Reprinted from JOURNAL OF THE AERONAUTICAL SCIENCES

OCTOBER, 1955 VOLUME 22, No. 10

Iteration in Semidefinite Eigenvalue Problems

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SUMMARY

General rules are derived for setting up iteration matrices yielding convergence towards the elastic modes of semidefinite self-adjoint vibration problems. They are classified according to the methods used for a preliminary reduction of the problem to a finite number of degrees of freedom. These include the Rayleigh-Ritz, the complementary energy, and the integral equation approach. In the last case it is shown how extended influence coefficients are related to isostatic reference frames. A numerical example is treated and a procedure suggested for improving the higher frequency modes obtained from the lumped mass models which are provided by the application of numerical integration and collocation to the integral equation.

INTRODUCTION

NORMAL VIBRATION modes of continuous elastic structures form an infinite set with a discrete frequency spectrum. Restraining the freedom of the system to a few of the normal coordinates offers an adequate approximation to its elastic and inertial behavior in some frequency range. Most of the requirements of aeroelasticity are for the low range from zero to some limit above which the higher frequency modes may be assumed not to be appreciably excited.

It is always possible in theory to obtain the modes and their eigenvalues from the differential equations and boundary conditions of the problem. This approach, however, is impractical except in the simplest situations where the general solution of the differential equations are known. In this connection a new method due to W. E. Milne[1] deserves attention. As an alternative, iteration methods are useful but present difficulties in the case of unrestrained systems, like structures of aircraft in free flight.

Such structures have no Green's function or static influence function in the accepted meaning of displacement at one point due to unit load applied at another, and their vibration problem lacks the usual integral equation formulation.

To each degree of freedom remaining in the structure after complete rigidification, there corresponds a vibration mode of zero frequency. More generally, the presence of eigensolutions with zero eigenvalues, which characterizes semidefinite eigenvalue problems, requires an extension in the concept of the Green's function.

From this standpoint, integral equations involving extended Green's functions are briefly treated by Courant and Hilbert;[2] an example in the aeronautical field is given by R. L. Bisplinghoff, G. Isakson and T. H. H. Pian.[3] H. A. Fettis[4] applies iteration directly to the differential system of those problems.

Setting aside the very exceptional cases where the integrations may be carried out in closed analytical form, both of these approaches imply some type of numerical integration, whereby a system of linear algebraic equations approximates either the differential system or the integral equation. In the present paper, reduction to a vectorial space of finite dimensions is accepted as the initial step from which developments necessary to provide convergent iterations are left to the care of matrix algebra.

This approach allows in particular a general formulation of the ideas on which extended Green's functions are based. The various rules obtained remain of standard application in very general situations; all have in common the central role played by a matrix operator for orthogonal projection in the vectorial subspace of the elastic modes.

(1) EQUATIONS OF MOTION AND NORMAL MODES

Consider the eigenvalue problem exemplified by the natural modes of vibration of a system having a finite number of degrees of freedom

$$(C - \omega^2 M)x = 0 \tag{1}$$

where C stands for a symmetrical semidefinite positive matrix of stiffness coefficients, M for a symmetrical positive definite matrix of inertia coefficients, ω for the circular frequency and x for the one column matrix or vector of the vibration amplitudes.

Eq. (1) is obtained from the general equation of motion

$$M\ddot{q} + Cq = p \tag{2}$$

by letting the vector p of generalized forces, corresponding to the generalized coordinates vector q vanish and by separation of the time variable $q = e^{i\omega t}x$.

To each eigenvalue ω_r^2, root of the frequency equation

$$det(C - \omega^2 M) = 0 \tag{3}$$

there corresponds a normal mode $x_{(r)}$ satisfying the relation

$$Cx_{(r)} = \omega_r^2 Mx_{(r)} \tag{4}$$

The assumption that C is semidefinite is equivalent to stating that $Cq = 0$ admits of non-zero solutions. Let $u_{(i)}$ $(i = 1, 2 \ldots m)$ denote a complete linearly independent set of these free modes so that

$$Cu_{(i)} = 0 \qquad (i = 1, 2 \ldots m) \tag{5}$$

Received August 26, 1954.

