submitted to white noise parametric and forcing excitations of small intensities. It is observed that the asymptotic solution of the Pontryagin equation, at leading order, provides a good approximation of the solution when excitations are small. The derivation highlighted the dependence of the expected first passage time on the Hamiltonian, globally, and not on the position and velocity separately. In the undamped case, three different regimes have been highlighted, namely the incubation regime (I), the multiplicative regime (M) and the additive regime (A). These three regimes exhibit different features and the behavioral responses of the system, mainly the linear or log scaling of the first passage time with the target energy level, was thoroughly discussed and analyzed with the help of the two dimensionless groups  $H_0^*$  and  $\Delta H^*$ , see Section 2.

As a first variant, the first order solution is improved by addition of the second order solution. In spite of the complexity of the analytical solutions, it is possible to derive a boundary layer solution that develops for target energy levels that are only slightly larger than the initial energy level. Unlike the first order solution, the second order one depends on both the

initial kinetic and potential energies, i.e. position and velocity separately. It is also noted that the correlation between the forced and parametric excitations does not influence the leading order solution and appears in the second order solution. Although the smallness of the excitations is required for the asymptotic expansion, comparison with Monte Carlo simulations showed very good accuracy of the first-order theoretical solutions for values of  $\varepsilon$  as large as 0.1. For the considered example, second-order solutions virtually match Monte Carlo simulation results for values of  $\varepsilon$  as large as 0.5 (or more).

Finally, the first passage time is also developed for a damped oscillator. It was shown that the three regimes remain, with little influence of the damping in the incubation regime. Positive damping expectedly tends to increase the mean first passage times, in all regimes, and eventually induces such a large dissipation that the first passage time is not finite, on average.

At each step, analytical solutions were validated with Monte Carlo simulations in order to demonstrate the accuracy of the closed form solutions. The dependence of the first passage time with the initial and target energy is fully described by the complete solution as well as some particular cases.

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### Appendix A. Variant 1: Second order and boundary layer solutions

The Itô formulation (10) corresponding to the energy-phase variables  $k$  and  $\theta$  reads

$$d\tilde{\mathbf{x}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t)dt + \tilde{\mathbf{b}}(\tilde{\mathbf{x}}, t)d\mathbf{B}, \tag{A.1}$$

where  $\tilde{\mathbf{x}} = \begin{bmatrix} k \\ \theta \end{bmatrix}$ ,  $\tilde{\mathbf{f}} = \begin{bmatrix} \delta_1 \\ 1 + \delta_2 \end{bmatrix}$  and  $\tilde{\mathbf{b}} = \begin{bmatrix} -k\cos\theta \sin\theta \frac{\cos\theta}{2} \\ \sin^2\theta - \frac{\sin\theta}{2k} \end{bmatrix}$  and the Wong-Zakai correction terms ([14,35,10]) are given by

$$\delta = \sum_{j=1}^2 \sum_{k=1}^2 \frac{1}{2} \sigma_{ij} \frac{\partial^2 \tilde{\mathbf{x}}}{\partial x_i \partial x_j} = \varepsilon \left( \nu_w - 4k\nu_{uw} \sin\theta + 4k^2\nu_u \sin^2\theta \right) \begin{bmatrix} \frac{\sin^2\theta}{8k} \\ \frac{\cos\theta \sin\theta}{4k^2} \end{bmatrix}. \tag{A.2}$$

From this transformation follows that

$$\begin{aligned} \tilde{\sigma} &= \varepsilon \tilde{\mathbf{b}}(\tilde{\mathbf{x}}, t) \nu \tilde{\mathbf{b}}^T(\tilde{\mathbf{x}}, t) \\ &= \varepsilon \begin{bmatrix} k^2\nu_u \cos^2\theta \sin^2\theta - k\nu_{uw} \cos^2\theta \sin\theta + \nu_w \frac{\cos^2\theta}{4} k\nu_u \cos\theta \sin^3\theta + \nu_{uw} \cos\theta \sin^2\theta - \nu_w \frac{\cos\theta \sin\theta}{4k} \\ -k\nu_u \cos\theta \sin^3\theta + \nu_{uw} \cos\theta \sin^2\theta - \nu_w \frac{\cos\theta \sin\theta}{4k} & \nu_u \sin^4\theta - \nu_{uw} \frac{\sin^3\theta}{k} + \nu_w \frac{\sin^2\theta}{4k^2} \end{bmatrix}. \end{aligned} \tag{A.3}$$

