Anticoherence measures for spin states

Dorian Baguette and John Martin

Institut de Physique Nucléaire, Atomique et de Spectroscopie, CESAM, Université de Liège, 4000 Liège, Belgium

Coherent vs anticoherent spin-j states

Coherent state

\[ |\psi_j\rangle \text{ is a spin-j coherent state } |n\rangle \text{ if it is an eigenstate of } J \cdot n \text{ for some } n \text{ with the highest eigenvalue } jh. \]

\[ \langle J \rangle = jhn \]

Anticoherent state

\[ |\psi_j\rangle \text{ is a spin-j } t\text{-anticoherent state if } [1] \]

\[ \langle \psi_j | (J \cdot n)^k | \psi_j \rangle \]

\[ \langle J \rangle = 0 \]

does not depend on \( n \) for \( k = 1, \ldots, t \).

One-to-one mapping and geometrical representation

One-to-one mapping

\[ \begin{align*}
\text{spin-j state} & \leftrightarrow \text{symmetric state of } 2j \text{ spin-}j^\frac{1}{2} \\
\text{symmetric state of } N = 2j \text{ qubits} & \leftrightarrow \text{symmetric state of } 3 \text{ spin-}j^\frac{1}{2}
\end{align*} \]

Any \( N \)-qubit symmetric state \( |\psi_S\rangle \) can be expressed in the Dicke basis \( \{|D_N^{(k)}\rangle\} \) with \( |D_N^{(k)}\rangle = N^{\frac{k}{2}} \sum_{\text{perm. } |0_{N-k}, 1_k\rangle} \), where \( a_\ell \) is a string of \( \ell \) characters 0, 1, \( N \) is the number of qubits and \( k \) the excitation number:

\[ |\psi_S\rangle = \sum_{k=0}^N d_k |D_N^{(k)}\rangle, \quad \sum_{k=0}^N |d_k|^2 = 1 \]

There is a formal correspondence between the standard \( |j, m\rangle \) basis (common eigenstates of \( J_z \) and \( J^2 \)) and the Dicke basis:

\[ |j, m\rangle \leftrightarrow |D_N^{(j-m)}\rangle \quad |\psi_j\rangle \leftrightarrow |\psi_S\rangle \]

(1)

Majorana representation

\[ \begin{align*}
\text{symmetric state of } N = 3 \text{ qubits} & \leftrightarrow \text{symmetric state of } 3 \text{ points on the Bloch sphere} \\
N = 3 \text{ single-qubit states} & \leftrightarrow N = 3 \text{ points on the Bloch sphere}
\end{align*} \]

Majorana points can be located at the same position on the Bloch sphere, and the largest number of degenerated points is denoted by \( d_{\text{max}} \) \( (\leq N) \).

- Spin-j coherent states are characterised by \( d_{\text{max}} = 2j \).
- Anticoherent states can only exist if \( d_{\text{max}} \leq j \).

Conditions for anticoherence

A multiquit symmetric state \( |\psi_S\rangle \leftrightarrow |\psi_j\rangle \) is \( t\)-anticoherent if all \( t\)-qubit states are maximally mixed in the symmetric subspace \([2,3]\), or if the expectation value of spin operators fulfill the following conditions [4]:

\[ \rho_{\ell t+1}^{\text{tr}} = \frac{1}{t+1} \quad \Longleftrightarrow \quad \langle \psi_j | J_{\ell t+1} J_{\ell t+1}^T | \psi_j \rangle = \frac{\text{Tr}(J_{\ell t+1}^2)}{2j+1} \delta_{0\ell} \]

for \( r = 0, \ldots, t, q = 0, \ldots, t - r \), where \( N \) is the number of qubits and \( \delta_{0\ell} \) the Kronecker symbol.

Measures of \( t\)-anticoherence

Anticoherence conditions only allow to determine whether a state is anticoherent or not. In order to quantify the amount of anticoherence of a state, we introduce measures of anticoherence on the set of pure spin-j states \( |\psi_j\rangle \). We require a measure of \( t\)-anticoherence, \( A_t \), to fulfill the following conditions:

i. \( A_t(|\psi_j\rangle) = 0 \leftrightarrow |\psi_j\rangle \) is coherent

ii. \( A_t(|\psi_j\rangle) = 1 \leftrightarrow |\psi_j\rangle \) is \( t\)-anticoherent

iii. \( A_t(|\psi_j\rangle) \in [0, 1] \) for any state \( |\psi_j\rangle \)

iv. \( A_t(|\psi_j\rangle) \) is invariant under global phase changes and arbitrary spin rotations

Measures of anticoherence based on purity

A measure satisfying all conditions above, based on the mapping (1) and the purity of the \( t\)-qubit reduced density matrix of \( |\psi_S\rangle\langle \psi_S|, \quad R_t \equiv \text{Tr}(\rho_t^2) \), reads

\[ A_t(|\psi_j\rangle) = \frac{t+1}{2j+1} \left( 1 - R_t \right) \]

(2)

- \( t = 1 \)

In this case, (2) is a linear function of the total variance [5], \( V = \sum_{i=x,y,z} (J_i^2) - \langle J_i \rangle^2 \), and reads

\[ A_1(|\psi_j\rangle) = \frac{(V - j/j^2} \]

- \( t = 2 \)

In this case, (2) involves two-point correlators of spin operators, and reads

\[ A_2(|\psi_j\rangle) = \left( \frac{W + \alpha}{\beta} \right)^\beta \]

where \( \alpha = \frac{j(s^2 - 2s + 2)}{2(j-1)} \), \( \beta = (2j-1)/2j \), and

\[ W = V - \sum_{i,x,y,z} \langle J_i J_x J_y J_z \rangle / (2j(j-1)) \]

Exemples: GHZ and W states

\[ A_1((|j, j\rangle + |j, -j\rangle)/\sqrt{2}) = 1 \quad A_1((|j, j-1\rangle) = (2j-1)/j^2 \]

\[ A_2((|j, j\rangle + |j, -j\rangle)/\sqrt{2}) = 3/4 \quad A_2((|j, j-1\rangle) = 3(j-1)/j^2 \]

Anticoherence and degeneracy of Majorana points

Anticoherence measures can be numerically optimized to find anticoherent states of given \( j \) and \( d_{\text{max}} \).

Fig.: Maximum degeneracy allowed for the existence of \( t\)-anticoherent states with a spin quantum number \( j \).