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From Eq. (4) it appears that the free modes are normal modes of zero eigenvalue. They involve no elastic deformation energy and are consequently easy to determine by simple inspection of the degrees of freedom remaining in the structure after complete rigidification. Elementary examples of free modes are the vertical, longitudinal, and lateral translations, the pitching, rolling, and yawing motions of a rigid airplane in free flight. To those might be added the relative control surface motions provided they are not effectively spring-restrained.

The notation $x_{(r)}$ ($r = 1, 2 \ldots n$) will qualify elastic modes only; those are modes calling for elastic deformation energy and consequently associated with non-zero eigenvalues. The following orthogonality relations are essential:*

$$u_{(i)}'Mu_{(j)} = 0 \qquad i \neq j \tag{6a}$$

$$u_{(i)}'Mx_{(r)} = x_{(r)}'Mu_{(i)} = 0 \tag{6b}$$

$$x_{(r)}'Mx_{(s)} = 0 \qquad r \neq s \tag{6c}$$

$$x_{(r)}'Cx_{(s)} = 0 \qquad r \neq s \tag{6d}$$

They hold as a consequence of Eqs. (4) and (5) between any two modes of different eigenvalues and may be assumed to hold between two distinct modes of the same eigenvalue as the result of an orthogonalization process. The modes

$$u_{(i)} \qquad (i = 1, 2 \ldots m) \qquad x_{(r)} \qquad (r = 1, 2 \ldots n)$$

then form a complete orthogonal set in terms of which any vector admits a unique expansion. For example, the expansion of the displacement vector

$$q = \sum_1^m \eta_i u_{(i)} + \sum_1^n \xi_r x_{(r)}$$

introduces the normal coordinates $\eta_i(t)$ and $\xi_r(t)$ and may be substituted in the equations of motion Eq. (2). Multiplying then to the left by $u_{(i)}'$ and $x_{(r)}'$ in turn, using Eqs. (4), (5), and (6), the equations of motion are obtained in normal form

$$\ddot{\eta}_i = \frac{u_{(i)}'p}{u_{(i)}'Mu_{(i)}} \tag{7a}$$

$$\omega_r^2\ddot{\xi}_r + \xi_r = \frac{x_{(r)}'p}{x_{(r)}'Mx_{(r)}} \tag{7b}$$

The simplification resulting from the absence of inertial and elastic coupling between normal coordinates justifies the interest attached to the computation of the normal modes.

(2) Projection Operators in Restrained Systems

Let the normal mode problem of a restrained system be cast in the form

$$(C^{-1}M - \lambda E)x = 0 \tag{8}$$

*The exchange between rows and columns or transposition is denoted by a prime. u' and x' are therefore one-rowed matrices. In transposing a product each factor is transposed and the order reversed.

where C^{-1} is the reciprocal of the nonsingular matrix C, E the unit matrix, and

$$\lambda = 1/\omega^2 \tag{9}$$

the characteristic value parameter. With the transformations

$$M = LL' \qquad L'x = y \tag{10}$$

It reduces to the characteristic value problem

$$(F - \lambda E)y = 0$$

of a symmetrical matrix

$$F = L'C^{-1}L = F'$$

The computation of L is always possible and elementary when it is assumed to be lower (or upper) triangular, L' then being upper (or lower) triangular. This triangular factorization has been proposed by Cholesky in connection with the solution of symmetrical linear equations systems.[5]

Let $y_{(r)}$ be the modal columns and $\lambda_1 > \lambda_2 \ldots > \lambda_n$ the characteristic values. To eliminate the mode $y_{(1)}$ yielded by the iteration with F we form the scalar product $y_{(1)}'y_{(1)}$, the degenerate square matrix $y_{(1)}y_{(1)}'$, and the resulting projection matrix

$$A_1 = E - [y_{(1)}y_{(1)}'/y_{(1)}'y_{(1)}]$$

In view of the orthogonality relations between modes

$$y_{(1)}'y_{(r)} = 0 \qquad r \neq 1$$

this operator has the obvious selective properties

$$\begin{cases} A_1y_{(1)} = 0 \\ A_1y_{(r)} = y_{(r)} \qquad r \neq 1 \end{cases}$$