The invariant operators  $\mathcal{L}_1$  and  $\mathcal{L}_2$  become

$$\mathcal{L}_1\{\cdot\} = \frac{\partial}{\partial\theta} \quad ; \quad \mathcal{L}_2\{\cdot\} = \delta_1 \frac{\partial}{\partial k} + \delta_2 \frac{\partial}{\partial\theta} + \frac{1}{2} \tilde{\sigma}_{11} \frac{\partial^2}{\partial k^2} + \frac{1}{2} \tilde{\sigma}_{22} \frac{\partial^2}{\partial\theta^2} + \tilde{\sigma}_{12} \frac{\partial^2}{\partial k \partial\theta}. \tag{A.4}$$

Eq. (22) provides the following expression:

$$\begin{aligned} \mathcal{L}_1\{u_1\} &= \frac{\partial u_1}{\partial\theta} = -1 - \mathcal{L}_2\{u_0\} = \langle \mathcal{L}_2\{u_0\} \rangle - \mathcal{L}_2\{u_0\} \\ &= \left[ k^2\nu_u \cos 2\theta + 2\nu_{uw} \cos\theta \right] \frac{\partial u_0}{\partial H} + k^2 \left( k^2\nu_u \cos 4\theta + 8k\nu_{uw} \cos^2\theta \sin\theta - \nu_w \cos 2\theta \right) \frac{\partial^2 u_0}{\partial H^2} \end{aligned} \tag{A.5}$$

Integration of expression (A.5) with respect to  $\theta$  provides a decomposition of  $u_1$  into two components with the constant of integration  $u_{12}(k)$ :

$$u_1(k, \theta) = u_{11}(k, \theta) + u_{12}(k), \tag{A.6}$$

with

$$\begin{aligned}
 u_{11}(k, \theta) &= \frac{k^2 \nu_u \sin 2\theta}{2(k^2 \nu_u + \nu_w)^2} \left[ (\cos 2\theta - 2)k^2 \nu_u - 3\nu_w + \frac{8}{3k} \frac{\sin^2 \frac{\theta}{2}}{\sin 2\theta} \nu_{uw} \left( 2k^2(2 \cos \theta + \cos 2\theta) - 6 \frac{\nu_w}{\nu_u} \right) \right] \\
 &= \frac{k^2 S_u \sin 2\theta}{2(k^2 S_u + S_w)^2} \left[ (\cos 2\theta - 2)k^2 S_u - 3S_w + \frac{8}{3k} \frac{\sin^2 \frac{\theta}{2}}{\sin 2\theta} S_{uw} \left( 2k^2(2 \cos \theta + \cos 2\theta) - 6 \frac{S_w}{S_u} \right) \right].
 \end{aligned}
 \tag{A.7}$$

Provided  $u_{12}$  is dependent on the Hamiltonian only, the averaging of Eq. (23) over one period of the unperturbed motion gives:

$$\begin{aligned}
 \langle \mathcal{L}_2\{u_{12}\} \rangle &= \left( \frac{H}{2} \nu_u + \frac{1}{2} \nu_w \right) \frac{\partial u_{12}}{\partial H} + \left( \frac{H^2}{4} \nu_u + \frac{H}{2} \nu_w \right) \frac{\partial^2 u_{12}}{\partial H^2} \\
 &= -\langle \mathcal{L}_2\{u_{11}\} \rangle = -\frac{\nu_{uw} \left( k^6 \nu_u^3 - k^4 \nu_u^2 \nu_w + 27k^2 \nu_u \nu_w^2 - 3\nu_w^3 \right)}{12k \left( k^2 \nu_u + \nu_w \right)^3}
 \end{aligned}
 \tag{A.8}$$

The resolution of Eq. (A.8) provides:

$$u_{12}(k) = \frac{4k\nu_{uw} \left( k^2 \nu_u + 3\nu_w \right)}{3 \left( k^2 \nu_u + \nu_w \right)^2} + \frac{C_1}{\nu_w} \ln \left( \frac{k^2}{k^2 \nu_u + \nu_w} \right) + C_2.
 \tag{A.9}$$

The first constant of integration  $C_1$  is equal to zero in order to respect the solvability condition  $|U(0)| < \infty$ . The second constant  $C_2$  will be determined together with the boundary layer solution in order to respect the boundary condition  $U(H_c) = 0$ .

The boundary layer solution is the solution of  $\mathcal{L}\{G_n\} = 0$ . The coordinate stretching is classical in the boundary layers problems [8]. Therefore, the boundary layer solution  $G_n$  is written as a function of the stretched coordinate  $\zeta = \frac{H-H_c}{\sqrt{\varepsilon}}$

$$G_n(\zeta, \theta) = g_1(\zeta, \theta) + \sqrt{\varepsilon} g_2(\zeta, \theta) + \dots + \varepsilon^{\frac{n-1}{2}} g_n(\zeta, \theta).
 \tag{A.10}$$

