NONLINEAR DYNAMICS OF FLEXIBLE BODIES

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ABSTRACT

The mean motion of a flexible body is usually taken to satisfy the TISSERAND conditions of zero relative momentum and angular momentum associated to a minimum of the relative kinetic energy.
The paper proposes a choice minimizing the mean square of relative displacements. It preserves the zero momentum condition but linearizes the angular momentum condition in such a way that the relative displacements are representable exactly by an expansion in natural elastic vibration modes.
HAMILTON'S principle is used to derive all equations of motion, including the mean one, by using the concept of quasi-coordinates. Gravitational potential and thrust vectors, as locally oriented by the body motion and deformation, are accounted for.
Distortions may be large provided strains remain small.
Keywords : Flexible bodies, Nonlinear dynamics, Mean motion.

1. INERTIAL and DYNAMIC REFERENCE FRAMES

A material point of the body has a position vector $\vec{y}$ in a reference configuration C_o and $\vec{w}$ in the actual configuration C at time t, as measured from the origin G_o of an inertial reference frame.
If the body does not deform, we may think of the dynamic axes as frozen in the body and carried by its motion. If it does deform we can still define dynamic axes by some useful property that would result for the relative displacements $\vec{u}$. Figure 1 illustrates the vectorial relationship between the displacement vector $\vec{z}$ in the inertial system and $\vec{u}$

$$\vec{w} = \vec{y} + \vec{z} = \vec{a} + \vec{r} = \vec{a} + \vec{x} + \vec{u} \qquad (1)$$

The relative position of the two reference frames is fully described by the position vector $\vec{a}(t)$ of the origin of dynamic axes and a matrix rotation operator U(t)

$$U\,U^T = U^T U = I \text{ identity matrix} \qquad (2)$$
$$\det U = 1$$

If for an arbitrary vector $\vec{v}$, we denote the column matrix of its inertial components by $\hat{v}$ and of its dynamic components by v, we have by definition of the rotation operator,

$$v = U\,\hat{v} \qquad \hat{v} = U^T v \qquad (3)$$

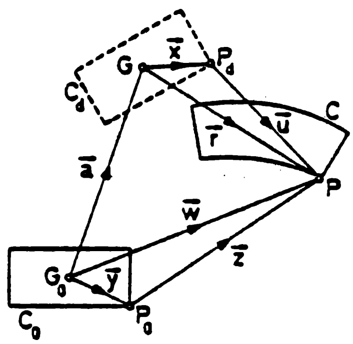

FIG. 1

The dynamic reference configuration C_d is the reference configuration C_o carried by the change of axes and a position vector $\vec{y}$ is carried into the position vector $\vec{x}$ so that

$$\hat{y} = x \qquad (4)$$

From (1)

$$\hat{w} = \hat{a} + \hat{x} + \hat{u} = \hat{a} + U^T(x + u)$$

or, premultiplying by U,

$$x + u = U\,(\hat{w} - \hat{a}) \qquad (5)$$

As $\vec{x}$ and $\vec{u}$ are normally observed in the dynamic frame, while $\hat{w}$ and $\hat{a}$ are data observed in the inertial frame, (5) is the most useful form of relation (1).

u is a function $u(x,t)$, $\hat{y}$ a function $\hat{y}(\bar{y},t)$, and, as already seen $\hat{a}$ and U depend only on time.

2. MEASURE OF STRAIN and STRAIN ENERGY

Denote by

$$A = \{\frac{\partial u_m}{\partial x_n}\} \qquad (6)$$

the matrix of gradients of relative displacements in the dynamic frame. An exact measure of strain is provided by the GREEN tensor

$$\Gamma = \{\gamma_{mn}\} = \frac{1}{2}(A + A^T + A^T A) \qquad (7)$$

or, explicitly,

$$\gamma_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_n}\right) \qquad (8)$$

Those quantities will be assumed to be small and the strain energy to be

$$W = \frac{1}{2}\int C^{pq}_{mn}\, \gamma_{mn}\, \gamma_{pq}\, \frac{d\mu}{\rho} \qquad (9)$$

where C^{pq}_{mn} are the elastic moduli, ρ the mass per unit volume and $d\mu$ an elementary mass of the body. If in addition to small strains, the body deforms only slightly (this may not be the case for thin elongated parts) the major part of the local material rotations experienced between the configurations C_o and C can be absorbed by the rotation of the dynamic axes. The relative material rotations are then small and the measure of strain may be linearized by dropping the last term in (8).

