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STRUCTURAL DYNAMICS

par

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1. INTRODUCTION

Linear structural dynamics is one of the many field problems of engineering that can receive a variational formulation. The classical approach is the kinematical one, and the discretization of the Hamiltonian variational principle in finite elements results from a polynomial approximation of the displacement field inside each separate region. Continuity will be secured through identification of a suitable set of generalized interface displacements, in which case the kinematical elements are said to be conforming. Integrating the kinetic and potential energies of the finite elements leads to lagrangian, or so-called coherent, mass and stiffness matrices.

The first really satisfactory formulation of a dual principle, in which the kinetic energy is transformed through satisfaction of the dynamic equilibrium equations in a functional expressed in terms of an impulse field, is due to Toupin [38]. Similar approaches where followed by Crandall, Yu Chen, Gladwell and Zimmermann [6, 41, 24]. Some numerical applications to beam and plate problems were presented by Tabarrok, Sakaguchi and Karnopp [35, 36] Its use as an efficient tool in finite element applications is however very recent [19,21] and has still to be generalized to the three-dimensional linear elasticity.

This presentation gives a logical derivation of the dual dynamic principle through the canonical form in the spirit of the Friedrichs transformation. Its discretization results from a polynomial approximation of the stress field within each separate region. The diffusion of the boundary tractions will be preserved if choosing a suitable system of generalized boundary loads that may be defined uniquely in terms of the parameters of the stress field inside each element.

This paper also discusses the general procedure for assembling equilibrium, or statically admissible finite elements, in order to implement the dual principle. It will appear that, when starting from an assumed stress field, the expression of the kinetic energy of an element involves only a small number of interior parameters; the equilibrium approach leads thus in a more natural way to an "eigenvalue economizer". Zero frequency modes are associated with the other parameters, which

improve the representation of the strain energy without increasing the order of the eigenvalue problems. Moreover, it is shown that there is no advantage in using the dual principle together with the requirement of forcing orthogonality with respect to all zero frequency modes. On the contrary, experience proves that, by ignoring this unnecessary requirement, the computed eigenvalues generally converge to the exact values by lower bounds, hence giving precious accuracy estimates by comparison with the displacement approach which always give upper bounds.

2. THE VARIATIONAL PRINCIPLES OF ELASTODYNAMICS

2.1. Hamilton's principle

Hamilton's principle, or displacement variational principle, states that for time fixed end values of the displacement field u_i, the Lagrangian action

$$f[u] = \int_{t_1}^{t_2} (T - V)dt \qquad (2.1.1)$$

of a conservative system takes a stationary value on the true trajectory of the motion.

T denotes the kinetic energy of the system :

$$T = \frac{1}{2} \int_R \rho \dot{u}_i \dot{u}_i \, d R \quad , \qquad (2.1.2)$$

and V, its potential energy, can be split into different distinct parts :

The strain energy

$$V_1 = \int_R W(\varepsilon) \, dR \quad , \qquad (2.1.3)$$

results from the integration of the strain energy density $W(\varepsilon)$ in the domain R. If we restrict ourselves to the infinitesimal strains and rotations of a linear elastic material, W can be written as a positive definite quadratic form

$$W(\varepsilon) = \frac{1}{2} C_{ij}^{mn} \varepsilon_{ij} \varepsilon_{mn} \qquad (2.1.4)$$

of the symmetric strain tensor :

$$\varepsilon_{ij} = \frac{1}{2} (D_i u_j + D_j u_i) \quad . \qquad (2.1.5)$$

The set of elastic moduli defining the tensor C_{ij}^{mn} verifies the symmetry conditions

$$C_{ij}^{mn} = C_{ji}^{mn} = C_{ij}^{nm} = C_{mn}^{ij} \quad . \qquad (2.1.6)$$

The relations (2.1.5) show the strain energy to be a functional of the first order derivatives of the displacement field :

$$V_1 = \int_R W\,(D\,u)\,dR \qquad . \tag{2.1.7}$$

According to the symmetry properties (2.1.6), we obtain the more explicit expression

$$V_1 = \frac{1}{2}\int_R C_{ij}^{mn}\, D_i\, u_j\, D_m\, u_n\, dR \qquad . \tag{2.1.8}$$

Finally, we deduce from the strain energy density W the stress tensor σ_{ij} conjugated to the strain tensor ϵ_{ij} through the energy relation

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = C_{ij}^{mn}\,\epsilon_{mn} = C_{ij}^{mn}\, D_m\, u_n \quad , \tag{2.1.9}$$

under the condition that the formal distinction between ϵ_{ij} and ϵ_{ji} in the case $i \neq j$ be kept.

Another contribution to the potential energy, V_2, is a potential energy associated to the displacements on the part $\partial_2 R$ of the boundary :

$$V_2 = -\int_{\partial_2 R} F(u,t)\, d\partial R \qquad . \tag{2.1.10}$$

On the remaining part $\partial_1 R$ of the boundary, we shall assume that the displacements are prescribed time functions

$$u_j = \bar{u}_j\,(t) \qquad \text{on } \partial_1 R \qquad . \tag{2.1.11}$$

For simplicity, we do not take into consideration other potential energies like that associated with conservative external body forces functions of position (for example, gravity or electromagnetic forces).

Applying the principle

$$\delta\, f[u] = 0 \tag{2.1.12}$$

with the Hamilton's principle rules

- at the extremities of the time interval :

$$\delta u_j = 0 \qquad \text{for} \qquad t = t_1 \quad \text{and} \quad t = t_2 \qquad (2.1.13)$$

- on that part of the boundary on which displacements are imposed :

$$\delta u_j = 0 \qquad \text{at all times on } \partial_1 R \ , \qquad (2.1.14)$$

the following variational derivatives are obtained

$$- \rho \ddot{u}_j + D_i \sigma_{ij} = 0 \qquad \text{in } R \ . \qquad (2.1.15)$$

They are the dynamic equilibrium conditions written in the d'Alembert's form. In terms of the displacement field alone, they become

$$- \rho \ddot{u}_j + D_i \left(C_{ij}^{mn} D_m u_n \right) = 0 \ . \qquad (2.1.15')$$

The natural boundary conditions supplementing the essential conditions (2.1.11) are found to be

$$- n_i \sigma_{ij} + \frac{\partial F}{\partial u_j} = 0 \qquad \text{on } \partial_2 R \ , \qquad (2.1.16)$$

where n_i denotes the direction cosines of the outward normal. They state that the surface tractions

$$t_j = n_i \sigma_{ij} \ . \qquad (2.1.17)$$

are prescribed functions of displacement and time

$$t_j = \frac{\partial F}{\partial u_j} = \bar{t}_j \ (u,t) \qquad \text{on } \partial_2 R \ . \qquad (2.1.18)$$

An important particular case is that of a boundary potential energy which is a linear function of the displacement field :

$$F(u,t) = u_j \ \bar{t}_j \ (t) \ . \qquad (2.1.19)$$

The natural boundary conditions reduce then to the imposition of the surface tractions

$$t_j = \bar{t}_j \ (t) \qquad \text{on } \partial_2 R \quad . \tag{2.1.20}$$

2.2. The canonical variational principle

We follow the way indicated by Friedrichs[2] to transform Hamilton's principle into a canonical and, later, dual form. To this purpose we introduce a dislocation potential Δ into the functional (2.1.1) that may then be written as

$$\hat{f}\ [\ u,v,\varepsilon,\lambda,\alpha,\mu\ ] = \int_{t_1}^{t_2} (T - V + \Delta)dt \tag{2.2.1}$$

The dislocation potential consists of three parts :
a first one,

$$\Delta_1 = \int_R \lambda_{ij} \{ \varepsilon_{ij} - \tfrac{1}{2} (D_i u_j + D_j u_i) \} \, dR \quad , \tag{2.2.2}$$

that removes the compatibility relations (2.1.5) as essential conditions of the variational equations for stationarity of (2.2.1) by means of a tensor of lagrangian multipliers λ_{ij}. Correspondingly, the potential energy is now expressed as

$$V_1 = \int_R W(\varepsilon) \, dR \quad . \tag{2.2.3}$$

The second part,

$$\Delta_2 = \int_{\partial_1 R} \alpha_j \{ u_j - \bar{u}_j(t) \} \, d\partial R \quad , \tag{2.2.4}$$

incorporates the essential boundary constraints (2.1.11) as natural boundary conditions through the vector of lagrangian multipliers α_j. Finally,

$$\Delta_3 = \int_R \mu_j \, (\dot{u}_j - v_j) \, dR \tag{2.2.5}$$

introduces the velocity field, v_j, as an independent field, and the kinetic energy is now expressed as

$$T = \frac{1}{2} \int_R \rho \, v_j \, v_j \, dR \qquad (2.2.6)$$

The introduction of the dislocation potentials Δ_1 and Δ_2 is a classical procedure used in statics to provide a logical approach to the two-field variational principles, one of which is the Reissner-Hellinger principle. Its dual form is given in reference 15.
On the other hand, the introduction of the dislocation potential Δ_3 is an essential step in the logical transformation from Hamilton's principle to the eulerian variational principles of fluid mechanics

The first step in the simplification of the functional (2.2.1) consists in identifying the multipliers λ_{ij} and μ_j. This results immediatly from setting the variational derivatives of ε_{ij} and v_j equal to zero :

$$\lambda_{ij} - \frac{\partial W}{\partial \varepsilon_{ij}} = 0 \qquad (2.2.7)$$

and $$\mu_i - \rho \, v_i = 0 \qquad \text{in } R \,, \qquad (2.2.8)$$

and $$\alpha_j - n_i \, \sigma_{ij} = 0 \qquad \text{on } \partial_1 R \,. \qquad (2.2.9)$$

Hence the λ_{ij} tensor is identified with the stress tensor σ_{ij} related to the strain tensor by the constitutive equation (2.1.9); similarly, α_j is identified through (2.1.18) with the surface tractions t_j on the part $\partial_1 R$ of the boundary.
On the other hand, (2.2.8) shows that the μ_i vector corresponds to the momentum per unit mass.

When (2.2.7) is substitued into the functional (2.2.1), the expression

$$\varepsilon_{ij} \frac{\partial W}{\partial \varepsilon_{ij}} - W \qquad (2.2.10)$$

that appears in the integrand of the functional is treated as a Legendre transformation introducing the complementary energy density $\phi(\sigma)$, a function of the elements of the stress tensor. This transformation is possible whenever the strains can be expressed in terms of the stresses, in which case, through differentiation of

$$\varepsilon_{ij}\,\sigma_{ij} - W = \phi(\sigma) \quad , \tag{2.2.11}$$

the dual constitutive equations

$$\varepsilon_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}} \quad . \tag{2.2.12}$$

follow immediatly.
We thus obtain

$$\Delta_1 - V_1 = U_1 - \Gamma_1 \tag{2.2.13}$$

with the complementary strain energy

$$U_1 = \int_R \phi(\sigma)\, dR \quad , \tag{2.2.14}$$

and

$$\Gamma_1 = \int_R \sigma_{ij}\, D_i\, u_j\, dR \tag{2.2.15}$$

where account was taken of the symmetry of the stress tensor.
Also the substitution of (2.2.8) yields

$$T + \Delta_3 = - T + \Gamma_3 \tag{2.2.16}$$

where

$$\Gamma_3 = \int_R \rho\, v_j\, \dot{u}_j\, dR \quad . \tag{2.2.17}$$

The meaning (2.2.9) of the multipliers α_j on the part $\partial_1 R$ of the boundary is utilized in order to transform Δ_2 into

$$\Delta_2 = \int_{\partial_1 R} n_i\, \sigma_{ij}\, \{\, u_j - \bar{u}_j\,(t)\,\}\, d\partial R \quad . \tag{2.2.18}$$

The final result is a canonical variational principle in the Friedrichs sense, requiring the stationarity of the functional

$$c\,[u,v,\sigma] = \int_{t_1}^{t_2} (U_1 - \overset{1}{T} - V_2 + \Delta_2 - \Gamma_1 + \Gamma_3)dt \qquad (2.2.19)$$

in which the various integrands are given respectively by (2.2.14), (2.2.6), (2.1.10), (2.2.18), (2.2.15) and (2.2.17). It can be used to develop approximate solutions based on independent discretizations of displacements, velocities and stresses. Taking the variational derivatives of (2.2.19) restitutes the natural conditions of the canonical principle. The Euler equations are respectively :

- the dynamic equilibrium equations

$$D_i\,\sigma_{ij} - \rho\,\dot{v}_j = 0 \qquad \text{in } R \qquad (2.2.20)$$

and $$t_j = \bar{t}_j\,(u,t) \quad \text{on } \partial_2 R \qquad (2.2.21)$$

for the variations on displacements;

- the compatibility equations (2.1.5) written in terms of the stress tensor

$$\frac{\partial \Phi}{\partial \sigma_{ij}} - \frac{1}{2}\,(D_i\,u_j + D_j\,u_i) = 0 \qquad \text{in } R \qquad (2.2.22)$$

and $$u_i = \bar{u}_i(t) \qquad \text{on } \partial_1 R \qquad (2.2.23)$$

for the variations on stresses;

- the constraints between velocity and displacement fields

$$\rho(\dot{u}_i - v_i) = 0 \qquad (2.2.24)$$

for the variations on velocities.

Note that all boundary conditions are now natural. For finite element applications of the variational principles, it is important to recognize the nature of the transition conditions at interfaces. Because the only space derivatives contained in the functional are those affecting displacements, we must consider that physical continuity

$$(u_j)_+ = (u_j)_- \tag{2.2.25}$$

is an a priori requirement; the natural transition conditions provided by the principle and deduced from the common variations $(\delta u_j)_+ = (\delta u_j)_-$ at interfaces are then

$$(t_j)_+ + (t_j)_- = 0 \quad . \tag{2.2.26}$$

From that point of view, the canonical principle behaves as the classical principle of variation of displacements. Obviously the interface constraints could be incorporated into the principle through an additional dislocation potential and the corresponding multipliers identified. Then, just as was found to be the case for the boundary conditions, all transitional conditions would be cared for by the principle.

2.3. The complementary energy principle of elastodynamics.

The complementary energy principle of elastodynamics can easily be deduced from the canonical principle by specializations.
To this purpose, the functional (2.2.19) is integrated by parts with respect to the displacement field in both the terms Γ_1 and Γ_3 of the integrand.
The first transformation

$$\Gamma_1 = \int_{\partial R} n_i \, \sigma_{ij} \, u_j \, d\partial R - \int_R u_j \, D_i \, \sigma_{ij} \, dR \tag{2.3.1}$$

cancels a part of the contribution of Δ_2 :

$$- V_2 + \Delta_2 - \Gamma_1 = \int_{\partial_2 R} \{ F(u,t) - n_i \, \sigma_{ij} \, u_j \} d\partial R$$

$$- \int_{\partial_1 R} n_i \, \sigma_{ij} \, \bar{u}_j \, \partial dR + \int_R u_j \, D_i \, \sigma_{ij} \, dR \quad . \tag{2.3.2}$$

In the second transformation,

$$\int_{t_1}^{t_2} \Gamma_3 \, dt = \left[\int_R \rho \, v_j \, u_j \, dR \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_R \rho \, \dot{v}_j \, u_j \, dR \, dt \tag{2.3.3}$$

the term at the time limits appearing in the right-hand side is dropped on the basis of the Hamilton's rule (2.1.13), and the introduction of the new requirements

$$\delta v_j = 0 \quad \text{for } t = t_1 \text{ and } t = t_2 \quad . \tag{2.3.4}$$

The functional has thus been modified as follows :

$$\begin{aligned} & c \left[u, v, \sigma \right] = \\ & \int_{t_1}^{t_2} \left[\int_R \{ (D_i \sigma_{ij} - \rho \dot{v}_j) u_j \} \, dR + U_1 - T \right. \\ & \left. + \int_{\partial_2 R} \{ F(u,t) - t_j u_j \} \, d\partial R - \int_{\partial_1 R} t_j \bar{u}_j \, d\partial R \right] dt \quad , \end{aligned} \tag{2.3.5}$$

and may be simplified by making the assumption that the dynamic equilibrium equations (2.2.20) are a priori satisfied; this causes the first term to vanish. On the other hand, the integrand of the term $\partial_2 R$ is obviously related to another Legendre transformation

$$u_j \frac{\partial F}{\partial u_j} - F(u,t) = G(t_j, t) \quad , \tag{2.3.6}$$

which is defined whenever the surface tractions given by (2.1.19) depend on the displacement field, in which case the Hessian matrix

$$\left[\frac{\partial^2 F}{\partial u_i \, \partial u_j} \right] \tag{2.3.7}$$

does not vanish. The complementary function is then such that

$$\frac{\partial G}{\partial t_j} = u_j \quad . \tag{2.3.8}$$

With the notations

$$B_1 = \int_{\partial_1 R} t_j \, \bar{u}_j(t) \, d\partial R \tag{2.3.9}$$

and

$$B_2 = \int_{\partial_2 R} G(t_j, t) \, d\partial R \quad , \tag{2.3.10}$$

the canonical functional reduces now to one that depends only on the stress field :

$$g[\sigma] = \int_{t_1}^{t_2} (U_1 - T - B_1 - B_2) \quad , \tag{2.3.11}$$

provided the kinetic energy be also expressed in terms of the stress field. The dynamical equilibrium equations (2.2.20) that must be satisfied a priori allow precisely to express the velocity field in terms of stresses :

$$v_j = v_j \Big|_{t_1} + \frac{1}{\rho} D_i \int_{t_1}^{t} \sigma_{ij} \, dt \quad . \tag{2.3.12}$$

The time integration is avoided by introducing an impulse field, τ_{ij}, such that

$$v_j = \frac{1}{\rho} D_i \, \tau_{ij} \quad . \tag{2.3.13}$$

Hence we adopt the definition

$$\tau_{ij} = \tau_{ij} \Big|_{t_1} + \int_{t_1}^{t} \sigma_{ij} \, dt \tag{2.3.14}$$

which yields

$$\sigma_{ij} = \dot{\tau}_{ij} \quad , \tag{2.3.15}$$

and we have to replace the requirements on the velocity field at time limits by

$$\delta \tau_{ij} = 0 \qquad \text{for} \quad t = t_1 \quad \text{and} \quad t = t_2 \quad . \tag{2.3.16}$$

A convenient formulation of the complementary energy principle of elastodynamics is thus the stationarity of the functional

$$g[\tau] = \int_{t_1}^{t_2} \Big[\int_R \{ \phi(\dot{\tau}_{ij}) - \frac{1}{2\rho} D_i \tau_{ij} D_m \tau_{mj} \} dR$$

$$- \int_{\partial_1 R} t_j \bar{u}_j(t) \, d\partial R - \int_{\partial_2 R} g(t_j, t) d\partial R \Big] dt \quad . \tag{2.3.17}$$

It depends only on the impulse field τ_{ij}, since the associated surface tractions can be written as

$$t_j = n_i \dot{\tau}_{ij} \quad . \tag{2.3.18}$$

The variational derivatives of this principle are

$$- \frac{d}{dt} \frac{\partial \phi}{\partial \dot{\tau}_{ij}} + \frac{1}{2\rho} (D_i D_m \tau_{mj} + D_j D_m \tau_{mi}) = 0 \tag{2.3.19}$$

that, in view of (2.3.15), (2.2.12) and (2.2.20), are easily interpreted as the time derivatives of the compatibility conditions (2.1.5) :

$$\dot{\varepsilon}_{ij} - \frac{1}{2} (D_i v_j + D_j v_i) = 0 \quad . \tag{2.3.20}$$

The natural boundary conditions are seen to be

$$- \frac{1}{\rho} D_i \tau_{ij} + \frac{d}{dt} \bar{u}_j(t) = 0 \qquad \text{on } \partial_1 R \tag{2.3.21}$$

$$- \frac{1}{\rho} D_i \tau_{ij} + \frac{d}{dt} \frac{\partial G}{\partial \dot{\tau}_{ij}} = 0 \qquad \text{on } \partial_2 R \quad . \tag{2.3.22}$$

Both are easily interprated through (2.3.13) and (2.3.8) as expressing the continuity of the velocity field on boundaries.

In the special case where the imposed surface tractions (2.1.19) do not depend on the displacement field, the Legendre transformation (2.3.6) yields a G function that is identically zero; the boundary requirement

$$t_j = \bar{t}_j(t) \qquad \text{on} \quad \partial_2 R \tag{2.3.23}$$

becomes then an essential condition in the complementary energy formulation. Similarly, because the displacement field vanished completely, the continuity of the impulse field (2.2.26) becomes an a priori requirement.

2.4. Matrix formulation of the variational principles

Displacements, surface tractions, strains, stresses and impulses can be presented as row vectors :

$$\begin{aligned} u' &= (u_1\ u_2\ u_3) \\ t' &= (t_1\ t_2\ t_3) \\ \varepsilon' &= (\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{23}\ \gamma_{31}\ \gamma_{12}) \\ \sigma' &= (\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{31}\ \sigma_{12}) \\ \tau' &= (\tau_{11}\ \tau_{22}\ \tau_{33}\ \tau_{23}\ \tau_{31}\ \tau_{12}) \end{aligned} \tag{2.4.1}$$

with the classical definition of the shear angles :

$$\gamma_{ij} = \gamma_{ji} = \varepsilon_{ij} + \varepsilon_{ji} \tag{2.4.2}$$

Then, with the help of a matrix differential operator

$$D' = \begin{pmatrix} D_1 & 0 & 0 & 0 & D_3 & D_2 \\ 0 & D_2 & 0 & D_3 & 0 & D_1 \\ 0 & 0 & D_3 & D_2 & D_1 & 0 \end{pmatrix} \tag{2.4.3}$$

and a corresponding matrix of direction cosines for the outward normal

$$N' = \begin{pmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{pmatrix} , \qquad (2.4.4)$$

The basic equations of linear elasticity theory recalled in the preceding sections take the following forms

$$\varepsilon = D\,u \qquad \text{strain-displacement relations} \qquad (2.4.5)$$

$$D'\,\tau - \rho\,\ddot{u} = 0 \qquad \text{dynamic equilibrium equations} \qquad (2.4.6)$$

$$t = N'\sigma \qquad \text{surface equilibrium equations} \; . \qquad (2.4.7)$$

The linear stress-strain relations

$$\sigma = H\,\varepsilon \qquad \text{or} \qquad \varepsilon = H^{-1}\,\sigma \qquad (2.4.8)$$

involve a symmetric, positive-definite matrix H of elastic moduli.

Following these definitions, the functionals of both displacement and complementary energy principles may be written in matrix form :

$$f[u] = \int_{t_1}^{t_2} \Big[\frac{1}{2}\int_R \rho\,\dot{u}'\,\dot{u}\,dR - \frac{1}{2}\int_R (Du)'H(Du)dR + \int_{\partial_2 R} \bar{t}'\,u\,d\partial R\Big]\,dt \; , \qquad (2.4.9)$$

and

$$g[\tau] = \int_{t_1}^{t_2} \Big[\frac{1}{2}\int_R \dot{\tau}'\,H^{-1}\,\dot{\tau}\,dR - \frac{1}{2}\int_R \frac{1}{\rho}\,(D'\tau)'(D'\tau)dR - \int_{\partial_1 R} (N'\dot{\tau})'\,\bar{u}\,d\partial R\Big]\,dt \qquad (2.4.10)$$

in the particular case where the external potential energy results from the imposition of the surface tractions on $\partial_1 R$.

The matrix notation adopted here reveals more suitable than tensor notations when discretizing the functionals (2.4.9) and (2.4.10) by the finite element method.

3. EIGENVALUE ANALYSIS

The computation of the eigenvalues and associated eigenmodes of an elastic structure can be considered as one of the most important goals aimed at in the dynamic analysis of structures.

The benefit expected from a dual approach of the eigenvalue problem is the same as that guaranteed in statics : a bracketing of the exact solution between upper and lower bounds when using alternately the kinematical and equilibrium approaches. Keeping the eigenvalue problem under variational form is the easiest way to derive the respective properties of both analyses.

3.1. THE KINEMATICAL APPROACH

3.1.1. The Rayleigh quotient

Under the assumption that no external load is applied to the system, the Euler equation of the displacement variational principle reduces to

$$D' (H Du) - \rho \ddot{u} = 0 \tag{3.1.1}$$

under the a priori requirement

$$u = 0 \qquad \text{on } \partial_1 R \tag{3.1.2}$$

that defines the set of admissible solutions. The natural boundary conditions obtained from the principle are homogeneous

$$t = N' \sigma = 0 \qquad \text{on } \partial_2 R \tag{3.1.3}$$

As is well known, the general solution can be obtained by superposition of harmonic solutions of the form

$$u(x,t) = u(x) \cos (\omega t + \phi) \tag{3.1.4}$$

obtained by separation of the time variable.
The eigen-modes and eigenvalues of the corresponding self-adjoint differential system

$$\left.\begin{aligned} &D' (H Du) + \omega^2 \rho u = 0 \\ &u = 0 \end{aligned}\right\} \quad \text{on } \partial_1 R \tag{3.1.5}$$

are denoted by

$$\left.\begin{aligned} &u_{(1)} , u_{(2)} , \cdots , u_{(n)} \cdots \\ &\omega_1^2 \leqslant \omega_2^2 \leqslant \cdots \leqslant \omega_n^2 \cdots \end{aligned}\right\} \tag{3.1.6}$$

We can define the corresponding Rayleigh quotient

$$\left.\begin{aligned} \omega^2 = R(u) = \frac{\int_R (Du)' H (Du)\, dR}{\int_R \rho\, u'\, u\, dR} \\ u = 0 \qquad \text{on } \partial_1 R \end{aligned}\right\} , \tag{3.1.7}$$

and by differentiating it with respect to the displacement vector u, it follows immediatly that (3.1.7) takes a stationary value $\omega^2 = \omega_i^2$ for the corresponding eigenmode $u_{(i)}$.

We recall the well known orthogonality properties :

$$\int_R (Du_{(i)})' H (Du_{(j)})\, dR = \gamma_i \delta_{ij} \tag{3.1.8}$$

and $$\int_R \rho\, u'_{(i)}\, u_{(j)}\, dR = \mu_i \delta_{ij} \,. \tag{3.1.9}$$

The generalized stiffness and mass so introduced, γ_i and μ_i , are not independent of each other, since (3.1.7) gives

$$\omega_i^2 = \frac{\gamma_i}{\mu_i} \,. \tag{3.1.10}$$

A convenient choice of norm consists thus to impose

$$\mu_i = \int_R \rho\, u'_{(i)}\, u_{(i)}\, dR = 1 \tag{3.1.11}$$

so that $$\int_R (Du_{(i)})' H (Du_{(i)})\, dR = \omega_i^2 \,. \tag{3.1.12}$$

Note that the possible zero frequency modes of the structure have to be interpreted as either the rigid body modes or the mechanisms of the structure, since (3.1.11) shows that they involve no strain energy.

3.1.2. Independent, or maximum-minimum characterization of eigenvalues[2]

All minimum properties of the Rayleigh quotient in the displacement approach can be deduced from Courant's theorem, often called the "minimax principle".

To this purpose let us turn to the variational problem of minimizing the Rayleigh quotient (3.1.7) under r-1 arbitrary constraints taking the form of conditions of orthogonality :

$$\int_R v'_j \, u \, dR = 0 \qquad j = 1, \ldots, r-1 \qquad (3.1.13)$$

Courant's principle states that "the r^{th} eigenvalue of problem (3.1.5) is the maximum value which can be given to the minimum of the Rayleigh quotient (3.1.7) by varying the r - 1 constraints (3.1.13) . This maximum is reached for $v_j = \rho \, u_{(j)}$".

If we denote by $m(v_1, \ldots v_{r-1})$ the constrained minima of the Rayleigh quotient, Courant's principle can thus be written as

$$\omega_r^2 = \max_{v_j} m\,(v_1, \ldots v_{r-1}) = m(\rho \, u_{(1)}, \ldots, \rho \, u_{(r-1)}) \; . \qquad (3.1.14)$$

To prove it, express the admissible solutions as expansions in eigenmodes of the unconstrained system :

$$u = \sum_{i=1}^{\infty} a_i \, u_{(i)} \qquad . \qquad (3.1.15)$$

Thanks to the orthonorming properties (3.1.8 to 3.1.12), the problem is transformed into

$$\omega^2 = \frac{\sum_1^{\infty} a_i^2 \, \omega_i^2}{\sum_1 a_i^2} \qquad \text{minimum} \qquad (3.1.16)$$

under the constraints

$$\sum_{i=1}^{\infty} c_{ij}\, a_i = 0 \qquad j = 1, 2 \ldots r-1 \qquad (3.1.17)$$

where $$c_{ij} = \int_R v'_j\, u_{(i)}\, dR \qquad (3.1.18)$$

Or, equivalently,

$$m = \sum_{1}^{\infty} a_i^2\, \omega_i^2 \qquad \text{minimum} \qquad (3.1.19)$$

under the constraints (3.1.17) and the additional norming constraint

$$\sum_{1}^{\infty} a_i^2 = 1 \qquad (3.1.20)$$

Under given constraints (3.1.17) m takes the value

$$m = \sum_{1}^{r} a_i^2\, \omega_i^2 \leq \omega_r^2 \sum_{1}^{r} a_i^2 = \omega_r^2 \qquad (3.1.21)$$

for the particular set of values

$$a_i = 0 \qquad i > r$$

and values of a_i for $i < r$ satisfying the constraints which are then reduced to

$$\begin{cases} \sum_{1}^{r} c_{ij}\, a_i = 0 & j = 1, 2 \ldots r-1 \qquad (3.1.22) \\ \sum_{1}^{r} a_i^2 = 1 & \qquad (3.1.23) \end{cases}$$

The homogeneous system (3.1.22) has always a non trivial solution (the number of unknowns exceeds by one the number of equations), that can be scaled to satisfy (3.1.23).

The result (3.1.21) being independent of the constraints, it can be concluded that, whatever those may be, the constrained minimum cannot exceed the r^{th} eigenvalue ω_r^2. Courant's principle is proved if a set of constraints can be produced for which the minimum actually reaches ω_r^2; it is then the maximum of all minimums. This is the case for the choice

$$v_j = \rho\, u_{(j)} \qquad\qquad j = 1, 2 \ldots r-1$$

for then we find from the orthonorming properties

$$c_{ij} = \delta_{ij} \qquad \text{and, in particular} \qquad c_{ij} = 0 \quad \text{for} \quad i \geq r \ .$$

Thus the constraints (3.1.17) require simply that

$$a_i = 0 \qquad\qquad i = 1, 2 \ldots r-1$$

The choice $a_r = 1$ and $a_i = 0$ $\quad i > r$.

satisfies the norming condition (3.1.20) and we find

$$m = \omega_r^2 \ .$$

3.1.3. Recursive characterization of eigenvalues

As a direct consequence of the minimax principle, we consider the restricted class of admissible solutions which consists only of displacement modes orthogonal to the first r - 1 eigenvectors

$$\int_R \rho\, u'_{(j)}\, u\, dR = 0 \qquad j = 1, \ldots r-1 \ . \qquad (3.1.24)$$

Then $$\min_u R(u) = \omega_r^2 \ , \qquad (3.1.25)$$

and this minimum is reached for $u = u_{(r)}$.

