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Crankshaft-Propeller Vibration Modes as Influenced by the Torsional Flexibility of the Engine Suspension

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ABSTRACT

The present investigation is concerned with the fact that crankshaft-propeller torsional oscillations are coupled to the torsional oscillations of the engine as a whole. The basic phenomenon was discussed by Den Hartog and Butterfield[1] in the elementary case of a single radial engine with a single resonant frequency on the suspension.

A general method of calculation is presented for long crankshaft engines. It is an extension of Biot's method,[2] where, in addition to the usual dynamic moduli at both crankshaft ends, the solution contains the dynamic modulus of the engine suspension plus air frame with respect to the harmonic torque exerted by the crankcase.

(1) INERTIA COUPLING BETWEEN CRANK ASSEMBLY AND CRANKCASE MOTION

LET ϕ_x BE the angular coordinate of the xth crankpin with respect to the crankcase and ψ be the angular coordinate of the crankcase. The complete expression for the kinetic energy of the crank assembly is a quadratic, homogeneous expression in the angular velocities $\dot\phi_x$ and $\dot\psi$

$$T_x = (1/2)\dot\phi_x^2 I(\phi_x) + \dot\psi\dot\phi_x M(\phi_x) + (1/2)\dot\psi^2 V(\phi_x)$$

The coefficients are periodic functions, of which only the mean values I, M, and V will be considered.

For instance, in the case of a single piston per crankpin, these coefficients are:

$$M = I_c + bR^2[1 - (H/L)]$$

$$I_c = M + \frac{1}{2}R^2\left(1 + \frac{R^2}{4L^2}\right) \times \left[p + b\frac{H}{L} + \frac{J - bH(L - H)}{L^2}\right]$$

where

R = radius of the crankpin
I_c = moment of inertia of crank section about crank axis
p = mass of piston
b, L, and H = respectively, mass, length, and distance of center of gravity to crank axis for the connecting rod
J = moment of inertia of connecting rod about the c.g. axis

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It comes out that, contrary to V, the value of M is the same for any location of the oscillation axis of the crankcase. This makes parts of M, due to connecting rod and piston, additive in the case of several centrally connected pistons per crankpin (e.g., V-12 engines). The value of V is not given, since it depends on the oscillation axis and will be discussed later.

The formulas are most easily obtained after a preliminary reduction to concentrated masses and corrective inertias. Even the apparently involved case of the radial engine with angular eccentricities is tractable that way.[4] Referring to Tables 3 and 4 of this paper, where the formulas appropriate to the calculation of two coefficients U_0 and V_0 are given, the following correspondence exists:

$$I = 2R^2U_0/g, \quad M = R^2V_0/g$$

(2) EQUATIONS OF MOTION

Fig. 1 shows an equivalent mechanical model of a long crankshaft engine with constant values of I and M and a spring constant k. Let θ_x denote the amplitude of vibration of the xth crankpin with respect to the crankcase and let α denote the amplitude of vibration of the crankcase.

The natural vibrations having the same frequency, ω, and the same phase angle, $\theta_x + \alpha$, will be the amplitude of absolute motion of the xth crankpin.

The equations of motion are then: for the xth crankpin,

$$-\omega^2 I\theta_x - \omega^2 M\alpha = k(\theta_{x-1} + \theta_{x+1} - 2\theta_x) \qquad (1)$$

for the first crank,

$$-\omega^2 I\theta_1 - \omega^2 M\alpha = k(\theta_2 - \theta_1) + M_0 \qquad (2)$$

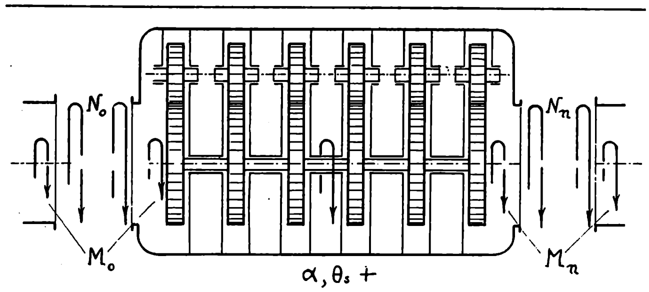

FIG. 1. Equivalent mechanical model of long crankshaft engine with constant inertia factors, including positive angle and torque conventions.

for the last crank,

$$-\omega^2 I\theta_n - \omega^2 M\alpha = k(\theta_{n-1} - \theta_n) - M_n \qquad (3)$$

for the crankcase,

$$-\omega^2(I_f + nV) - \omega^2 M \sum_1^n \theta_x = M_0 - M_n + N_0 - N_n + N \qquad (4)$$

where I_f is the moment of inertia of the crankcase separated from the propeller and blower gearing casings about the oscillation axis and where N is the torque amplitude due to the engine suspension.