3. ABSOLUTE VELOCITY and KINETIC ENERGY

Differentiation of (5) with respect to time produces

$$\frac{\partial u}{\partial t} = \dot{U}(\hat{y}-\hat{a}) + U\left(\frac{\partial \hat{y}}{\partial t} - \frac{d\hat{a}}{dt}\right)$$

Substituting $\hat{y} - \hat{a} = U^T(x + u)$ and noting that

$$\hat{v}_a = \frac{\partial \hat{y}}{\partial t} \qquad \hat{v}_g = \frac{d\hat{a}}{dt}$$

are respectively the absolute velocity of a material point and the velocity of the origin of dynamic axes

$$v_a = v_g + [\omega](x + u) + \frac{\partial u}{\partial t} \qquad (10)$$

where

$$[\omega] = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = -\dot{U}U^T = U\dot{U}^T \qquad (11)$$

is a skew symmetric matrix built up, as the notation suggests, from the dynamic components of the pseudo-vector angular velocity ω :

$$\omega^T = (\omega_1\ \omega_2\ \omega_3)$$

The skew symmetric character, explicitit in the last equality of (11), is immediatly verified by time differentiation of (2). The second term on the right hand side of (10) is the vector of dynamic components of a vector product $\vec{\omega} \times \vec{r}$. It may thus be written indifferently

$$[\omega](x + u) = -[x + u]\,\omega$$

From (10), the kinetic energy of the body can be decomposed as follows

$$\begin{aligned} T &= \frac{1}{2}\int v_a^T v_a\, d\mu \\ &= \frac{1}{2} v_g^T v_g \int d\mu + v_g^T \int \frac{\partial u}{\partial t} d\mu + v_g^T [\omega] \int (x + u) d\mu \\ &+ \frac{1}{2}\int \frac{\partial u^T}{\partial t}\frac{\partial u}{\partial t} d\mu + \omega^T \int [x+u] \frac{\partial u}{\partial t}\, d\mu + \frac{1}{2}\omega^T R\, \omega \end{aligned} \qquad (12)$$

where

$$R = \int [x + u]\, [x + u]^T\, d\mu \qquad (13)$$

is the matrix of inertia moments of the body about the dynamic axes in its true configuration. We recognize in the 3 last terms of (12) the relative kinetic energy, the contribution of gyroscopic and centrigugal forces. Moreover

$\int \frac{\partial u}{\partial t} d\mu$ is the vector of relative momentum

$\int [x + u] \frac{\partial u}{\partial t} d\mu$ the vector of relative angular momentum.

4. VIRTUAL WORK and QUASI COORDINATES

The decomposition (10) of the absolute velocity must, by analogy with virtual velocity, give birth to an analogous decomposition of virtual displacement.
From (5) again

$$\delta u = \delta U(\hat{y}-\hat{a}) + U(\delta\hat{y} - \delta\hat{a})$$

$$\hat{y} - \hat{a} = U^T(x+u)$$

$U\,\delta\hat{y} = \delta h$ absolute virtual displacement

$U\,\delta\hat{a} = \delta p$ vitual displacement of the origin of dynamic axes

both in dynamic components. Hence

$$\delta h = \delta p + [\delta\alpha](x+u) + \delta u \qquad (14)$$

where

$$[\delta\alpha] = \begin{pmatrix} 0 & \delta\alpha_3 & -\delta\alpha_2 \\ -\delta\alpha_3 & 0 & \delta\alpha_1 \\ \delta\alpha_2 & -\delta\alpha_1 & 0 \end{pmatrix} = -\delta U.U^T = U.\,\delta U^T \qquad (15)$$

The elements of δp are also known as the quasi-coordinates of translation and those of $\delta\alpha$, the quasi-coordinates of rotation.

Let us consider a perturbed rotation as composed of the unperturbed one followed by a small additional rotation, that will not differ much from the identity

$$U + \delta U = (I - [\delta\alpha]\ )\ U$$

this is equivalent to

$$\delta U = - [\delta\alpha]\ \ U$$

or the definition (15). The elements of $\delta\alpha$ are the small angles of rotation of the dynamic frame about its own axes, when passing from the unperturbed to the perturbed position. This agrees with the definitions of the elements of ω , the angular velocities of the dynamic frame about its own axes.
The virtual work of a force $\vec{f}$ applied at a material point is thus, as seen in the dynamic frame

$$f^T \delta h = f^T(\delta p + [\delta\alpha]\ (x+u) + \delta u) \qquad (16)$$

In the application of HAMILTON'S principle, the variation of the kinetic energy involves the computation of

$$\delta v_g = \delta(U \frac{d\hat{a}}{dt}) = \delta U \frac{d\hat{a}}{dt} + U \frac{d}{dt}\ \delta\hat{a}$$
$$= \delta U . U^T v_g + U \frac{d}{dt}\ (U^T\ \delta p)$$

or finally, after expanding the last term,

$$\delta v_g = - [\delta\alpha]\ v_g + [\omega]\ \delta p + \frac{d}{dt}\ \delta p \qquad (17)$$

It involves also the computation of $\delta\omega$. Starting from

$$\delta\hat{\omega} = \frac{d}{dt}\ \ \delta\hat{a} = \frac{d}{dt}\ (U^T\ \delta\alpha) = \dot{U}^T \delta\alpha + U^T \frac{d}{dt}\ \delta\alpha$$

and premultiplying by U, there follows

$$\delta\omega = [\omega]\ \delta\alpha + \frac{d}{dt}\ \delta\alpha \qquad (18)$$

The variational derivatives associated to δp and $\delta\alpha$ will be respectively the equations of mean translation and mean rotation about the dynamic axes.