This well known result defines an efficient procedure for obtaining eigenvalues recursively by minimization techniques [18].

3.1.4. Minimum of the Rayleigh quotient under constraints [25]

The maximum-minimum property of eigenvalues also allows to predict how the eigenspectrum will be modified by imposing a set of independent constraints :

$$\int_R g'_j \, u \, dR = 0 \qquad j = 1, \ldots r-1 \qquad (3.1.26)$$

As a first consequence of Courant's principle, ω_r^2 is the maximum value which can be given to the minimum of R(u) when varying the r-1 arbitrary constraints (3.1.13) :

$$\omega_r^2 = \max_v m\,(v_1, \ldots v_{r-1}) \qquad . \qquad (3.1.27)$$

Next denote by $\tilde{\omega}_r^2$ the r^{th} eigenvalue of the constrained Rayleigh quotient. We may also write

$$\tilde{\omega}_r^2 = \max_v m\,(v_1 \ldots v_{r-1} ; g_1, \ldots g_s) \qquad , \qquad (3.1.28)$$

which is obviously bounded from below by the corresponding eigenvalue of the unrestrained minimum problem.

Hence $$\omega_r^2 \leq \tilde{\omega}_r^2 \qquad . \qquad (3.1.29)$$

Now let us also vary the s imposed constraints (3.1.26) : the maximum value that would be reached by the minimum of R(u) if all r + s - 1 constraints could be varied is equal to

$$\omega_{r+s}^2 = \max_{v,g} m\,(v_1, \ldots v_{r-1} ; g_1, \ldots g_s) \qquad (3.1.30)$$

Hence we obtain the second inequality

$$\tilde{\omega}_r^2 \leqslant \omega_{r+s}^2 \quad . \qquad (3.1.31)$$

We have thus established Rayleigh's theorem on constraints :
"If s arbitrary constraints are imposed on a vibrating system of which eigenvalues are given by (3.1.6), then the new eigenvalues $\tilde{\omega}_r^2$ separate the old ones in the sense that

$$\omega_r^2 \leqslant \tilde{\omega}_r^2 \leqslant \omega_{r+s}^2 \qquad (3.1.32)$$

for every $r \leqslant n-s$".

Note that the discretization of the eigenvalue differential system (3.1.5), as it reduces the class of admissible functions to a finite set of displacement modes, may be considered as an imposition of additional constraints to the eigenvalue problem. Rayleigh's theorem guarantees thus the upper bound character of eigenfrequencies computed from a kinematical approach.

3.2. THE EQUILIBRIUM APPROACH

3.2.1. Self-stressing and vibration modes

The variational derivative of the complementary principle (2.4.10) is

$$- H^{-1} \ddot{\tau} + D \left(\frac{1}{\rho} D' \tau\right) = 0 \qquad (3.2.1)$$

and its natural boundary condition

$$- \frac{1}{\rho} D' \tau + \frac{d}{dt} \bar{u} = 0 \qquad \text{on } \partial_1 R \qquad (3.2.2)$$

to which we must add the a priori boundary condition

$$t = N' \dot{\tau} = 0 \qquad \text{on } \partial_2 R \quad . \qquad (3.2.3)$$

In the spectral analysis problem we set

$$\tau\ (x,t) = \tau(x)\ \cos(\omega t + \phi) \tag{3.2.4}$$

whereby the Euler equation (3.2.1) takes the form

$$\omega^2\ H^{-1}\ \tau + D\ (\frac{1}{\rho}\ D'\ \tau) = 0 \tag{3.2.5}$$

The Rayleigh-type quotient appears in the form

$$\omega^2 = R(\tau) = \frac{\int_R \frac{1}{\rho}\ (D'\tau)'\ D'\tau\ dR}{\int_R \tau'\ H^{-1}\ \tau\ dR} \tag{3.2.6}$$

in which the amplitude vector $\tau(x)$ must satisfy the a priori boundary condition

$$N'\ \tau = 0 \qquad \text{on} \quad \partial_2 R \quad . \tag{3.2.7}$$

As in the kinematical approach $\omega^2 = 0$ belongs to the spectrum of eigenvalues of the eigenvalue problem (3.2.5) and (3.2.6). It appears clearly on (3.2.6) that the eigenmodes pertaining to this eigenvalues satisfy

$$D'\ \tau = 0 \tag{3.2.8}$$

together with (3.2.7). They are statical self-stressing modes, that is stress distributions in equilibrium without external forces; thus capable of existing without inertia forces. In contrast however to the kinematical approach, their number is infinite for a continuum (finite but large after a discretization), while the kinematical modes are finite in number and even non existent if the structure is at least isostatically supported.

The self-stressing modes will be denoted by $\sigma_{(i)}$ and will be supposed to be referred to an orthonormal basis

$$\int_R \sigma'_{(i)} H^{-1} \sigma_{(j)} \, dR = \delta_{ij} \qquad (3.2.9)$$

The other eigenmodes will be denoted by $\tau_{(r)}$ and ranged in increasing order of their eigenvalues

$$\begin{cases} \tau_{(1)} \, , \; \tau_{(2)} \, , \; \cdots \quad \tau_{(n)} \quad \cdots \\ \omega_1^2 \leqslant \omega_2^2 \quad \cdots \leqslant \omega_n^2 \quad \cdots \end{cases} \qquad (3.2.10)$$

We may assume without loss of generality that, in addition to (3.2.9), we dispose of the orthonorming properties

$$\int_R \sigma'_{(i)} H^{-1} \tau_{(r)} \, dR = 0 \qquad (3.2.11)$$

$$\int_R \tau'_{(r)} H^{-1} \tau_{(s)} \, dR = \delta_{rs} \qquad (3.2.12)$$

$$\int_R (D' \tau_{(r)})' D' \tau_{(s)} \, dR = \omega_r^2 \delta_{rs} \qquad (3.2.13)$$

3.2.2. Properties of the Rayleigh quotient [13, 20]

As a consequence of Courant's principle, the Rayleigh quotient will theoretically furnish an upper bound to the eigenvalue ω_1^2, provided τ is orthogonal to all the self-stressing modes of zero frequency.

It is of interest to note that it is quite possible to construct impulse distributions that are orthogonal to all the self-stressing modes. Orthogonality means

$$\int_R \sigma'_{(i)} \hat{\varepsilon} \, dR = 0 \qquad (3.2.14)$$

where $\sigma_{(i)}$ is any self-stressing mode, and $\hat{\varepsilon}$ is the strain distribution associated with the orthogonal stress distribution through the constitutive equations. A sufficient condition to implement (3.2.14) is to make the strain distribution compatible :

$$\hat{\varepsilon} = D\,\hat{u} \qquad \text{in } R \qquad (3.2.15)$$

with the homogeneous boundary condition

$$\hat{u} = 0 \qquad \text{on } \partial_1 R \quad . \qquad (3.2.16)$$

Indeed one obtains then from integration by parts

$$\int_R \sigma'_{(i)}\,\hat{\varepsilon}\,dR = \int_R \sigma'_{(i)}\,(D\,\hat{u})\,dR$$

$$= \int_{\partial R} (N'\sigma_{(i)})'\hat{u}\,d\,R - \int_R (D'\sigma_{(i)})'\hat{u}\,dR \quad ,$$

and the right-hand side vanishes in view of (3.2.7) and (3.2.8). and (3.2.16)
The condition is also necessary as can be shown by satisfying (3.2.8) with the help of stress functions, integrating (3.2.14) by parts and using the arbitrariness in the stress functions[34] to show that the strains have to satisfy the integrability conditions for the existence of displacements.
In view of

$$\hat{\tau} = H\,\hat{\varepsilon} = H\,(D\,\hat{u}) \quad , \qquad (3.2.17)$$

the Rayleigh quotient becomes expressible entirely in terms of the auxiliary displacement field $\hat{u}$:

$$R\,(\hat{u}) = \frac{\int_R \frac{1}{2}\,(D'\,H\,D\,\hat{u})'(D'\,H\,D\,\hat{u})\,dR}{\int_R (D\,\hat{u})'(D\,\hat{u})\,dR} \quad . \qquad (3.2.18)$$

This new Rayleigh quotient, that leads immediatly to the non zero eigenvalues of the problem, has the same kinematical boundary conditions to be satisfied ab initio as (3.1.7), but in addition the transformed boundary conditions (3.2.7)

$$N' H (D \hat{u}) = 0 \qquad \text{on } \partial_2 R \quad . \tag{3.2.19}$$

An interesting inequality[13] relates (3.1.7) to (3.2.18).
To prepare its proof consider the following obvious equality

$$\int_R (\sqrt{\rho}\, a - \frac{\lambda b}{\sqrt{\rho}})' (\sqrt{\rho}\, a - \frac{\lambda b}{\sqrt{\rho}}) \, dR > 0 \tag{3.2.20}$$

valid for any value of the scalar λ . Expanding it gives

$$\int_R \rho\, a' a \, dR - 2\lambda \int_R a'b \, dR + \lambda^2 \int_R \frac{1}{\rho} b'b \, dR \geq 0 \quad , \tag{3.2.21}$$

and the minimum of the left-hand side is reached for

$$\lambda = \frac{\int_R a' b \, dR}{\int_R \frac{1}{\rho} b'b \, dR} \quad . \tag{3.2.22}$$

Hence, after substitution

$$\int_R \frac{1}{2} \rho\, a'a \, dR - \{ \int_R \frac{1}{2} a'b \, dR \}^2 / \int_R \frac{1}{2} \rho\, b'b \, dR \} \geq 0 \tag{3.2.23}$$

Setting $a = \hat{u}$ (3.2.24)

and $b = D' (H \quad D \hat{u})$,

we already identify two of the integrals as the denominator of (3.1.7) and the numerator of (3.2.18).
For the third integral

$$\int_R \frac{1}{2} a'b \, dR = \int_R \frac{1}{2} \hat{u}' \, D' \, (H \, D\hat{u}) \, dR \qquad (3.2.26)$$

$$= \int_{\partial R} \frac{1}{2} (N \, \hat{u})' H (D \, \hat{u}) \, dS - \int_R \frac{1}{2} (D\hat{u})' H (D\hat{u}) dR$$

$$= - \int_R \frac{1}{2} (D\hat{u})' H (D\hat{u}) \, dR$$

since the surface integral vanishes on account of (3.2.16) and (3.2.19). We obtain thus the inequality

$$\omega_R^2 \leq \hat{\omega}_R^2$$

where ω_R^2 denotes the classical Rayleigh quotient (3.1.7), and $\hat{\omega}_R^2$, the quotient (3.2.12). This means that whenever a displacement field is chosen that satisfies both the kinematical and the stress boundary conditions, the classical Rayleigh quotient is always a better approximation than the one derived from the stress approach with orthogonality to all zero frequency modes. However, as numerical experience shows, direct applications of (3.2.18) usually under-estimate the frequency and converge through lower bounds when the number of degrees of freedom is increased. This is of course due to the fact that the assumed stress modes do not satisfy orthogonality with respect to all the self-stressing modes. Unfortunately there is no guarantee of this property of lower boundeness, and further theoretical research is necessary to be able to incorporate this into the formalism.

4. FINITE ELEMENT MODELS

4.1. The displacement approach [12, 21]

Element stiffness and matrices

Consider the simply connected domain E of an element bounded by its surface ∂E on which it will be convenient to assume, for the time being, that all surface tractions are specified.

The displacement field within the element will in general be discretized in terms of polynomials contained in a 3 x n(a) matrix P(x)

$$u(x,t) = P(x)\, a(t) \tag{4.1.1}$$

where a(t) is a column matrix of unknown time dependent coefficients. It is important for purposes of connections between elements to distinguish between the displacement field along the boundary of the element and in its interior. The boundary displacement field will be determined unambiguously by a set q of generalized boundary coordinates. When those are chosen, we always obtain their values in terms of the set of coefficients

$$q(t) = M\, a(t) \tag{4.1.2}$$

We shall assume, temporarily, that the homogeneous adjoint equation

$$M'\, g = 0$$

has but the trivial solution zero. This insures that the q can be chosen independently and that the general inverse of (4.1.2) is of type

$$a(t) = Q\, q(t) + B\, b(t) \tag{4.1.3}$$

where the first term is a particular solution, the second is the general solution of Ma = 0 and contains the arbitrary column matrix b. Substitution of (4.1.3) into (4.1.1) yields

$$u(x,t) = Q(x)\, q(t) + B(x)\, b(t) \tag{4.1.4}$$

with $$Q(x) = P(x)\, Q \qquad B(x) = P(x)\, B \tag{4.1.5}$$

The Q(x) are the "shaping functions" the B(x) the "bubble functions" so-called because they vanish at the boundary, since the boundary displacements are, by assumption, uniquely determined by the q coordinates. When there are bubble functions, the shaping functions are not unique, they can be modified by arbitrary additions of bubble functions; this is however not essential for the developments to be presented here. The strain field can be represented indifferently by

$$\varepsilon(x,t) = D\,u\,(x,t) = (D\,P(x)\,)\,a(t)$$

$$= (D\,Q(x)\,)\,q(t) + (D\,B(x)\,)\,b(t) \qquad (4.1.6)$$

Discretization of the strain energy is similarly

$$V_1 = \int_E W(\varepsilon)\,dE = \frac{1}{2}\int_E \varepsilon'\,H\,\varepsilon\,dE$$

$$= \frac{1}{2}\,a'\,K_{aa}\,a \qquad (4.1.7)$$

$$= \frac{1}{2}\,q'\,K_{qq}\,q + q'\,K_{qb}\,b + \frac{1}{2}\,b'\,K_{bb}\,b \qquad (4.1.8)$$

In general

$$K_{aa} = \int_E (D\,P(x)\,)'\,H\,(D\,P(x)\,)\,dE \qquad (4.1.9)$$

is easier to compute and

$$\begin{cases} K_{qq} = \int_E (DQ(x))'\,H\,(DQ(x))\,dE \\ K_{qb} = \int_E (DQ(x))'H(DB(x))\,dE \\ K_{bb} = \int_E (DB(x))'\,H\,(DB(x))\,dE \end{cases} \qquad (4.1.10)$$

instead of being calculated from the derivatives of shaping and bubble functions can also be obtained from (4.1.3) and K_{aa}

$$V_1 = \frac{1}{2} (b'B' + q'Q') K_{aa} (Q q + B b) \qquad (4.1.11)$$

$$K_{qq} = Q'K_{aa} Q \qquad K_{qb} = Q'K_{aa} B = K'_{bq}$$

$$K_{bb} = B'K_{aa} B$$

Similarly the kinetic energy discretization

$$T = \frac{1}{2} \int_E \rho \dot{u}' u \, dE = \frac{1}{2} \dot{a}'(t) M_{aa} \dot{a}(t) \qquad (4.1.12)$$

$$M_{aa} = \int_E \rho P'(x) P(x) \, dE \qquad (4.1.13)$$

instead of being expressed in terms of a consistent mass matrix M_{aa}, can also be expressed in terms of consistent mass matrices for the boundary velocities and internal velocities :

$$T = \frac{1}{2} \dot{q}'(t) M_{qq} \dot{q}(t) + \dot{q}'(t) M_{qb} \dot{b}(t) + \frac{1}{2} \dot{b}'(t) M_{bb} \dot{b}(t) \qquad (4.1.14)$$

$$M_{qq} = \int_E \rho Q'(x) Q(x) \, dE = Q' M_{aa} Q$$

$$M_{qb} = \int_E \rho Q'(x) B(x) \, dE = Q' M_{aa} B = M'_{bq} \qquad (4.1.15)$$

$$M_{bb} = \int_E \rho B'(x) B(x) \, dE = B' M_{aa} B$$

While M_{aa} and $\begin{pmatrix} M_{qq} & M_{qb} \\ M_{bq} & M_{bb} \end{pmatrix}$

are certainly positive definite,

K_{aa} and $\begin{pmatrix} K_{qq} & K_{qb} \\ K_{bq} & K_{bb} \end{pmatrix}$

are only non negative because the rigid body modes of the element, which must essentially be included in (4.1.1), are not associated with any strain energy.

Discretization of the external potential energy introduces a natural definition of the "generalized loads"

$$\int_{\partial E} u' \, \bar{t} \, d\partial E = q' \, g^* \qquad (4.1.16)$$

when $$g^* = \int_{\partial E} Q'(x) \, \bar{t} \, d\partial E \; . \qquad (4.1.17)$$

There are no generalized loads conjugate to the bubble coordinates b, because the bubble functions precisely vanish along ∂E.
The starred notation for the generalized loads conjugate to q indicates that they are linear functionals of the surface traction distribution and consequently provide only "weak" information about this distribution.

The functional (2.4.9) of the displacement principle can now be displayed in its completely discretized form for the element as

$$\int_{t_1}^{t_2} \frac{1}{2} \{ q'(K_{qq} \, q + K_{qb} \, b) + b' \, (K_{bq} \, q + K_{bb} \, b) - \dot{q}'(M_{qq} \, \dot{q} + M_{qb} \, \dot{b}) - \dot{b}'(M_{bq} \, \dot{q} + M_{bb} \, \dot{b}) \} \, dt - \int_{t_1}^{t_2} q' \, g^* \, dt \; . \qquad (4.1.18)$$

Its variational derivatives with respect to q and b are

$$K_{qq} \, q + K_{qb} \, b + M_{qq} \, \ddot{q} + M_{qb} \, \ddot{b} = g^* \qquad (4.1.19)$$

$$K_{bq} \, q + K_{bb} \, b + M_{bq} \, \ddot{q} + M_{bb} \, \ddot{b} = 0 \; . \qquad (4.1.20)$$

In view of their contribution to the kinetic energy, the bubble coordinates are not as easily eliminated as in statics, where we can always write

$$b = - K_{bb}^{-1} K_{bq} q \qquad (4.1.21)$$

because bubble functions necessarily represent independent deformation modes of the element and involve a positive definite K_{bb} matrix.
If however, as is generally the case, we are mainly interested in the low frequency dynamic behavior of the structure, we are entitled to assume that the bubble coordinates follow statically the boundary motions as in (4.1.21). Then, substituting (4.1.21), not in equation (4.1.19), but directly into the kinetic and potential energies, we obtain equations of motion of type

$$K_E q_E + M_E \ddot{q}_E = g^*_E \qquad (4.1.22)$$

with "reduced" stiffness and mass matrices[21] :

$$K_E = K_{qq} - K_{qb} K_{bb}^{-1} K_{bq} \qquad (4.1.23)$$

$$M_E = M_{qq} - M_{qb} K_{bb}^{-1} K_{bq} - K_{bq} K_{bb}^{-1} M_{bq} + K_{qb} K_{bb}^{-1} M_{bb} K_{bb}^{-1} K_{bq}$$

This procedure is a particular case of a general reduction method to be presented in section 7. In (4.1.22) q_E and g^*_E stand respectively for q and g^*. When the bubble coordinates are not eliminated we still consider (4.1.22) as representing the equations of motion of the element, but here

$$q'_E = (q' \quad b') \qquad g^{*\prime}_E = (g^{*\prime} \quad 0)$$

and

$$K_E = \begin{pmatrix} K_{qq} & K_{qb} \\ K_{bq} & K_{bb} \end{pmatrix} \qquad M_E = \begin{pmatrix} M_{qq} & M_{qb} \\ M_{bq} & M_{bb} \end{pmatrix}$$

4.2. Structural stiffness and mass matrices

The principle of assembling the elements consists in stating that corresponding boundary displacements should have common interface values, implementing as shown earlier the exact transition conditions

$$u_+ = u_- \qquad \text{along interfaces.}$$

If w denotes the column matrix of all independent generalized displacements at the structural level, the identification of displacements is achieved by means of Boolean, or incidence, matrices L_E (often called the element localizing matrix) addressing the elements of q_E to the proper ones of w :

$$q_E = L_E\, w \tag{4.1.24}$$

Equating the sum of the virtual work of all external forces acting on each element to the virtual work performed by the forces y^* conjugate to w, which are external to the assembled structure

$$\sum_E q'_E\, g^*_E = w' \sum_E L'_E\, g^*_E = w'\, y^*$$

and observing that this must hold for arbitrary w :

$$y^* = \sum_E L'_E\, g^*_E \tag{4.1.25}$$

Substitution of (4.1.22) yields finally

$$K\, w + M\, \ddot{w} = y^* \tag{4.1.26}$$

with

$$K = \sum_E L'_E\, K_E\, L_E \tag{4.1.27}$$

$$M = \sum_E L'_E\, M_E\, L_E \tag{4.1.28}$$

4.3. Dependent boundary displacements. Superelements

We turn to the case where equation (4.1.2), relating the time dependent coefficients of a(t) and the boundary displacements q(t) required to implement correctly the transition conditions, is such that

$$M' g = 0 \qquad \text{admits non trivial solutions.}$$

Let $g = Y c$ with arbitrary c denote its general solution (the columns of Y are independent). Then

$$Y' q = 0 \tag{4.3.1}$$

is a necessary and sufficient condition for the inversion of (4.1.2) and the boundary coordinates q are no more independent.
This case typically presents itself in trying to set up conforming plate bending elements respecting the Kirchhoff-Love assumption of zero transverse strain [9, 33].

It can be solved by constructing a superelement, that is assembling a small number of elements, each of which suffers from dependency relations between its boundary displacements, in such a way that independence is obtained for the boundary displacements at the assembled level. Suppose that the coordinates of each component element are adressed either to the boundary coordinates $q_{(b)}$ of the superelement or to its interface coordinates $q_{(i)}$

$$q_E = F_E\, q_{(b)} + G_E\, q_{(i)} \tag{4.3.2}$$

and consider the dependency relations

$$Y'_E\, q_E = (Y'_E\, F_E)\, q_{(b)} + (Y'_E\, G_E)\, q_{(i)} = 0 \qquad E = 1, 2, \ldots, N \tag{4.3.3}$$

Then, the complete set of dependancy relations

$$F\, q_{(b)} + G\, q_{(i)} = 0 \qquad F = \begin{pmatrix} Y'_1 F_1 \\ \hline\hline Y'_N F_N \end{pmatrix} \quad G = \begin{pmatrix} Y'_1 G_1 \\ \hline\hline Y'_N G_N \end{pmatrix} \tag{4.3.4}$$

must be solvable for $q_{(i)}$. Thus the rows of G must be linearly independent. Then we can express

$$q_{(i)} = W\, q_{(b)} + B\, b \tag{4.3.5}$$

where the columns of B represent possible "assembled" bubble modes for the superelement.
From the inversion properties

$$F + G\, N = 0 \qquad \text{and} \qquad G\, B = 0 \tag{4.3.6}$$

we deduce

$$Y'_E\, F_E + Y'_E\, G_E\, W = 0 \qquad \text{and} \qquad Y'_E\, G_E\, B = 0 \qquad E = 1, 2, \ldots, N \tag{4.3.7}$$

so that $$q_E = (F_E + G_E\, W)\, q_{(b)} + G_E\, B\, b \qquad E = 1, 2, \ldots, N \tag{4.3.8}$$

obviously satisfies the dependency relation (4.3.3).
A particular solution of

$$M_E\, a_E = (F_E + G_E\, W)\, q_{(b)} + G_E\, B\, b$$

is thus available for arbitrary $q_{(b)}$ and b

$$a_E = Q_E\, q_{(b)} + P_E\, b + B_E\, b_E \tag{4.3.9}$$

the last term containing the eventual bubble modes of the component element.

The situation is now comparable to that of a simple element. The only difference is the necessity of extending the integrals required to compute stiffness and mass to the union of the domain E, or to the boundary of this union.

4.4. The equilibrium approach. Element flexibility and inverse-mass matrices [12, 13, 21]

The impulse field is discretized as follows

$$\tau(x,t) = R(x)\ c(t) + S(x)\ s(t) \qquad (4.4.1)$$

The stress distributions adopted are thus divided in two types :

a) In S(x) each column is a set of stresses in equilibrium without body loads. Thus

$$D'\ S(x) = 0 \qquad (4.4.2)$$

b) In R(x) each column corresponds to a stress distribution that requires a non zero distribution of body loads

$$D'\ R(x) = v(x) \qquad (4.4.3)$$

In general R(x) and S(x) together constitute a complete representation of $\tau(x,t)$ in terms of polynomials up to a given degree and v(x) contains at least the inertia loading due to a rigid body motion of the element.

The intensities c(t) and s(t) of the stress distributions are the unknown time functions to determine so that deformation compatibility be satisfied in some "best" sense. Extremization of the functional (2.4.10) is a convenient tool to use for that purpose.

Along each facet $\partial_\alpha E$ of the boundary of an element, the assumption (4.4.1) generates a set of independent surface traction modes. To each surface traction mode a time dependent intensity is attached.
The set of intensities constitutes a column vector $g_\alpha(t)$.
Hence if N_α denotes the direction cosines operator for $\partial_\alpha E$ and $T_\alpha(x)$ the identified surface traction modes, the following identity must hold for arbitrary $\dot{c}(t)$ and $\dot{s}(t)$

$$N'_\alpha\ R(x)\ \dot{c} + N'_\alpha\ S(x)\ \dot{s} \equiv T_\alpha(x)\ g_\alpha(t) \qquad \text{on } \partial_\alpha E \qquad (4.4.4)$$

The elements of $g_\alpha(t)$ are called generalized loads; in contrast to the kinematical approach they give strong information about the surface tractions distribution, that can be reconstitued from the knowledge of their values. Each time a choice is made for the measure of intensity of a surface traction mode, this measure becomes related to the values of $\dot{c}$ and $\dot{s}$. Consequently we dispose of matrix relations

$$g_\alpha(t) = G_\alpha \dot{c}(t) + C_\alpha \dot{s}(t) \qquad \text{for } \partial_\alpha E$$

Denoting by $g(t)$ the column of all independent $g_\alpha(t)$ defined on the partial boundaries, we finally obtain

$$g(t) = G \dot{c}(t) + C \dot{s}(t) \qquad (4.4.5)$$

and this equation plays a role very similar to (4.1.2) in the kinematical models. The matrices G and C are called the load connexion matrices.

The virtual work of surface tractions on prescribed boundary displacements can now be subjected to a discretization coherent with (4.4.1) and (4.4.4). On the partial boundary $\partial_\alpha E$

$$\int_{\partial_\alpha E} \bar{u}' \, t \, d\partial E = \int_{\partial_\alpha E} \bar{u}' \, T_\alpha(x) \, g_\alpha(t) \, d\partial E = q^*_\alpha{}' \, g_\alpha(t) \qquad (4.4.6)$$

where $$q^*_\alpha = \int_{\partial_\alpha E} T'_\alpha(x) \, \bar{u} \, d\partial E \qquad (4.4.7)$$

Again, collecting in $q^*(t)$ the independent $q^*(t)$ defined on partial boundaries, the virtual work of surface tractions receives a canonical scalar product form

$$\int_{\partial E} \bar{u}' \, t \, d\partial E = g'(t) \, q^*(t) \qquad (4.4.8)$$

The quantities defined by (4.4.7) are linear functionals of the boundary displacement field and provide only a weak knowledge of this field. It will be observed that from the viewpoint of strong and weak knowledge the roles of displacements and forces is here reversed as compared to the kinematical models.

We are now in a position to discretize completely the complementary variational principle (2.4.10):

$$\frac{1}{2}\int_E \dot{\tau}' H^{-1} \dot{\tau}\, dE = \frac{1}{2}\dot{c}' F_{cc} \dot{c} + \dot{c}' F_{cs} \dot{s} + \frac{1}{2}\dot{s}' F_{ss} \dot{s} \qquad (4.4.9)$$

with a positive definite flexibility matrix

$$F = \begin{pmatrix} F_{cc} & F_{cs} \\ F_{sc} & F_{ss} \end{pmatrix} \qquad (4.4.10)$$

$$F_{cc} = \int_E R'(x)\, H^{-1}\, R(x)\, dE$$

$$F_{cs} = \int_E R'(x)\, H^{-1}\, S(x)\, dE$$

$$F_{ss} = \int_E S'(x) H^{-1} S(x)\, dE$$

and

$$\frac{1}{2}\int_E \frac{1}{\rho}(D'\tau)'(D'\tau)\, dE = \frac{1}{2} c' N c \qquad (4.4.11)$$

which introduces a positive definite "inverse-mass" matrix

$$N = \int_E \frac{1}{\rho}(D' R(x))' D' R(x)\, dE = \int_E \frac{1}{\rho} v'(x)\, v(x)\, dE \qquad (4.4.12)$$

Substituting (4.4.9), (4.4.11) and (4.4.8) into the variational principle (2.4.10), the variational derivatives, with δc and δs vanishing at the time limits, yield

$$-(F_{cc} \ddot{c} + F_{cs} \ddot{s}) - Nc + G' \dot{q}^* = 0 \qquad (4.4.13)$$

$$-(F_{sc} \ddot{c} + F_{ss} \ddot{s}) + C' \dot{q}^* = 0 \qquad (4.4.14)$$

They are the compatibility conditions looked for.

4.5. Solution of equilibrium approach in terms of unknown displacements

The use of unknown generalized displacements for assembling finite elements at the structural level by localization is a convenient procedure that is also applicable to equilibrium types of elements. The only difference is that the generalized displacements are "weak" and are essentially of the interface type. Hence interface identification of weak displacements will result in equilibrium of conjugate generalized loads. Since those however are "strong" they will enforce complete interface equilibrium of surface tractions, which was a requirement of the complementary variational principle at the structural level. The weak displacements on the boundary, $q^*(t)$, were already defined but we have still to attach weak conjugate displacements to the body loads generated by (4.4.3). This is again obtained by virtual work considerations. The body loads are given by

$$- D' \dot{\tau} = - D' R(x) \dot{c} = - v(x) \dot{c} \quad . \qquad (4.5.1)$$

Their virtual work, put in canonical form,

$$- \int_E u' D' \dot{\tau} \, dE = p^{*\prime} \dot{c} \qquad (4.5.2)$$

yields $$p^* = - \int_E v'(x) \, dE \qquad (4.5.3)$$

linear functionals of the internal displacement field.