Similarly, the operator  $\mathcal{L}$  is transformed via the asymptotic expansion of the functions  $\tilde{\alpha}_{ij}$  and  $\tilde{\delta}_i$  in the neighborhood of  $H = H_c$ :

$$\tilde{\alpha}_{ij}(H, \theta) = \tilde{\alpha}_{ij}(H_c, \theta) + \sqrt{\varepsilon} \zeta \tilde{\alpha}_{ij}^{(1)}(H_c, \theta) + \dots
 \tag{A.11}$$

$$\tilde{\delta}_i(H, \theta) = \tilde{\delta}_i(H_c, \theta) + \sqrt{\varepsilon} \zeta \tilde{\delta}_i^{(1)}(H_c, \theta) + \dots
 \tag{A.12}$$

Taking into account  $\frac{\partial}{\partial k} = \frac{4k}{\sqrt{\varepsilon}} \frac{\partial}{\partial \zeta}$  and  $\frac{\partial^2}{\partial k^2} = \frac{4}{\sqrt{\varepsilon}} \frac{\partial}{\partial \zeta} + \frac{16k^2}{\varepsilon} \frac{\partial^2}{\partial \zeta^2}$ , the Backward-Kolmogorov operator becomes

$$\mathcal{L}\{\cdot\} = \frac{\partial}{\partial \theta} + 4H_c \tilde{\alpha}_{11}(H_c, \theta) \frac{\partial^2}{\partial \zeta^2} + \sqrt{\varepsilon} \left[ 12\sqrt{2H_c} \tilde{\delta}_1(H_c, \theta) + 2\tilde{\alpha}_{11}(H_c, \theta) \frac{\partial}{\partial \zeta} \right] + \dots = \Lambda_0\{\cdot\} + \sqrt{\varepsilon} \Lambda_1\{\cdot\} + \varepsilon \Lambda_2\{\cdot\} + \dots
 \tag{A.13}$$

so that the governing equation becomes

$$\Lambda G_n = (\Lambda_0 + \sqrt{\varepsilon} \Lambda_1 + \varepsilon \Lambda_2 + \dots) \{ g_1 + \sqrt{\varepsilon} g_2 + \dots \} \Lambda_0 \{ g_1 \} + \sqrt{\varepsilon} \Lambda_0 \{ g_2 \} + \Lambda_1 \{ g_2 \} + \dots = 0
 \tag{A.14}$$

Balancing again the similar powers of  $\varepsilon$  provides the expression of the functions  $g_i(\zeta, \theta)$ . In particular, the first order solution  $g_1(\zeta, \theta)$  is the solution of the following diffusion equation

$$\Lambda_0 \{ g_1 \} = \frac{\partial g_1}{\partial \theta} + 4H_c \tilde{\alpha}_{11}(H_c, \theta) \frac{\partial^2 g_1}{\partial \zeta^2} = 0
 \tag{A.15}$$

with the boundary conditions  $g_1(0, \theta) = -u_1(H_c, \theta)$  and  $g_1(\zeta, \theta) \rightarrow 0$  when  $\zeta \rightarrow -\infty$ . The solution of this equation is given by

$$g_1(\zeta, \theta) = b_0 + \sum_{n=1}^{\infty} b_n e^{\sqrt{\frac{nC_1}{2}} \zeta} \cos \left( n\alpha(\theta) - \sqrt{\frac{nC_1}{2}} \zeta \right) + \sum_{n=1}^{\infty} a_n e^{\sqrt{\frac{nC_1}{2}} \zeta} \sin \left( n\alpha(\theta) - \sqrt{\frac{nC_1}{2}} \zeta \right)
 \tag{A.16}$$

with

$$\begin{aligned}
 c_1 &= 2\pi \int_0^T 4H_c \tilde{\delta}_{11}(H_c, s) ds \\
 \alpha(\theta) &= c_1 \int_0^\theta 4H_c \tilde{\delta}_{11}(H_c, s) ds \\
 b_0 &= -\frac{1}{2\pi} \int_0^{2\pi} u_1(H_c, \theta(\alpha)) d\alpha \\
 b_n &= -\frac{1}{\pi} \int_0^{2\pi} u_1(H_c, \theta(\alpha)) \cos(n\alpha) d\alpha \\
 a_n &= -\frac{1}{\pi} \int_0^{2\pi} u_1(H_c, \theta(\alpha)) \sin(n\alpha) d\alpha
 \end{aligned} \tag{A.17}$$

Because of the second boundary condition, it follows that

$$b_0 = -\frac{1}{2\pi} \int_0^{2\pi} u_1(H_c, \theta(\alpha)) d\alpha = 0. \tag{A.18}$$

Finally, averaging (A.6) with respect to variable  $\alpha$  and accounting that  $b_0 = 0$ , the constant of integration  $C_2$  is obtained:

$$-\frac{1}{2\pi} \int_0^{2\pi} u_{12}(k_c) d\alpha = \frac{4k_c \nu_{uw} (k_c^2 \nu_u + 3\nu_w)}{3 (k_c^2 \nu_u + \nu_w)^2} + C_2 = \frac{1}{2\pi} \int_0^{2\pi} u_{11}(k_c, \theta(\alpha)) d\alpha = \frac{4k_{uw} (k_c^2 \nu_u + 3\nu_w)}{3 (k_c^2 \nu_u + \nu_w)^2}. \tag{A.19}$$

It finally follows from (A.19) that the second constant of equation is equal to zero.