5. MINIMUM KINETIC ENERGY IN RELATIVE MOTION

Assuming the absolute velocity of particles to be given, analyze such motions of dynamical axes that minimize

$$\frac{1}{2} \int (\frac{\partial u}{\partial t})^T \frac{\partial u}{\partial t}\ d\mu \qquad \min$$

The minimum requires that

$$\int (\frac{\partial u}{\partial t})^T\ \Delta\ \frac{\partial u}{\partial t}\ d\mu = 0$$

for all changes in relative velocity due only to changes in the motion of dynamical axes; the position of the axes themselves may be kept unperturbed at the epoch of comparison. Hence we take $\Delta u = 0$ and $\Delta U = 0$, from which follows

$$\Delta v_a = \Delta(U\ \hat{v}_a) = 0$$

From (10) follows then

$$\Delta \frac{\partial u}{\partial t} = - \Delta v_g - [\Delta\omega] \qquad (x+u)$$
$$= - \Delta v_g + [x+u]\ \Delta\omega$$

This, replaced in the minimizing condition gives, in view of the arbitrariness of Δv_g and $\Delta\omega$

$$\int \frac{\partial u}{\partial t}\ d\mu = 0 \qquad (19)$$

the condition of zero relative momentum, and

$$\int [x + u]\ \frac{\partial u}{\partial t}\ d\mu = 0 \qquad (20)$$

the condition of zero relative angular momentum. Axes which move in such manner that these conditions are satisfied are TISSERAND axes. While (19) is linear, (20) has the disadvantage of being non linear in the relative displacements.
On the other hand they simplify markedly the formula (12) for the global kinetic energy.

6. LEAST SQUARES OF RELATIVE DISPLACEMENTS

A choice of dynamic axes minimizing at all times the functional

$$\frac{1}{2} \int u^T u\ d\mu \qquad \min \qquad (21)$$

will now be investigated. The minimizing condition

$$\int u^T\ \delta u\ \ d\mu\ = 0$$

must hold for all variations due to a change in position of the dynamic axes. Such variations can be taken directly from (14), setting $\delta h = 0$, since $\hat{v}$ is a given field and $\delta\hat{v} = 0$.
Thus

$$\delta u = - [\delta\alpha](x + u) - \delta p = [x + u]\delta\alpha - \delta p$$

The variations δp and $\delta\alpha$ being free and independent, the minimizing conditions become, noting that $[u]\, u = 0$,

$$\int u\ d\mu = 0 \qquad (22)$$

$$\int [x + u\ ] u\ d\mu = \int [x]\ u\ d\mu\ = 0 \qquad (23)$$

For the inertial observer, condition (22) implies

$$U \int (\hat{v} - \hat{a})\ d\mu = \int x\ d\mu\ = 0$$

if the origin of axes in the reference configuration is taken at the center of mass. This equation yields then the explicit position vector of the origin of dynamic axes, given the actual configuration of the body

$$\hat{a} \int d\mu = \int \hat{v}\ d\mu \qquad (24)$$

Similarly, when (23) is expressed in terms of observables in the inertial system, the consequence

$$\int [x] \; d\mu = 0 \qquad (25)$$

of our choice of the origin and the fact that $[x] \; x = 0$, reduce it to the following equation for the computation of the rotation operator

$$\int [x] \; U \, \hat{\omega} \, d\mu = 0 \qquad (26)$$

To solve it we introduce a RODRIGUES-HAMILTON representation of the rotation operator

$$U = I + 2\beta \; [b] + 2 \; [b] \; [b] \qquad (27)$$

and recall the significance of the vector b and the parameter β.

1. As $[b] \; b = 0$ we observe that $Ub = b$ and this vector, invariant under the operator must be aligned with the axis of rotation;

2. Any vector u perpendicular to b remains perpendicular after transformation by the operator :

$$b^T Uu = b^T u + 2\beta \, b^T \, [b] + 2 \, b^T \, [b] \; [b] = 0$$

if $b^T u = 0$, as also $b^T [b] = 0$.

3. Any vector perpendicular to b must be rotated by the same angle θ; we must have

$$u^T Uu = u^T u \cos\theta$$

The left-hand side is computed using the properties

$$u^T \, [b] \, u = 0 \qquad [b] \; [b] = - b^T b \, I + bb^T$$

and compared to the right, giving

$$1 - 2 \, b^T b = \cos\theta \qquad \text{or} \qquad b^T b = \sin^2 \frac{\theta}{2} \qquad (28)$$

4. Finally the condition $U^T U = I$ imposes the constraint

$$\beta^2 + b^T b - 1 = 0 \qquad (29)$$

from which follows

$$\beta^2 = \cos^2 \frac{\theta}{2} \qquad (30)$$

The substitution of (27) into (26) leads to an algebraic equation in b and β that is rather difficult to manipulate.