We have accordingly for the deflated matrix[6]

$$F_1 = A_1F = FA_1 = F - \lambda_1[y_{(1)}y_{(1)}'/y_{(1)}'y_{(1)}]$$

the relations

$$\begin{cases} Fy_{(1)} = 0 \\ Fy_{(r)} = \lambda_r y_{(r)} \qquad r \neq 1 \end{cases}$$

proving that F_1 retains the modes of the original F matrix. But, whereas the $y_{(r)}$ ($r \neq 1$) retain their original characteristic values λ_r, the characteristic value of $y_{(1)}$ has been reduced to zero. The very first iteration with the deflated matrix F_1 will then remove any first mode component and will converge toward the mode of next highest characteristic value. Pre- or postmultiplication by the successive projection matrices allows repetition of the iteration process without the growing complications sometimes encountered with other methods like "sweeping."[7]

The transformation (10) is not an essential step and may be avoided altogether. The projection matrix to be used is then

$$A_1 = E - \frac{x_{(1)}x_{(1)}'M}{x_{(1)}'Mx_{(1)}}$$

with the selective properties

$$A_1x_{(1)} = 0 \qquad A_1x_{(r)} = x_{(r)} \qquad r \neq 1$$

resulting from the orthogonality relations Eq. (6).

The original problem Eq. (8) is replaced by the problem

$$(C^{-1}MA_1 - \lambda E)x = 0$$

in which $\lambda = 0$ is now the characteristic value pertaining to $x_{(1)}$.

When interpreted as the substitution of a deflated inertia matrix

$$M_1 = MA_1 = A_1'M = M - \frac{Mx_{(1)}[Mx_{(1)}]'}{x_{(1)}'Mx_{(1)}}$$

in place of the original M, the method reduces to zero the effective inertia of the mode to be eliminated.

$$x_{(1)}'M_1x_{(1)} = x_{(1)}'MA_1x_{(1)} = 0$$

thereby raising its frequency to infinity, while the other effective inertias are left unaltered

$$x_{(r)}'M_1x_{(r)} = x_{(r)}'MA_1x_{(r)} = x_{(r)}'Mx_{(r)} \qquad r \neq 1$$

(3) Inertial Deflation of Free Modes and the Rayleigh-Ritz Method

The Rayleigh-Ritz method uses a finite expansion in assumed displacement functions for the continuous structure, each multiplier being conceived as a generalized coordinate. The strain energy is computed from the strains obtained by differentiation of the displacements and is ultimately a quadratic homogeneous form in the generalized coordinates with matrix C. The kinetic energy is a quadratic homogeneous form of matrix M in the time derivatives of the generalized coordinates. As a consequence the vibration problem is obtained in the form presented by Eq. (1). From the positive definite nature of the kinetic energy the matrix M is never singular and suggests the use of an iteration process based on the modified equation

$$(M^{-1}C - \omega^2E)x = 0$$

with inherent convergence toward the mode of highest frequency. Several procedures are available[5] for shifting convergence to the lower modes. Wielandt's method for instance[9] is applicable and enjoys the advantage of determining the modes independently and unaffected by round-off errors due to the elimination of other modes. It requires, however, some a priori knowledge of the eigenvalues and costly matrix inversions. In the case of unrestrained systems the more promising formulation Eq. (8) is unavailable, since, in view of Eq. (6), the C matrix is singular and its reciprocal, the matrix of static influence coefficients, does not exist.

However, such a formulation exists in the vectorial subspace of the elastic modes and will be shown to follow from inertial deflation of the free modes.

The projection matrix used here is

$$A = E - \sum_1^m \frac{u_{(i)}u_{(i)}'M}{u_{(i)}'Mu_{(i)}} \tag{11}$$

with the selective properties resulting from Eqs. (6)

$$Au_{(i)} = 0 \qquad (i = 1, 2 \ldots m) \tag{12}$$

$$Ax_{(r)} = x_{(r)} \qquad (r = 1, 2 \ldots n) \tag{13}$$

In view of Eqs. (13) the elastic modes of the modified eigenvalue problem

$$(C - \omega^2MA)x = 0 \tag{14}$$

are identical in shape and eigenvalue to those of the original Eq. (1). On the other hand, the free modes are now modal vectors of Eq. (14) with arbitrary eigenvalues, since they satisfy both Eqs. (5) and (12). There follows that MA presents the same type of degeneracy as C, which indicates the possibility of a simultaneous reduction to the vectorial subspace of the elastic modes. For definiteness we assume that in the application of the Rayleigh-Ritz expansion to the unrestrained structure a complete set of m independent rigid body displacements is used as representing the m first assumed functions, the n remaining ones involving compatible elastic deformations.