(3) Coupling Due to Propeller and Blower Gearings

The following linear and symmetrical relations hold between the torque amplitudes N_0 and M_0 and the corresponding absolute angular amplitudes α and $\theta_1 + \alpha$:

$$\left.\begin{aligned} -M_0 &= K_0(\theta_1 + \alpha) + G_0\alpha \\ -N_0 &= G_0(\theta_1 + \alpha) + L_0\alpha \end{aligned}\right\} \qquad (5)$$

The coefficients are dynamic moduli whose expressions as functions of the frequency are to be set up in each particular case. With regard to the fixed frame problem ($\alpha = 0$), K_0 is seen to correspond to Biot's K_g.[2]

The structures of the dynamic moduli in the case of a spur gear reductor are obtained as follows (Fig. 2): Let

$$K_p = -m_p/(\theta_p + \alpha)$$

be the dynamic modulus at the end of the propeller shaft. The amplitudes θ_p and θ_d, relative to the gear casing and the torque amplitudes m_p and m_d, due to teeth pressures, are related by the equations

$$\theta_d = \tau_0\theta_p, \quad m_p = \tau_0 m_d$$

(τ_0 negative here) while

$$N_0 = (\tau_0 - 1)m_d + \omega^2 I_{r0}\alpha$$

I_{r0} is the moment of inertia of the whole reductor group about the oscillation axis, the moment of inertia of the shaft groups about their own axis being ignored.

On the other hand,

$$\begin{aligned} m_d &= Z_{dd}(\theta_d + \alpha) - Z_{d0}(\theta_1 + \alpha) \\ M_0 &= Z_{0d}(\theta_d + \alpha) - Z_{00}(\theta_1 + \alpha) \end{aligned}$$

with $Z_{d0} = Z_{0d}$

The Z's are the dynamic moduli of the drive shaft "other end clamped." Solving for the angular amplitudes,

$$\begin{aligned} \theta_d + \alpha &= (m_d/K_{dd}) - (M_0/K_{d0}) \\ \theta_1 + \alpha &= (m_d/K_{d0}) - (M_0/K_{00}) \end{aligned}$$

The K's are the dynamic moduli of the drive shaft "other end free"; they are simply related to the Z.

$$Z_{00}K_{dd} = Z_{d0}K_{0d} = Z_{dd}K_{00} = Z_{dd}Z_{00} - Z_{0d}^2$$

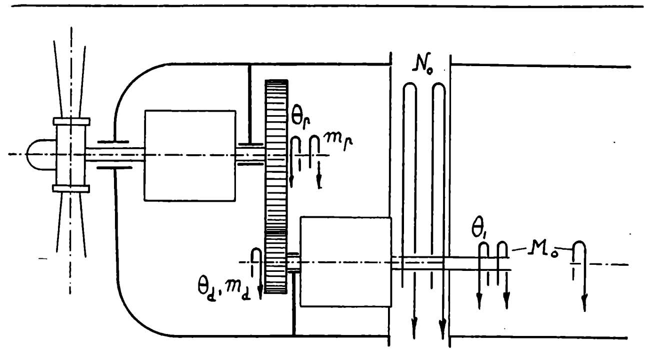

Fig. 2. Symbols used to set up dynamic moduli of spur gear type reductors.

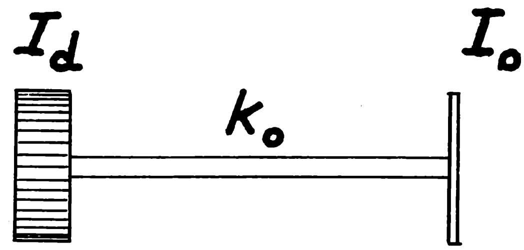

Fig. 3. Simple structure of driving shaft for spur gear reductor.

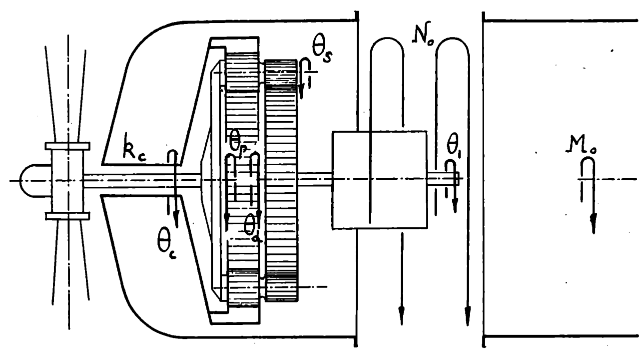

Fig. 4. Symbols used to set up dynamic moduli of epicyclic type reductor.

These equations yield in succession

$$\begin{aligned} \theta_d + \alpha &= -(\tau_0 - 1)\alpha - (\tau_0^2 m_d/K_p) \\ \Delta m_d &= -Z_{d0}\theta_1 - [Z_{d0} + (\tau_0 - 1)Z_{dd}]\alpha \end{aligned}$$

where

$$\Delta = 1 + (\tau_0^2 Z_{dd}/K_p)$$

and, finally,

$$\begin{aligned} K_0 &= Z_{00}[1 + (\tau_0^2 K_{dd}/K_p)]\Delta^{-1} \\ G_0 &= (\tau_0 - 1)Z_{0d}\Delta^{-1} \\ L_0 &= (\tau_0 - 1)^2 Z_{dd}\Delta^{-1} - \omega^2 I_{r0} \end{aligned}$$

If the drive shaft has the simple structure of Fig. 3,

$$\begin{aligned} Z_{dd} &= k_0 - \omega^2 I_d \\ Z_{d0} &= k_0 \\ Z_{00} &= k_0 - \omega^2 I_0 \end{aligned}$$