The situation is much clearer if we begin by translating the functional (21) in terms of the unknowns and the data

$$\frac{1}{2} \int (U(\hat{\omega} - \hat{a}) - x)^T \; (U(\hat{\omega} - \hat{a}) - x) d\mu \quad \min$$

Expanding the integral, taking (25) into account, and keeping only the terms that depend on the unknowns $\hat{a}$ and U :

$$- 2\hat{a}^T \int \hat{\omega} \, d\mu + \hat{a}^T \hat{a} \int d\mu - 2 \int x^T \, U \, \hat{\omega} \, d\mu \quad \min.$$

The minimization with respect to $\hat{a}$ gives (24) again and we are left to determine U with the problem

$$\phi = - \int x^T U \, \hat{\omega} \, d\mu \qquad \text{minimum} \qquad (31)$$

We now introduce the representation (27) that, after some commutations of factors in vector products gives to ϕ the structure of a second degree form in the unknowns b and β

$$\phi = - \int x^T \, \hat{\omega} \, d\mu + 2\beta \, b^T \hat{m} + b^T \, \hat{M} \, b$$

$$\hat{m} = \int [x] \; \hat{\omega} \, d\mu \qquad (32)$$

$$\hat{M} = \int (\, [x] \; [\hat{\omega}]^T + [\hat{\omega}] \; [x]^T \,) \, d\mu \qquad (33)$$

The minimizing conditions obtained by equating to zero the partial derivatives of the augmented functional

$$\psi = \phi - \lambda \, (\beta^2 + b^T b - 1)$$

can be presented in the form of a self-adjoint eigenvalue problem

$$\begin{pmatrix} \hat{M} & \hat{m} \\ \hat{m}^T & 0 \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} b \\ \beta \end{pmatrix} \qquad (34)$$

For any eigenvalue λ and associated eigenvector , normed by the constraint (29), there follows from the minimizing equations that

$$\phi = - \int x^T \, \hat{\omega} \, d\mu + \lambda$$

Hence the minimum of the functional is associated to the smallest eigenvalue of problem (34). As there are always 4 real eigenvalues and corresponding eigenvectors the problem of the unicity of the minimizing solution arises.

6. DEGENERACY OF THE LEAST SQUARES PROBLEM

Assume the smallest eigenvalue and one associated eigenvector of problem (34) to be determined by one of the standard algorithms. Consider a new set of inertial axes obtained by the corresponding rotation operator U (so that the optimal dynamic axes will coincide in orientation with the inertial axes at the epoch considered). Denote by $w = U^{-1}\hat{\omega} = a + x + u$ the absolute displacement in the new inertial axes. Then, the other eigensolutions will be investigated as solutions of

$$- \int x^T \, Vw \, d\mu \qquad \text{stationary}$$

where V is the rotation operator exploring the new orientations of dynamic axes with respect to the optimal ones.

(VU is of course the rotation operator with respect to the original inertial axes). The eigenvalue problem becomes

$$\begin{pmatrix} M & m \\ m^T & 0 \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} b \\ \beta \end{pmatrix} \qquad (35)$$

$$m = \int [x] \quad w \, d\mu = \int [x] u \, d\mu \qquad (36)$$

$$M = \int ([x] \; [x+u]^T + [x+u][x]^T) \, d\mu \qquad (37)$$

with the advantage that one eigensolution is known

$$V = I \rightarrow b = 0 \quad \text{and} \quad \beta = 1.$$

Substitution of this solution in (35) shows that

1. $m = 0$, already known as minimizing property (23)

2. $\lambda = 0$, the smallest eigenvalue has been shifted to zero. This has the effect of splitting the eigenvalue problem in two parts

$$Mb = \lambda \, b \qquad (38)$$

$$\lambda\beta = 0 \qquad (39)$$

If the 3 other eigenvalues are strictly positive, M is positive definite and conversely. The minimum is unique. Matrix M can be put in the form

$$M = 2 \left\{ \int [x+\tfrac{u}{2}][x+\tfrac{u}{2}]^T \, d\mu - \int [\tfrac{u}{2}][\tfrac{u}{2}]^T \, d\mu \right\}$$

of a difference of moments of inertia, the first associated to a sort of half-way configuration, the second to a configuration where the position vectors are reduced to half the displacements of the masses.

Obviously M will be positive definite if the optimal relative displacements are not unduly large.

Moreover (39) shows that the other eigenvectors have their β component zero: they are 180° rotations about axes that will be perpendicular, since the eigenvectors b of (38) will be orthogonal.

The identity operator V=I and the three 180° rotation V_1, V_2, V_3 form an Abelian group.