$\dot{p}^* = -\int_E v'(x)\dot{u}\,dE = \left\{-\int_E \frac{1}{\ell} v'(x)v(x)\,dE\right\}\dot{c} = -N\dot{c}$ follows from

Now $- D' \tau$ also represents the inertia forces $- \rho \ddot{u}$. $D'\tau = v(x)a(t) = q\ddot{u}$

Consequently, considering the kinetic energy,

$$\frac{d}{dt} \frac{1}{2} c' N c = \dot{c}' N c = \int_E (\rho \ddot{u})' \dot{u} \, dE = \int_E (D'\ddot{u})' \dot{u} \, dE$$

$$= \dot{c}' \int_E v'(x) \dot{u} \, d\tau = - \dot{c}' \dot{p}^*$$

and comparing

$$\dot{p}^* = - N c \qquad (4.5.4)$$

The introduction of p^* as conjugate to $\dot{c}$ also furnishes an interpretation to

$$\mathbf{c} = -N^{-1}\,\dot{p}^* \tag{4.5.5}$$

In view of (4.4.18) the compatibility equation (4.4.13) and (4.4.14) can be rewritten together as

$$F_E \begin{pmatrix} \ddot{c} \\ \ddot{s} \end{pmatrix}_E = C'_E \begin{pmatrix} \dot{q}^* \\ \dot{p}^* \end{pmatrix}_E \tag{4.5.6}$$

in terms of the elements complete flexibility matrix (4.4.10) and a complete load connexion matrix

$$C_E = \begin{pmatrix} G & C \\ I & O \end{pmatrix} \tag{4.5.7}$$

Noting that (4.4.5) can be rewritten in the form

$$C_E \begin{pmatrix} \dot{c} \\ \dot{s} \end{pmatrix}_E = \begin{pmatrix} \mathbf{g} \\ \dot{\mathbf{c}} \end{pmatrix}_E \tag{4.5.8}$$

differentiating and substituting in this the solution of (4.5.6), we find

$$\begin{pmatrix} \dot{g} \\ \ddot{c} \end{pmatrix}_E = K_E \begin{pmatrix} \dot{q}^* \\ \dot{p}^* \end{pmatrix}_E \tag{4.5.9}$$

with $$K_E = C_E\, F_E^{-1}\, C'_E \tag{4.5.10}$$

the stiffness matrix of the equilibrium element.
Introducing now the mass matrix of the element, defined as

$$M_E = \begin{pmatrix} O & O \\ O & N^{-1} \end{pmatrix} \tag{4.5.11}$$

The system (4.6.9) is finally presented in the same form as that (4.1.26) of a kinematical element

$$K_E \begin{pmatrix} \dot{q}^* \\ \dot{p}^* \end{pmatrix}_E + M_E \begin{pmatrix} \ddot{q}^* \\ \ddot{p}^* \end{pmatrix}_E = \begin{pmatrix} \dot{g} \\ 0 \end{pmatrix}_E \qquad (4.5.12)$$

However, while in the former case it was a discretized form of the dynamic equilibrium equations, in the present case it is a discretized form of the compatibility conditions (2.4.5).

4.6. Kinematical freedoms of equilibrium elements

A set of weak displacements

$$\begin{pmatrix} q^* \\ p^* \end{pmatrix} = u^*$$

that produces no strain energy is one for which

$$u^{*\prime} K_E u^* = (C'_E u^*)' F_E^{-1} (C'_E u^*) = 0$$

Since F_E is positive definite, such weak displacements are all found as non trivial solutions of the homogeneous system

$$C'_E u^* = 0 \qquad (4.6.1)$$

or, in view of the structure (4.4.21) of C_E,

$$C' q^* = 0 \qquad p^* = - G' q^* \qquad (4.6.2)$$

Attention can thus be focused on the first of equations (4.6.2). For any non trivial solution it furnishes for the boundary part q^*, the second equation furnishes the corresponding internal part p^*.

Rigid body modes of the element must evidently produce solutions. They can be found by inserting rigid body displacements fields for $\bar{u}$ into the definitions (4.4.7). Any non trivial solution other than rigid body modes is a kinematical freedom of the element; it is an undesirable feature, the exact converse of the non independence of generalized boundary displacements in kinematical models. Indeed it can be viewed as an undesirable dependence constraint on the generalized loads. Equations (4.6.1) is the homogeneous adjoint of (4.5.8) and provides the necessary and sufficient conditions for (4.5.8) to be invertible. More simply, the non trivial solutions q^* of the first of equations (4.6.2) provide the conditions

$$u^{*\prime} (g - G \dot{c}) = 0 \qquad (4.6.3)$$

for $\dot{s}$ to be expressible in terms of g and $\dot{c}$ in (4.4.5).

As long as u^* represents a rigid body mode, the conditions (4.6.3) merely express the required global equilibrium between forces applied at the surface of the element and inertia loads. But if there are additional kinematical freedoms the set of g forces is submitted to further restrictions.
It is also interesting to note in (4.4.5) the possible existence of non trivial $\dot{s} = h$ vectors, such that

$$C h = 0 \qquad (4.6.4)$$

The corresponding states of stress

$$\dot{\tau} = S(x) h \qquad (4.6.5)$$

do not produce surface tractions and are in equilibrium without body forces. They are really self-stressing states within the element.
By analogy with the kinematical models they could be called stress bubble modes.

4.7. Structural assembling of equilibrium models

Turning back to equation (4.5.9) after partionning the stiffness matrix

$$\dot{g}_E = K_{qqE}\,\dot{q}^*_E + K_{qpE}\,\dot{p}^*_E \tag{4.7.1}$$

$$\ddot{c}_E = K_{pqE}\,\dot{q}^*_E + K_{ppE}\,\dot{p}^*_E \tag{4.7.2}$$

we substitute

$$c_E = -\,N_E^{-1}\,\dot{p}^*_E \tag{4.7.3}$$

and localize the weak boundary displacements by

$$q^*_E = L_E\,w^* \tag{4.7.4}$$

From (4.7.2) we then obtain a first set of dynamic equations

$$-\,N_E^{-1}\,\ddot{p}^*_E = K_{pqE}\,L_E\,\dot{w}^* + K_{ppE}\,\dot{p}^*_E \qquad E = 1,2\ldots N \tag{4.7.5}$$

If there are no interface and external boundary inertia loads

$$\sum_E L'_E\,g_E = 0 \tag{4.7.6}$$

Time differentiation of this and substitution of (4.7.1) and (4.7.4) produces

$$\left(\sum_E L'_E\,K_{EE}\,L_E\right)\dot{w}^* + \sum_E L'_E\,K_{qpE}\,\dot{p}^*_E = 0 \tag{4.7.7}$$

The structural eigenvalue problem is finally reduced to the standard form

$$(\mathring{K} - \omega^2\,\mathring{M})\,\mathring{x} = 0 \tag{4.7.8}$$

where $\mathring{x}$ is a vector of amplitudes of the weak displacements p_1^*, p_2^*, ..., p_N^*, w^* and

$$\overset{\circ}{K} = \begin{bmatrix} K_{pp1} & 0 & 0 & 0 & K_{pq1} L_1 \\ 0 & K_{pp2} & 0 & 0 & K_{pq2} L_2 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & K_{ppN} & K_{pqN} L_N \\ L_1' K_{qp1} & L_2' K_{qp2} & 0 & L_N' K_{qpN} & \sum_E L_E' K_{qqE} L_E \end{bmatrix}$$

$$\overset{\circ}{M} = \begin{bmatrix} N_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & N_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & N_N^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because of the absence of any inertia attached to the w^* part of the displacements (the last row and column of $\overset{\circ}{M}$ are zero), it is possible to eliminate w^* statically by using the equation (4.7.7). The degrees of freedom of the eigenvalue problem are thus reduced to the set of all internal degrees of freedom p_E^*. Because this number is usually quite low compared to a kinematical approach of equivalent idealization, the equilibrium approach to dynamics appears as a natural "eigenvalue economizer".

5. EIGENVALUE ANALYSIS IN THE PRESENCE OF KINEMATICAL MODES

5.1. Introduction

The problem of determining the natural frequencies and mode shapes of structures by matrix iteration on the linear system

$$K x = \omega^2 M x \qquad (5.1.1)$$

becomes complicated when the stiffness matrix K, instead of being positive definite, is only non negative. This situation prevails in free-free beams, for instance, or, more generally, whenever the structure is capable of undergoing displacement modes without storing deformation energy. Such modes will be indifferently referred to as kinematical or rigid body modes. The matrix K being singular, a commonly proposed procedure [3] consists in applying a spectral shift α and solve the modified problem

$$(K + \alpha M) x = (\omega^2 + \alpha) M x \qquad (5.1.2)$$

In an improved version of this procedure[14] the spectral shift is obtained by separate elementary modifications to the stiffness and mass matrices in order to eliminate the presence of the kinematical modes.

The technique discussed here consists in generalizing the concept of a flexibility matrix to positive semi-definite stiffness matrices. It was first proposed in [8] and can be modified in order to obtain a symmetrical iteration matrix. When applied to an analysis in finite elements, it requires the non-trivial preliminary operation of determining the kinematical modes. This can be achieved by applying the Gauss inversion algorithm to the original stiffness matrix.

Such a numerical procedure, using a Choleski factorization, was first applied by Craig and Bampton [5]. However advantage can be gained from the Gaussian inversion ability to preserve the symmetry of the original matrix while selecting at each stage the largest pivot on the diagonal. Moreover the latter procedure is characterized by a minimal growth of round-off errors [39, 40].

5.2. Kinematical modes and deformation modes

K is positive but only semi-definite if there exist non trivial solutions of

$$K\,x = 0 \tag{5.2.1}$$

They may obviously be considered as modal shapes of problem (5.1.1) associated to a zero eigenvalue. We denote a fundamental set (linearly independent and complete) of such solutions by $x = u_{(i)}$ $(i = 1,2...\rho)$ and refer to them as the kinematical modes.
The other eigensolutions of problem (5.1.1) are modal shapes associated to non zero eigenvalues. A complete set of those will be denoted by $x = x_{(r)}$ $(r = 1,2...\sigma)$; they are referred to as the deformation modes.
Modal shapes belonging to different eigenvalues are known to be orthogonal with respect to both the stiffness and the mass matrix. Modes belonging to the same eigenvalue can always be rendered orthogonal to one another. Consequently and without loss of generality

$$\begin{aligned} u'_{(i)}\,M\,u_{(j)} &= \delta_{ij} & x'_{(s)}\,M\,x_{(r)} &= \delta_{sr} \\ u'_{(i)}\,M\,x_{(r)} &= 0 & x'_{(s)}\,K\,x_{(r)} &= \omega_r^2\,\delta_{rs} \end{aligned} \tag{5.2.2}$$

Relations like $x'_{(r)}\,K\,u_{(i)} = 0$ and $u'_{(j)}\,K\,u_{(i)} = 0$ are only weak consequences of the equation

$$K\,u_{(i)} = 0 \tag{5.2.3}$$

satisfied by the kinematical modes. Equations (5.2.2) show additionally that the modal shapes were normed to unity with respect to the mass matrix.

Introduction of the modal matrices

$$U = \{\, u_{(1)} \cdots u_{(\rho)} \,\} \qquad X = \{\, x_{(1)} \cdots x_{(\sigma)} \,\}$$

allows to present the same information in matrix form

$$K\,U = 0 \tag{5.2.4}$$

$$U'\,M\,U = E_\rho \tag{5.2.5}$$

$$X'\,M\,X = E_\sigma \tag{5.2.6}$$

$$U'\,M\,X = X'\,M\,U = 0 \tag{5.2.7}$$

$$X'\,K\,X = \Delta \tag{5.2.8}$$

where E_ρ and E_σ are identity matrices of dimensions $\rho \times \rho$ and $\sigma \times \sigma$ respectively, while Δ is a diagonal matrix of the eigenvalues ω_r^2.

(U X) being a base matrix (square with linearly independent columns) any vector x admits of a unique modal expansion

$$x = U\,a + X\,b \tag{5.2.9}$$

whose columns of coefficients are immediatly obtained by application of the orthogonality relations as

$$a = U'\,M\,x \qquad\qquad b = X'\,M\,b \tag{5.2.10}$$

Similarly, M being positive definite, (M U M X) is a natural base matrix for the unique expansion of an arbitrary force amplitude vector

$$p = M\,U\,c + M\,X\,d \tag{5.2.11}$$

with

$$c = U'\,p \qquad\qquad d = X'\,p \tag{5.2.12}$$

The modal expansion (5.2.11) is in terms of the inertia force distribution pertaining to each modal shape.

5.3. Static equilibrium conditions

Let us momentarily turn to the more general equations of motion

$$K\,q + M\,\ddot{q} = p(t) \tag{5.3.1}$$

from which problem (5.1.1) is derived by setting $p(t) = 0$ and assuming harmonic free vibrations ($q = x \cos\omega t$). Introducing normal coordinates

$$q(t) = \sum_{1}^{\rho} \eta_i(t)\, u_{(i)} + \sum_{1}^{\sigma} \xi_r(t)\, x_{(r)} \tag{5.3.2}$$

equations (5.3.1) are transformed in a set of uncoupled single degree of freedom oscillators, governed by the "normal equations"

$$\ddot{\eta}_i = u'_{(i)}\, p(t) \qquad (i = 1,\ldots,\rho) \tag{5.3.3}$$

$$\omega_r^2\, \xi_r + \ddot{\xi}_r = x'_{(r)}\, p(t) \qquad (r = 1,\ldots,\sigma) \tag{5.3.4}$$

If the external load vector p is static (independent of time) the normal equations show clearly that the necessary and sufficient conditions for the existence of a static response are

$$u'_{(i)}\, p = 0 \qquad (i = 1,\ldots,\rho) \tag{5.3.5}$$

Each condition expresses the nullity of the virtual work performed by the static load on a kinematical displacement mode.
They are a reminder of the fundamental theorem of statics through virtual work and are fully equivalent to the conditions of global equilibrium.

5.4. The projection operator A. Pseudo-inversion of K

The static equilibrium conditions (5.3.5), that can be summarized in the matrix equation

$$U' p = 0 \tag{5.4.1}$$

show, by reference to (5.2.12), that the modal expansion of the static load p should not contain the part pertaining to the inertia loads of the kinematical modes ($c = 0$).
Then, to any arbitrary loading vector p corresponds a modified loading vector $\hat{p}$ satisfying the global equilibrium conditions (5.3.1)

$$\hat{p} = p - M U c = p - M U U' p$$

or
$$\hat{p} = A' p \tag{5.4.2}$$

with
$$A' = E - M U U' \tag{5.4.3}$$

(E denotes here the identity matrix for the complete vector space of dimensions $\rho + \sigma$).
As a verification we find

$$U' \hat{p} = U' A' p = 0 \tag{5.4.4}$$

because
$$U' A' = U' - (U' M U) U' = U' - U' = 0 \tag{5.4.5}$$

Moreover we find

$$X' \hat{p} = X' A' p = X' p \tag{5.4.6}$$

because
$$X' A' = X' - (X' M U) U' = X' \tag{5.4.7}$$

(5.4.5) and (5.4.7) are equivalent to the selective properties of the projector operator A :

$$A U = 0 \quad \text{or} \quad A u_{(i)} = 0 \quad (i=1,\ldots\rho) \tag{5.4.8}$$

$$A X = 0 \quad \text{or} \quad A x_{(r)} = x_{(r)} \quad (r=1,\ldots,\sigma) \tag{5.4.9}$$

Moreover, we have

$$A' M U = 0 \qquad \text{or} \qquad A' M u_{(i)} = 0 \tag{5.4.10}$$

$$A' M X = M X \qquad \text{or} \qquad A' M x_{(r)} = M x_{(r)} \tag{5.4.11}$$

The relationship (5.4.2) has a well known dynamical interpretation that becomes apparent when it is reformulated as

$$\hat{p} = p - \sum_i (u'_{(i)} p) M u_{(i)}$$

and use is made of the dynamic equations (5.3.3)

$$\hat{p} = p - \sum_i \ddot{\eta}_i M u_{(i)}$$

The application of the arbitrary static loading p induces accelerations in the kinematical degrees of freedom. Suppressing the deformation modes, the addition of the induced inertia loadings to p, as in the right-hand side, yields according to d'Alembert's principle a self-equilibrated system.

By introducing (5.4.2) we have now insured the existence of a solution to the modified static problem

$$K q = A' p \tag{5.4.12}$$

for arbitrary p. The solution is however not unique; to any particular solution $q = F_o p$, we can add a general solution to the corresponding homogeneous equation. Thus

$$q = F_o p + U g \tag{5.4.13}$$

where g is arbitrary. As any g can always be represented in the form

$$g = R' p$$

by a suitable matrix R', the general solution can be placed in the form

$$q = (F_o + U R') p \qquad (5.4.14)$$

This solution is a generalized pseudo-inverse of the singular static problem

$$K q = p \qquad (5.4.15)$$

$$F = F_o + U R' \qquad (5.4.16)$$

is a flexibility matrix that is a generalized pseudo-inverse of the singular stiffness matrix K.
In [8] it was called a matrix of extended influence coefficients.
Substituting

$$q = F p \qquad (5.4.17)$$

into (5.4.12) and noting that p is arbitrary

$$K F = A' \qquad (5.4.18)$$

This pseudo-inverse relationship replaces the classical inversion

$$K K^{-1} = E$$

valid in the absence of kinematical freedoms.

5.5. Unicity of a pseudo-inverse. Isostaticity constraints

The whole problem of finding a pseudo-inverse to K consists in obtaining a particular solution F_o to equations (5.4.12).
The practical answer to this problem is precisely related to the introduction of additional requirements that remove the arbitrariness in F. Suppose that we require of the pseudo-inversion (5.4.17) that the solution q be orthogonal to the kinematical modes for arbitrary p :

$$U' M q = 0 \qquad (5.5.1)$$

Then, substituting(5.4.13)and using (5.2.5)

$$U' M F_o p + g = 0$$

hence $$q = (F_o - U U' M F_o) p$$

This solution of (5.4.12) will be denoted by

$$q = G p \tag{5.5.2}$$

and the pseudo-inverse

$$G = F_o - U U' M F_o = A F_o \tag{5.5.3}$$

will be shown to be independent of the particular choice F_o.
Indeed it is readily apparent from (4.16) that any other particular choice F_1 is related to F_o by

$$F_1 = F_o + U R'$$

where R' is some fixed matrix. But then, in view of (5.4.8)

$$A F_1 = A F_o$$

The pseudo-inverse G is not only unique, it is also symmetrical.
A simple proof consists in transposing relation (5.4.18) and post-multiplying it by F, then

$$F' K F = A F = G$$

and in view of the structure of the left-hand side, symmetry is obvious.
Except for the fact that orthogonality is here defined in terms of a positive definite matrix M and not the identity matrix, G coincides with the mathematical pseudo-inverse of K as defined by Penrose[31].
It can also be shown [8] to coincide with the spectral expansion of the dynamic influence coefficients matrix $(K - \omega^2 M)^{-1}$, amputated of the terms corresponding to the kinematical mode shapes :

$$G = \sum_{1}^{\sigma} \frac{1}{\omega_r^2} x_{(r)} \, x'_{(r)} \qquad . \qquad (5.5.4)$$

The unicity of (5.4.17) follows more generally from a set of ρ additional constraints of type

$$S' \, q = 0 \qquad (5.5.5)$$

provided $S' \, U$ be a non singular $\rho \times \rho$ matrix.
Indeed, substituting (5.4.13) we must have

$$S' \, F_o \, p + S' \, U \, g = 0$$

and then $g = - (S' \, U)^{-1} \, S' \, F_o \, p$

whereby $q = F_o - U \, (S' \, U)^{-1} \, S' \, F_o \, p$

and a more general pseudo-inverse

$$F = (E - U \, (S' \, U)^{-1} \, S') \, F_o = P \, F_o \qquad (5.5.6)$$

Unicity is proved by the same technique

$$P \, F_1 = P \, F_o + P \, U \, R' = P \, F_o .$$

because $P \, U = (E - U \, (S' \, U)^{-1} \, S') \, U = 0$

Constraints such as (5.5.5) can be qualified by their property to suppress the kinematical freedoms; for if we restrict displacements to the kinematical modes

$$q = U \, a$$

and apply the constraints, there follows

$$S' \, U \, a = 0 \qquad , \text{ implying } \quad a = 0 \, ,$$

since S'U is non singular. Consequently we can conceive of (5.5.5) as physical constraints linking the structure to a solid reference frame in an isostatic manner. The structure can then accept an arbitrary static loading p and a unique flexibility or influence coefficients matrix G_{iso} can be produced such that

$$q = G_{iso}\, p$$

G_{iso} is symmetrical and has for instance the elementary structure

$$G_{iso} = \begin{pmatrix} K_{\sigma\sigma}^{-1} & . & 0 \\ & & \\ 0 & & 0 \end{pmatrix} \qquad (5.5.7)$$

when assuming that suitable constraints are obtained by preventing the ρ last displacement components to take place.
An automated selection of the constraints will be presented in section 5.7.
When the arbitrary loading p is replaced by $A'p$, the structure becomes self-equilibrated without the reaction loads due to the constraints. As a matter of fact, since they are isostatically determined, they can but vanish and we can conclude that

$$q = G_{iso}\, A'\, p$$

is a possible displacement vector of the unconstrained structure.
The role played by the constraints is reduced to the determination of a particular set of displacements without the indeterminacy contained in (5.4.16); the unicity of pseudo-inverses of type (5.5.6) is thus physically clarified. Moreover the solution found

$$F = G_{iso}\, A' \qquad (5.5.8)$$

is now quite easily obtainable. It also furnishes a new direct proof of the symmetry of

$$G = A\, F = A\, G_{iso}\, A'. \qquad (5.5.9)$$

5.6. Iteration in semi-definite eigenvalue problems

The deformation modes $x_{(r)}$, which satisfy

$$K\, x_{(r)} = \omega_r^2\, M\, x_{(r)}$$

will also satisfy the system

$$F'\, K\, x_{(r)} = \omega_r^2\, F'\, M\, x_{(r)}$$

where F' is the transpose of a pseudo-inverse of K.
Considering the transpose of the property (5.4.18) and using (5.4.9), this can be written in the form

$$F'\, M\, x_{(r)} = \lambda_r\, x_{(r)} \tag{5.6.1}$$

$$\lambda_r = 1/\omega_r^2 \tag{5.6.2}$$

and suggests the use of F'M in a power iteration scheme to solve the new eigenvalue problem

$$F'\, M\, x = \lambda\, x \tag{5.6.3}$$

Because of (5.6.1) this problem admits precisely the unknown deformation modes as eigensolutions. There remains however to investigate the nature of the ρ still missing eigensolutions. They can be termed "parasitic solutions" because the kinematical modes, though eigensolutions of the original problem (5.1.1), are generally no more solutions of (5.6.3). It is a remarkable property of pseudo-inverses F obtained through the use of isostaticity constraints that they induce in (5.6.3) parasitic solutions of zero characteristic value λ, which are thus wiped out at the very first iteration. To this purpose we must prove that

$$F'\, M\, w = 0 \tag{5.6.4}$$

admits ρ non trivial linearly independent solutions.
Indeed, when the construction of F is based on (5.5.5), we have obviously

$$S' F = 0 \qquad \text{or} \qquad F' S = 0 \tag{5.6.5}$$

a property that is also easily verified on (5.5.6).
Thus

$$w = M^{-1} S s \tag{5.6.6}$$

is a non trivial solution of (5.6.4) for arbitrary s and, the ρ columns of S being linearly independent the proof is completed.
Observe that when F is taken to be the symmetrical G matrix, corresponding to the constraints (5.5.1), the parasitic modes

$$w = M^{-1} (U' M)' s = U s$$

are the kinematical modes themselves, whose natural frequency zero has thus been shifted to infinity.
As a final conclusion the problem (5.6.3) in which F is of type (5.5.8) or

$$A G_{iso} M x = \lambda x \tag{5.6.7}$$

is well adapted to power iteration and will provide initial convergence towards the deformation mode of largest characteristic number λ (or smallest natural frequency). Classical deflation algorithms [11,21] are applicable to the successive determination of modes of higher frequency.

5.7. Numerical computation of kinematical modes

Suppose that rows and columns of the original stiffness matrix are ordered in such a manner that suppression of the ρ last degrees of freedom, q_ρ, suppresses the kinematical degrees of freedom. The other $\sigma = n - \rho$ degrees of freedom may thus be interpreted as generalized deformation coordinates.

Hence the singular static problem (5.4.15) admits of the partitioned form

$$\begin{pmatrix} K_{\sigma\sigma} & K_{\sigma\rho} \\ K_{\rho\sigma} & K_{\rho\rho} \end{pmatrix} \begin{pmatrix} q_\sigma \\ q_\rho \end{pmatrix} = \begin{pmatrix} p_\sigma \\ p_\rho \end{pmatrix} \quad ; \qquad (5.7.1)$$

the static equilibrium condition to be verified by the right-hand side vector will be explicited further.

Performing σ Gaussian inversion steps leads to the set of equations

$$q_\sigma = K_{\sigma\sigma}^{-1} \, (p_\sigma - K_{\sigma\rho} \, q_\rho) \quad , \qquad (5.7.2)$$

and substitution of this into the second equation (5.7.1) transforms the initial system as follows :

$$\begin{bmatrix} K_{\sigma\sigma}^{-1} & - K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho} \\ K_{\rho\sigma} . K_{\sigma\sigma}^{-1} & K_{\rho\rho} - K_{\rho\sigma} . K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho} \end{bmatrix} \begin{bmatrix} p_\sigma \\ q_\rho \end{bmatrix} = \begin{bmatrix} q_\sigma \\ p_\rho \end{bmatrix} . \qquad (5.7.3)$$

From a computational point of view, it is essential to point out at this stage that the symmetry of the initial matrix will be preserved by the following changes of sign

$$\begin{bmatrix} - K_{\sigma\sigma}^{-1} & K_{\sigma\sigma}^{-1} . K_{\sigma\rho} \\ K_{\rho\sigma} . K_{\sigma\sigma}^{-1} & K_{\rho\rho} - K_{\rho\sigma} . K_{\sigma\sigma}^{-1} . K_{\sigma\rho} \end{bmatrix} \begin{bmatrix} -p_\sigma \\ -q_\rho \end{bmatrix} = \begin{bmatrix} q_\sigma \\ -p_\rho \end{bmatrix} . \qquad (5.7.4)$$

Next consider the submatrix $K_{\rho\rho} - K_{\rho\sigma} \cdot K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho}$ which, in view of the positive semi-definite character of the initial matrix, is expected to vanish. The proof follows by setting the linear homogeneous problem (5.2.1) into the partitioned form

$$\begin{bmatrix} K_{\sigma\sigma} & K_{\sigma\rho} \\ K_{\rho\sigma} & K_{\rho\rho} \end{bmatrix} \begin{bmatrix} q_\sigma \\ q_\rho \end{bmatrix} = 0 \quad , \qquad (5.7.5)$$

the solution of which are the kinematical mode shapes. Indeed the first equation (5.7.5) associates to any arbitrary choice of the kinematical degrees of freedom q_ρ, the set of dependent generalized coordinates

$$q_\sigma = - K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho} \; q_\rho \quad . \tag{5.7.6}$$

Introducing (5.7.6) into the second equation (5.7.5) yields

$$(K_{\rho\rho} - K_{\rho\sigma} \cdot K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho}) \; q_\rho = 0$$

for any q_ρ, terminating the proof.

The static problem (5.7.4) reduces thus to the form

$$\begin{bmatrix} - K_{\sigma\sigma}^{-1} & K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho} \\ K_{\rho\sigma} \cdot K_{\sigma\sigma}^{-1} & 0 \end{bmatrix} \begin{bmatrix} - p_\sigma \\ - q_\rho \end{bmatrix} = \begin{bmatrix} q_\sigma \\ - p_\rho \end{bmatrix} \quad , \tag{5.7.7}$$

from which one deduces the expected global equilibrium conditions

$$K_{\rho\sigma} \cdot K_{\sigma\sigma}^{-1} \; p_\sigma = p_\rho \quad . \tag{5.7.8}$$

As another consequence of (5.7.6), the modal matrix

$$\begin{pmatrix} - K_{\sigma\sigma}^{-1} \cdot K_{\sigma\rho} \\ E_\rho \end{pmatrix} \tag{5.7.9}$$

defines the linearly independent set of kinematical modes associated to a unit displacement of each kinematical degree of freedom successively. These modes still have to be orthonormed in order to verify the assumed properties (5.2.5).

On the other hand, setting the ρ arbitrary independent coordinates q_ρ equal to zero provides a particular isostatic reference frame. Hence (5.7.7) also yields

$$G_{iso} = \begin{pmatrix} K_{\sigma\sigma}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad . \qquad (5.7.10)$$

The numerical inversion procedure will thus be organized as follows :

1. The pivot element is chosen at each inversion step as the largest term of the diagonal which has not yet been utilized : say

$$k_{ii} = \max_{j} k_{jj} \qquad j = 1, \ldots n \quad . \qquad (5.7.11)$$

2. The other elements are transformed according the following rules :

- $$k_{ii}^{*} = -\frac{1}{k_{ii}} \qquad (5.7.12)$$

- $$k_{ij}^{*} = k_{ji}^{*} = \frac{k_{ij}}{k_{ii}} = \frac{k_{ji}}{k_{ii}} \qquad j \neq i \qquad (5.7.13)$$

- $$k_{j\ell}^{*} = k_{\ell j}^{*} = k_{j\ell} - \frac{k_{ji} \cdot k_{i\ell}}{k_{ii}} \qquad j,\ell \neq i \qquad (5.7.14)$$

$$= k_{\ell j} - \frac{k_{\ell i} \cdot k_{ij}}{k_{ii}}$$

3. The procedure is restarted until all non zero terms on the diagonal have been inverted.

The symmetry of the initial stiffness matrix has obviously been preserved by the sequence of operations (5.7.11) to (5.7.14), and the algorithm may thus be performed even when limiting the memorization of the K matrix to its lower - or upper-half triangular part. Note that other procedures like Choleski factorization [5] or Gauss-Jordan elimination [21] do not permit the pivot selection anywhere on the diagonal when storing the initial matrix under triangular form . As a consequence of this symmetry property of the Gaussian inversion algorithm, the associated growth of round-off errors is minimized when choosing the best pivot element at each inversion step [39,40].

At the end of the inversion procedure, the kinematical modes are obtained on rows and columns associated with the zero diagonal terms. Setting then equal to zero the same rows and columns gives the symmetric matrix G_{iso}. Both operations needed have thus been performed together.

5.8. Obtention of a symmetric iteration matrix

Consider first the change of variables

$$\bar{x} = T\, x \quad , \qquad (5.8.1)$$

where T denotes the upper-triangular matrix resulting from the Choleski decomposition of the mass matrix :

$$M = T'\, T \quad . \qquad (5.8.2)$$

The transformed kinematical modal matrices

$$\bar{U} = T\, U$$

verify the simpler orthonormality relations

$$\bar{U}'\, \bar{U} = E_p \quad , \qquad (5.8.3)$$

and the eigenvalue problem

$$A\, G_{iso}\, A'\, M\, x = \lambda\, x \qquad (5.8.4)$$

has to be rewritten as

$$T\, A\, G_{iso}\, A'\, T'\, \bar{x} = \lambda\, \bar{x} \qquad (5.8.5)$$

If we still introduce the new projection operator

$$\bar{A} = I - \bar{U}\, \bar{U}' \quad , \qquad (5.8.6)$$

(5.8.5) changes into

$$\overline{A}\,\overline{G}_{iso}\,\overline{A}'\,\overline{x} = \lambda\,\overline{x} \qquad (5.8.7)$$

with the transformed flexibility matrix

$$\overline{G}_{iso} = T\,G_{iso}\,T' \qquad (5.8.8)$$

Note that the matrix product (5.8.8) does not involve a larger number of operations than (5.6.5) if taking into account the triangular form of T. Finally, the pré- and post-multiplication by the projection operator reduces the operation of obtaining the iteration matrix to a sum of dyadic products

$$\begin{aligned} S &= \overline{A}\,\overline{G}_{iso}\,\overline{A}' \\ &= \overline{G}_{iso} - \overline{U}\,\overline{U}'\,\overline{G}_{iso} - \overline{G}_{iso}\,\overline{U}\,\overline{U}' \\ &\quad + \overline{U}\,.\,(\overline{U}'\,G_{iso}\,\overline{U})\,.\,\overline{U}' \end{aligned} \qquad (5.8.9)$$

The use of (5.8.9) extends to hypostatic structures the possibility of reducing the memorization of the dynamic flexibility matrix to its upper or lower triangular part [21].