If, furthermore, the inertias I_d and I_0 are neglected,

$$Z_{dd} = Z_{d0} = Z_{00} = k_0, \quad K_{dd} = 0$$

and there follows an exceedingly simple structure conveniently referred to as the "purely elastic coupling":

$$1/K_0 = (1/k_0) + (\tau_0^2/K_p)$$
$$G_0 = (\tau_0 - 1)K_0, \quad L_0 = (\tau_0 - 1)^2K_0 - \omega^2 I_{r0}$$

One should be reminded that, owing to the flexibility of the propeller blades, K_p and, hence, K_0 will also be functions of the mean angular velocity of the crankshaft.[3]

The case of the epicyclic reduction gear is slightly more complicated. Fig. 4 shows elements similar to the spur gear reductor and similarly denoted. In addition to those, a crown wheel, of moment of inertia I_c', is connected to the gear casing through an elastic shaft of spring constant k_c that allows in a crude fashion for the flexibility of the gear casing. Planet gears have a total moment of inertia about their own axis equal to I_s.

The relations between the different angles are

$$\theta_s = -\mu_0\theta_p + (1 + \mu_0)\theta_c$$
$$\theta_d = \tau_0\theta_p - (\tau_0 - 1)\theta_c$$

The same relations hold naturally when α is added to each relative angle. Using these relations in a kinetic energy equation, such as

$$I_s(\theta_s + \alpha)^2 = A(\theta_c + \alpha)^2 + B(\theta_p + \alpha)^2 + C(\theta_d + \alpha)^2$$

values of A, B, and C are found that satisfy this equation. It means that the total inertia I_s of the planet gears may be transferred as follows: an amount

$$A = (1 + \mu_0)\,[1 + (\mu_0/\tau_0)]I_s$$

goes to the crown wheel, whose fictitious total inertia becomes $I_c' + A = I_c$; an amount

$$B = -\mu_0[(\mu_0 + \tau_0)/(\tau_0 - 1)]I_s$$

goes to the planet carrier at the end of the propeller shaft; and an amount

$$C = \mu_0(1 + \mu_0)/\tau_0(\tau_0 - 1)$$

goes to the sun gear.

In the expressions of the dynamic moduli that will follow it is understood that this transfer has taken place. There comes, in succession,

$$\theta_c = \frac{\tau_0 - 1}{K_c} m_d + \left(\frac{k_c}{K_c} - 1\right)\alpha$$

$$\theta_d + \alpha = -\left[\frac{(\tau_0 - 1)^2}{K_c} + \frac{\tau_0^2}{K_p}\right] m_d - (\tau_0 - 1)\frac{k_c}{K_c}\alpha$$

with

$$K_c = k_c - \omega^2 I_c$$

and

$$\Delta m_d = -Z_{d0}\theta_1 - [Z_{d0} + (\tau_0 - 1)Z_{dd}(k_c/K_c)]\alpha$$

with

$$\Delta = 1 + Z_{dd}\left[\frac{(\tau_0 - 1)^2}{K_c} + \frac{\tau_0^2}{K_p}\right]$$

and, finally,

$$K_0 = Z_{00}\left\{1 + K_{dd}\left[\frac{(\tau_0 - 1)^2}{K_c} + \frac{\tau_0^2}{K_p}\right]\right\}\Delta^{-1}$$
$$G_0 = (\tau_0 - 1)Z_{0d}(k_c/K_c)\,\Delta^{-1}$$
$$L_0 = (\tau_0 - 1)^2 Z_{dd}\left(\frac{k_c}{K_c}\right)^2\Delta^{-1} - \omega^2 I_c\frac{k_c}{K_c} - \omega^2 I_{r0}$$

Similar formulas will hold for the other crankshaft end, eventually geared to blowers and (or) to exhaust turbines.

$$\left.\begin{aligned} M_n &= K_n\theta_n + G_n\alpha \\ M_n + N_n &= G_n\theta_n + L_n\alpha \end{aligned}\right\} \qquad (6)$$

(4) The Fixed Crankcase Problem

Treated by Biot,[2, 3] this will correspond to the present problem where α is put equal to zero. The general equation of motion for the xth crankpin is satisfied by putting

$$\theta_x = C\sin(x\mu + \phi) \quad (x = 1, 2 \ldots n) \qquad (7)$$

the frequency being related to the parameter μ by the equation

$$\omega = 2\sqrt{k/I}\sin(\mu/2) \qquad (8)$$

Eqs. (2) and (3) are reduced to the following:

$$\left.\begin{aligned} [2\cos\mu - 1 + (K_0/k)]\theta_1 - \theta_2 &= 0 \\ [2\cos\mu - 1 + (K_n/k)]\theta_n - \theta_{n-1} &= 0 \end{aligned}\right\} \qquad (9)$$

and are used to fix the unknowns ϕ and μ.

The substitution of the general solution [Eq. (7)] transforms them in a pair of linear and homogeneous equations in the unknowns $\sin\phi$ and $\cos\phi$. The condition of compatibility or "frequency equation" appears in the form of a determinant,

$$\begin{vmatrix} (K_0/k - 1)\cos\mu + 1 & (K_0/k - 1)\sin\mu \\ (K_n/k - 1)\cos n\mu + \cos(n + 1)\mu & (K_n/k - 1)\sin n\mu + \sin(n + 1)\mu \end{vmatrix} = 0 \qquad (10)$$

The dynamic moduli K_0 and K_n are functions of ω and, hence, of μ through Eq. (8).