Assume now $\lambda = 0$ to be a double root. It implies that a unit vector n will exist, such that

$$Mn = 0 \qquad n^T n = 1$$

n is parallel to the axis of the rotation V and, since $\lambda = 0$, β may be taken arbitrarily (between -1 and +1). It is easily verified that ϕ remains minimum under the one parameter (θ) family of rotations

$$b = n \sin\frac{\theta}{2} \qquad \beta = \cos\frac{\theta}{2}$$

$$V = I + (nn^T - I)(1 - \cos\theta) + \sin\theta \, [n]$$

7. AN EXAMPLE OF DEGENERACY

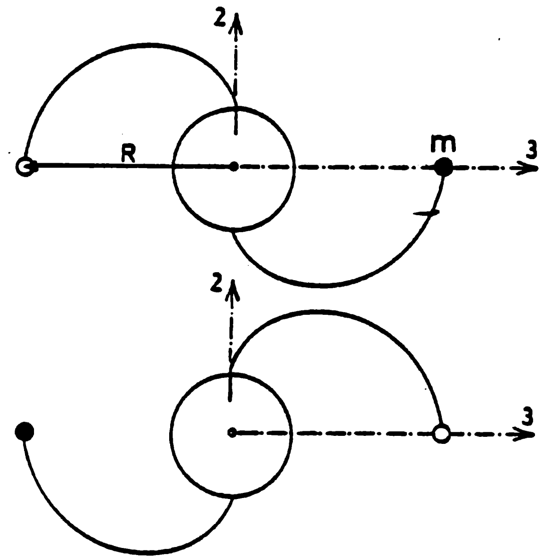

FIG. 2

The example, illustrated on figure 2, shows the possibility of non uniqueness of the minimizing choice when large elastic displacements are involved. A satellite with a rigid central axisymmetric body has two flexible massless appendages terminated by concentrated masses. In its reference configuration the matrix of moments of inertia is

$$\begin{pmatrix} A + 2 m R^2 & 0 & 0 \\ 0 & B + 2 m R^2 & 0 \\ 0 & 0 & B \end{pmatrix}$$

Suppose that in the deformed configuration the central body has not moved but that the masses of the appendages have been interchanged. We find from (32) and (33)

$$\hat{m} = 0 \qquad \hat{M} = 2 \begin{pmatrix} A - 2 m R^2 & 0 & 0 \\ 0 & B - 2 m R^2 & 0 \\ 0 & 0 & B \end{pmatrix}$$

The eigenvalues of problem (34) are :

$\lambda_0 = 0$ (associated to $b = 0$ $\beta = 1$ and the identity operator)

$$\lambda_1 = 2(A - 2mR^2) \qquad \lambda_2 = 2(B - 2mR^2) \qquad \lambda_3 = 2B$$

Take the case $B > A$, so that λ_1 is the smallest of the last three.
If $2mR^2 < A$ the smallest eigenvalue is λ_o and is unique.
The original axes are also the ones that minimize the functional. The rotations V_1, V_2 and V_3 are of 180° respectively about the first, second and third cartesian axis; V_3 maximizes the functional.
If $2mR^2 > A$, the smallest eigenvalue is λ_1 and is again unique. The rotation V_1 of 180° about the axis of symmetry of the central body brings the displacements of the concentrated masses of the appendages back to zero, but displaces all the masses of the central body. This operation achieves the relative configuration minimizing the functional.
In the limiting case $2mR^2 = A$, $\lambda_o = \lambda_1 = 0$ and the smallest eigenvalue is a doublet. The functional remains invariant and minimum under rotations of arbitrary amplitude about the axis of symmetry.

8. DISCRETIZATION OF BODY FLEXIBILITY

The time derivative of (22) coincides with the zero relative momentum TISSERAND condition (19). But the time derivative of (23) is a linearized form of the zero angular momentum condition (20) and is easier to apply to the discretization of body flexibility. The discretization by means of a modal analysis is classical in the case of small deformations [1,3]. In the case of large deformations it is simple to apply in relation to the choice of axes suggested in section 6. With the advent of powerful computer programs for modal analysis and accurate testing methods [4,5], it is also practical.
A modal analysis of the small amplitude vibrations of the free body produces the following expansion for small relative displacements, in which the summation convention on repeated indices is used and e_{ipn} denotes the alternating tensor

$$u_i(x,t) = v_i(t) + e_{ipn}\, \alpha_p(t)\, x_n + q_\beta(t)\, f_i^\beta(x)$$

The unknowns are the rigid body translation amplitudes $v_i(t)$, the small rigid body rotation amplitudes $\alpha_p(t)$ and a denumerable set of vibration amplitudes $q_\beta(t)$.
As the set of functions describing the displacement field is complete, the expansion can be used even for large relative displacements, in which case large $\alpha_p(t)$ terms induce strains because of the non linear terms in the exact strain measures (6). However, the minimizing conditions (22) and (23) are precisely satisfied by keeping the q_β terms alone.
Indeed the modal function $f_i^\beta(x)$ have the properties

$$\int f_i^\beta\, d\mu = 0 \qquad (i = 1,2,3) \text{ all } \beta \qquad (40)$$

$$e_{jqi} \int x_q\, f_i^\beta\, d\mu = 0 \qquad (j = 1,2,3) \text{ all } \beta \qquad (41)$$

expressing their inertial orthogonality with respect to the small rigid body modes (which are natural modes of zero frequency).
Thus, considering that the origin of the reference configuration is at the center of mass,

$$\int x_n\, d\mu = 0 \qquad (n = 1,2,3)$$

the minimizing conditions (22)

$$\int u_i\,(x,t)\, d\mu = 0 \qquad (i = 1,2,3)$$

reduce to

$$v_i(t) \int d\mu = 0 \qquad (i = 1,2,3)$$

and are satisfied by setting $v_i(t) = 0$. The minimizing conditions (23)

$$e_{jqi} \int x_q\, u_i\, d\mu = 0 \quad (j = 1,2,3)$$

reduce to

$$\alpha_p(t)\, e_{ipn}\, e_{jqi} \int x_n x_q\, d\mu = \alpha_p(t) \int (x_n x_n \delta_{jp} - x_j x_p)\, d\mu = 0$$