Adapted to relation (A.17) and provided the expected parity of the boundary layer with respect to  $\theta$ , only the even coefficients  $a_n$  are non-zero and can be obtained via numerical integration.

### Appendix B. Variant 2: Damped oscillator

The mean first passage time of the damped oscillator governed by Equation (50) is obtained following the same methodology. The Itô formulation of the differential equation now reads

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t) dt + \mathbf{b}(\mathbf{x}, t) dB, \tag{B.1}$$

where  $\mathbf{x} = \begin{bmatrix} q \\ p \end{bmatrix}$ ,  $\mathbf{f} = \begin{bmatrix} p \\ -q - 2\varepsilon \nu p \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 0 & 0 \\ -q & -1 \end{bmatrix}$ .

The Backward-Kolmogorove operator  $\mathcal{L}$  is now defined by

$$\mathcal{L}\{\cdot\} = \mathcal{L}_1\{\cdot\} + \varepsilon \mathcal{L}_2\{\cdot\}, \tag{B.2}$$

where

$$\begin{cases}
 \mathcal{L}_1\{\cdot\} = p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p} \\
 \mathcal{L}_2\{\cdot\} = \frac{1}{2} (q^2 \nu_u + \nu_w - 2q \nu_{uw}) \frac{\partial^2}{\partial p^2} - 2\nu_\xi p \frac{\partial}{\partial p}
 \end{cases} \tag{B.3}$$

The first operator  $\mathcal{L}_1\{\cdot\}$  is independent of the damping and Eq. (21) still provides the information that the first order solution  $u_0$  is a function of the Hamiltonian only.

Averaging Eq. (22) assuming transformation (27) successively provides:

$$\begin{aligned}
 \langle \mathcal{L}_2\{u_0\} \rangle &= -1, \\
 \left[ \frac{H}{2} \nu_u + \frac{1}{2} \nu_w - 2 H \nu_\xi \right] \frac{du_0}{dH} + \left[ \frac{H^2}{4} \nu_u + \frac{H}{2} \nu_w \right] \frac{d^2 u_0}{dH^2} &= -1, \\
 \alpha(H) \frac{du_0}{dH} + \beta(H) \frac{d^2 u_0}{dH^2} &= -1.
 \end{aligned} \tag{B.4}$$

Similarly to the undamped oscillator, the governing equation is independent of the correlation between the parametric and forced excitations.

A first integration provides

$$\frac{du_0}{dH} = -e^{c(H)} \int_0^H \frac{1}{\beta(y)} e^{c(y)} dy + C_1 e^{-c(H)}. \tag{B.5}$$



with

$$c(H) = \int \frac{\alpha(H)}{\beta(H)} dH = \ln(H) + \left(1 - \frac{8\nu_\xi}{\nu_u}\right) \ln(H\nu_u + 2\nu_w) \quad (\text{B.6})$$

The solvability condition implies  $C_1 = 0$  so that

$$\frac{du_0}{dH} = \frac{2}{\nu_w} \frac{(1 + H^*)^{a-1} - 1}{H^*(1-a)}, \quad (\text{B.7})$$

with  $a = \frac{8\nu_\xi}{\nu_u} = \frac{8\xi}{S_u}$  and  $H^* = \frac{H\nu_u}{2\nu_w} = \frac{HS_u}{2S_w}$ . A second integration accounting for  $u_0(H_c) = 0$  provides

$$\begin{aligned} u_0(H_0) &= \int_0^H \frac{du_0}{dH} dH + C_2 \\ &= \frac{4}{\nu_u(1-a)} \left[ \ln\left(1 + \frac{\Delta H^*}{H_0^*}\right) + \frac{(1 + H_0^* + \Delta H^*)^a - (1 + H_0^*)^a}{a} - \int_{H_0^*}^{H_0^* + \Delta H^*} \frac{(1+t)^a}{t} dt \right] \end{aligned} \quad (\text{B.8})$$

so that the mean first passage time  $U_0$  is finally given by

$$U_0(H_0) = \frac{4}{S_u(1-a)} \left[ \ln\left(1 + \frac{\Delta H^*}{H_0^*}\right) + \frac{(1 + H_0^* + \Delta H^*)^a - (1 + H_0^*)^a}{a} - \int_{H_0^*}^{H_0^* + \Delta H^*} \frac{(1+t)^a}{t} dt \right]. \quad (\text{B.9})$$

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