Instead of plotting the left-hand side as a transcendental oscillating function of μ to find its roots by an interpolation process, Biot suggests constructing a complex quantity of which it is to be the imaginary part. It is useful, in view of the generalization of this process to the present problem, to draw attention to a convenient method for this construction. Any determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is the imaginary part of either

$$(a - ib)\,(c + id)$$

or

$$-(a + ib)\,(c - id)$$

As a result, the complex quantity is immediately resolved in factors; in the case of Eq. (10),

$$e^{in\mu}[e^{i\mu/2} + (K_0/k - 1)e^{-i\mu/2}] \times [e^{i\mu/2} + (K_n/k - 1)e^{-i\mu/2}] = A_0 A_n e^{i(n\mu + \phi_0 + \phi_n)} \quad (11)$$

An equivalent statement to the condition that the imaginary part should vanish consists in equating the argument of the complex quantity to a multiple of π. Hence,

$$n\mu + \phi_0 + \phi_n = \text{multiple of } \pi$$

where

$$\left.\begin{aligned} \tan\phi_0 &= (2k/K_0 - 1)\tan(\mu/2) \\ \tan\phi_n &= (2k/K_n - 1)\tan(\mu/2) \end{aligned}\right\} \quad (12)$$

(5) Limitations of the Solution

It appears from Eq. (8) that the proposed solution yields only those natural frequencies that are inferior to the cutoff frequency

$$\omega_c = 2\sqrt{k/I}$$

of the mechanical filter represented by the long crankshaft. This cut frequency is indeed the natural frequency of an elementary cell of that filter (Fig. 5).

A corresponding process, yielding the natural frequencies, if any, above the cutoff frequency, would consist in putting

$$\theta_x = (-1)^x C \sinh(x\mu + \phi) \quad (x = 1, 2 \ldots n)$$

the general equation of motion being again satisfied, provided

$$\omega = 2\sqrt{k/I}\cosh(\mu/2)$$

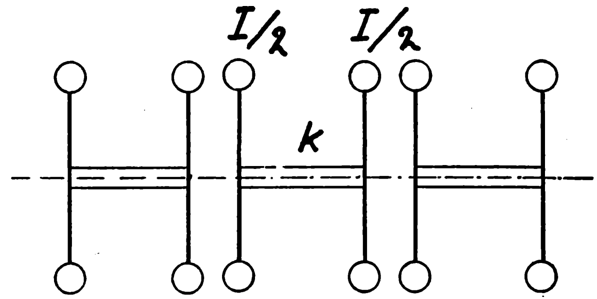

Fig. 5. Elementary cells of long crankshaft considered as a mechanical filter. The cut frequency is the natural frequency of a cell.

However, it seems doubtful whether the consideration of these higher frequencies is worth while, especially since their accurate value will be impaired by the simplifications involved in the actual concentrated masses model.

The same remarks concerning the limitation in frequency range will apply to the treatment of the general problem.

(6) The Oscillating Frame Problem

Returning to the general equation [Eq. (1)], it is seen to admit of a solution

$$\left.\begin{aligned} \theta_x &= C[\sin(x\mu + \phi) + \gamma] \\ \alpha &= -C(I/M)\gamma \end{aligned}\right\} \quad (13)$$

C is again an arbitrary amplitude coefficient, and the frequency is still given by Eq. (8). The three parameters μ, ϕ, and γ will follow from the equations governing the motions or the end crankpins and of the crankcase.

The investigation is limited to the case of "purely elastic coupling" on both ends of the crankshaft. The general case involves no special difficulties but is more cumbersome.

Substitution of Eqs. (13) and (5) in Eqs. (2) and (3) expressing the motions of the end crankpins yields

$$\left.\begin{aligned} (K_0/k - 1)\sin(\mu + \phi) + \sin\phi - \epsilon_0(K_0/k)\gamma &= 0 \\ (K_n/k - 1)\sin(n\mu + \phi) + \\ \sin[(n+1)\mu + \phi] - \epsilon_n(K_n/k)\gamma &= 0 \end{aligned}\right\} \quad (14)$$

where coupling coefficients appear

$$\epsilon_0 = \tau_0(I/M) - 1, \quad \epsilon_n = \tau_n(I/M) - 1 \quad (15)$$

Multiply Eq. (4) throughout by $I(CkM)^{-1}$.

When substituting the solution, note that

$$-\omega^2 \frac{I}{k}\sum_1^n \theta_x = 2(\cos\mu - 1)\sum_1^n \theta_x = 2(\cos\mu - 1)n\gamma + \sin\phi - \sin(\mu + \phi) - \sin(n\mu + \phi) + \sin[(n+1)\mu + \phi]$$

The following additional notations are introduced:

$$F = I_f + n\gamma + I_{r0} + I_{rn}$$

total moment of inertia of the engine about the oscillation axis; this makes a discussion about the exact value of V practically unnecessary, since the value actually included in an experimental measurement of F will, at least for a sufficient number of pistons, be independent of the angular position of the crankshaft and represent accurately the required mean

$$K_f = -N/\alpha$$

dynamic modulus representing the response of the engine mounting and air frame when submitted to a harmonic torque from the crankcase. This dynamic modulus is part of a more complicated expression that will be called the dynamic modulus of the suspension:

$$h = -\omega^2 nI - (I/M)^2[K_f - \omega^2 F] \tag{16}$$

The coupling coefficients ϵ_0 and ϵ_n are again put in evidence if Eqs. (14) multiplied, respectively, by $\tau_0(I/M)$ and $\tau_n(I/M)$ are substracted:

$$\epsilon_0[\sin(\mu + \phi) - \sin\phi] + \epsilon_n\{\sin(n\mu + \phi) - \sin[(n+1)\mu + \phi]\} + (h/k)\gamma = 0 \tag{17}$$