The matrix of integrals is that of the inertia moments of the body in its reference configuration. It is positive definite, and the conditions can only be satisfied by taking $\alpha_p(t) = 0$.
Thus, an expansion of relative displacements limited to the natural modes of non zero frequency

$$u_i(x,t) = q_\beta\,(t)\, f_i^\beta\,(x) \qquad (42)$$

satisfies automatically the principle of minimum square average of relative displacements and the equations of mean motion will be those associated to the corresponding choice of mean axes.
Another advantage of the discretization in natural modes is of course the existence of the orthogonality properties

$$\int f_i^\beta\, f_i^\gamma\, d\mu = \delta_\gamma^\beta \int d\mu \qquad (43)$$

the squared norm of a mode, or generalized mass, being here conventionally equated to the total mass of the body, and then, with $D_m = \partial/\partial x_m$,

$$\int C_{mn}^{pq}\, D_m\, f_n^\beta\, D_p\, f_q^\gamma\, \frac{d\mu}{\rho} = 0 \qquad \text{if } \gamma \neq \beta$$
$$= \lambda_{(\beta)}^2 \int d\mu \text{ if } \gamma = \beta \qquad (44)$$

The natural circular frequencies $\lambda_{(\beta)}$ are assumed to be ordered by increasing values. In practice, as interest is primarily centered on the low frequency response of the body, the expansion is truncated, leaving only a finite number of degrees of freedom.

9. INERTIA TERMS OF THE EQUATIONS OF MOTION

The expansion (42) is used to compute the absolute velocity (10)

$$v_{ai} = v_{gi} + e_{imn}\,\omega_m\,(x_n + q_\beta f_n^\beta) + \dot q_\beta\, f_i^\beta \qquad (45)$$

and produces for the kinetic energy the expression

$$\begin{aligned} 2\,T &= \int v_{ai}\, v_{ai}\, du \\ &= (v_{gi}\, v_{gi} + \dot q_\beta\, \dot q_\beta) \int du \\ &+ \omega_m\, \omega_m \left(\int x_n x_n du + 2 q_\beta F_{nn}^\beta + q_\beta\, q_\beta \int du \right) \\ &- \omega_m\, \omega_n \left(\int x_m x_n du + 2 q_\beta F_{mn}^\beta + q_\beta q_\gamma S_{mn}^{\beta\gamma} \right) \\ &+ 2\, A_m^{\gamma\beta}\, \omega_m \dot q_\beta q_\gamma \end{aligned} \qquad (46)$$

This formula contains three types of coupling coefficients resulting from the modal analysis

$$F_{mn}^\beta = \int x_m\, f_n^\beta\, du = F_{nm}^\beta \qquad (47)$$

the symmetry with respect to the lower indices being a consequence of (41),

$$S_{mn}^{\beta\gamma} = \frac{1}{2} \int (f_m^\beta\, f_n^\gamma + f_n^\beta\, f_m^\gamma)\, du \qquad (48)$$

symmetrical in both pairs of indices and finally a set of skew symmetric matrices, governing the gyroscopic terms

$$A_m^{\gamma\beta} = e_{imn} \int f_n^\gamma\, f_i^\beta\, du = -\,A_m^{\beta\gamma} \qquad (49)$$

The inertia terms of the equations of motion follow from the computation of the variation

$$\delta \int_{t_1}^{t_2} T\, dt = \int_{t_1}^{t_2} \left(\frac{\partial T}{\partial v_{gi}}\, \delta v_{gi} + \frac{\partial T}{\partial \omega_i}\, \delta\omega_i + \frac{\partial T}{\partial q_\beta}\, \delta q_\beta + \frac{\partial T}{\partial \dot q_\beta}\, \delta \dot q_\beta \right) dt$$

Substitution of (17) and (18) and integration by parts yield the following inertia terms

Mean translation (coefficient of δp_i under the integral sign)

$$-\frac{d}{dt}\, \frac{\partial T}{\partial v_{gi}} + e_{mni}\, \omega_n\, \frac{\partial T}{\partial v_{gm}} \quad (i=1,2,3) \qquad (50)$$

Mean rotation (coefficient of $\delta\alpha_i$)

$$-\frac{d}{dt}\, \frac{\partial T}{\partial \omega_i} + e_{mni}\, \omega_n\, \frac{\partial T}{\partial \omega_m} \quad (i=1,2,3) \qquad (51)$$

Deformation mode of index β (coefficient of δq_β)

$$-\frac{d}{dt}\, \frac{\partial T}{\partial \dot q_\beta} + \frac{\partial T}{\partial q_\beta} \qquad (52)$$