6. THE REDUCTION METHODS

6.1. INTRODUCTION

It is economically unrealistic to solve eigenvalue problems involving more than a few hundred degrees of freedom on present day computers. As acceptable finite element idealizations of aerospace and civil engineering structures generally involve thousands of degrees of freedom, it becomes necessary to reduce the size of the eigenvalue problems by methods that preserve the accuracy of the low frequency spectrum.
A first approach outlined in this paper uses the classical elimination procedure[27,43] based on the assumption that a certain number of degrees of freedom may be reduced by static considerations.
Use was made of the Kato and Temple theorems[28,37] on bounding of eigenvalues to improve the algorithms by adding the possibility of computing bounds to the error introduced by condensation [17].
Coupled with the finite element method, the elimination algorithm allows an easy step by step build-up of the whole structure (substructures in series). The results obtained in treating large scale applications show that the loss of accuracy for the lower frequency modes is negligible.

An alternative, briefly discussed in this paper, concerns the method of coupling substructures in parallel and consists in treating the whole structure as an array of several sub-regions already separately condensed.

In both cases it is agreed to limit either the deformation modes of the structure, or those of its constitutive parts, to some known modes, $r_{(i)}$, smaller in number than the one n of the degrees of freedom of the initial idealization. They are chosen as the most representative of the low frequency behavior of the structure.

The possible motions of the structure, q , are thus restricted to the form

$$q = R\,a \qquad (6.1.1)$$

where R denotes the (n x m) matrix collecting the selected modes $r_{(i)}$.

Expressing that the potential and kinetic energies associated with a displacement mode are restrained by (1.1) gives

$$U = \frac{1}{2} a' (R' K R) a \qquad (6.1.2)$$

$$= \frac{\omega^2}{2} a' (R' M R) a \quad , \qquad (6.1.3)$$

where K and M denote the structural stiffness and mass matrices respectively. One thus introduces the reduced matrices

$$\overline{K} = R' K R \qquad (6.1.4)$$

and

$$\overline{M} = R' M R \qquad (6.1.5)$$

defined in the new system of generalized displacements a .

The various reduction methods [4,23,26] (elimination of variables, coupling of substructures, branch modes) encountered in the litterature do not essentially differ from each other, except by the choice of the deformation modes $r_{(i)}$. The influence of this choice can however be a deciding factor on the accuracy of the process, as well as on its flexibility in dealing with practical applications of various sizes.

6.2. STATIC CONDENSATION OF VARIABLES

Consider the matrix equation governing the natural frequencies and modal shapes of the global structure in the form

$$K q = \omega^2 M q \qquad (6.2.1)$$

One possible technique, due to IRONS [27] , for reducing the size of the matrices involved consists in making a choice of a subset q_C of coordinates to be eliminated; the complementary subset being denoted by q_R (coordinates to be retained). Equation (6.2.1) is thus partitioned as follows

$$K_{RR} q_R + K_{RC} q_C = \omega^2 (M_{RR} q_R + M_{RC} q_C) \qquad (6.2.2)$$

$$K_{CR}\, q_R + K_{CC}\, q_C = \omega^2 \,(M_{CR}\, q_R + M_{CC}\, q_C) \tag{6.2.3}$$

Imagine q_C to be split into two contributions

$$q_C = q_S + q_D \tag{6.2.4}$$

where the "static" part q_S is given by

$$q_S = - K_{CC}^{-1}\, K_{CR}\, q_R \tag{6.2.5}$$

as if in (6.2.3) one neglected the inertia forces, and a "dynamic" part q_D which is then governed by the transformed (6.2.3) equations

$$(K_{CC} - \omega^2\, M_{CC}) q_D = \omega^2\, \bar{M}_{CR}\, q_R \tag{6.2.6}$$

with $$\bar{M}_{CR} = M_{CR} - M_{CC}\, K_{CC}^{-1}\, K_{CR} \tag{6.2.7}$$

The static condensation method consists in neglecting q_D and substituting directly for q_C the right-hand side of (6.2.5) into the kinetic and strain energies, which then become quadratic forms in q_R only with "reduced" stiffness and mass matrices

$$\bar{K}_{RR} = K_{RR} - K_{RC}\, K_{CC}^{-1}\, K_{CR} \tag{6.2.8}$$

$$\bar{M}_{RR} = M_{RR} - M_{RC}\, K_{CC}^{-1}\, K_{CR} - K_{RC}\, K_{CC}^{-1}\, M_{CR} + K_{RC}\, K_{CC}^{-1}\, M_{CC}\, K_{CC}^{-1}\, K_{CR} \tag{6.2.9}$$

$$= M_{RR} - M_{RC}\, K_{CC}^{-1}\, K_{CR} - K_{RC}\, K_{CC}^{-1}\, \bar{M}_{CR} \tag{6.2.10}$$

The eigenvalue problem is reduced to the condensed one

$$(\bar{K}_{RR} - \omega^2\, \bar{M}_{RR})\, q_R = 0 \tag{6.2.11}$$

The procedure is of course a particular case of (6.1.4) and (6.1.5)

with $$R = \begin{pmatrix} I \\ \\ - K_{CC}^{-1}\, K_{CR} \end{pmatrix} \tag{6.2.12}$$

First order corrections [2]

The validity of the condensation algorithm depends of course on the extent to which q_D is really negligible. To investigate the conditions under which this is true, consider the eigenvalue problem

$$(K_{CC} - \mu^2 M_{CC})q_C = 0 \qquad (6.2.13)$$

to which the original problem reduces by applying the contraints

$$q_R = 0 \qquad (6.2.14)$$

Let $\mu_1^2 \leq \mu_2^2 \leq \ldots \leq \mu_m^2$

denote its eigenvalues with attendant orthonormed modal shapes

$$c_1 , c_2 \ldots c_m$$

$$c'_i M_{CC} c_j = \delta_{ij} \qquad (6.2.15)$$

An arbitrary vector q_c as a unique expansion

$$q_C = \sum_1^m \alpha_i c_i$$

with coefficients determined from (6.2.15) as

$$\alpha_i = c'_i M_{CC} q_C$$

Thus $$q_C = (\sum_1^m c_i c'_i M_{CC}) q_C$$

and, since q_C is arbitrary, we obtain the spectral expansion of the identity matrix

$$I = \sum_1^m c_i c'_i M_{CC} = \sum_1^m c_i (M_{CC} c_i)' \qquad (6.2.16)$$

Noting that the eigensolutions verify

$$\mu_i^2 M_{CC} c_i = K_{CC} c_i \qquad (i = 1, 2, \ldots m)$$ we have also

$$I = \sum_1^m \frac{1}{\mu_i^2} c_i c'_i K_{CC} \tag{6.2.17}$$

The following results derived from (6.2.16) or (6.2.17) will be of interest :

$$\omega^2 K_{CC}^{-1} M_{CC} = \sum_1^m \frac{\omega^2}{\mu_i^2} c_i c'_i M_{CC} \tag{6.2.18}$$

$$K_{RC} = \sum_1^m (K_{RC} c_i)(M_{CC} c_i)' \tag{6.2.19}$$

$$\omega^2 M_{RC} = \sum_1^m \frac{\omega^2}{\mu_i^2} (M_{RC} c_i)(K_{CC} c_i)' \tag{6.2.20}$$

We are now able to show that the range of validity of the condensation algorithm is that for which holds

$$\frac{\omega^2}{\mu_1^2} = \varepsilon << 1 \tag{6.2.21}$$

ε being the order of magnitude of the errors we are prepared to accept in the low frequency modal shapes. In other words the low frequency spectrum of (6.2.1) will be accurately represented by the eigensolution of the condensed problem (6.2.11) up to eigenvalues ω^2 satisfying (6.2.21). To show this we discuss the solutions of (6.2.6), which we put in the form

$$(I - \omega^2 K_{CC}^{-1} M_{CC}) q_D = \omega^2 K_{CC}^{-1} \bar{M}_{CR} q_R \tag{6.2.22}$$

and (6.2.2) in which we substitute (6.2.4) and (6.2.5) to bring it into the final form

$$(\bar{K}_{RR} - \omega^2 \bar{M}_{RR})q_R = (\omega^2 M_{RC} - K_{CR})q_D + \omega^2 K_{RC} K_{CC}^{-1} \bar{M}_{CR} q_R \tag{6.2.23}$$

If ω^2 is such that (6.2.21) holds we can observe by comparing (6.2.18) to (6.2.16) that $\omega^2 K_{CC}^{-1} M_{CC}$ is a matrix whose elements are of order ε compared to the unit matrix. As a matter of fact (6.2.21) with $\varepsilon < 1$ is sufficient for the convergence of the expansion

$$(I - \omega^2 K_{CC}^{-1} M_{CC})^{-1} = I + \omega^2 K_{CC}^{-1} M_{CC} + (\omega^2 K_{CC}^{-1} M_{CC})^2 + \ldots$$

Comparing (6.2.19) and (6.2.20) and making the natural assumption that the vectors

$$M_{CC}\, c_i \qquad \text{and} \qquad M_{RC}\, c_i$$

are of the same order of magnitude, and similarly for

$$K_{CC}\, c_i \qquad \text{and} \qquad K_{RC}\, c_i$$

then, under (6.2.21)

$$\omega^2 M_{RC} \quad \text{of order } \varepsilon \quad \text{compared to } K_{RC} \tag{6.2.24}$$

and also $\omega^2 K_{CC}^{-1} M_{CC} = O(\varepsilon)$ and $\omega^2 K_{CC}^{-1} \bar{M}_{CR} = O(\varepsilon)$ (6.2.25)

The zero order solution of problem (6.2.22), (6.2.23), fully equivalent to (6.2.1), is then given by

$$(\bar{K}_{RR} - \bar{\omega}^2 \bar{M}_{RR})\bar{q}_R = 0 \qquad \bar{q}_D = 0 \tag{6.2.26}$$

where $\bar{\omega}^2$ is some eigenvalue of the condensed problem satisfying the low frequency requirement (6.2.21) and $\bar{q}_R$ the associated modal shape of the retained coordinates, while
$\bar{q}_C = - K_{CC}^{-1} K_{CR} \bar{q}_R = \bar{q}_S$.

Equating now the terms of order ε in both equations

$$\delta q_D = \bar{\omega}^2 K_{cc}^{-1} \bar{M}_{cR} \bar{q}_R \tag{6.2.27}$$

$$(\bar{K}_{RR} - \bar{\omega}^2 \bar{M}_{RR}) \delta q_R = \delta\omega^2 \bar{M}_{RR} \bar{q}_R - K_{Rc} \delta q_D + \bar{\omega}^2 K_{Rc} K_{cc}^{-1} \bar{M}_{cR} \bar{q}_R \tag{6.2.28}$$

where δq_D and δq_R are the first order corrections for modal shape and $\delta\omega^2$ an eventual first order correction to the eigenvalue. The first order value of q_D is explicit in (6.2.27). Observing that it causes the last two terms of (6.2.28) to cancel

$$(\bar{K}_{RR} - \bar{\omega}^2 \bar{M}_{RR}) \delta q_R = \delta\omega^2 \bar{M}_{RR} \bar{q}_R \tag{6.2.29}$$

This is a singular system to calculate δq_R and the necessary and sufficient condition the right-hand side has to satisfy for a solution to exist is obtained by premultiplying by $\bar{q}'_R$, which will make the left-hand side vanish by virtue of (6.2.26)

$$0 = \delta\omega^2 \, \bar{q}'_R \bar{M}_{RR} \bar{q}_R$$

$\bar{M}_{RR}$ being positive definite, the conclusion is that

$$\delta\omega^2 = 0$$

The conclusion that the eigenvalue correction is at most of second order was to be expected by virtue of the stationary character of the eigenvalues obtained from Rayleigh quotients.
The only solution of (6.2.29) is now

$$\delta q_R = \delta\alpha \, \bar{q}_R$$

where $\delta\alpha$ is a, still undetermined, small scalar. There is no loss of generality in taking $\delta\alpha = 0$ since this will merely change the scale of the eigensolution, correct to second order, by a factor $(1 + \delta\alpha)^{-1}$ and since

$$q_D(1+\delta\alpha)^{-1} = \delta q_D(1+\delta\alpha)^{-1} = \delta q_D - \delta\alpha\ \delta q_D \ldots \tilde{=} \delta q_D$$

neglecting second order terms, this will not change (6.2.27).
To summarize : if the spectrum of problem (6.2.1) is to be determined accurately up to some definite frequency, the coordinates to be condensed should be numbered and chosen in such a way that the smallest eigenvalue μ_1^2 of the constrained problem (6.2.11) sufficiently dominates the highest eigenvalue $\bar{\omega}^2$ of the condensed problem (6.2.7) that fits into the required spectrum. The ratio

$$\bar{\omega}^2 \ / \ \mu_1^2 = \varepsilon \qquad (6.2.30)$$

determines the order of approximation to which the modal shapes of the condensed problem represent the exact modal shapes within the required spectrum. The first order correction on the modal shapes is given explicitly by

$$\delta q_R = 0 \qquad \delta q_C = \delta q_D \text{ given by (6.2.27).}$$

The eigenvalue correction is of second order.
As a last observation , if the structure has kinematical freedoms (rigid body modes or mechanisms) they should not be inhibited by the choice of condensed variables.

6.3. BOUND ALGORITHMS [29, 30]

In problem (6.2.1) define the successive iterates of an arbitrary starting vector q_o

$$q_{p+1} = K^{-1} M q_p \qquad (6.3.1)$$

and the corresponding Rayleigh quotients

$$\rho_{2p} = \frac{q'_p K q_p}{q'_p M q_p} = \frac{q'_p M q_{p-1}}{q'_p M q_p} > 0 \qquad (6.3.2)$$

Further developments necessitate the definitions of the zero order Schwarz quotients

$$\rho_{2p+1} = \frac{q'_p \, M \, q_p}{q'_p \, M \, q_{p+1}} > 0 \qquad (6.3.3)$$

and of the bilinear forms

$$A_p(\alpha,\beta) = \frac{(q_p - \alpha q_{p+1})' K (q_p - \beta q_{p+1})}{q'_{p+1} \, K \, q_{p+1}} \qquad (6.3.4)$$

$$B_p(\alpha,\beta) = \frac{(q_p - \alpha q_{p+1})' \, M \, (q_p - \beta q_{p+1})}{q'_{p+1} \, M \, q_{p+1}} \qquad (6.3.5)$$

We have

$$A_p(\alpha,\beta) = \rho_{2p} \cdot \rho_{2p+1} - (\alpha+\beta)\rho_{2p+1} + \alpha\beta \qquad (6.3.6)$$

$$B_p(\alpha,\beta) = \rho_{2p+1} \cdot \rho_{2p+2} - (\alpha+\beta)\rho_{2p+2} + \alpha\beta \qquad (6.3.7)$$

Noting that $A_p(\alpha,\alpha)$ and $B_p(\alpha,\alpha)$ are positive definite

$$\rho_{2p} \cdot \rho_{2p+1} - 2\alpha \rho_{2p+1} + \alpha^2 \geq 0 \qquad (6.3.8)$$

$$\rho_{2p+1} \cdot \rho_{2p+2} - 2\alpha \rho_{2p+2} + \alpha^2 \geq 0 \qquad (6.3.9)$$

and those expressions take their minimum value respectively for

$$\alpha = \rho_{2p+1} \qquad \text{and} \qquad \alpha = \rho_{2p+2}$$

For those values the inequalities (6.3.8) and (6.3.9) go into the well known Schwartz inequalities

$$\rho_{2p} \geq \rho_{2p+1} \geq \rho_{2p+2} \qquad (6.3.10)$$

Krylov-Bogoliubov bounds

Let the unique expansion of q_p in eigenmodes be

$$q_p = \sum_n a_n q_{(n)}$$

Then, since

$$K^{-1} M q_{(n)} = \frac{1}{\omega_n^2} q_{(n)}$$

the expansion of the next iterate will be

$$q_{p+1} = \sum_n \frac{a_n}{\omega_n^2} q_{(n)}$$

and, under the orthonormality relations

$$q'_{(m)} M q_{(n)} = \delta_{mn} \qquad q'_{(m)} K q_{(n)} = \omega_n^2 \delta_{mn}$$

we find

$$A_p(\alpha,\beta) = \frac{\sum_n \alpha_n^2 (\omega_n^2 - \alpha)(\omega_n^2 - \beta)}{\sum_n \alpha_n^2} \qquad (\alpha_n = \frac{a_n}{\omega_n}) \tag{6.3.11}$$

Setting $\beta = \alpha$ and comparing with (6.3.8)

$$\rho_{2p}\rho_{2p+1} - 2\alpha\rho_{2p+1} + \alpha^2 = \frac{\sum_n \alpha_n^2(\omega_n^2 - \alpha)^2}{\sum_n \alpha_n^2} \geqslant \min_n(\omega_n^2-\alpha)^2$$

whatever be α, there is an eigenvalue ω_j^2, closest to α, and corresponding to the right-hand side minimum.

Setting $\alpha = \rho_{2p+1}$, which is known to give its minimum value to the left-hand side, we find

$$\rho_{2p+1}(\rho_{2p} - \rho_{2p+1}) \geqslant (\omega_j^2 - \rho_{2p+1})^2$$

which yields the Krylov and Bogoliubov bounds

$$\rho_{2p+1} - \sqrt{\rho_{2p+1}(\rho_{2p} - \rho_{2p+1})} \leq \omega_j^2 \leq \rho_{2p+1} + \sqrt{\rho_{2p+1}(\rho_{2p} - \rho_{2p+1})} \tag{6.3.12}$$

In particular for any trial vector q_o

$$\rho_1 - \sqrt{\rho_1(\rho_o - \rho_1)} \leq \omega_j^2 \leq \rho_1 + \sqrt{\rho_1(\rho_o - \rho_1)} \tag{6.3.13}$$

and if q_o is known to be a reasonable approximation to a modal shape (6.3.13) furnishes bounds to the corresponding exact eigenvalue of the discretized structure. Since discretization is known to give itself upper bounds to the eigenvalues of the continuous structure, the upper bound of (6.3.13) is valid for the continuous structure. There is no similar guarantee for the lower bound.

Iterates with large values of p, based on (6.3.1), will produce close bounds (6.3.12) for the first eigenvalue ω_1^2 of the discretized structure, since the iteration process converges to this value. However, applying deflation techniques that shift already computed eigenvalues to zero, Krylov sequences of iterates can be produced whose quotients ρ_{2p} and ρ_{2p+1} converge towards the next ω_j^2. In this case (6.3.12) can produce close bounds for it. It should however be observed that in this case the upper bound is clearly worse than the simpler one

$$\omega_j^2 \leq \rho_{2p+1} \tag{6.3.14}$$

Temple-Kato bounds

Let α and β lie in the interval between two consecutive eigenvalues, then it is clear from (6.3.11) that

$$A_p(\alpha,\beta) = \rho_{2p} \cdot \rho_{2p+1} - (\alpha + \beta)\rho_{2p+1} + \alpha\beta > 0 \tag{6.3.16}$$

Take first $\alpha = \omega_i^2$ and

$$\omega_i^2 \leq \rho_{2p+1} < \beta \leq \omega_{i+1}^2 \tag{6.3.17}$$

Substitution into (6.3.16) produces the bounds

$$\rho_{2p+1} \frac{\beta - \rho_{2p}}{\beta - \rho_{2p+1}} < \omega_i^2 \leqslant \rho_{2p+1} \qquad (6.3.18)$$

If, for instance, ρ_{2p+1} comes from a Krylov sequence that converges to ω_i^2 the upper bound is "naturally" verified and all that is needed to obtain the lower bound to ω_i^2 is a knowledge of some lower bounds β to the next eigenvalue ω_{i+1}^2. The Krylov sequence can be pushed far enough for $\beta - \rho_{2p+1} > 0$ to be satisfied.
If q_o is merely an approximate modal shape associated to ω_i^2, the inequality $\omega_i^2 \leqslant \rho_o$ is satisfied but, for $i > 1$, there is no guarantee that the iterate $q_1 = K^{-1} M q_o$ will yield $\omega_i^2 < \rho_1$. Thus if we only work with iterates based on the original $K^{-1} M$ dynamic matrix it is preferable to consider q_o to be the approximate modal shape in order to secure $\omega_i^2 < \rho_1$ and work backwards to compute ρ_o from $q_o = M^{-1} K q_1$. Even in this case it will not always be possible to know a lower bound β to ω_{i+1}^2 that verifies $\beta > \rho_1$ in order to have the lower bound

$$\rho_1 \frac{\beta - \rho_o}{\beta - \rho_1} < \omega_i^2$$

Had we worked with $B_p(\alpha,\beta)$ instead of $A_p(\alpha,\beta)$ we would have found that (6.3.18) holds also with $2p+1$ replaced by $2p+2$ and $2p$ replaced by $2p+1$. In other words if

$$\omega_i^2 \leqslant \rho_{q+1} < \beta < \omega_{i+1}^2$$

$$\rho_{q+1} \frac{\beta - \rho_q}{\beta - \rho_{q+1}} < \omega_i^2 < \rho_{q+1}$$

holds for any two consecutive Schwartz quotients.
Take next $\beta = \omega_i^2$ and

$$\omega_{i-1}^2 \leqslant \alpha < \rho_{2p+1} \leqslant \omega_i^2$$

Then, (6.2.16) gives

$$\rho_{2p+1} \leq \omega_i^2 < \rho_{2p+1} \frac{\rho_{2p} - \alpha}{\rho_{2p+1} - \alpha} \tag{6.3.20}$$

and this bounding algorithm also holds for any two consecutive Schwartz quotients. It is of a less practical value. It produces an upper bound to a given eigenvalue, from the knowledge of one of its lower bounds (ρ_{2p+1}) and of an upper bound (α) to the preceding eigenvalue. A simple way of obtaining the inequalities (6.3.19) under which (6.3.20) holds is of course to take for α a higher Schwartz quotient from a Krylov sequence converging to ω_{i-1}^2. Then, provided p and q are high enough

$$\omega_i^2 < \rho_{p+1} \frac{\rho_p - \rho_{p+q}}{\rho_{p+1} - \rho_{p+q}} \tag{6.3.21}$$

Introducing the convergence estimator

$$k_q = \sqrt{\frac{\rho_q}{\rho_{q+1}}} - 1 \tag{6.3.22}$$

of a Krylov sequence.

The Temple-Kato bounds yielded by (6.3.18) for the lower and (6.3.20) for the upper, can be put in the form

$$\rho_{q+1} \left(1 - \frac{k^2 \rho_{q+1}}{\beta - \rho_{q+1}}\right) < \omega_i^2 < \rho_{q+1} \left(1 + \frac{k^2 \rho_{q+1}}{\rho_{q+1} - \alpha}\right) \tag{6.3.23}$$

6.4. THE SUBSTRUCTURE TECHNIQUE [16, 21]

The concept of substructure

The method of substructures may be conceived in two different manners, as shown on figure 6.1.
If the substructures are considered in parallel, reductions are processed separately in each of them. One then expresses transition conditions at interfaces in order to assemble the whole structure. It is thus required, when analizing the separate substructures, to keep the degrees of freedom in terms of which connections are expressed. The procedure is illustrated in figure 6.1.a. : in fact, it has to be interpreted as a method of coupling substructures that will be discussed briefly further.
It may reveal successful when transition conditions between substructures involve a relatively small number of degrees of freedom.

On the other hand, substructures in series are defined in a slightly different manner, as shown in figure 1.c.
A substructure results here from the addition to the preceeding one of a certain number of finite elements. A degree of freedom can be eliminated when it is no longer required for assembling the remaining elements.
One generally prefers this later technique, since it gives a smaller maximal band with when analyzing structures that do not split naturally into strongly uncoupled subregions.

Organization of the elimination algorithm

The N^{th} substructure is defined as that part of the whole structure that has been assembled at the end of the preceding assembling operation.
In the last step it represents the condensed form of the whole structure.
Let us denote by :

- n_e , the number of elements to be inserted in the N^{th} substructure;
- k_i and m_i , the element stiffness and mass matrices;
- ℓ_i , the localization operators of the elements;
- $\bar{K}_{N-1}$ and $\bar{M}_{N-1}$, the reduced stiffness and mass matrices of the preceding structure;
- L_{N-1} , the localization matrix of the preceding substructure formally considered as a "super-element" for the next assembling sequence.

For each substructure the following set of operations, which is also described by the first flow chart, defines the elimination algorithm :

1) The K and M matrices of the preceding substructure are readdressed attending to their localization operator :

$$\bar{K}_N = L'_{N-1}\ \bar{K}_{N-1}\ L_{N-1}$$
$$M_N = L'_{N-1}\ \tilde{M}_{N-1}\ L_{N-1} \qquad (6.4.1)$$

2) The n_e new elements are added to the substructure :

$$K_N = K_N + \sum_i \ell'_i\ k_i\ \ell_i$$
$$M_N = M_N + \sum_i \ell'_i\ m_i\ \ell_i \qquad (6.4.2)$$

3) Reduction is achieved using the matrix R defined by (6.2.12) :

$$\bar{K}_{RR} = R'\ K\ R$$
$$M_{RR} = R'\ M\ R \qquad (6.4.3)$$

It reduces the size of the K and M matrices, allowing space for the assemblage of the elements of the next substructure.

At this stage, a displacement q can be eliminated if it satisfies the following conditions :

- it has not been specified in the list of the remaining displacements q_R
- all elements in which this displacement appears have already been included.

The last reduction operation produces the reduced matrices $\bar{K}_{RR}$ and $\bar{M}_{RR}$ of the whole structure, for which we solve the eigenvalue problem (6.2.4).

The approximate modes obtained can be restituted into the whole set of structural displacements by recalling for each substructure, in reverse order, the matrix $K_{CC}^{-1} \cdot K_{CR}$. These are stored on a disk unit during the elimination procedure,

$$q_C = - K_{CC}^{-1}\ K_{CR}\ q_R \qquad (6.4.4)$$

Error measure : computation of the first iterated vector [17, 21]

Bounds to the error produced by the condensation are obtained by applying the algorithms (6.3.13) and (6.3.23), provided the Rayleigh and Schwarz quotients can be computed for the approximation of each eigenmode $q_{(i)}$.

First the Rayleigh quotient ρ_o associated with the approximation q_o of $q_{(i)}$

$$q_o = R\, q_{(i)} \tag{6.4.5}$$

is simply equal to the eigenvalue $\bar{\omega}_i^{-2}$ of the reduced system :

$$\rho_o = \frac{q'_o\, K\, q_o}{q'_o\, M\, q_o} = \frac{q'_{R(i)}\, \bar{K}_{RR}\, q_{R(i)}}{q'_{R(i)}\, \bar{M}_{RR}\, q_{R(i)}} = \bar{\omega}_i^{-2} \quad . \tag{6.4.6}$$

Next, the computation of the Schwarz quotient requires the formation of the first iterated vector, to which a slightly different definition is given from that of section 6.3. :

$$q_1 = \rho_o\, K^{-1}\, M\, q_o \tag{6.4.7}$$

The Schwarz quotient takes thus the form

$$\rho_1 = \rho_o \frac{q'_o\, M\, q_o}{q'_o\, M\, q_1} \quad . \tag{6.4.8}$$

The static problem (6.4.7) should obviously be solved without assembling again the whole structure. This is possible if, when assembling the whole structure, we store on peripheric devices all the elements needed to compute (6.4.7).

Indeed the first iterate q_1 is the solution of the linear system

$$\begin{pmatrix} K_{RR} & K_{CR} \\ K_{RC} & K_{CC} \end{pmatrix} \begin{pmatrix} q_{1R} \\ q_{1C} \end{pmatrix} = \begin{pmatrix} g_R \\ g_C \end{pmatrix} \quad , \tag{6.4.9}$$

where the loads g appearing in the right-hand side of (6.4.9) are the inertia loads associated with the approximate mode q_o :

$$g = \begin{pmatrix} g_R \\ g_C \end{pmatrix} = \bar{\omega}^2 \begin{pmatrix} M_{RR} & M_{RC} \\ M_{CR} & M_{CC} \end{pmatrix} \begin{pmatrix} q_{oR} \\ q_{oC} \end{pmatrix} \quad . \qquad (6.4.10)$$

From the second equation in (6.4.9) we obtain the condensed displacements q_{1C} of the first iterated vector in terms of the q_R :

$$q_{1C} = K_{CC}^{-1} (g_C - K_{CR}\, q_{1R}) \quad . \qquad (6.4.11)$$

By introducing equation (6.4.11) into the first equation in (6.4.9), we obtain :

$$(K_{RR} - K_{RC}\, K_{CC}^{-1}\, K_{CR})\, q_{1R} = g_R - K_{RC} \cdot K_{CC}^{-1}\, g_C \qquad (6.4.12)$$

or $$\bar{K}_{RR}\, q_{1R} = \bar{\omega}^2\, M_{RR}\, q_{oR} \quad , \qquad (6.4.13)$$

by use of equations (6.4.10) and (6.4.4).

Therefore the displacements retained take the same value for the first iterated vector q_1 and for the fundamental solution q_o :

$$q_{1R} = q_{oR} \quad , \qquad (6.4.14)$$

this has already been pointed out in the error analysis of the elimination process. The formula (6.4.11) used to compute the condensed displacements shows that their expression is corrected by the influence of the inertia loads g_C that were neglected in the zero order approximation. In order to compute the Schwartz quotient (6.4.8), we also need the inertia loading

$$g = \rho_o\, M\, q_o \qquad (6.4.15)$$

for the whole structure. It will be restituted by the following recurrence process : if we denote by $\bar{g}_R$ the reduced inertial load

$$\bar{g}_R = \bar{\omega}^2 \, \bar{M}_{RR} \, q_{oR} \quad , \qquad (6.4.16)$$

the loads g_C and g_R are given successively by

$$g_C = \bar{\omega}^2 \, (M_{CR} \, q_{oR} + M_{CC} \, q_{oC}) \quad (6.4.17)$$

and

$$g_R = \bar{g}_R + K_{RC} \, K_{CC}^{-1} \, g_C \quad . \qquad (6.4.18)$$

To compute the first iterate (6.4.7) and the error measure coefficient

$$k^2 = \frac{\rho_o}{\rho_1} - 1 \quad , \qquad (6.4.19)$$

it will thus be necessary to recall the matrices K_{CC}^{-1} , $K_{CC}^{-1} \, K_{CR}$, M_{CR} and M_{CC} for each substructure. This is achieved at best using direct access data set.

The second flow chart sequence gives the detailed organization of the program sequence that achieves the recursive procedure described by equations (6.4.4) and (6.4.9) to (6.4.19). We define q_o as the eigenvector restituted in the complete set of degrees of freedom. Its components that are retained or eliminated in a specified substructure are collected in the vectors q_{oR} and q_{oC} respectively; the same notations hold for the first iterate q_1 and the inertia load g. The transfer operations from q_o to q_{oR} and q_{oC} (denoted by an arrow in the flow chart) are performed by recalling the localization vector of the substructure : it gives the address of each component of q_{oR} or q_{oC} in q_o.