Eqs. (14) and (17) form a linear and homogeneous set in the unknowns $\sin\phi$, $\cos\phi$, and γ; hence, they yield a compatibility condition or frequency equation:

$$\begin{vmatrix} (K_0/k - 1)\cos\mu + 1 & (K_0/k - 1)\sin\mu & -\epsilon_0 K_0/k \\ (K_n/k - 1)\cos n\mu + \cos(n+1)\mu & (K_n/k - 1)\sin n\mu + \sin(n+1)\mu & -\epsilon_n K_n/k \\ \epsilon_0(\cos\mu - 1) + \epsilon_n[\cos n\mu - \cos(n+1)\mu] & \epsilon_0 \sin\mu + \epsilon_n[\sin n\mu - \sin(n+1)\mu] & h/k \end{vmatrix} = 0$$

The development of this determinant is started through the elements of the last column, each determinantal cofactor being replaced by a complex quantity of which it is the imaginary part according to the process indicated for the fixed frame problem. Some terms, which are real, are conveniently dropped, and after some grouping of the others there comes:

$$e^{in\mu}\left[\frac{h}{k}A_0A_n e^{i(\phi_0 + \phi_n)} + 2(1 - \cos\mu)\left(\epsilon_0^2\frac{K_0}{k} + \epsilon_n^2\frac{K_n}{k}\right) - (1 - e^{-i\mu})(\epsilon_0^2 + \epsilon_n^2)\frac{K_0K_n}{k^2}\right] - 2\epsilon_0\epsilon_n\frac{K_0K_n}{k^2}e^{-i\mu} \div \text{real} \tag{18}$$

where

$$A_0 = \frac{K_0}{k}\frac{\cos(\mu/2)}{\cos\phi_0} \quad \text{and} \quad A_n = \frac{K_n}{k}\frac{\cos(\mu/2)}{\cos\phi_n}$$

In contrast to the fixed frame problem, $e^{in\mu}$ is no longer a factor of the whole expression; hence, the primitive idea of equating the total argument to a multiple of π will no longer give a satisfactory result, since the argument would depend upon circular functions of the angle $n\mu$, which means that it would go itself through many oscillations.

However, it is possible to bring the vanishing condition of the imaginary part to a reduced form:

$$\sin(n\mu + \psi) + \sin\zeta = 0 \tag{19}$$

where the angles ψ and ζ are independent of $n\mu$ and vary smoothly. To this effect, introduce the dimensionless flexibilities

$$\lambda = \frac{2k}{h}, \quad \alpha_0 = \frac{2k}{K_0} - 1, \quad \alpha_n = \frac{2k}{K_n} - 1 \tag{20}$$

then put

$$\tan\psi = a/b \tag{21}$$

where

$$a = \alpha_0 + \alpha_n - (\epsilon_0^2 + \epsilon_n^2)\lambda \tag{22}$$

$$b = \cot\frac{\mu}{2} + \tan\frac{\mu}{2}[\lambda(\epsilon_0^2\alpha_n + \epsilon_n^2\alpha_0) - \alpha_0\alpha_n] \tag{23}$$

The angle ψ is the argument of the bracket in Eq. (18), which is written accordingly:

$$\frac{h}{k}\frac{K_0K_n}{k^2}\left[\frac{a\sin\mu}{2\sin\psi}e^{i(n\mu + \psi)} - \lambda\epsilon_0\epsilon_n e^{-i\mu}\right] \div \text{real}$$

Finally, defining ζ through one of expressions

$$\sin\zeta = \frac{2\epsilon_0\epsilon_n\lambda}{a}\sin\psi = \frac{2\epsilon_0\epsilon_n\lambda}{b}\cos\psi = \frac{2\epsilon_0\epsilon_n\lambda}{\sqrt{a^2 + b^2}} \tag{24}$$

yields the reduced form [Eq. (19)] of the frequency equation.

TABLE 1

$\mu°$	$\phi_0°$	$\phi_n°$	$n\mu + \phi_0 + \phi_n$	Natural Frequencies $\mu°$	Natural Frequencies $\omega/2\pi$, Cycles per Sec.
0	−90	−90	−180		
1	−8.97	−68.05	−71.02		
1.2	−4.915	−40.15	−37.865		
1.3	−3.175	−7.74	−3.115	1.31	14.435
1.4	−1.58	+27.38	+34.20		
2.0	+6.14	+75.15	+93.29		
5.0	+29.72	+86.34	+146.06		
7.0	+40.28	+87.47	+169.75	7.96	87.65
8.0	+44.69	+87.79	+180.48		
9.0	+48.52	+88.06	+190.58		
10	+51.815	+88.26	+200.075		
15	+64.13	+88.86	+242.99		
30	+82.785	+89.44	+352.225		
45	+101.83	+89.64	+461.47	31	337.5
				51.5	548.6
60	+208.055	+89.74	+657.795		
				66.5	692.4
90	+256.95	+89.85	+886.80		
				92.5	912.2
120	+263.53	+89.91	+1,073.44		
				121	1,099
150	+267.14	+89.96	+1,257.1		
				150	1,220
180	+270	+90	+1,440		
				Cut frequency	1,263

Fixed frame solution of the example shown in Fig. 6.