10. ELASTIC RESTORING TERMS OF THE EQUATIONS OF MOTION

Under the expansion in modes the deformation tensor (3) becomes

$$2\,\gamma_{mn} = q_\beta (D_m f_n^\beta + D_n f_m^\beta) + q_\beta q_\gamma D_m f_i^\beta D_n f_i^\gamma \qquad (53)$$

and the strain energy (9), due account being taken of (44)

$$W = \frac{1}{2}\, q_\beta q_\beta \lambda^2_{(\beta)} \int du + \frac{1}{2}\, q_\beta q_\eta q_\zeta\, \Gamma^\beta_{\eta\zeta} + \frac{1}{3}\, q_\beta q_\gamma q_\eta q_\zeta\, \Gamma^{\beta\gamma}_{\eta\zeta} \qquad (54)$$

The following coupling coefficients were introduced

$$\Gamma^\beta_{\eta\zeta} = \int C^{pq}_{mn}\, D_m\, f_n^\beta\, D_p\, f_i^\eta\, D_q\, f_i^\zeta\, \frac{du}{\rho} = \Gamma^\beta_{\zeta\eta} \qquad (55)$$

$$\Gamma^{\beta\gamma}_{\eta\zeta} = \int C^{pq}_{mn}\, D_m\, f_i^\beta\, D_n\, f_i^\gamma\, D_p\, f_j^\eta\, D_q\, f_j^\zeta\, \frac{du}{\rho} \qquad (56)$$

the last one presenting the same type of symmetry as the elastic moduli

$$\Gamma^{\beta\gamma}_{\eta\zeta} = \Gamma^{\beta\gamma}_{\zeta\eta} = \Gamma^{\gamma\beta}_{\eta\zeta} = \Gamma^{\eta\zeta}_{\beta\gamma} \qquad (57)$$

The generalized elastic restoring forces appear in the deformation mode equations only as

$$-\frac{\partial W}{\partial q_\beta} = -\,\lambda^2_{(\beta)}\, q_\beta \int du - \frac{1}{2}\, q_\eta q_\zeta (\Gamma^\beta_{\eta\zeta} + 2\,\Gamma^\zeta_{\beta\eta}) - \frac{1}{4}\, q_\gamma q_\eta q_\zeta (\Gamma^{\eta\zeta}_{\beta\gamma} + \Gamma^{\gamma\eta}_{\zeta\beta}) \qquad (53)$$

For small deformations only the first term needs to be retained.

11. GRAVITATIONAL TERMS OF THE EQUATIONS OF MOTION

In the case of the gravitational potential use is made of the fact that the body dimensions are usually small compared to a characteristic length of the gravitational gradient.
A truncated Taylor expansion of the specific gravitational potential is then considered, centered at the origin of dynamic axes

$$G = G(\hat a) - \hat g_m\, \hat r_m - \frac{1}{2}\, \hat H_{mn}\, \hat r_m\, \hat r_n \qquad (59)$$

where

$$\hat{g}_m(\hat{a}) = -\frac{\partial G(\hat{a})}{\partial \hat{a}_m} \quad (60)$$

is the gravitational acceleration at the origin, and

$$\hat{H}_{mn}(\hat{a}) = -\frac{\partial^2 G(\hat{a})}{\partial \hat{a}_m \partial \hat{a}_n} \quad (61)$$

the local gravity gradient tensor.
In dynamical axes this becomes

$$G = G(\hat{a}) - g_m (x_m + u_m) - \frac{1}{2} H_{mn} (x_m + u_m)(x_n + u_n)$$

and, integrated over the mass of the body, produces a potential

$$P = G(\hat{a}) \int d\mu - H_{mn} \{\frac{1}{2} \int x_m x_n d\mu + q_\beta F^\beta_{mn} + \frac{1}{2} q_\beta q_\gamma S^{\beta\gamma}_{mn}\} \quad (62)$$

Unlike the inertial and strain-energy forces, the gravitational forces have fixed orientations in the inertial axes; in other terms the dynamic components H_{mn} depend on the orientation U of the dynamic axes

$$H = U \hat{H} U^T$$

and, using (15)

$$\delta H = \delta U \hat{H} U^T + U \hat{H} \delta U^T = -[\delta\alpha] H - H[\delta\alpha]^T$$

or $$\delta H_{mn} = -e_{mij} \delta\alpha_i H_{jn} - e_{nij} \delta\alpha_i H_{jm}$$

On the other hand

$$\delta G(\hat{a}) = - \hat{g}_i \delta\hat{a}_i = - g_i \delta p_i$$

The contributions of the gravitational forces to the equations of motion are then established as the respective coefficients of δp_i, $\delta\alpha_i$ and δq_β appearing in $-\delta P$.