Finally it is important to note that the error measure coefficient, k^2, will only be used to produce lower bounds to the exact eigenfrequencies; indeed we have established in section 6.2 that the computed eigenfrequencies $\bar{\omega}^2$ are the most accurate upper bounds to the eigenvalues of equation (2.1.1).

6.5. THE COUPLING METHODS

When solving approximately the eigenvalue problem of large structures by synthetising the results of a parallel analysis of the constitutive parts, the difficulty lies in the choice of the deformation modes to be adopted for representing the motion of the different subregions.
Indeed these modes must satisfy two essential conditions :

- to give an accurate representation of the internal behavior for each subregion;
- to allow an easy expression of transition conditions along interfaces.

When using the method of substructures in parallel, the generalized displacements of each substructure are split in two categories :

- the boundary degrees of freedom, q_B,
- the interior degrees of freedom, q_I.

By analogy with the assembling process of kinematically admissible elements, an accurate representation of the static behavior of a substructure interacting with the adjacent ones implies that the conformity of the displacement field along the interface is preserved.
In other words, we have to include all deformation modes generated by any sollicitation of interfaces.
They are collected in the modal matrix

$$R_L = \begin{pmatrix} I \\ \Phi_L \end{pmatrix} = \begin{pmatrix} I \\ - K_{II}^{-1} \cdot K_{IL} \end{pmatrix} \quad . \qquad (6.5.1)$$

We superimpose to the boundary modes (6.5.1) a certain number of interior deformation modes selected as the most representative of the substructure considered. They may be restricted without loss of generality to the form

$$R_I = \begin{pmatrix} 0 \\ \Phi_I \end{pmatrix} \qquad (6.5.2)$$

which is obtained by imposing zero boundary displacements.

Hurty's method [26]

When the interior deformation modes ϕ_I chosen are part of the vibration modes ϕ_N of the substructure, one obtains the method of substructures coupling proposed by HURTY [26] and used in conjunction with the finite element method by CRAIG and BAMPTON [5].

The normal modes ϕ_N are solutions of the interior eigenvalue problem

$$K_{II}\,\phi_N = \omega^2\, M_{II}\,\phi_N \quad , \qquad (6.5.3)$$

and the reduction matrix takes the form

$$R = \begin{pmatrix} I & 0 \\ \phi_C & \phi_N \end{pmatrix} .$$

One obtains for each substructure the reduced stiffness and mass matrices

$$\bar{K} = \begin{pmatrix} \bar{K}_{LL} & 0 \\ 0 & \bar{K}_{NN} \end{pmatrix} \text{ and } \bar{M} = \begin{pmatrix} \bar{M}_{LL} & 0 \\ 0 & \bar{M}_{NN} \end{pmatrix} \qquad (6.5.4)$$

with the sub-matrices

$$\begin{aligned}
\bar{K}_{LL} &= K_{LL} - K_{LI}\, K_{II}^{-1}\, K_{IL} \\
\bar{K}_{NN} &= \phi'_N\, K_{II}\, \phi_N \\
\bar{M}_{LL} &= M_{LL} + \phi'_L\, M_{II}\, \phi_L + M_{LI}\, \phi_L + (M_{LI}\, \phi_L)' \\
\bar{M}_{LN} &= \bar{M}'_{NL} = \phi'_L\, M_{II}\, \phi_N + M_{LI}\, \phi_N \\
\bar{M}_{NN} &= \phi'_N\, M_{II}\, \phi_N
\end{aligned} \qquad (6.5.5)$$

Hurty's method seems particularly well adapted to the eigenvalue computation of for instance a complete aircraft structure as a combination of the eigenmodes of each part analyzed independently. Indeed, the eigenmodes of each part of the whole structure are strongly uncoupled from each other, as it is generally observed experimentally from vibration tests. This is due to the fact that the eigenfrequencies of the fuselage are very different from those of the lifting surfaces.

The use of static modes

The vibration modes used in Hurty's method are specific of the high frequency behavior of the substructure. It can therefore be objected that the method does not lead to a correct representation of the quasi-static behavior of a substructure considered as a part of the whole structure. Refining the subdivision into substructures reinforces this weakness.

Hence another way of reducing the internal discretization of the substructure consists to admit, as a first approximation, that the internal deformation modes selected are those obtained by static application of the inertia loads generated under rigid body motion.

In other words, the set of normal modes proposed by Hurty are replaced by the set of static modes obtained by application to the substructure of the inertia forces conjugated to its rigid body modes.

This later procedure also presents the advantage of avoiding the solution of the eigenvalue problem for each substructures successively, and replacing it by a static analysis, the cost of which is much reduced.

Its application to plate like structures [22] has shown that an important reduction of the number of degrees of freedom can be achieved without any significant alteration of the lower frequency spectrum.

THE METHOD OF SUBSTRUCTURES
a) THE WHOLE STRUCTURE
b) SUBSTRUCTURES IN PARALLEL
c) SUBSTRUCTURES IN SERIES
1
2
3
4
remaining d.o.f q_R
condensed d.o.f q_C
fixed d.o.f q_F
boundaries between substructures

FIGURE 6.1

FLOW CHART NR1: THE ASSEMBLING SEQUENCE

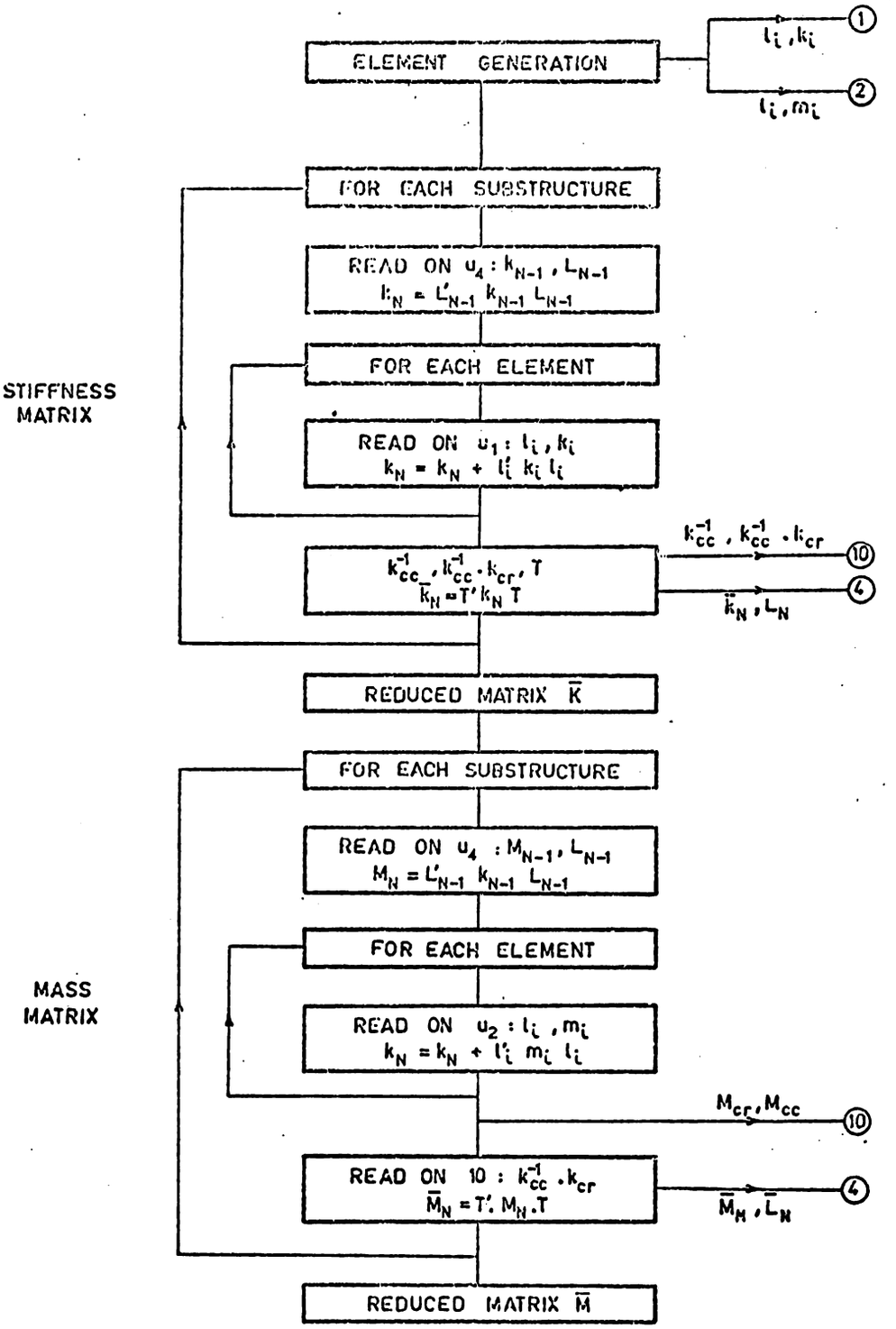

FIGURE 6.2

FLOW CHART NR2 : THE RESTITUTION SEQUENCE

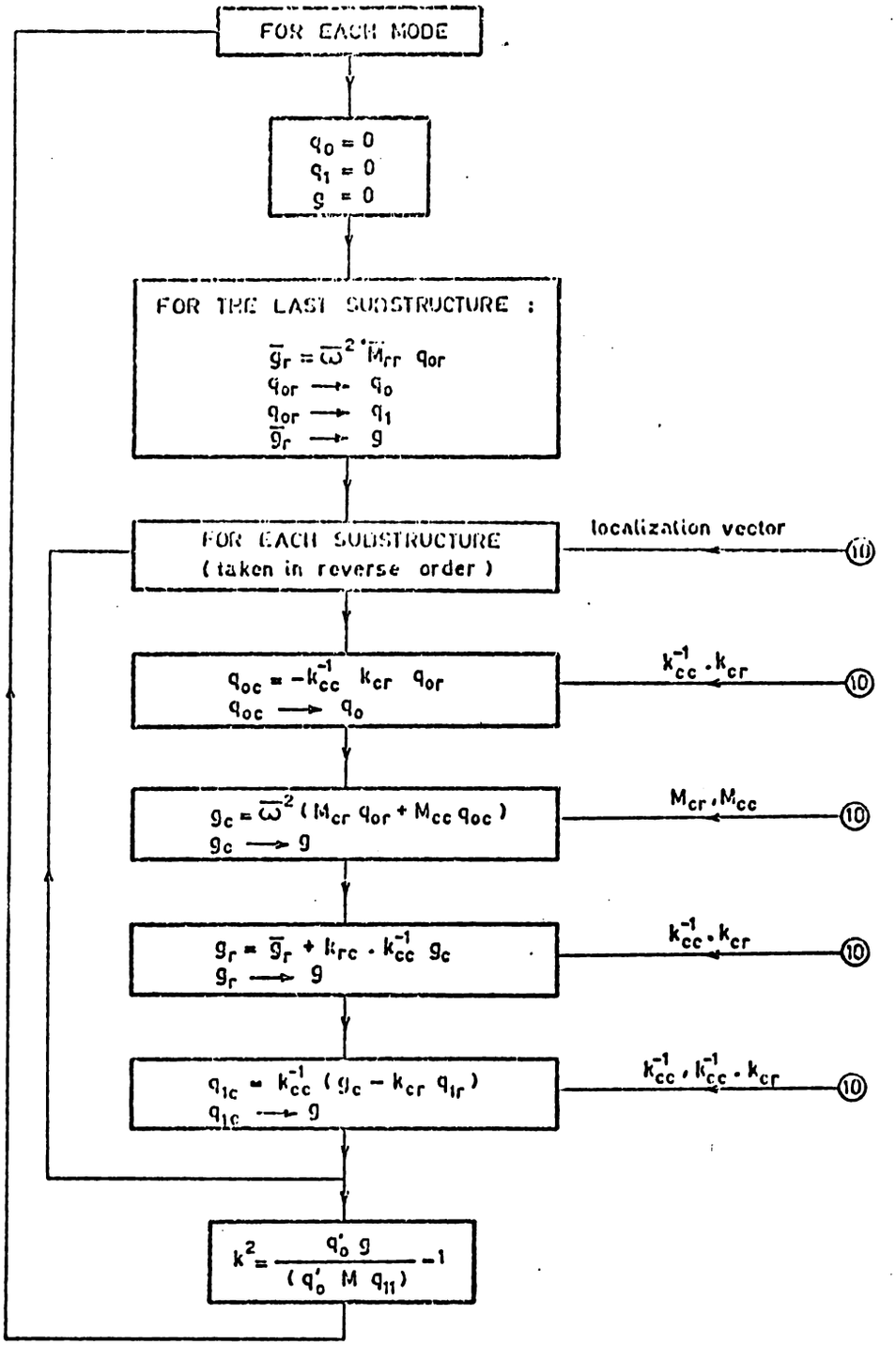

FIGURE 6.3

7. NUMERICAL APPLICATIONS OF THE DUAL ANALYSIS TO PLATE LIKE STRUCTURES

7. NUMERICAL APPLICATIONS OF THE DUAL ANALYSIS TO PLATE LIKE STRUCTURES

The numerical applications presented are limited to the dual analysis of thin plate structures. The corresponding plate bending finite elements are illustrated on figure 7.1. The conforming displacement elements are derived as shown by figure 7.2 by assembling triangular subregions to form a super element[9,33] . In each subregion a cubic polynomial represents the deflection. The degrees of freedom are the deflection and two slopes at each vertex, and the slopes normal to the external interfaces.

In the equilibrium models, parallelogram and triangle shapes are used. A linear variation of the bending moment field is assumed, as shown by the figure 7.3. The variables are concentrated loads at vertices, bending moments and Kirchhoff shear load along interfaces, plus the internal degrees of freedom corresponding to the inertia forces which correspond to the three rigid body modes of the element.

7.1. Cantilever square plate

Figure 7.4 illustrates the application of the dual analysis principle to a cantilever plate of uniform thickness. It has been analysed first with only one element for the whole plate, and then using finer and finer grid up to 36 elements which corresponds roughly to 200 degrees of freedom in the kinematical approach conforming quadrilateral element denotes CQ . Two different subdivisions of the elementary square mesh into equilibrium triangular elements (EQT) have been adopted for the dual approach. The finest idealization - note that it corresponds to the 3 x 3 subdivision - involves 600 degrees of freedom, from which 216 only contribute to the kinetic energy, and hence are retained in the eigenvalue problem.

One can appreciate the monotonic convergence properties of the finite element solution, and the effectiveness and usefullness of the bounds computed to the eigenfrequencies. With relatively few elements, good approximations of the five first eigenfrequencies are obtained, and the gap between bounds has been reduced so that their average value represents for practical purposes an exact solution (difference

less than 0,2 % for the five computed eigenfrequencies).
It is also worth pointing that equilibrium models yield for crude idealization better results than compatible elements.

Figures 7.5 and 7.6 represent the eigenmodes obtained from both approaches, as well as the convergence associated with both types of models when increasing the number of degrees of freedom.

The modes obtained by using equilibrium models correspond to a "weak knowledge" of the displacement field. It is thus necessary to compute (from the knowledge of the generalized displacements conjugated to the inertia forces only) an average displacement of the element as a rigid body. The discontinuities of the nodal lines translate the mathematical concept of dislocation which is specific of equilibrium models.

7.2. Point supported plates

As the point support represents for plate bending problem a severe test due to the singularity at these bending points, the accuracy obtained by finite elements has therefore been tested on a square plate point supported in the center of each quarter plate (figures 7.7 and 7.8). Two grids of finite elements have been used : 4 x 4 and 8 x 8 in the case of equilibrium models, 4 x 4 and 12 x 12 in the case of conforming models. The finer CQ subdivision involves 541 degrees of freedom, and account was taken of the symmetry to solve the eigenvalue problem. The same remark holds for the second equilibrium approach, since it counts 2133 generalized displacements, from which 768 contribute to the kinetic energy.

The very good agreement between the dual analysis, which yield very narrow brackets for the exact eigenvalue solution, allows to be very confident in the behavior of these elements in the case of point support.

7.3. Cantilever skew plates

Figures 7.9 to 7.14 illustrate the dual solution of a cantilever plate for various skew angles.

In the displacement analysis by CQ elements, the finest grid corresponds as before to 231 degrees of freedom, from which 27 are fixed. The results collected in the figure 7.9 confirm the upper boundeness guaranteed by the kinematical approach. Note that the finite element solution deteriorates for increasing skewness.

The equilibrium approach (figure 7.10) underlines the same phenomenon. The finest mesh uses now 50 elements and 420 free generalized displacements. Only 150 of them appear in the eigenvalue problem. Note that in some cases the computed frequency appears now to be an upper estimation. Hence we conclude that the lower bound properties expected from the equilibrium models can fail when the discretization adopted reveals too crude for a correct representation of inertia forces.

An other important factor is the type of subdivision adopted for triangular elements : its influence increases with the skewness of the plate.
All these remarks strike up from the examination of the figure 7.11 and 7.12 which represent, for modes 1 and 2, the convergence as a function of three factors :

- the number of degrees of freedom in terms of which the eigenvalue problem is solved;
- the skewness of the plate;
- for triangular equilibrium models, the type of subdivision adopted.

In order to facilitate the interpretation of the diagrams, the curves associated to different skew angles have been shifted horizontally.

The deterioration of the eigenvalue solution with the skewness of the plate becomes evident. Note also the considerable influence of the type of subdivision adopted.

The figure 7.13 represents the modes 1 and 2 for $\theta = 45°$; those

corresponding to equilibrium models have to be interpreted in the same way as described for the square plate (section 6.1).

Finally the results reported on figure 7.14 allow a comparison with analyses realized by other authors. Among others, note that the experimental eigenfrequencies given by BARTON [1], even after the correction compensating the additional inertia due to aerodynamic forces, are not bracketed by the bounds obtained from the dual analysis. The advantage offered by a dual analysis appears clearly in this case, as it allows a better understanding of the discrepancies between experimental results and an analysis. The difference between experiment and numerical approach is very likely due to the difficulty to achieve a perfectly clamped support.

The comparison with other numerical results also underlines that non conforming kinematical elements [43] do not guarantee upper bounds to exact eigenfrequencies.

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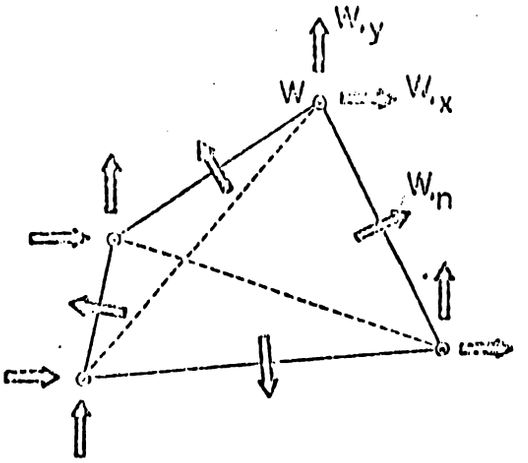
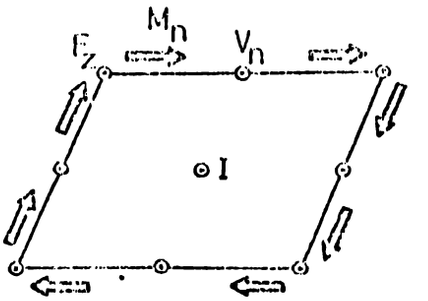
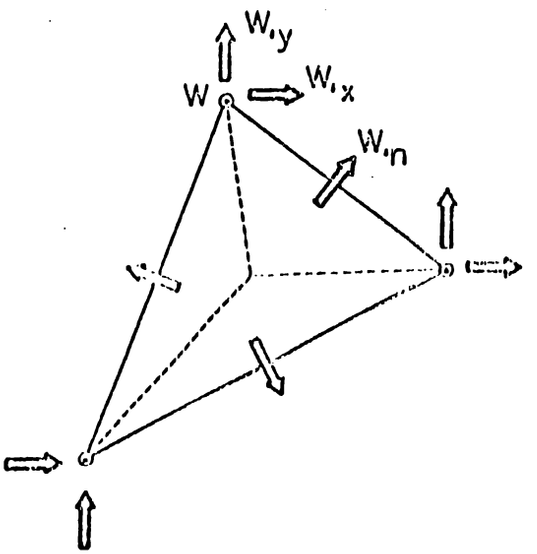
PLATE BENDING FINITE ELEMENTS FOR DUAL ANALYSIS



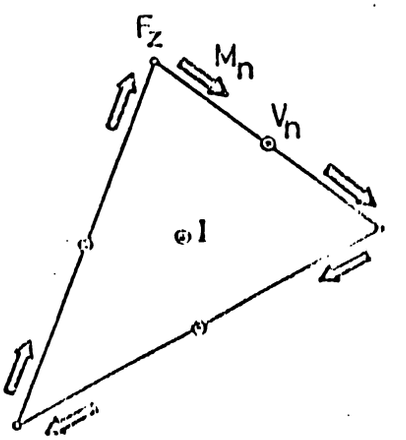
COMPATIBLE QUADRILATERAL



EQUILIBRIUM PARALLELOGRAM



COMPATIBLE TRIANGLE



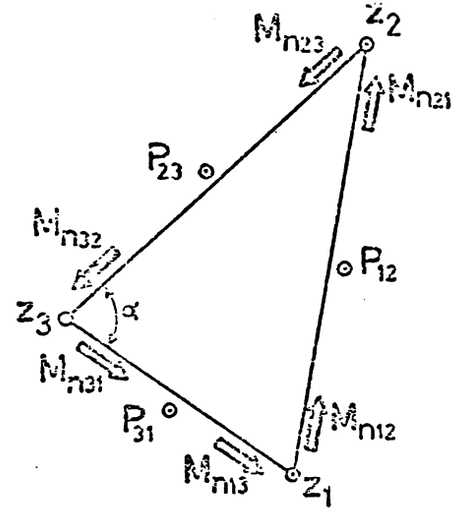
EQUILIBRIUM TRIANGLE

FIGURE 7.1

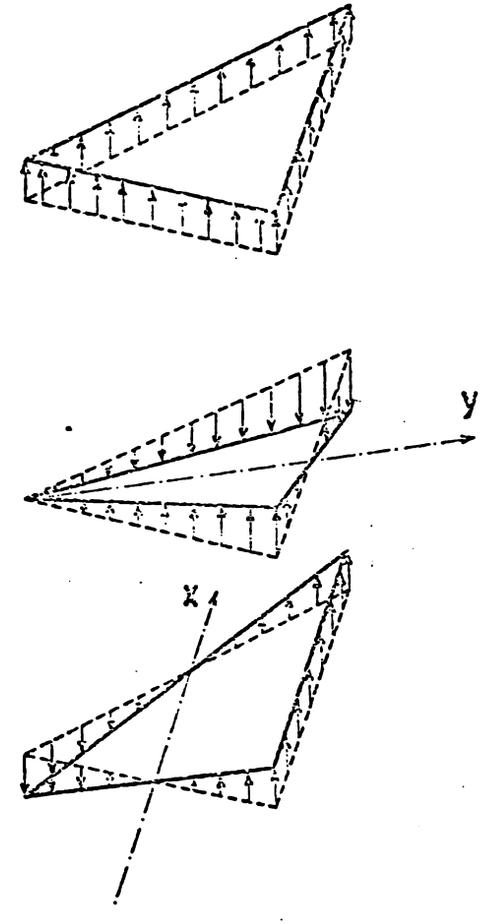
KINEMATICAL PLATE ELEMENT AS AN ASSEMBLING OF THREE TRIANGULAR REGIONS

FIGURE 2.2

EQUILIBRIUM PLATE ELEMENT



PARTICULAR SOLUTION



CONJUGATED GENERALIZED DISPLACEMENTS :

W_i = DEFLECTION AT VERTICES

$\overline{W}_{ik}$ = MEAN DEFLECTION

$\overline{W}_{nik}$ ET $\overline{W}_{nik}$ = MEAN SLOPES

} ON EDGE i-k

$\ddot{W}$ = MEAN DEFLECTION

$\ddot{W}_x$ ET $\ddot{W}_y$ = MEAN SLOPES

} ON THE AREA

FIGURE 7.3

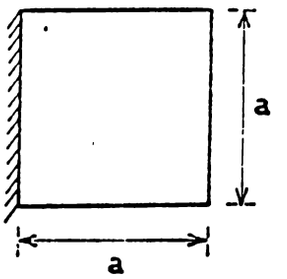
SQUARE CANTILEVER PLATE

non dimensional circular frequency :

$$\bar{\omega} = \frac{\omega a^2}{\sqrt{\frac{D}{m}}}$$

m = mass/unit of area

D = bending rigidity

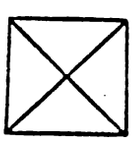


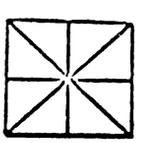
DISPLACEMENT APPROACH

MODE N°	1 X 1	2 X 2	3 X 3	4 X 4	5 X 5	6 X 6
1	3.489	3.485	3.479	3.476	3.474	3.473
2	9.222	8.606	8.535	8.518	8.513	8.511
3	26.96	21.50	21.38	21.33	21.31	21.30
4	35.29	27.63	27.39	27.28	27.24	27.22
5	43.75	31.97	31.27	31.04	31.02	30.99

EQUILIBRIUM APPROACH

2 SUBDIVISIONS :

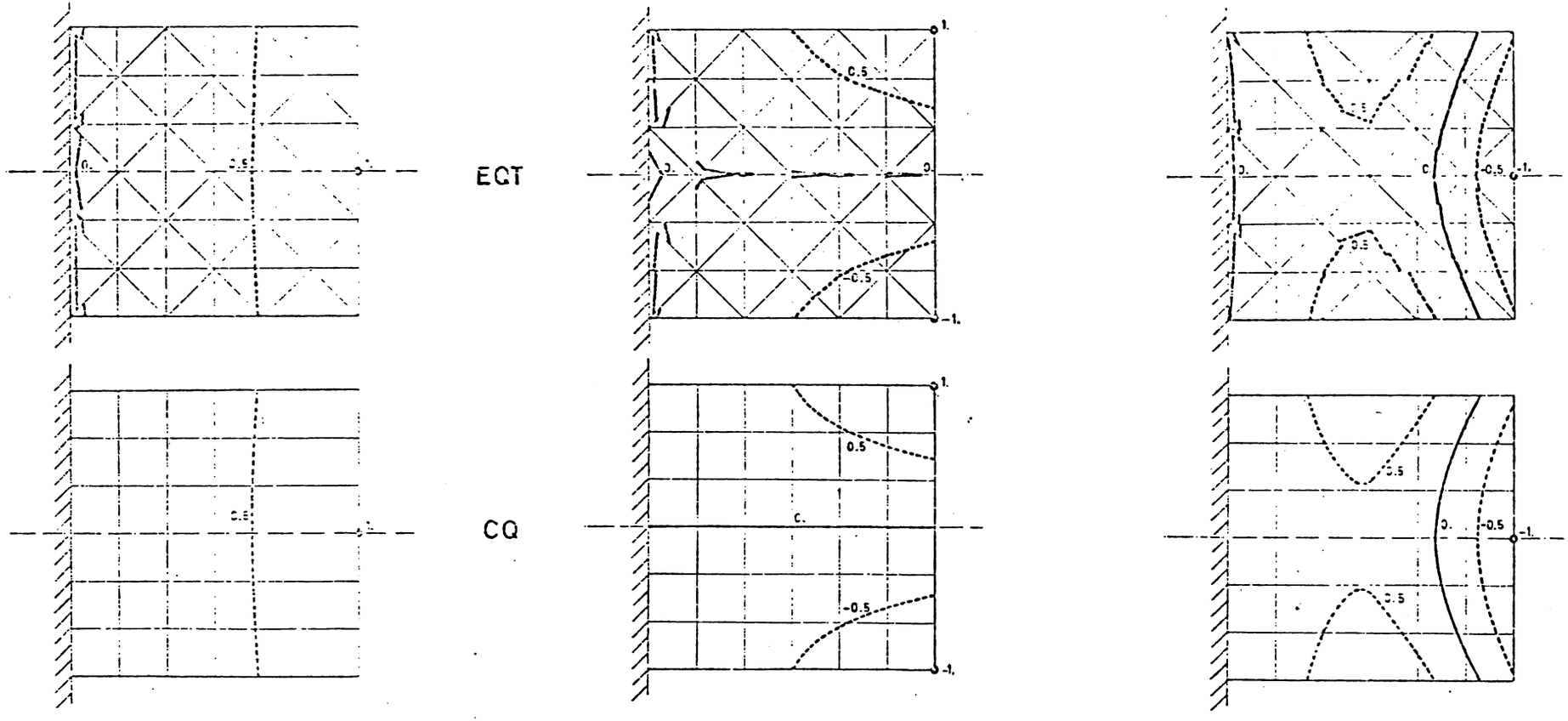


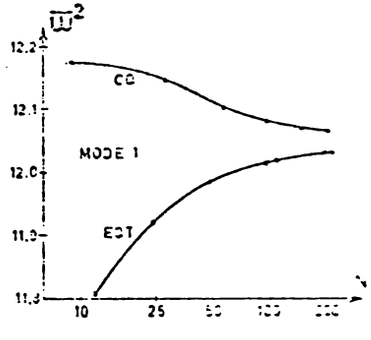


	1 X 1		2 X 2		3 X 3		4 X 4
MODE N°	⊠	⊠	⊠	⊠	⊠	⊠	⊠
1	3.447	3.453	3.462	3.466	3.467	3.469	3.468
2	7.850	8.246	8.410	8.470	8.475	8.494	8.492
3	20.74	20.76	21.16	21.21	21.24	21.26	21.26
4	25.07	26.73	26.89	27.07	27.10	27.16	27.16
5	28.36	28.92	30.14	30.72	30.75	30.89	30.87

FIGURE 7.