TABLE 2

$\omega_s/2\pi$ = 88.63 Cycles per Sec. ($\mu°$ = 8.05)					Natural Frequencies	
$\mu°$	$\psi°$	$\zeta°$	$n\mu + \psi + \zeta$	$n\mu + \psi - \zeta$	$\mu°$	$\omega/2\pi$, Cycles per Sec.
0	−180	0	−180	−180		
1.0	−77.26	−0.07	−71.33	−71.19		
					1.31	14.435
1.5	+49.08	−0.18	+57.90	+58.26		
2.0	+80.44	−0.09	+92.35	+92.53		
3.0	97.10	−0.07	+115.03	+115.17		
4.0	106.33	−0.07	+130.26	+130.40		
5.0	113.21	−0.08	+143.12	+143.29		
7.0	134.36	−0.15	+176.21	+176.51		
					7.14	78.63
8.0	168.31	−0.73	+215.58	+217.04		
					8.96	98.63
9.0	308.44	−0.165	+362.08	+362.41		
10.0	316.14	−0.065	+367.70	+376.80		
15.0	332.46	−0.01	+422.45	+422.47		
30	351.94	...	+531.94	+531.94		
					31	337.5
45	371.44	...	+641.44	+641.44		

TABLE 3

$\omega_s/2\pi$ = 14.325 Cycles per Sec.					Natural Frequencies	
$\mu°$	$\psi°$	$\zeta°$	$n\mu + \psi + \zeta$	$n\mu + \psi - \zeta$	$\mu°$	$\omega/2\pi$, Cycles per Sec.
0	−180	0	−180	−180		
1	−38.24	−5.40	−37.64	−26.84		
					1.15	12.68
1.2	+27.745	−16.59	+18.355	+51.535		
1.3	+82.43	−22.39	+67.84	+112.62		
1.4	+136.98	−18.72	+126.66	+164.10		
					1.44	15.87
1.5	+174.16	−12.20	+170.96	+195.36		
1.6	+197.16	−7.95	+198.81	+214.71		
1.7	+212.26	−5.49	+216.97	+227.95		
1.8	+222.895	−4.00	+229.70	+237.70		
1.9	+230.85	−3.05	+239.20	+245.30		
2.0	+237.13	−2.41	+246.72	+251.54		
2.5	+256.29	−1.02	+270.27	+272.31		

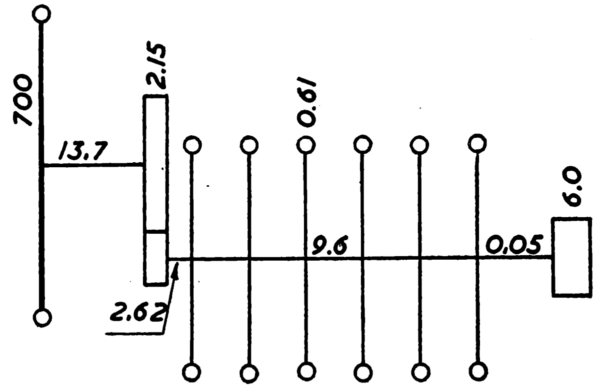

FIG. 6. Inertias in lbs. in. sec.² and rigidities in lbs. in. per microrad. of typical V-12 engine with spur gear reductor. Further data used:

$\tau_0 = -2.38 \qquad I/M = 1.3$
$\tau_n = +0.14 \qquad F = 700$ lbs. in. sec.²

There are two solutions to this equation

$$\left.\begin{aligned} n\mu + \psi + \zeta &= \text{even multiple of } \pi \\ n\mu + \psi - \zeta &= \text{odd multiple of } \pi \end{aligned}\right\} \qquad (25)$$

The process is thus reduced to the plotting of two smooth curves. To sum up: Plot the three expressions (20), the last two being already necessary in the fixed frame problem; then plot Eqs. (22) and (23), extract the angle ψ from Eq. (21), and, finally, extract ζ from one of Eqs. (24).

(7) Special Frequencies

Some particular values of μ are important to discuss.

(a) If $K_n = 0$, then,

$$\tan\psi = \cot(\mu/2)[-\alpha_0 + \epsilon_0^2\lambda]^{-1}$$

and $\sin\zeta = 0$.

The two curves have a common point if this condition occurs for a particular value of μ (resonance at the rear dynamic modulus), or they are identical if the condition is permanent (rear crankshaft end free).

The symmetrical case exists for resonant values of the front dynamic modulus.

(b) If h tends to zero, λ tends to infinity.