Mean translation

$$g_i \int d\mu \qquad (i=1,2,3) \quad (63)$$

Mean rotation

$$e_{mij} H_{jn}(\int x_m x_n d\mu + 2 q_\beta F^\beta_{mn} + q_\beta q_\gamma S^{\beta\gamma}_{mn}) \quad (64)$$

$$(i=1,2,3)$$

Deformation mode of index β

$$H_{mn} \{ F^\beta_{mn} + q_\gamma S^{\beta\gamma}_{mn} \} \quad (65)$$

In aircraft applications, as long as the velocity of flight V is small compared to the orbital velocity $\sqrt{R_o g_o}$ (R_o mean earth radius, g_o modulus of gravitational acceleration at this distance from the center of the earth) it is common practice to accept a "flat earth" approximation. The gravitational field is considered to be uniform and oriented as the third inertial axis. In this case the dynamical components g_i are, in terms of the usual choice of Euler angles.

$$- g_o \sin\theta \quad , g_o \sin\phi \cos\theta \quad , g_o \cos\phi \cos\theta$$

along the roll, pitch and yaw axes respectively.
In satellite applications the inertial axes are usually centered on the attracting body and oriented towards "fixed stars".
Let R be the distance at which the gravitational acceleration has a known modulus g_R; then, neglecting harmonics, the potential of a unit mass is

$$G(\hat{a}) = - g_R \frac{R^2}{\rho} \qquad \rho = \sqrt{\hat{a}^T\hat{a}} = \sqrt{a^T a}$$

From which follows easily

$$g_i = - \frac{a_i}{\rho^3} g_R R^2 \qquad H_{mn} = \{ \frac{3}{\rho^5} a_m a_n - \frac{1}{\rho^3} \delta_{mn} \} g_R R^2$$

12. THRUST FOLLOWER FORCES

Gravitational forces are so-called "dead" loads; they have components determined in inertial space and oriented independently of the deformation of the body. Propulsion forces generated by air-breathing engines or rocket thrusters are, generally speaking, "followers".
Attached rigidly to a rigid body their components remain fixed with respect to the dynamic axes. Mounted flexibly on a flexible body they are moreover influenced by the deformations.
To take this into account, we assume that the thrust axis of a given propulsion unit passes through a given material point x of the body and is oriented by the local material rotation prevailing at this point.
Let dx denote a differential step taken from the point x of the thrust axis in the direction of the thrust in the reference configuration, so that

$$n = \frac{dx}{\sqrt{dx^T dx}}$$

are the direction cosines of the thrust vector f in the reference configuration. In the deformed state the convected unit vector will be

$$n' = \frac{dx + du}{\sqrt{(dx+du)^T (dx+du)}}$$

Introduce the matrix A of displacement gradients, as defined in (6)

$$du = A\, dx$$

and

$$n' = \frac{(I+A)dx}{\sqrt{dx^T dx + 2dx^T \Gamma dx}}$$

Under the assumption that the strains remain very small $2dx^T\Gamma dx$ is negligible before $dx^T dx$ and

$$n' = (I + A)\, n$$

In other words the local Jacobian matrix, governing the local neighborhood transformation, represents with a good approximation the local rotation operator. The dynamic components of the thrust in the deformed configuration are thus given by

$$f' = (I + A)\, f$$

The virtual work, computed from the general formula developped in section 4, is now expressible in terms of the known thrust vector of the reference state

$$(\delta p^T + \delta\alpha^T \left[x{+}u \right] + \delta u^T)(I + A)\, f$$

From this the contributions of a thrust follower to the different equations of motion is obtained; it requires the computation of the deformation modes and their derivatives at the local attachment point.

Mean translation

$$(\delta_{mi} + q_\beta D_m f_i^\beta)\, f_m \qquad (i=1,2,3) \qquad (66)$$

Mean rotation

$$e_{pin}\, (x_n + q_\beta f_n^\beta)(\delta_{pm} + q_\gamma D_m f_p^\gamma)\, f_m \qquad (67)$$

$$(i=1,2,3)$$

If the deformations are small enough, only the linear terms in the amplitudes q_β may be retained.

Deformation mode of index β

$$f_i^\beta\, (\delta_{im} + q_\gamma D_m f_i^\gamma)\, f_m \qquad (68)$$

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DISCUSSION

A. Liégeois (Laboratoire d'Automatique): By using your method, the choice of the reference axes is based on the minimisation of the summed squares of the deviations. Have you tried to use another norm, by using a weightless matrix (different from the unit matrix)?

Author: No, I believe it would only complicate matters, because the linearised version of the Tisserand conditions that emerges from the simple norm is ideally suited to simplify the expansion in normal modes.

P. Likins (UCLA): I note that you began with a concept of a mean motion frame which is new to me, that of a frame which minimises the relative displacement norm, and showed that this is equivalent to a frame that I have seen used before, that of a frame for which relative angular momentum is zero in the linear approximation. In the course of your research, which came first?

Author: I was aware that others, such as Professor Buckens, have used the frame defined by zero relative angular momentum in linear approximation. However, I began with the realisation that the Tisserand frame minimised the relative kinetic energy, and sought instead to minimise the relative displacement norm, finding only after extensive analysis that the result was equivalent to the frame which gives zero relative angular momentum in linear approximation.