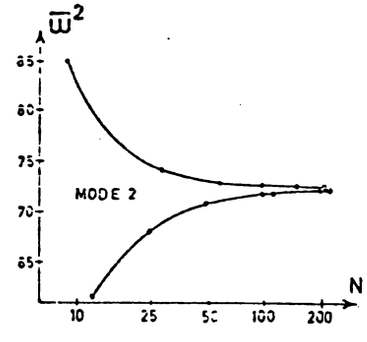
SQUARE CANTILEVER PLATE





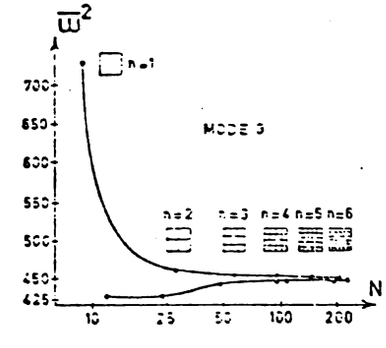
MODE 1

$3.469 < \overline{\omega}_1 < 3.473$



MODE 2

$8.494 < \overline{\omega}_2 < 8.511$

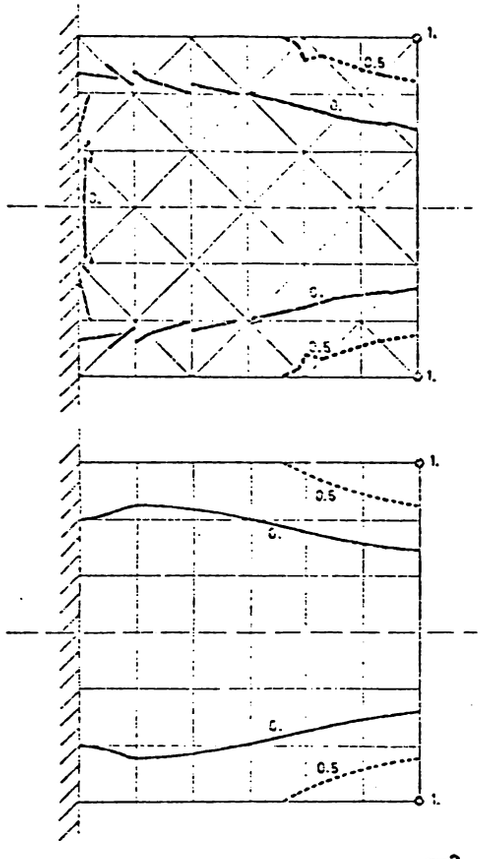


MODE 3

$21.28 < \overline{\omega}_3 < 21.31$

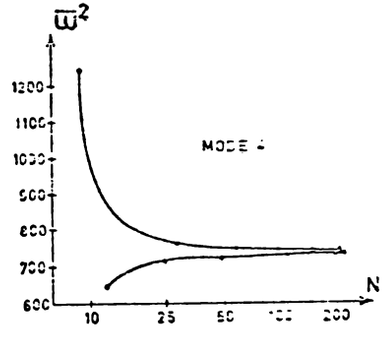
FIGURE 7.5

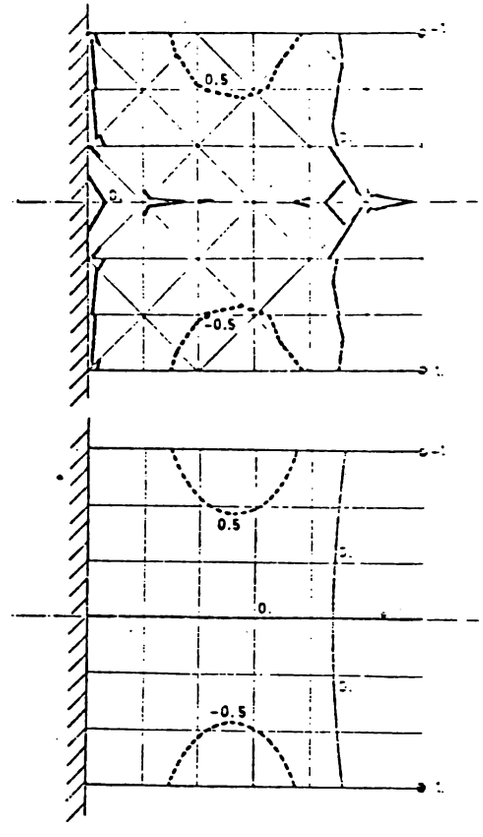
SQUARE CANTILEVER PLATE



MODE 4

$27.18 < \overline{\omega}_4 < 27.22$





MODE 5

$30.89 < \overline{\omega}_5 < 30.99$

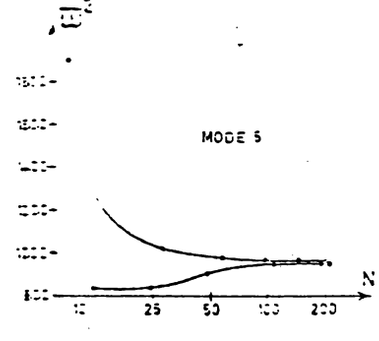


FIGURE 7.6

BOUNDS TO EIGENFREQUENCIES BY DUAL ANALYSIS

POINT SUPPORTED SQUARE PLATE

		EQUILIBRIUM		CONFORMING	
MODE	TYPE	4 x 4	8 x 8	12 x 12	4 x 4
1	S-S	19.594	19.596	19.596	19.616
2	S-S	23.335	23.368	23.380	23.426
3	S-A	31.540	32.323	32.642	33.175
4	A-A	33.650	34.659	35.074	35.469
5	S-A	35.007	35.214	35.283	35.917
6	S-S	—	54.077	55.365	59.714

NON DIMENSIONAL FREQUENCY $\frac{\omega a^2}{\sqrt{\frac{D}{m}}}$

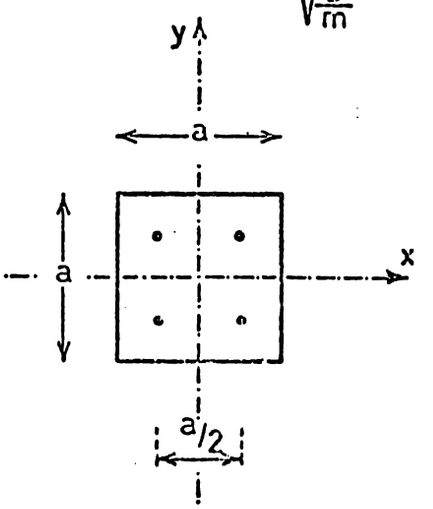
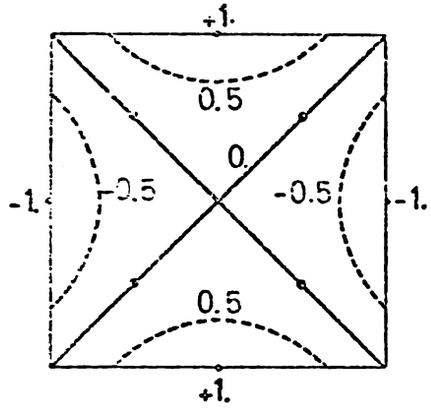
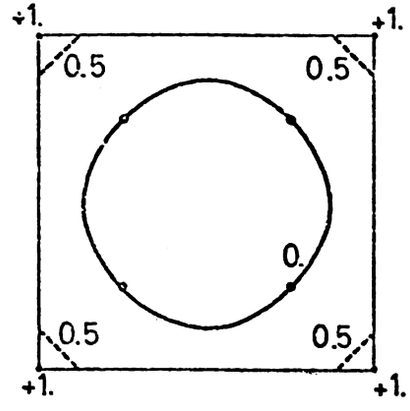


FIGURE 7.7

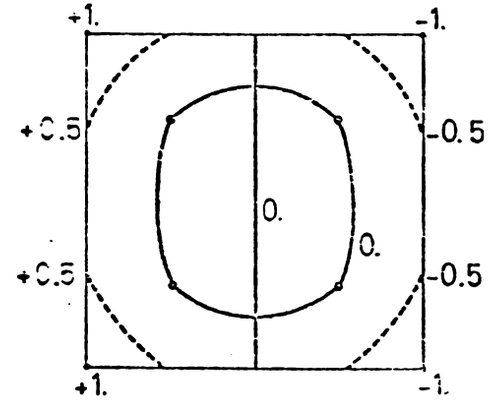
VIBRATION MODES OF A POINT SUPPORTED SQUARE PLATE



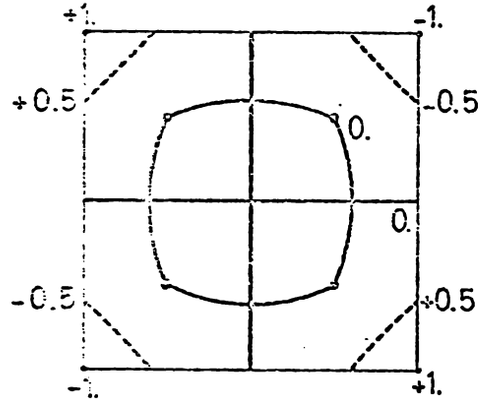
MODE 1



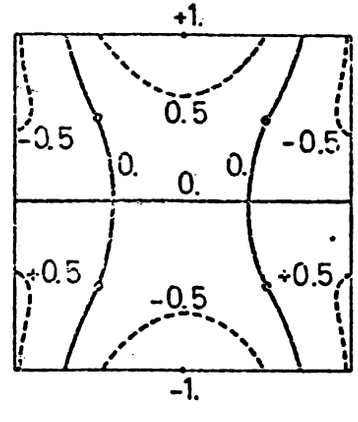
MODE 2



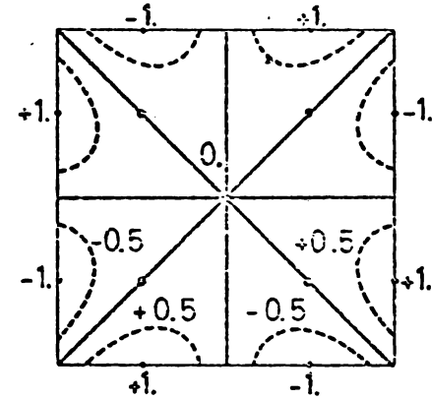
MODE 3



MODE 4



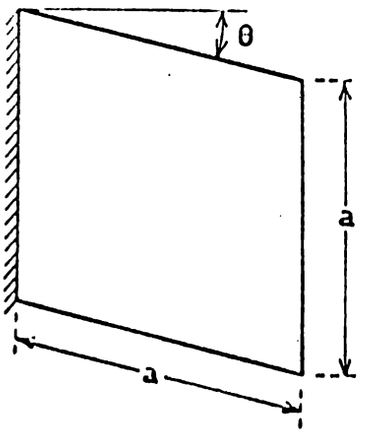
MODE 5



MODE 7

FIGURE 7.9

SKEW CANTILEVER PLATES



non dimensional circular frequency

$$\bar{\omega} = \frac{\omega a^2}{\sqrt{\frac{D}{m}}}$$

m = mass/unit of area

D = bending rigidity

DISPLACEMENT APPROACH

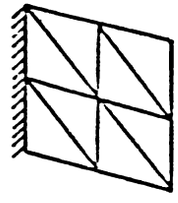
	1x1	2x2	3x3	4x4	5x5	6x6
15°	3.652	3.602	3.591	3.587	3.586	3.585
	9.376	8.872	8.765	8.731	8.717	8.710
	27.38	22.66	22.42	22.32	22.28	22.26
	35.21	26.78	26.52	26.41	26.37	26.36
	47.42	35.14	34.38	34.10	33.99	33.94
30°	4.159	3.988	3.954	3.943	3.938	3.935
	10.12	9.683	9.532	9.480	9.456	9.443
	28.72	26.22	25.75	25.56	25.47	25.41
	40.09	27.08	26.36	26.12	26.03	25.99
	57.33	43.03	42.06	41.67	41.52	41.45
45°	5.070	4.722	4.624	4.582	4.560	4.547
	12.19	11.47	11.36	11.32	11.30	11.29
	31.42	28.43	27.65	27.37	27.25	27.18
	48.87	34.15	32.62	32.18	31.98	31.86
	72.04	56.91	52.35	51.38	51.04	50.90

FIGURE 7.9

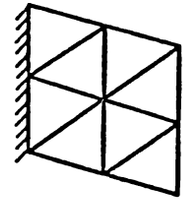
SKEW CANTILEVER PLATES

EQUILIBRIUM APPROACH

3 TYPES OF SUBDIVISION



2×2



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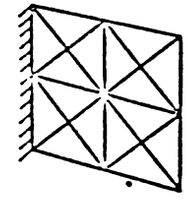


2×2

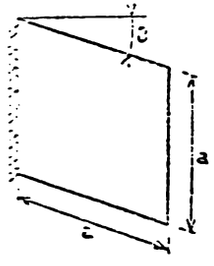
	1 X 1			2 X 2			3 X 3			4 X 4		5 X 5
	3.378	3.400	3.502	3.537	3.549	3.567	3.567	3.565	3.570	3.57[illegible]	3.574	3.579
	7.374	7.733	8.096	8.377	8.478	8.583	8.571	8.637	8.651	[illegible]	[illegible]	[illegible]
15°	11.82	16.02	21.40	21.23	21.[illegible]	21.97	21.84	22.00	22.13	22.[illegible]	22.11	22.16
	22.35	20.18	24.03	25.60	25.58	25.98	25.98	26.12	26.24	26.[illegible]	26.25	26.29
	33.37	47.97	32.46	31.17	32.[illegible]	33.03	32.89	33.32	33.59	33.[illegible]	33.60	33.71
	3.374	3.562	3.709	3.758	3.[illegible]8	3.830	3.857	3.892	3.906	3.[illegible]	3.907	3.914
	7.917	8.081	8.684	8.929	9.[illegible]9	9.257	9.166	9.297	9.333	9.[illegible]	9.336	9.365
30°	10.95	19.47	22.89	22.07	24.[illegible]5	24.56	23.91	24.76	24.97	24.[illegible]7	24.97	25.09
	23.11	21.08	25.58	25.10	25.33	25.53	25.32	25.63	25.73	25.[illegible]	25.73	25.85
	37.92	43.15	36.79	34.88	40.[illegible]2	40.33	39.14	40.82	41.03	40.[illegible]	41.03	41.13
	3.344	3.942	4.084	4.002	4.[illegible]1	4.370	4.235	4.406	4.432	4.[illegible]	4.441	4.460
	8.302	10.26	10.43	10.34	11.[illegible]	11.06	10.62	11.14	11.16	11.[illegible]	11.13	11.20
45°	11.00	23.35	26.27	22.69	25.[illegible]7	26.01	24.07	26.35	26.53	25.[illegible]5	26.60	26.72
	22.13	26.83	31.56	25.83	30.[illegible]7	30.49	28.54	30.84	30.98	29.[illegible]7	31.06	31.17
	40.91	43.91	40.71	35.32	48.[illegible]9	48.79	46.37	49.97	50.27	48.[illegible]6	50.41	50.56

FIGURE 7.10

FIGURE 7.11

DUAL ANALYSIS OF SKEW CANTILEVER PLATES

CONVERGENCE TO MODE Nº 1

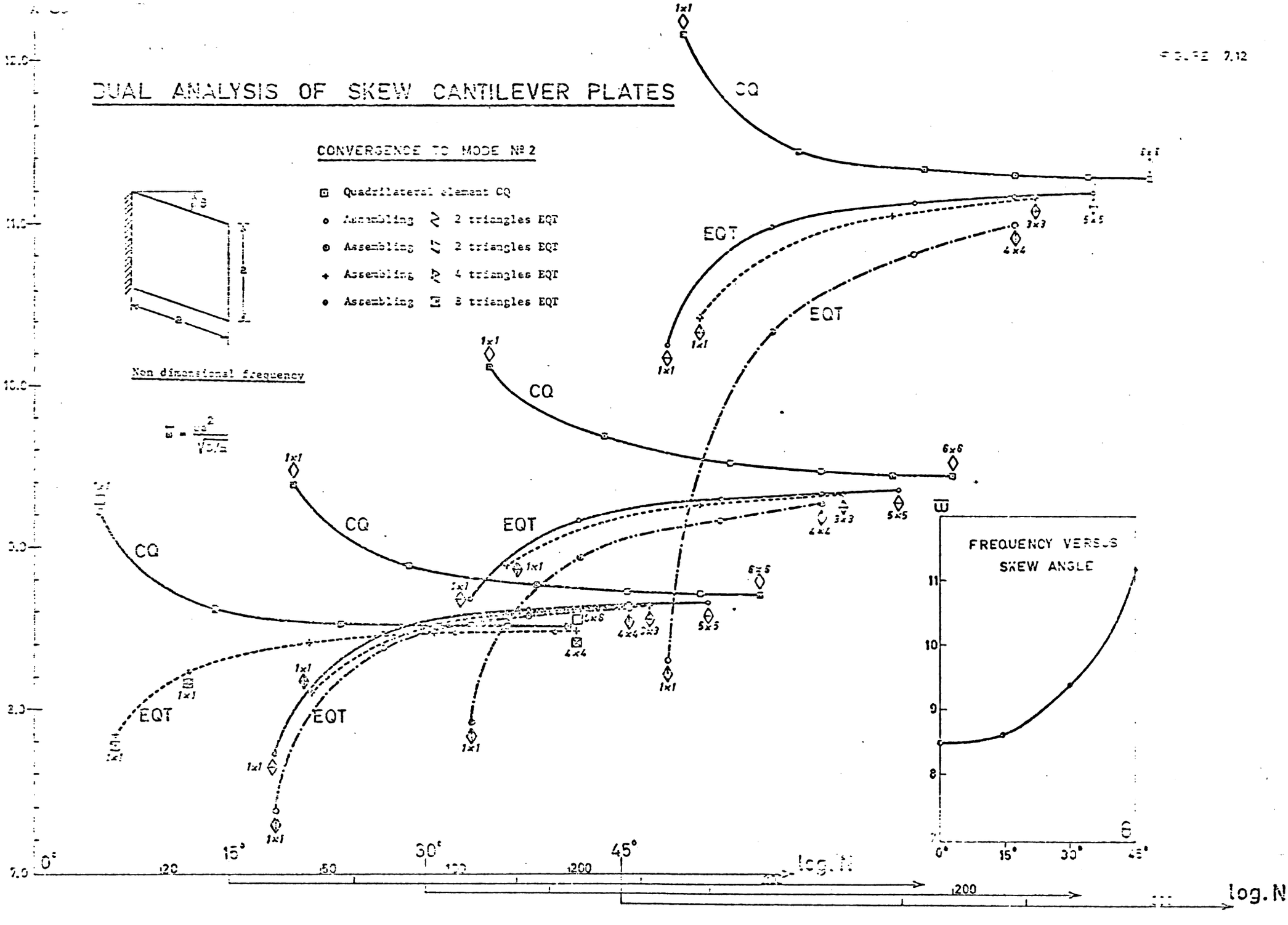


Non dimensional frequency :

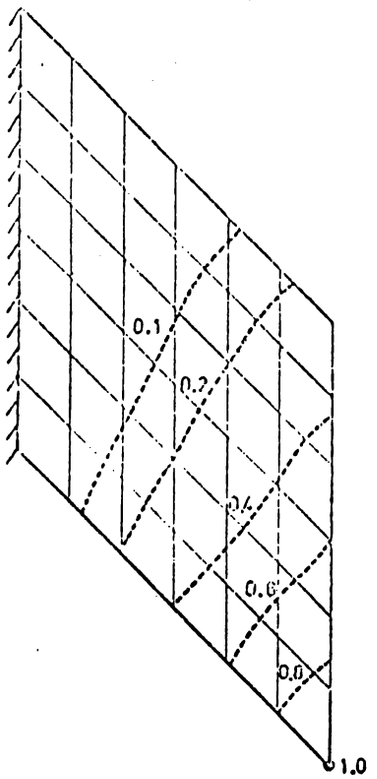
$$\bar{\omega} = \frac{\omega a^2}{\sqrt{D/\rho}}$$

- Quadrilateral element CQ
- Assembling 2 triangles EQT
- Assembling 2 triangles EQT
- Assembling 4 triangles EQT
- Assembling 8 triangles EQT

DUAL ANALYSIS OF SKEW CANTILEVER PLATES



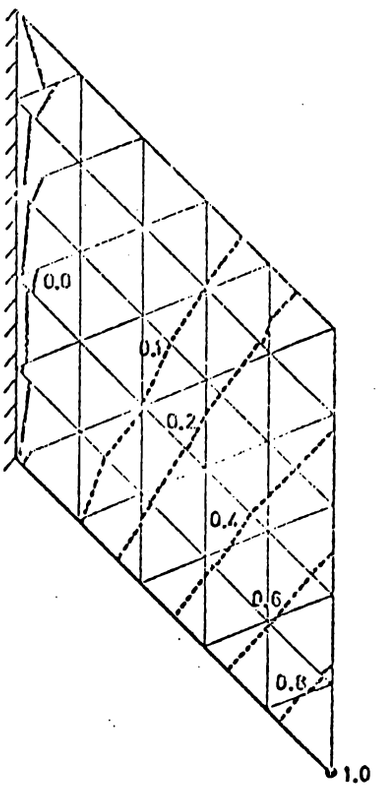
SKEW CANTILEVER PLATE 45°



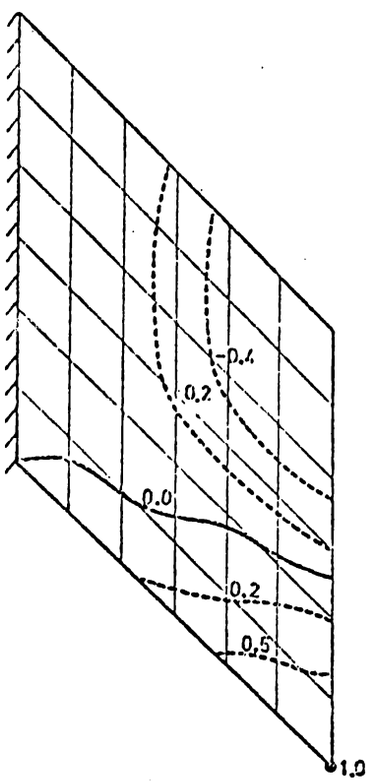
CQ 6×6

MODE 1

$4.450 < \bar{\omega}_1 < 4.547$



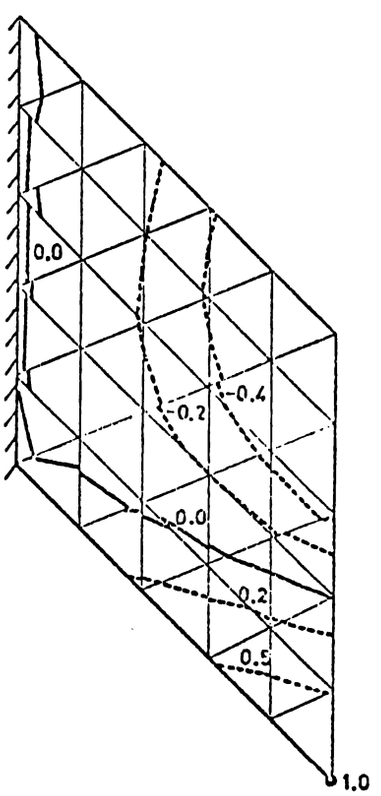
EQT 5×5



CQ 6×6

MODE 2

$11.20 < \bar{\omega}_2 < 11.29$



EQT 5×5

FIGURE 7.13

SKEW CANTILEVER PLATES

COMPARISON OF RESULTS

	THEORETICAL (1)	TEST (1)	CORRECTED TEST (1)	DAWE (2)	ZIENKIEWICZ (3)	CQ 6 x 6 (4)	EQT 5 x 5 (4)	ΔwZ (4)
15°	3.60	3.33	3.44	3.59	3.57	3.53	3.58	0,2
	8.87	8.63	8.68	8.71	8.60	8.71	8.67	0,5
	-	21.49	-	21.59	21.75	22.26	22.16	0,5
	-	26.04	-	-	-	26.36	26.29	0,3
	-	33.01	-	-	-	33.94	33.71	0,7
30°	3.96	3.82	3.88	3.95	3.98	3.93	3.91	0,5
	10.19	9.23	9.33	9.42	9.19	9.44	9.37	0,8
	-	24.51	-	25.56	24.56	25.41	25.09	1,3
	-	25.54	-	-	-	25.99	25.85	0,6
	-	40.64	-	-	-	41.45	41.13	0,7
45°	4.82	4.26	4.33	4.59	4.67	4.55	4.46	2,0
	13.75	11.07	11.21	11.14	11.01	11.29	11.29	0,8
	-	26.52	-	27.43	27.56	27.13	26.72	1,7
	-	30.13	-	-	-	31.80	31.17	2,2
	-	50.19	-	-	-	50.90	50.56	0,7

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FIGURE 7.14

8. TRANSIENT RESPONSE METHODS BASED ON MODAL ANALYSIS

INTRODUCTION

When a linear elastic structure has been idealized into a discrete undamped system, by means of the finite element method or any other discretisation procedure, the basic equation governing its motion under external time dependent forces is :

$$M \ddot{q} + K q = f(t) \qquad (8.1)$$

where M and K are the symmetric mass and stiffness matrices, respectively; q and $\ddot{q}$ are the generalized displacements and acceleration vectors, respectively, and f(t) is the generalized force vector.

The mass matrix M is positive definite while the stiffness matrix K may be only non negative if the system has internal machanisms or rigid body degrees of freedom.

In order to solve the basic equation(8.1),two major classes of methods can be used :
The first alternative consists in the direct integration of the coupled equations of motion(8.1)by means of finite differences or any other numerical integration procedure.
The other one is based on a preliminary modal analysis of the system leading to the uncoupled equations of motion which may then be integrated separately in order to get the general solution of equation(8.1)by superposition of the displacements in each eigenmode shape.

In spite of strong difficulties relative to stability and convergence, especially when response is to be computed over a large time range, the direct method seems to be often preferred to the modal decomposition one. It avoids a rather extensive modal analysis that may reveal prohibitive, for large systems, mainly when high frequency excitation is significant.

The purpose of the present paper is to show that such a choice is not always justified and that methods, based on a partial modal analysis including a sufficient number of significant low frequency modes, can be used successfully in many transient response problems.

MODAL ANALYSIS OF LINEAR ELASTIC SYSTEMS

Free vibrations of a discrete linear elastic system are governed by the homogeneous form of basic equation (1) :

$$M \ddot{q} + K q = 0 \tag{8.2}$$

which has particular solutions of the form :

$$q = x \, e^{i(\omega t + \phi)} \tag{8.3}$$

leading to the eigenvalue and eigenvector equation :

$$(K - \omega^2 M) \, x = 0 \tag{8.4}$$

the solutions of which are the N eigenmodes and the N associated circular eigenfrequencies ω_i of the system, N being the number of degrees of freedom.

Mechanisms and rigid body modes are the m non trivial solutions of the equation

$$K u = 0 \tag{8.5}$$

and may be considered as particular eigenmodes with corresponding zero eigenvalues.

The following important orthogonality relations hold between the different normal modes :

$$\begin{aligned}
& u'_{(r)} \, M \, u_{(j)} = \mu_j \, \delta_{rj} && (j = 1, \ldots, m) \\
& K \, u_{(j)} = 0 && (r = 1, \ldots, m) \\
& x'_{(s)} \, M \, u_{(j)} = 0 && \begin{matrix}(j = 1, \ldots, m) \\ (s = m+1, \ldots, N)\end{matrix} \\
& x'_{(s)} \, M \, x_{(i)} = \mu_i \, \delta_{si} && (i = m+1, \ldots, N)
\end{aligned} \tag{8.6}$$

where μ_i and γ_i are the generalized mass and stiffness, respectively, associated with the i^{th} normal mode shape and are bounded by the relation :

$$\gamma_i = \omega_i^2 \mu_i \tag{8.7}$$

δ_{ij} being the Kronecker's symbol.

If the system has no mechanisms nor rigid body degrees of freedom, its stiffness matrix K is non singular and equation (8.4) may be rewritten as :

$$K^{-1} M x_{(i)} = \frac{1}{\omega_i^2} x_{(i)} \tag{8.8}$$

showing that the eigenmodes $x_{(i)}$ are the eigenvectors of the matrix

$$D = K^{-1} M \tag{8.9}$$

associated with the eigenvalues

$$\nu_i = \frac{1}{\omega_1^2}$$

The matrix D, which plays a very important role in structural dynamics, is called the dynamic flexibility matrix of the system

The extension of the notion of dynamic flexibility matrix to hypostatic systems was first carried out by FRAEIJS de VEUBEKE [1] who demonstrated that the extended dynamic flexibility matrix of an hypostatic system is given by :

$$D = G M \tag{8.10}$$

with $$G = A G_{iso} A' \tag{8.11}$$

where $$A = \left(E - \sum_{j=1}^{m} \frac{u_{(j)} u'_{(j)} M}{u'_{(j)} M u_{(j)}}\right) \tag{8.12}$$

is a projection matrix with selective properties :

$$\Lambda\, u_{(r)} = 0 \qquad (r = 1, \ldots, m)$$
$$\Lambda\, x_{(s)} = x_{(s)} \qquad (s = m+1, \ldots, N). \tag{8.13}$$

G_{iso} denotes a symmetrical static flexibility matrix obtained by imposing to the system any isostatic reference frame.
Assuming the symmetrical form, or equivalently, the orthogonality to all zero frequency modes, yields the particular extended static flexibility matrix G given by (8.11).

The following pseudo-inversion relations hold between the K and G matrices :

$$K\, G = \Lambda'$$
$$G\, K = \Lambda \tag{8.14}$$

Finally, equation (8.8) becomes, for an hypostatic system :

$$K\, u_{(j)} = 0 \qquad (j = 1, \ldots, m)$$
$$G\, M\, x_{(i)} = \frac{1}{\omega_i^2}\, x_{(i)} \qquad (i = m+1, \ldots, N) \tag{8.15}$$

THE MODE DISPLACEMENT METHOD

The orthogonality relations (8.6) show that the eigenmodes $(u_{(j)}, x_{(i)})$ are linearly independent and form a complete set in which the general solution of basic equation (8.1) admits a unique expansion :

$$q = \sum_{j=1}^{m} \zeta_j(t)\, u_{(j)} + \sum_{i=m+1}^{N} \eta_i(t)\, x_{(i)} \tag{8.16}$$

so that (8.1) can be rewritten as :

$$\sum_{j=1}^{m} \ddot{\xi}_j M u_{(j)} + \sum_{i=m+1}^{N} (\ddot{\eta}_i M x_{(i)} + \eta_i K x_{(i)}) = f(t) \qquad (8.17)$$

Successive premultiplication of (8.17) by $(u'_{(r)}, r = 1, \ldots, m)$, $(x'_{(s)}, s = m+1, \ldots, N)$, with use of the orthogonality relations (8.6), yields then the uncoupled normal equations of motion :

$$\begin{aligned} \ddot{\zeta}_r &= \frac{u'_{(r)} f(t)}{\mu_r} && (r = 1, \ldots, m) \\ \ddot{\eta}_s + \omega_s^2 \eta_s &= \frac{x'_{(s)} f(t)}{\mu_s} && (s = m+1, \ldots, N) \end{aligned} \qquad (8.18)$$

which may be easily solved, by use of the Laplace transform and convolution theorem, into :

$$\begin{aligned} \zeta_r &= \frac{u'_{(r)}}{\mu_r} \int_0^t (t - \tau) f(\tau) \, d\tau && (r = 1, \ldots, m) \\ \eta_s &= \frac{x'_{(s)}}{\omega_s \mu_s} \int_0^t \sin \omega_s (t - \tau) f(\tau) \, d\tau && (s = m+1, \ldots, N) \end{aligned} \qquad (8.19)$$

providing zero initial conditions are assumed :

$$\begin{aligned} \zeta_r(0) = \dot{\zeta}_r(0) &= 0 && (r = 1, \ldots, m) \\ \eta_n(0) = \dot{\eta}_n(0) &= 0 && (n = m+1, \ldots, N). \end{aligned} \qquad (8.20)$$

Hence, the general solution of (8.1) is given by :

$$q(t) = \sum_{j=1}^{m} \frac{u_{(j)} u'_{(j)}}{\mu_j} \int_0^t (t-\tau) f(\tau) \, d\tau + \sum_{i=m+1}^{N} \frac{x_{(i)} x'_{(i)}}{\omega_i \mu_i} \int_0^t \sin \omega_i (t-\tau) f(\tau) d\tau \qquad (8.21)$$

And the equivalent static force which would at each time, produce the actual stress field is :

$$g(t) = K\, q_{(t)} = \sum_{i=m+1}^{N} \frac{M\, x_{(i)} x'_{(i)}}{\mu_i}\, \omega_i \int_0^t \sin \omega_i (t-\tau)\, f(\tau)\, d\tau \qquad (8.22)$$

Equations (8.21) and (8.22) give the complete solution of basic equation (8.1) providing all the normal eigenmode-shapes of the system are known. If this is not the case, and it will never occur practically with large systems for which a complete modal analysis would be prohibitive, only the first n normal modes are known and can be used to compute dynamic response.