Values of ω for which this occurs are conveniently referred to as resonant frequencies for the engine suspension, although the engine inertia and the inertia coupling factor I/M are involved in the expression for

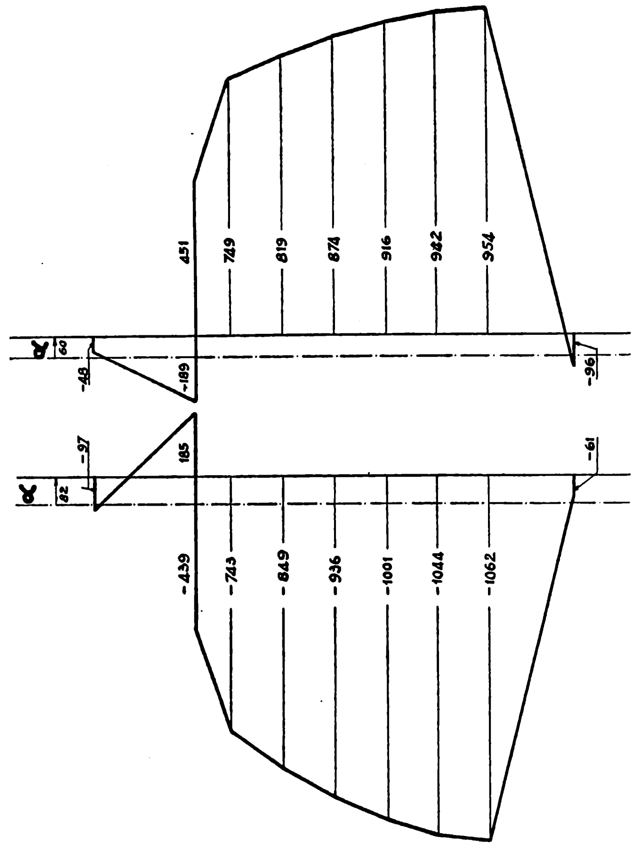

FIG. 7. Two modal shapes of the engine in Fig. 6, when the resonant frequency of the suspension almost equals the second natural frequency of the same engine with fixed crankcase (88.63 vs. 87.65 cycles per sec.). Frequency of first mode is 78.63 cycles per sec.; of second mode, 98.63 cycles per sec. Numbers refer to vibration amplitudes relative to the crankcase. To find absolute amplitudes, add values given for α.

h. There comes

$$\begin{aligned} \tan\psi &= -(\epsilon_0^2 + \epsilon_n^2)\cot(\mu/2)\,[\epsilon_0^2\alpha_n + \epsilon_n^2\alpha_0]^{-1} \\ \sin\zeta &= -[2\epsilon_0\epsilon_n/(\epsilon_0^2 + \epsilon_n^2)]\sin\psi \end{aligned}$$

(c) For antiresonant frequencies of the engine suspension, h tends to infinity, λ tends to zero, and

$$\begin{aligned} \tan\psi &= \frac{(\alpha_0 + \alpha_n)\cot(\mu/2)}{\cot^2(\mu/2) - \alpha_0\alpha_n} = \tan(\phi_0 + \phi_n) \\ \sin\zeta &= 0 \end{aligned}$$

So we are naturally brought back to Biot's fixed frame solution.

(8) Shape of the Vibration Modes

For each value of μ ensuring the compatibility of Eqs. (14) and (16), values of ϕ and γ should be obtained to fix the shape of the natural mode in question. An obvious algebraic method is as follows: Eliminate γ between Eqs. (14) and obtain linear and homogeneous equation in $\sin\phi$ and $\cos\phi$. From this, $\tan\phi$ is obtained, or, again, a corresponding complex equation is set up—e.g., after some reductions

$$\cos\frac{\mu}{2}\,\frac{K_0K_n}{k^2}e^{i[\phi+[(n+1)/2]\mu]}\left\{\frac{\epsilon_n}{\cos\phi_0}\,e^{-i[(n/2)\mu+\phi_0]} - \frac{\epsilon_0}{\cos\phi_n}\,e^{i[(n/2)\mu+\phi_n]}\right\} \div \text{real}$$

The argument χ of the bracket is given by

$$\tan \chi = \frac{(\epsilon_0 + \epsilon_n) \tan (n/2)\mu + (\epsilon_0\alpha_n + \epsilon_n\alpha_0) \tan (\mu/2)}{(\epsilon_0 - \epsilon_n) - (\epsilon_0\alpha_n - \epsilon_n\alpha_0) \tan (\mu/2) \tan (n\mu/2)}$$

and ϕ is determined by

$$\phi + [(n + 1)/2]\mu + \chi = 0$$

The determination chosen for ϕ is then used in one of Eqs. (14) to fix the corresponding value of γ. From the first one, for instance,

$$\epsilon_0\gamma = \cos (\mu/2) \sin [\phi + (\mu/2)] - \alpha_0 \sin (\mu/2) \cos [\phi + (\mu/2)]$$

(9) Numerical Example

Fig. 5 shows the numerical values of inertias and rigidities pertaining to a V-12 engine with spur gear reductor.

The interest being focused on the influence of the engine suspension, some simplifications were introduced. Flexibility of the propeller was not accounted for, and the engine suspension was treated as a torque spring connecting the engine to a perfectly rigid air frame of extremely large inertia—hence, for practical purposes, fixed in space. There is, accordingly, but

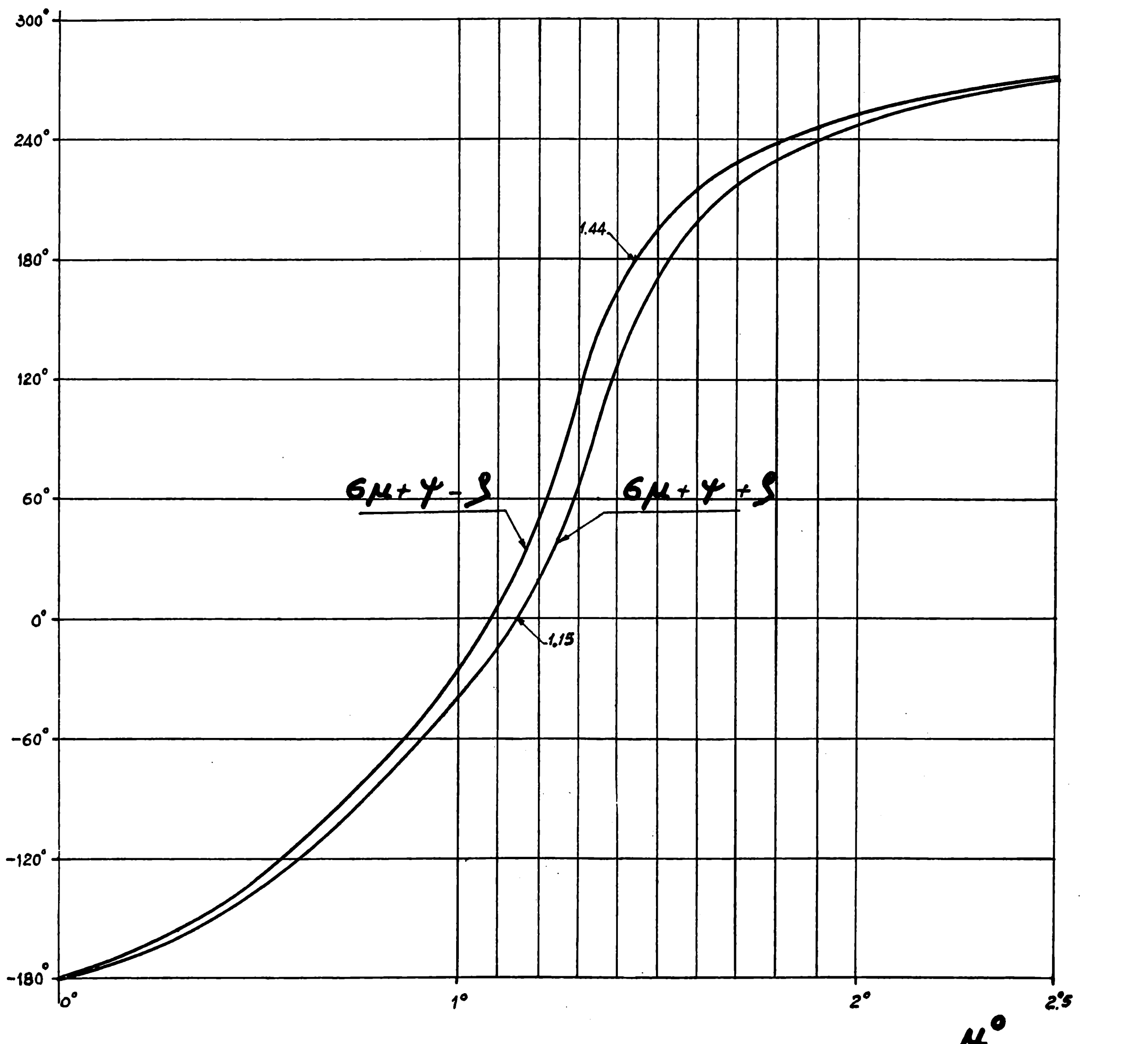

Fig. 8. Illustrating the determination of the two first natural frequencies of the engine in Fig. 6, when the resonant frequency of the suspension almost equals the first natural frequency of the same engine with fixed crankcase (14.325 vs. 14.445 cycles per sec.).

one resonant frequency for the engine suspension ($h = 0$), denoted by ω_s.

Table 1 shows the results for a perfectly rigid suspension (fixed frame solution). For flexible suspension, a resonant frequency of 88.63 cycles per sec., very near to the second natural of the fixed frame case, was first investigated, since the frequency shift effect was expected to be particularly large.

Table 2 summarizes the results of calculations for this case. The angle ψ follows closely the sum $\phi_0 + \phi_n$ except in the neighborhood of the resonant frequency, where it gains 180° due to the added degree of freedom and then follows again the sum at 180° distance (alternatively the ψ curve might be notched as in reference 3).

The angle ζ is practically negligible. It shows two maxima: the first between the zeros of α_0 and α_n; the second one near the resonant frequency of the suspension.

Figs. 6 and 7 show the calculated shapes of the two modes, one 10.3 per cent lower and one 12.5 per cent higher than the primitive second natural frequency. The other modes are practically undisturbed in frequency and shape.

The case just discussed may be said to correspond to a "hard" suspension. A case of "soft" suspension was next investigated by taking its resonant frequency at 14.325 cycles per sec., which is near the first natural. As should be expected, only blower oscillations are affected with small frequency shift.

The relatively large values of ζ, with the coalescence of the two previous maxima, are remarkable.

Table 3 and Fig. 8 summarize the numerical results.

In all cases ζ is exactly zero for $\mu = 37.160$, resonant frequency of the front dynamic modulus.

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