The problem is then to determine under which conditions an incomplete n-fold set of low frequency normal modes will be sufficient for computing of displacements and stresses with satisfactory accuracy by truncation of the expansions (8.21) and (8.22) to their n first terms.

To this purpose, let us suppose the transient external loads are of the type :

$$f(t) = p \cdot \phi(t) \qquad (8.23)$$

where p is a constant static force applied with time history $\phi(t)$ (it should be noted that this is not a restriction since any general load can be expressed by a superposition of such terms).

We have now to check the convergence of the expressions :

$$\sum_{i=m+1}^{n} \frac{x_{(i)}\, x'_{(i)}\, p}{\mu_i}\, \frac{1}{\omega_i} \int_0^t \sin \omega_i\, (t-\tau)\, \phi(\tau)\, d\tau \qquad (8.24)$$

and

$$\sum_{i=m+1}^{n} \frac{M\, x_{(i)} x'_{(i)} p}{\mu_i}\, \omega_i \int_0^t \sin \omega_i (t-\tau)\, \phi(\tau)\, d\tau \qquad (8.25)$$

when n is increased. It can be seen that each term of the expansions

(8.24) and (8.25) is a product of two different terms :
a static term

$$\frac{x_{(i)}\, x'_{(i)}\, p}{\mu_i} \quad \text{or} \quad \frac{M\, x_{(i)}\, x'_{(i)}\, p}{\mu_i}$$

which depends only on the static force p and a time dependent spectral term :

$\frac{1}{\omega_i} \sin \omega_i t * \phi(t)$ or $\omega_i \sin \omega_i t * \phi(t)$ which depends only of the spectrum of the time history $\phi(t)$.
Hence two types of convergence can be expected :

a) a pseudo-static convergence which will occur if n is large enough to allow a good representation of the static force p by means of its expansion in the n fold set of the known modes, this is to say that p must be quite orthogonal to the (N - n) unknown modes which will consequently not be excited.

b) A spectral convergence which will occur if the time history $\phi(t)$ is such that the convolution integrals $\frac{1}{\omega_i} \sin \omega_i t * \phi(t)$ or $\omega_i \sin \omega_i * \phi(t)$ converge to zero when i is increased so that the component of the response in the unknown high frequency modes may be neglected in comparison with that in the known low-frequency modes.

If we consider for instance the step loading

$$\phi(t) = \upsilon(t)$$

$$\frac{\sin \omega_i t * \phi(t)}{\omega_i} = \frac{1 - \cos \omega_i t}{\omega_i^2}$$

we obtain a spectral convergence with rate $\frac{1}{\omega_i^2}$ for the displacements but no spectral convergence for the equivalent static force and hence no spectral convergence for the stresses.

Let us now consider an harmonic loading :

$$\phi(t) = \sin \omega\, t$$

$$\frac{\sin \omega_i\, t \ast \phi(t)}{\omega_i} = \frac{\omega_i \sin \omega\, t - \omega \sin \omega_i\, t}{\omega_i\, (\omega_i^2 - \omega^2)}$$

provided ω_i is great compared to ω we get again a spectral convergence with rate $\frac{1}{2}$ for the displacements and, again, no spectral convergence for the ω_i stresses.

It should be outlined that only the quasi-static convergence is valid for both displacements and stresses calculation and that spectral convergence of the displacement solution does not at all imply convergence for the stresses.

If convergence is insured, we may write the mode displacement approximations for displacements and equivalent static force, which are :

$$q(t) = \sum_{j=1}^{m} \frac{u_{(j)} u'_{(j)}}{\mu_j} \int_0^t (t-\tau)\, f(\tau)\, d\tau + \sum_{i=m+1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i\, \mu_i} \int_0^t \sin\omega_i (t-\tau) f(\tau) d\tau \qquad (8.26)$$

$$g(t) = \sum_{i=m+1}^{n} \frac{x_{(i)} x'_{(i)}}{\mu_i} \omega_i \int_0^t \sin \omega_i\, (t-\tau)\, f(\tau)\, d\tau \qquad (8.27)$$

THE MODE ACCELERATION METHOD

This method, generally attributed to WILLIAMS [2], (see also [3,4]), was, in fact, already presented by Lord RAYLEIGH [5] (V,100) in a slightly different, but equivalent manner.

Basic equation (8.1) is a dynamic equilibrium equation in which three different forces are involved :

the applied force : $f(t)$

the equivalent static force : $g = K\,q$

the inertia force : $a = M\,\ddot{q}$

If inertia forces were removed from the system, the solution of (8.1) would be the quasi-static solution, which is, with assumed non singular stiffness matrix :

$$q(t) = K^{-1}\,f(t) \tag{8.28}$$

and can thus be obtained independently of any modal analysis.

If inertia forces are to be taken into consideration the exact solution is obtained by adding to(8.28)the static displacements they produce :

$$q(t) = K^{-1}\,f(t) - K^{-1}\,a \tag{8.29}$$

Thus, as pointed out by WILLIAMS, only the inertia forces need to be expanded in the incomplete n fold set of the known modes, but neither the applied force nor the equivalent static force, as done is the mode displacement method.
We shall now write, according to (8.16) :

$$q(t) = K^{-1}\left\{\, f(t) - \sum_{i=1}^{n} \ddot{\eta}_i\, M\, x_{(i)} \,\right\} \tag{8.30}$$

Using (8.18) and (8.19) we get :

$$\ddot{\eta}_i = \frac{x'_{(i)}}{\mu_i} \{ f(t) - \omega_i \int_0^t \sin \omega_i (t-\tau) f(\tau) d\tau \} \qquad (8.31)$$

and the new approximate solution of (8.1) becomes :

$$q(t) = K^{-1} \{ f(t) - \sum_{i=1}^{n} \frac{M x_{(i)} x'_{(i)}}{\mu_i} [f(t) - \omega_i \int_0^t \sin\omega_i (t-\tau) f(\tau) d\tau] \} \qquad (8.32)$$

that is with use of (8.8) :

$$q(t) = \sum_{i=1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i \mu_i} \int_0^t \sin\omega_i (t-\tau) f(\tau) d\tau + \{K^{-1} - \sum_{i=1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i^2 \mu_i} \} f(t) \qquad (8.33)$$

In a quite analogous way, Lord Rayleigh started from the uncoupled normal equations of motion(8.18) and pointed out that the normal inertia force may be neglected in comparison with the normal stiffness force, when the period of the considered mode becomes small when compared with that of the operating external loads. If this is true, except in the n first modes, we may rewrite the normal equations (8.18) in the form :

$$\ddot{\eta}_i + \omega_i^2 \eta_i = \frac{x'_{(i)} f(t)}{\mu_i} \qquad (i = 1, \ldots, n)$$

$$\omega_i^2 \eta_i \approx \frac{x'_{(i)} f(t)}{\mu_i} \qquad (i = n+1, \ldots, N) \qquad (8.34)$$

thus we get :

$$\eta_i = \frac{x'_{(i)}}{\omega_i \mu_i} \int_0^t \sin \omega_i (t-\tau) f(\tau) d\tau \quad (i = 1, \ldots, n)$$

$$\eta_i \simeq \frac{x'_{(i)}\, f(t)}{\omega_i^2\, \mu_i} \qquad (i = n+1,\ldots,N) \tag{8.35}$$

and the approximate solution of (8.1) is :

$$q(t) = \sum_{i=1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i\, \mu_i} \int_0^t \sin \omega_i\,(t-\tau) f(\tau)\, d\tau + \left\{ \sum_{i=n+1}^{N} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \right\} f(t) \tag{8.36}$$

Equivalence of (8.33) and (8.36) is obvious since the stiffness matrix K is equal to its spectral expansion in the complete N fold set of the normal modes :

$$K = \sum_{i=1}^{N} \frac{K\, x_{(i)}\, x'_{(i)}\, K}{x'_{(i)}\, K\, x_{(i)}} \tag{8.37}$$

and was assumed to be non singular. Thus, pre- and post-multiplication of (8.37) by K^{-1}, with use of relations (8.6) and (8.7) yields :

$$K^{-1} = \sum_{i=1}^{N} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \tag{8.38}$$

hence

$$K^{-1} - \sum_{i=1}^{n} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} = \sum_{i=n+1}^{N} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \tag{8.39}$$

Rayleigh's approach, which does not involve the static flexibility matrix K^{-1}, is directly extensible to hypostatic systems with singular stiffness matrix K, and we get in this case :

$$q(t) = \sum_{j=1}^{m} \frac{u_{(j)} u'_{(j)}}{\mu_j} \int_0^t (t-\tau) f(\tau)\, d\tau + \sum_{i=m+1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i\, \mu_i} \int_0^t \sin\omega_i (t-\tau) f(\tau)\, d\tau$$

$$+ \sum_{i=n+1}^{N} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \Big\} f(t) \tag{8.40}$$

But we may now write, remembering the definition of the extended static flexibility matrix G given in section I and using the analogy between equations (8.8) and (8.15) :

$$G = \sum_{i=m}^{N} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \tag{8.41}$$

So that the most general expression for the mode acceleration approximate solution of basic equation (8.1) with n known normal modes is :

$$q(t) = \sum_{j=1}^{m} \frac{u_{(j)} u'_{(j)}}{\mu_j} \int_0^t (t-\tau) f(\tau) d\tau + \sum_{i=m+1}^{n} \frac{x_{(i)} x'_{(i)}}{\omega_i\, \mu_i} \int_0^t \sin\omega_i (t-\tau) f(\tau) d\tau$$

$$+ \left\{ G - \sum_{i=m+1}^{n} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \right\} f(t) \tag{8.42}$$

The equivalent pseudo-static force being now given by :

$$g(t) = \sum_{i=m+1}^{n} \frac{M\, x_{(i)} x'_{(i)}}{\mu_i} \omega_i \int_0^t \sin\omega_i (t-\tau) f(\tau) d\tau + \left(A' - \sum_{i=m+1}^{n} \frac{M\, x_{(i)} x'_{(i)}}{\mu_i} \right) f(t)$$

(8.43)

(pseudo-static because A'f(t) contains the additional inertia forces involved by the rigid body accelerations).

If we compare the results of the mode displacement method (8.26) with those of the mode acceleration method (8.42), we see that they differ by the term :

$$u(t) = \left\{ G - \sum_{i=m+1}^{n} \frac{x_{(i)}\, x'_{(i)}}{\omega_i^2\, \mu_i} \right\} f(t) \tag{8.44}$$

which is the product of the applied load by the deflected extended flexibility matrix and represents the pseudo-static displacements in the (N-n) fold set of the unknown modes where inertia forces have been neglected.

The mode acceleration method will generally give better convergence than the mode displacement method since the expansion of the single inertia force can be expected to converge more rapidly than that of the whole operating forces. It can be seen that there is no longer problem with pseudo-static convergence since the method is statically exact. Furthermore, the mode acceleration correction (8.44) provides a useful criterion to check the pseudo-static convergence of the mode displacement method. In order to check the spectral convergence of the method, let us again consider the step loading :

$$\phi(t) = \upsilon(t)$$

$$\frac{\sin \omega_i t * \phi(t)}{\omega_i} = \frac{1 - \cos \omega_i t}{\omega_i^2}$$

it can be seen that in an unknown mode (i) the exact contribution $\frac{1 - \cos \omega_i t}{\omega_i^2}$ which has been neglected in the mode displacement method is now replaced by $\frac{1}{\omega_i^2}$ which represents its mean value over a period, but no significant improvement of the spectral convergence has been obtained.

For harmonic loading :

$$\phi(t) = \sin \omega t$$

$$\frac{\sin \omega_i t * \phi(t)}{\omega_i} = \frac{\omega_i \sin \omega t - \omega \sin \omega_i t}{\omega_i (\omega_i^2 - \omega^2)}$$

the difference between exact contribution of the unknown mode (i) and the assumed contribution $\frac{\sin \omega t}{\omega_i^2}$ is :

$$\frac{\omega}{\omega_i^2} \frac{\omega \sin \omega t - \omega_i \sin \omega_i t}{\omega_i^2 - \omega^2}$$

so that the spectral convergence rate is now $\frac{1}{3}$ for the displacements, hence $\frac{1}{\omega_i}$ for the stresses, provided ω_i is $^{\omega}i$ high enough compared with ω .
Thus, in the harmonic case, we get a significant improvement of the spectral convergence when using the mode acceleration method.

NUMERICAL RESULTS

Several examples have been chosen in order to illustrate the methods described here, including an extension of the mode acceleration method to systems with small orthogonal viscous damping (see appendix I). Results are compared with those of HOUBOLT's direct recurrence method [6] (see appendix II) or those of an exact analytical method. The modally uncoupled precise integration operator recently presented by DUNHAM, NICKELL and STICKLER [7] (see appendix III) has been checked and was found to be of great interest when analytical integration of the normal equations of motion (8.18) becomes prohibitive.

The first example deals with a three dimensional frame submitted to a concentrate step load which is removed after a short time $T = 10^{-2}$ sec. The frame (see figure 1) has been idealized with sixteen Euler's beam element of the third degree leading to a discrete system with fourty-eight degrees of freedom. Since the load is applied during a very short time, only the mode displacement method is suitable but the pseudo-static convergence can be checked by means of the mode acceleration correction (8.44). Figures 1 and 2 show the time histories of the most representative displacement and bending moment obtained by the mode displacement method, with twelve modes, and by the HOUBOLT's method with time step $\Delta t = 5.10^{-4}$ sec.
Agreement between the two methods is excellent. Figure 3 illustrates the convergence of the maximum strain energy (occuring at time $t = 0.035$ sec) versus time step Δt for the HOUBOLT's method and versus the number n of used modes for the mode displacement method : stabilization of the mode displacement solution with twelve modes and progressive bracketting when n is increased and Δt decreased are to be outlined.

The second example consists in a uniform pinned-pinned beam overcrossed by a concentrate load P with constant velocity v . Exact analytical solution has been computed by expansion of NAVIER's eigenfunctions and finite element analysis has been carried out by idealizing the beam with twenty EULER's beam element of the third degree leading to a system with fourty degrees of freedom. Discretization of the travelling load and integration of the normal equations of motion are somewhat laborious to be detailed here. A velocity equal to the second critical speed has been assumed. Figures 4 and 5 show the time history of the mid span deflection and bending moment while figures 6 and 7 give the deflection and bending moment distribution at the maximum mid span deflection time. All these results were computed analytically, by the HOUBOLT's method with time step $\Delta t = 5\ 10^{-4}$ sec, by the mode displacement method and by the mode acceleration method with the five first normal modes. Excellent agreement between the different methods can be again noticed. Furthermore, this problem has been chosen to check the modally uncoupled precise integration operator of DUNHAM, NICKELL and STICKLER : integration of the normal equations of motion by means of this operator was carried out with time step $\Delta t = 5\ 10^{-4}$ sec and no significant difference with analytical integration could be noticed over a zero to 0.5 sec time range.

The third example illustrates the extension of the mode acceleration method to hypostatic systems by means of a free-free beam under mid-span concentrated step loading. Analytical solution was obtained by expansion of DUNCAN's eigenfunctions while finite element analysis was made with twenty EULER's beam element of the third degree giving a system with fourty two degrees of freedom. Mode displacement and mode acceleration solutions were computed with the five first symmetric-symmetric modes. Figure 8 shows the mid-span deflection history over three periods and figure 9 the mid span bending moment history over the first period. Excellent agreement for the deflection and satisfactory bracketting of the bending moment can be noticed.

The fourth example is a simply supported square plate under mid point harmonic excitation with 3 % assumed orthogonal viscous damping ratio. Analytical solution was obtained by expansion of NAVIER's eigenfunctions. Finite elements idealization was made with sixteen KIRCHHOFF's plate elements of the third degree for a quater of the plate leading to a system with eighty degrees of freedom. Amplitude and phase of the forced harmonic responsewere computed by the mode acceleration method, with eleven symmetric-symmetric modes, for several excitation frequencies including resonance frequencies. Figure 10 shows the frequency history of the mid point deflection amplitude up to the third resonance peak and outlines again the good accuracy of the mode acceleration method.

CONCLUSION

As announced in the beginning methods based on modal analysis are seen to give very satisfactory transient responses althought only a generally small number of significant low frequency modes needs to be taken into consideration. Both mode displacement and mode acceleration methods have a simple finite element formulation and are easily extensible to singular systems.
The mode acceleration method was seen to be a simple improvement of the mode displacement method by means of the quasi-static contribution of the unknown modes which provides also a suitable criterion for checking the convergence : is this contribution small, so both methods are valid with advantage when using the mode acceleration method. Is this contribution not small so the number of known modes is unsufficient and both methods fail since the fact that the mode acceleration method will generally give a better result that the mode displacement method does no longer imply that this result will be good. When comparing the mode displacement or the mode acceleration method with direct methods, like HOUBOLT's method, it is seen that a substantial economy is get by the fact that only n second order linear differential equations are to be solved instead of iterating the complete (N x N) coupled system. Of course, the cost of the modal analysis must be taken into consideration but it must be kept in mind that a single modal analysis allows the treatment of any number of response problems.

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Integration operators for transient structural response
Computers and Structures, vol. 2, N° 1/2, February 1972

APPENDIX I

If viscous damping is assumed, basic equation (8.1) becomes :

$$M\ddot{q} + B\dot{q} + Kq = f(t) \tag{8.45}$$

where B is a symmetric non negative damping matrix. Assuming (8.16) leads then to the following coupled normal equations of motion :

$$\mu_r \ddot{\eta}_r + \sum_{i=1}^{N} \left[x'_{(r)} B x_{(i)} \right] \dot{\eta}_i + \gamma_r \eta_r = x'_{(r)} f(t) \tag{8.46}$$

These equations show that damping generally couples the normal modes of the associated undamped system. If damping is small and if coupling of modes may be neglected a great simplification of (8.46) is get when assuming the following additional orthogonality relations :

$$x'_{(r)} B x_{(i)} = 2 \varepsilon_r \omega_r \mu_r \delta_{ri} \tag{8.47}$$

where ε_i is the ratio of critical damping in the i^{th} mode.
It must be outlined that hypothesis (8.47) is rather restrictive, since neglecting coupling of the modes, and is to be considered only as an occasionally valid approximation.

If (8.47) be assumed, (8.46) becomes :

$$\ddot{\eta}_r + 2 \varepsilon_r \omega_r \dot{\eta}_r + \omega_r^2 \eta_r = \frac{x'_{(r)} f(t)}{\mu_r} \tag{8.48}$$

the solution of which is, with assumed zero initial conditions :

$$\eta_r(t) = \frac{x'_{(r)}}{\mu_r \sqrt{1-\varepsilon_r^2}\, \omega_r} \int_0^t e^{-\varepsilon_r \omega_r (t-\tau)} \sin\left[\sqrt{1-\varepsilon_r^2}\, \omega_r (t-\tau)\right] f(\tau) d\tau \tag{8.49}$$

leading to the following mode displacement response :

$$q(t) = \sum_{i=1}^{n} \frac{x_{(i)}x'_{(i)}}{\mu_i \sqrt{1-\varepsilon_i^2}\,\omega_i} \int_0^t e^{-\varepsilon_i\omega_i(t-\tau)} \sin\left[\sqrt{1-\varepsilon_i^2}\,\omega_i(t-\tau)\right] f(\tau)d\tau \qquad (8.50)$$

Now, if damping is small, both inertia and damping forces may be neglected for higher modes when compared with the corresponding stiffness and applied forces. Thus, by analogy with the mode acceleration method for undamped systems, we may use here a mode velocity method and improve (8.50), by means of the quasi-static contribution of the unknown modes, into :

$$q(t) = \sum_{i=1}^{n} \frac{x_{(i)}x'_{(i)}}{\mu_i \sqrt{1-\varepsilon_i^2}\,\omega_i} \int_0^t e^{-\varepsilon_i\omega_i(t-\tau)} \sin\left[\sqrt{1-\varepsilon_i^2}\,\omega_i(t-\tau)\right] f(\tau)d\tau$$

$$+ \left[K^{-1} \sum_{i=1}^{n} \frac{x_{(i)}x'_{(i)}}{\omega_i^2\,\mu_i} \right] f(t) \qquad (8.51)$$

APPENDIX II

If a cubic curve is assigned to pass through four successive ordinates with abcissa step h, the following difference equation holds :

$$\ddot{q}_n = \frac{1}{h^2}\,(2q_n - 5q_{n-1} + 4q_{n-2} - q_{n-3}) \tag{8.52}$$

Setting (8.52) into basic equation (8.1) with time step $\Delta t = h$ yields the HOUBOLT's backwards difference algorithm (see reference)

$$(M + \frac{h^2}{2}\,K)q_n = \frac{h^2}{2}\,f_n + M\,(\frac{5}{2}\,q_{n-1} - 2q_{n-2} + \frac{1}{2}\,q_{n-3}) \tag{8.53}$$

which is unconditionally stable but subject to spurious damping.

Iteration must be started with a special procedure using the derivaties at the third of the four points :

$$\begin{aligned} \dot{q}_n &= \frac{1}{6h}\,(2q_{n+1} + 3q_n - 6q_{n-1} + q_{n-2}) \\ \ddot{q}_n &= \frac{1}{h^2}\,(q_{n+1} - 2q_n + q_{n-1}) \end{aligned} \tag{8.54}$$

Hence, with zero initial conditions :

$$\begin{aligned} q_0 &= 0 \\ \dot{q}_0 &= 0 \end{aligned} \tag{8.55}$$

we get the following starting procedure

$$\begin{aligned} q_1 &= \frac{h^2}{6}\;(M + \frac{h^2}{6}\,K)^{-1}(f_1 + 2f_o) \\ q_0 &= 0 \\ q_{-1} &= h^2\,M^{-1}\,f_o - q_1 \end{aligned} \tag{8.56}$$

APPENDIX III

When numerical integration of the normal equations of motion (8.18) is required, the following modally uncoupled precise integration operator, due to DUNHAM, NICKELL and STICKLER (see reference) can be successfully used :

$$\begin{bmatrix} \eta_i(t+\Delta t) \\ \dot{\eta}_i(t+\Delta t) \\ \ddot{\eta}_i(t+\Delta t) \end{bmatrix} = A(\omega_i,\Delta t) \cdot \begin{bmatrix} \eta_i(t) \\ \dot{\eta}_i(t) \\ \ddot{\eta}_i(t) \end{bmatrix} + g(\omega_i,\Delta t) \cdot \frac{x'_{(i)} f(t+\Delta t)}{\mu_i} \qquad (8.57)$$

with

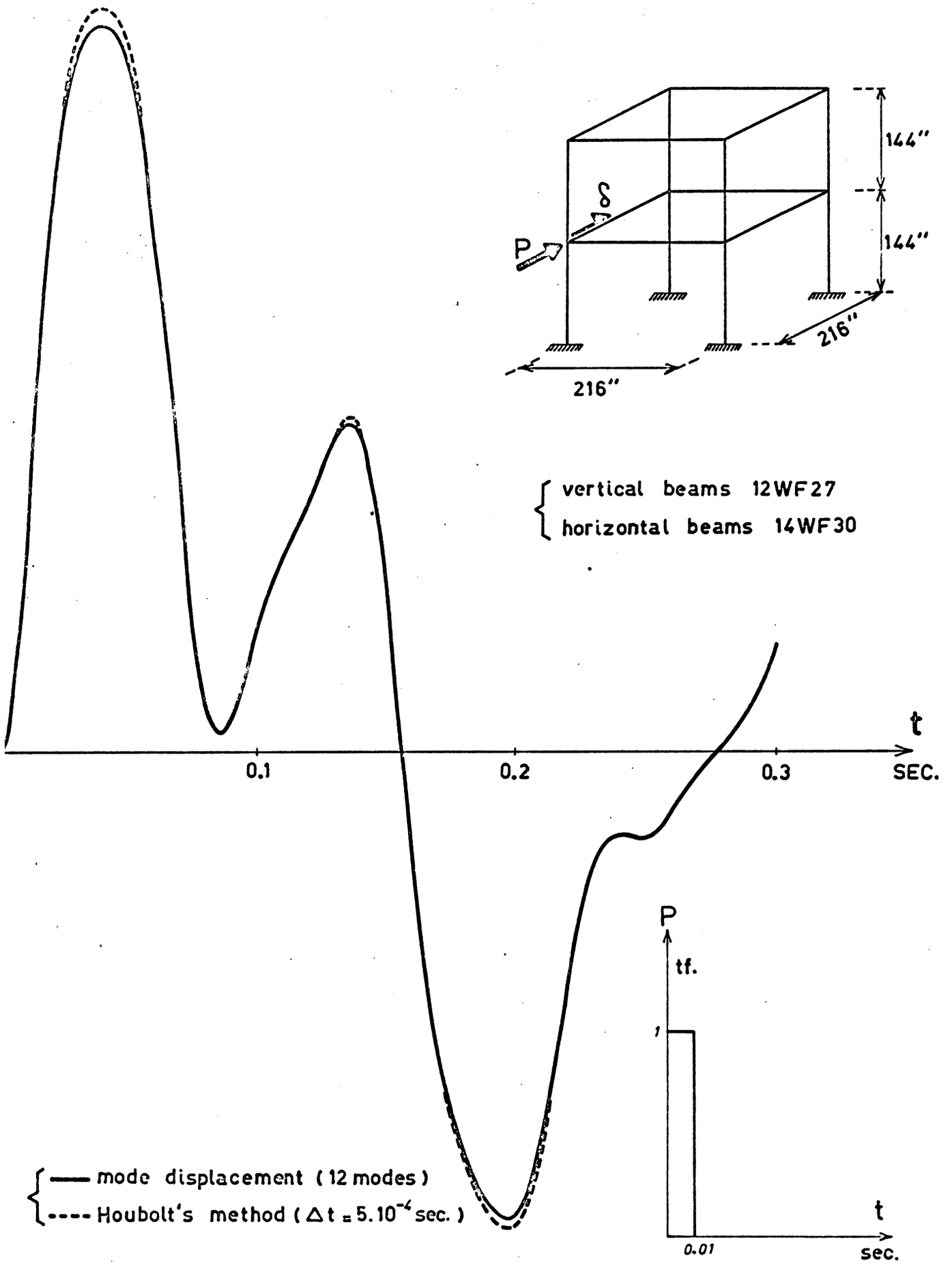
$$A(\omega_i,\Delta t) = \begin{bmatrix} \frac{\sin(\omega_i\Delta t)}{\omega_i\Delta t} & \frac{\sin(\omega_i\Delta t)}{\omega_i} & \frac{1}{\omega_i^2}\left[\frac{\sin(\omega_i\Delta t)}{\omega_i\Delta t} - \cos(\omega_i\Delta t)\right] \\ \frac{\cos(\omega_i\Delta t)-1}{\Delta t} & \cos(\omega_i\Delta t) & \frac{1}{\omega_i}\left[\sin(\omega_i\Delta t) + \frac{\cos(\omega_i\Delta t)-1}{\omega_i\Delta t}\right] \\ -\frac{\omega_i\sin(\omega_i\Delta t)}{\Delta t} & -\omega_i\sin(\omega_i\Delta t) & \cos(\omega_i\Delta t) - \frac{\sin(\omega_i\Delta t)}{\omega_i\Delta t} \end{bmatrix} \qquad (8.58)$$

and :

$$g(\omega_i,\Delta t) = \begin{bmatrix} \frac{1}{\omega_i^2}\left[1 - \frac{\sin(\omega_i\Delta t)}{\omega_i\Delta t}\right] \\ \frac{1}{\omega_i^2\Delta t}\left[1 - \cos(\omega_i\Delta t)\right] \\ \frac{\sin(\omega_i\Delta t)}{\omega_i\Delta t} \end{bmatrix} \qquad (8.59)$$

This operator is unconditionally stable with no spurious damping or error in vibrational period.

FIGURE 1



THREE DIMENSIONAL FRAME UNDER IMPULSE LOAD

TIME HISTORY OF THE LOADING POINT DEFLECTION

THREE DIMENSIONAL FRAME UNDER IMPULSE LOAD

TIME HISTORY OF THE MOMENT M

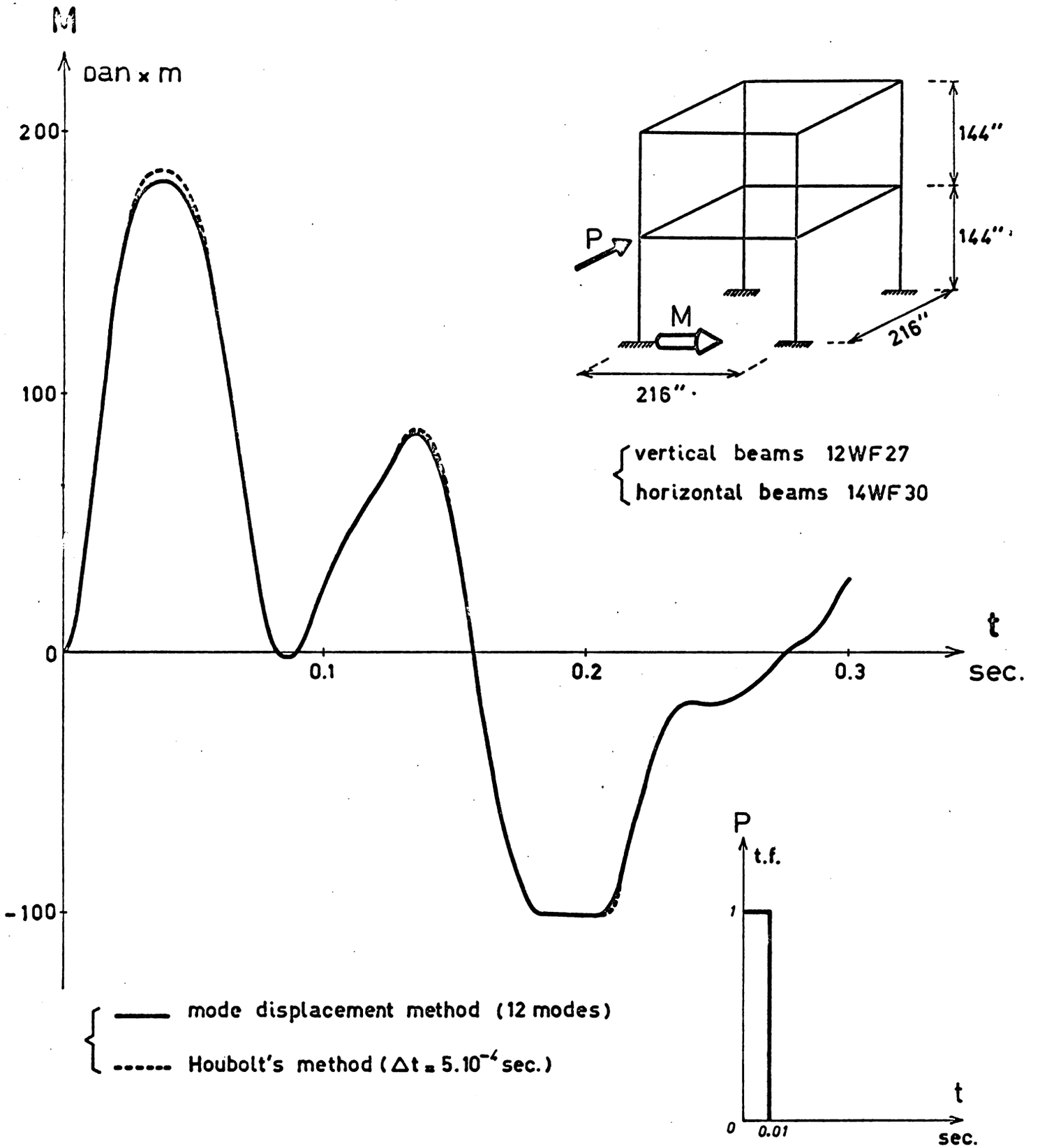


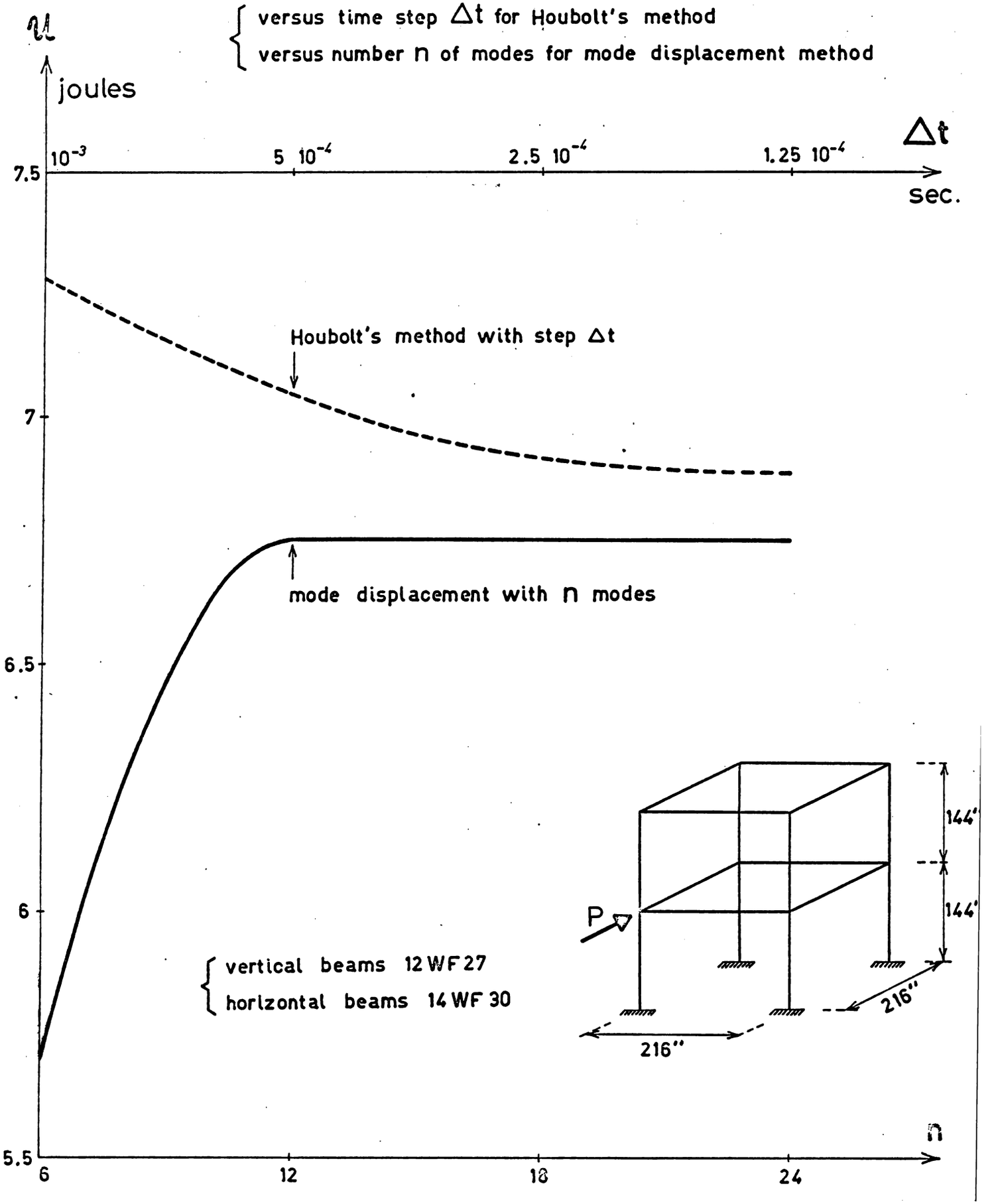
FIGURE 2

THREE DIMENSIONAL FRAME UNDER IMPULSE LOAD

CONVERGENCE OF THE MAXIMUM STRAIN ENERGY AT TIME $t = 0.035$ sec.

$\mathcal{U}$ versus time step Δt for Houbolt's method

versus number n of modes for mode displacement method



PINNED PINNED BEAM UNDER A LOAD TRAVELLING AT 2nd CRITICAL SPEED

TIME-HISTORY OF THE MID-SPAN DEFLECTION

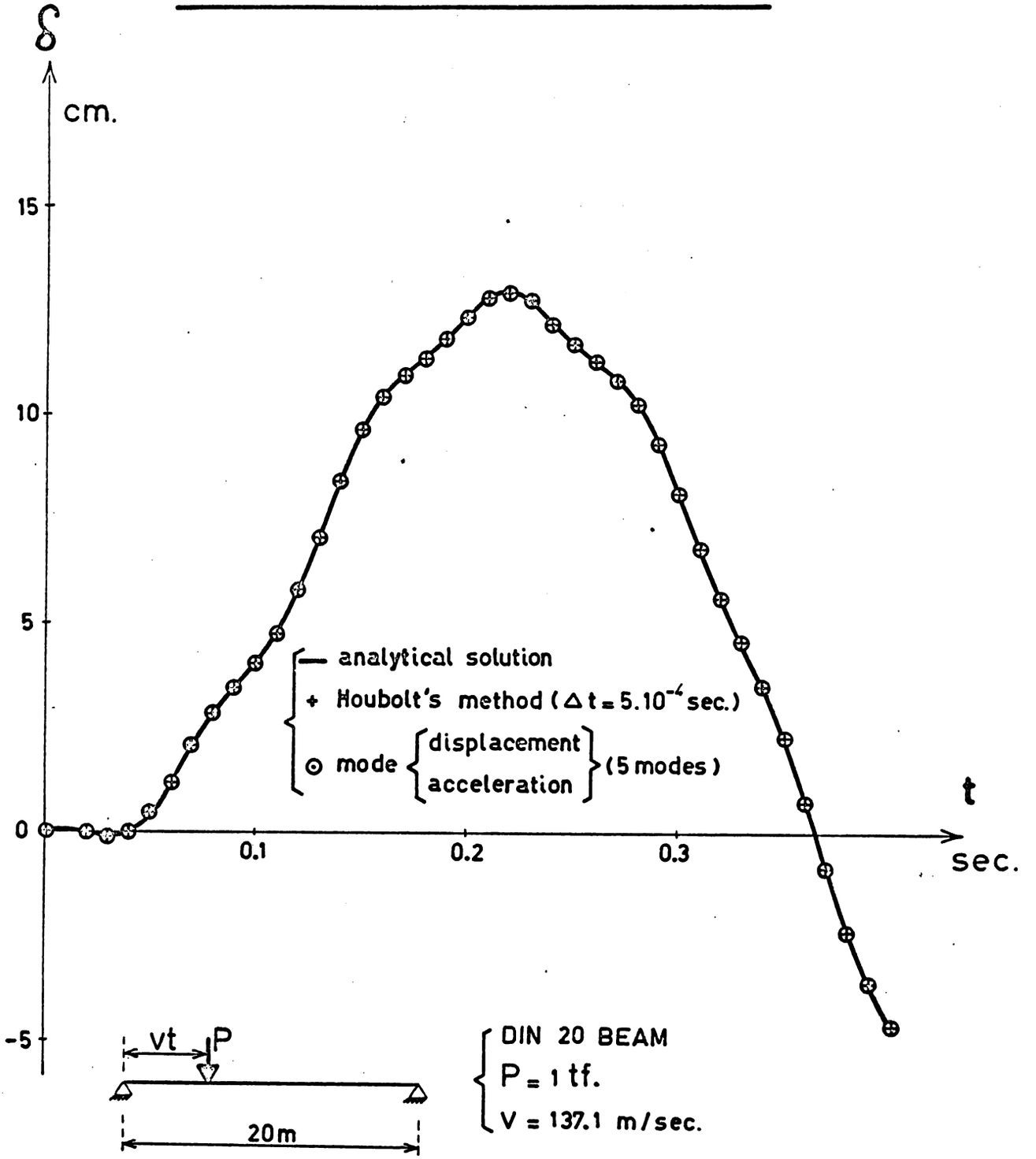
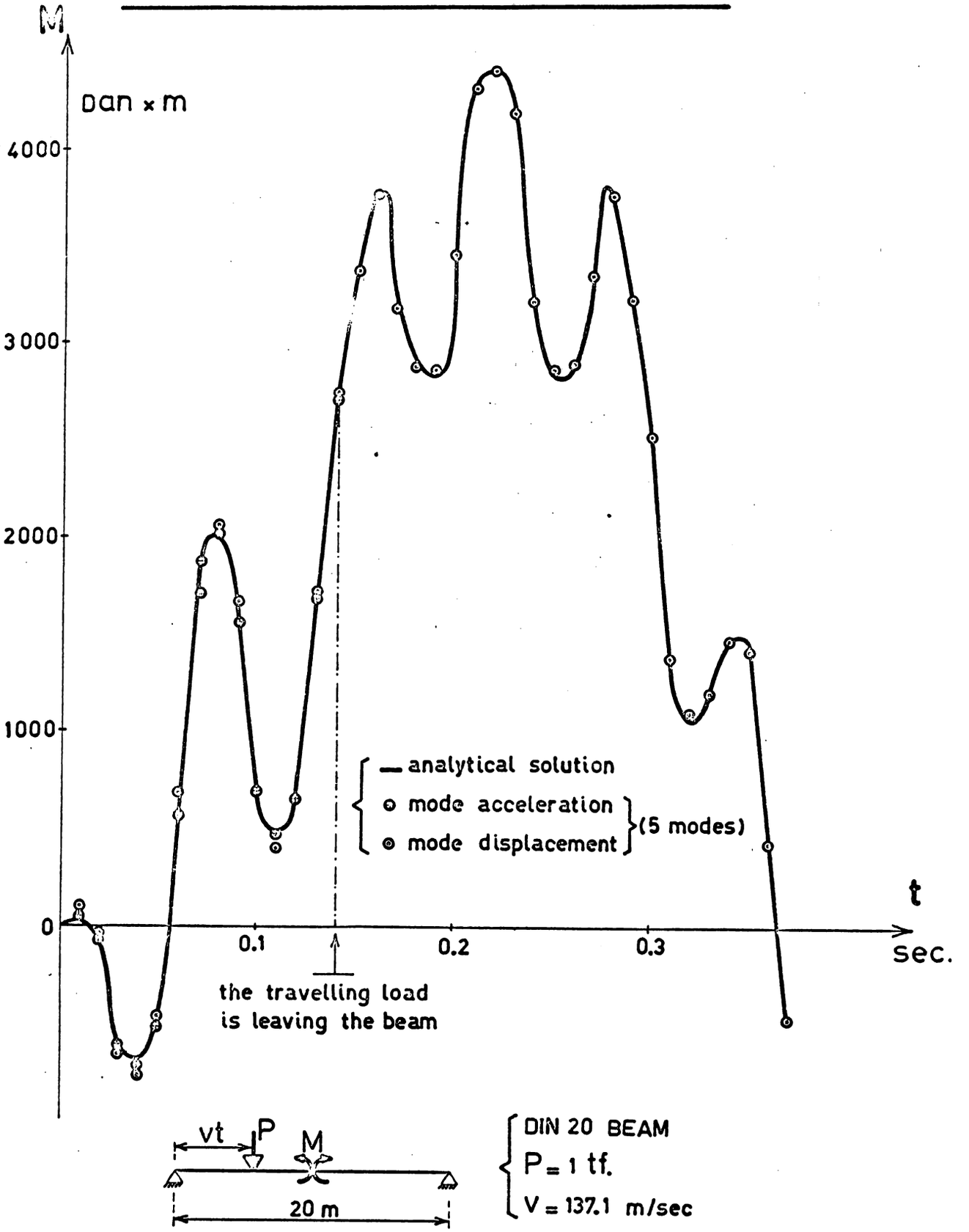


FIGURE 4

PINNED PINNED BEAM UNDER A LOAD TRAVELLING AT 2nd CRITICAL SPEED

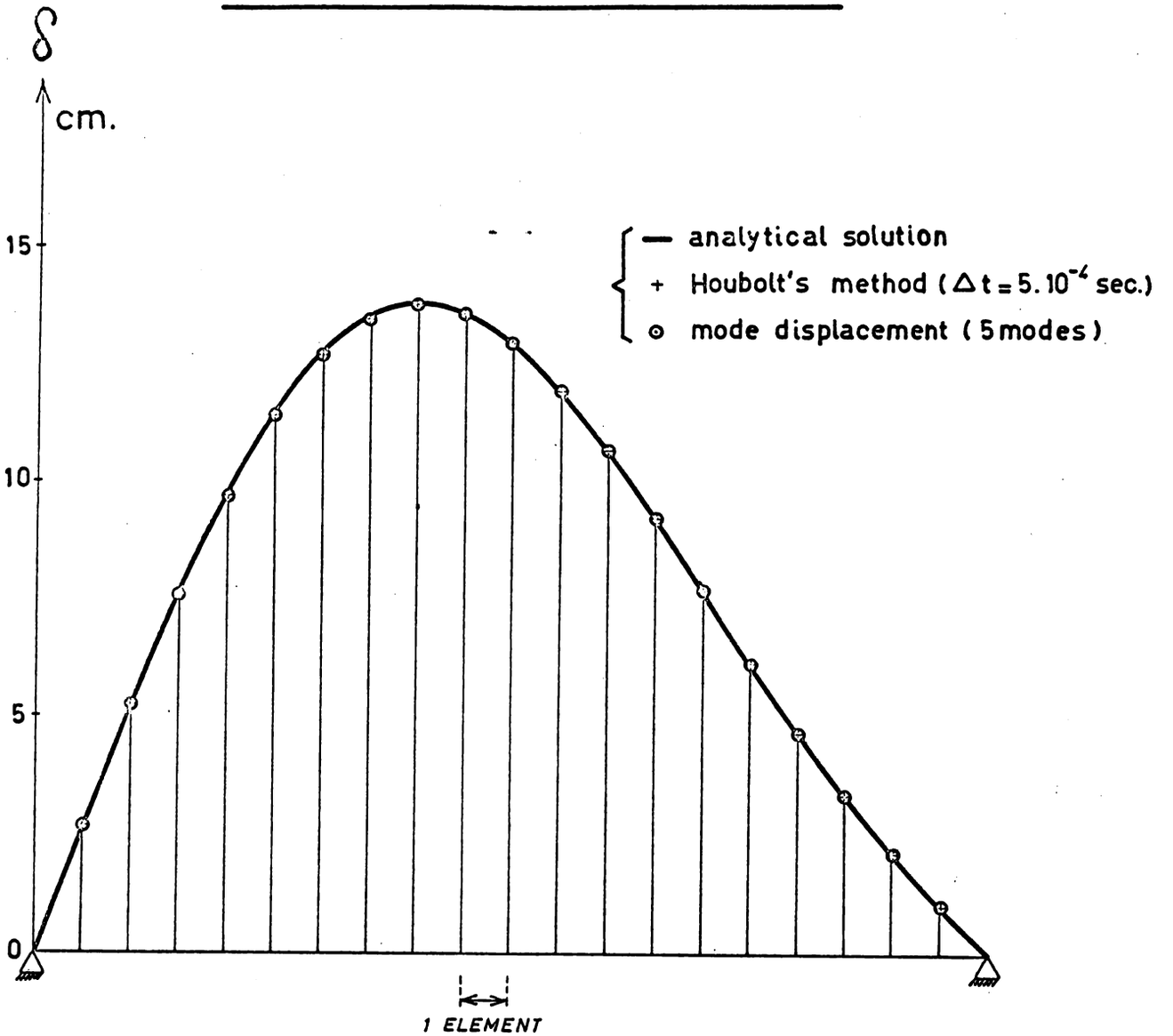
TIME-HISTORY OF THE MID-SPAN BENDING MOMENT

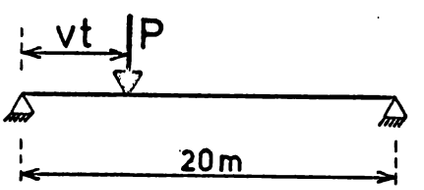


DIN 20 BEAM
P = 1 tf.
V = 137.1 m/sec

PINNED-PINNED BEAM UNDER A LOAD TRAVELLING AT 2nd CRITICAL SPEED

BEAM DEFLECTION AT TIME $t = 0.22$ sec.



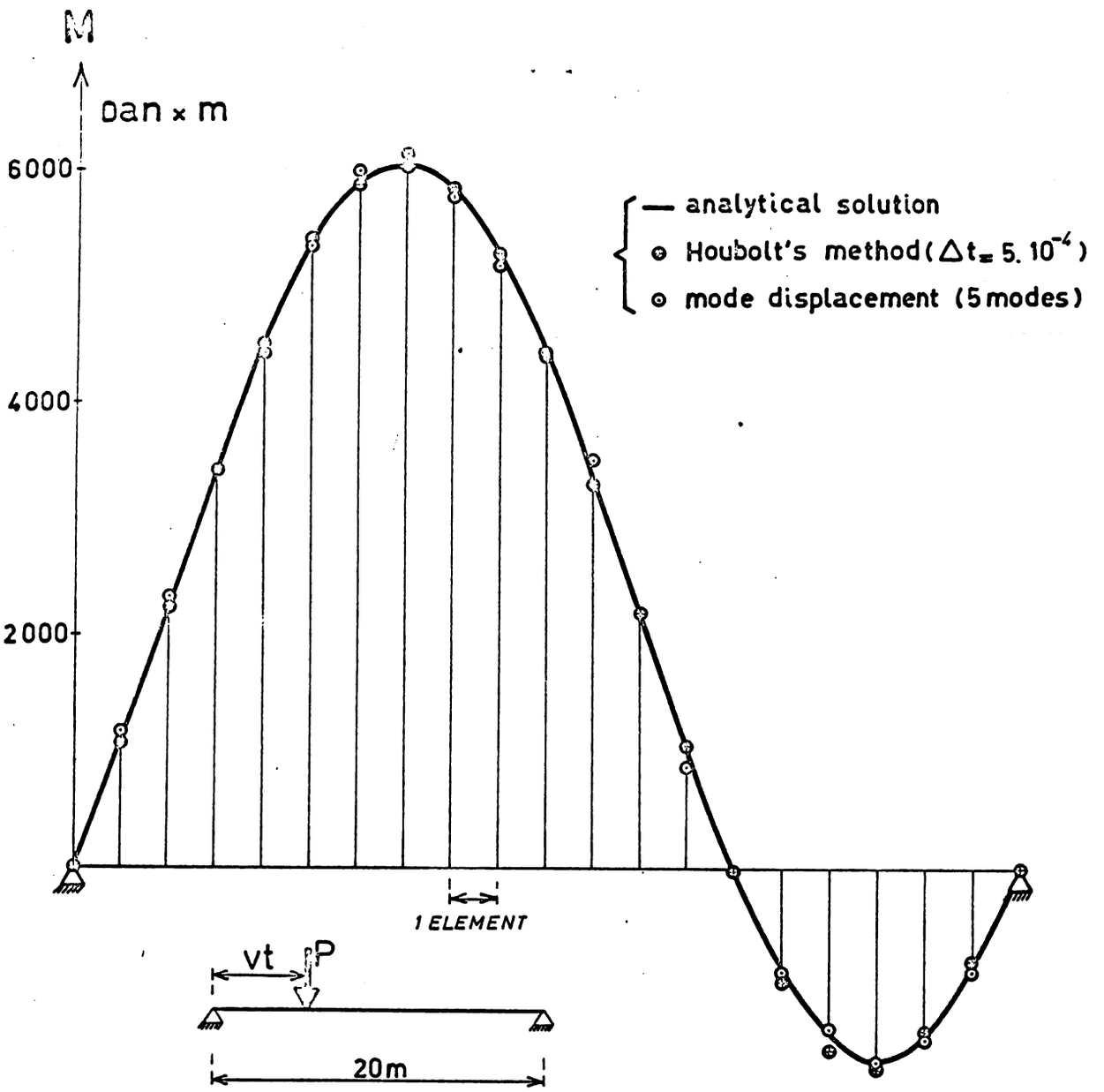


DIN 20 BEAM
$P = 1$ tf
$V = 137.1$ m/sec.

FIGURE 6

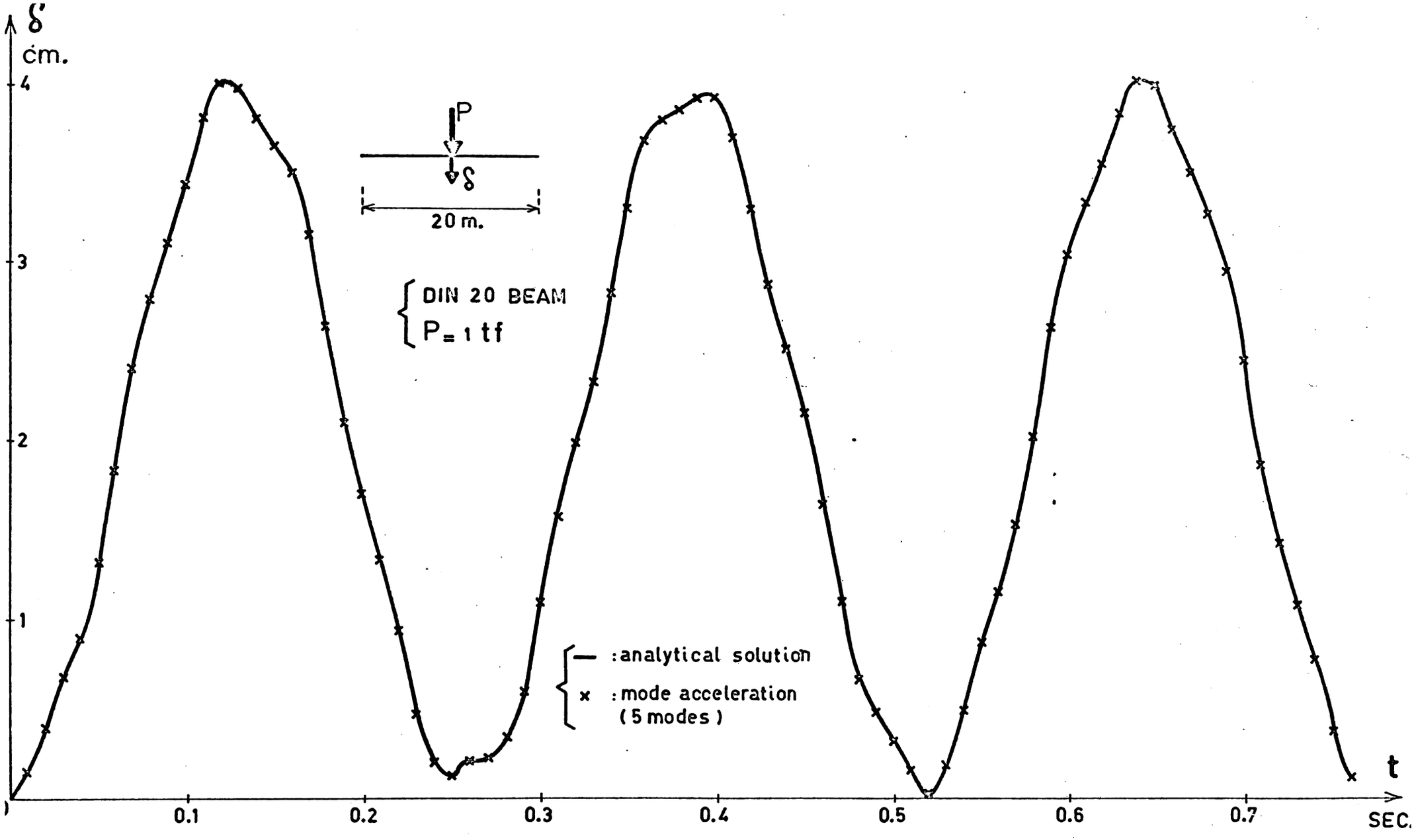
PINNED-PINNED BEAM UNDER A LOAD TRAVELLING AT 2nd CRITICAL SPEED

BENDING MOMENT DISTRIBUTION AT TIME $t = 0.22$ sec.



DIN 20 BEAM
$P = 1$ tf.
$V = 137.1$ m/sec.

FIGURE 7



FREE - FREE BEAM UNDER MID SPAN STEP LOADING

TIME HISTORY OF THE MID-SPAN DEFLECTION

FREE-FREE BEAM UNDER MID-SPAN STEP-LOADING

TIME HISTORY OF THE MID-SPAN BENDING MOMENT

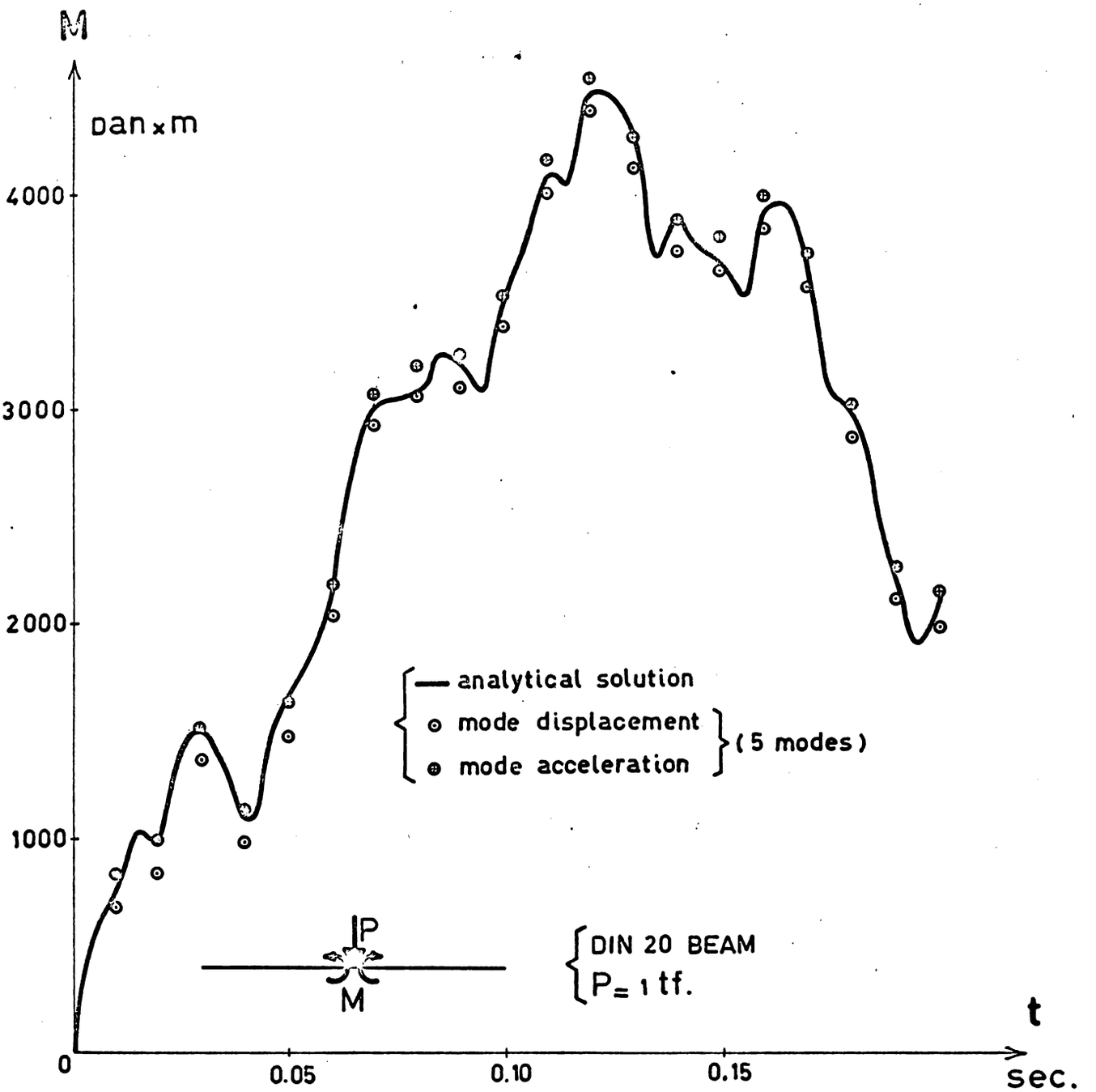
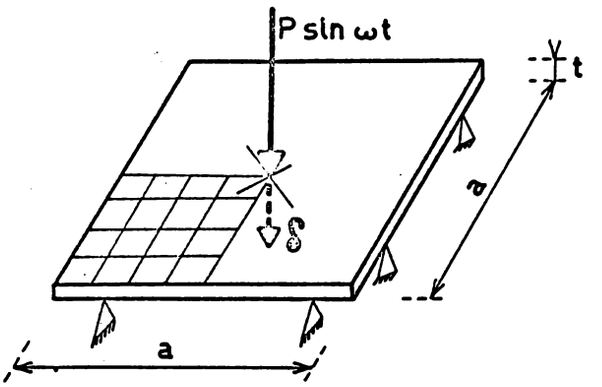


FIGURE 9

SIMPLY SUPPORTED SQUARE PLATE UNDER CONCENTRATED

HARMONIC EXCITATION AT THE MID-POINT

frequency history of the mid-point deflection amplitude
with 3% orthogonal viscous damping ratio



P = 250 kgf
a = 1 m.
t = 10 mm.
young's modulus: 7500 kgf/mm^2
poisson's ratio : 0.3
specific mass : 3000 kg/m^3

δ

mm.

100
50
10
5
1
0.5
0.1

⊙ analytical solution
\+ mode acceleration (11 modes)

ω

100 1000 10000 rad./sec.

FIGURE 10