The free modes are then represented in the vectorial space by the unit vectors $e_{(i)}$ $(i = 1, 2 \ldots m)$ defined as having all their elements equal to zero except the ith one which is equal to unity. Any orthogonalized set of free modes, from which the projection operator A should happen to be constructed, is made of linear homogeneous combinations of these unit vectors with a non-zero determinant. Hence from Eqs. (5) and (12) follow the equations

$$Ce_{(i)} = 0 \qquad Ae_{(i)} = 0 \qquad MAe_{(i)} = 0 \qquad (i = 1, 2 \ldots m)$$

indicating that the m first columns of the matrices C, A and MA are composed of zeros. Since further C and MA are symmetrical the matrices are partitioned as follows

$$C = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right| \qquad A = \left|\begin{array}{c|c} 0 & R \\ \hline 0 & B \end{array}\right| \qquad MA = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & N \end{array}\right|$$

K and N are symmetrical nonsingular (n, n) matrices, R and B respectively (m, n) and (n, n) matrices.

It is then natural to consider a similar partitioning of the modal vector

$$x = \left|\begin{array}{c} v \\ \hline z \end{array}\right|$$

where v contains the m first elements, z the n remaining ones. Eq. (14) thereby reduces to the elastic subspace formulation

$$(K - \omega^2N)z = 0$$

identical with that of a restrained system. Iteration may now be performed with $K^{-1}N$ to obtain the lowest frequency elastic mode and successive deflations applied on the basis of the orthogonality relations

$$z_{(r)}'Nz_{(s)} = 0 \qquad z_{(r)}'Kz_{(s)} = 0 \qquad r \neq s$$

holding in the subspace. The m first components of a modal vector may be obtained from the orthogonality conditions

$$e_{(i)}'Mx_{(r)} = 0 \qquad (i = 1, 2 \ldots m)$$

implied by Eqs. (6b), and also more directly from the selective property expressed by Eq. (13) in the form

$$v_{(r)} = Rz_{(r)} \tag{15}$$

(4) Rigid Body Equilibrium Conditions and the Complementary Energy Method

It is a feature of the complementary energy method that internal and external equilibrium should be verified a priori. No difficulties are to be expected from unrestrained systems since the rigid body equilibrium conditions, implied by each free mode, are used to eliminate the free mode coordinates. It is however instructive to show how the use of the projection matrix A is effective in carrying out the elimination.

The static problem

$$Cq = p \tag{16}$$

to which Eq. (2) reduces on the assumption of zero accelerations admits of a finite solution for q, but under restrictive conditions. From a classical theorem of algebra[10] it is necessary and sufficient that p should be orthogonal to all the solutions of the homogeneous adjoint equation

$$C'r = 0$$

Since in this case $C = C'$ the conditions are explicitly

$$u_{(i)}'p = p'u_{(i)} = 0 \qquad (i = 1, 2 \ldots m) \tag{17}$$

They state that the virtual work of applied forces must vanish when the displacements are in the ratios given by a free mode and they are consequently the expressions of all the rigid body equilibrium conditions of the problem. Now, since M is a nonsingular matrix, the vectors $Mu_{(i)}$ and $Mx_{(r)}$, each one proportional to the inertia force distribution associated with the mode, are linearly independent and form a complete set in terms of which any arbitrary vector p admits of a unique expansion

$$p = \sum_1^m \alpha_i Mu_{(i)} + \sum_1^n \beta_r Mx_{(r)} \tag{18}$$

Premultiplying by $u_{(i)}'$ or $x_{(r)}'$, and using Eqs. (6) again, the coefficients of the expansion are found to be

$$\alpha_i = \frac{u_{(i)}'p}{u_{(i)}'Mu_{(i)}} \qquad \beta_r = \frac{x_{(r)}'p}{x_{(r)}'Mx_{(r)}} \tag{19}$$

When these values are substituted in Eq. (18), and since p is arbitrary, the following useful expansion is obtained:

$$E = \sum^m \frac{Mu_{(i)}u_{(i)}'}{u_{(i)}'Mu_{(i)}} + \sum_1^n \frac{Mx_{(r)}x_{(r)}'}{x_{(r)}'Mx_{(r)}} \tag{20}$$

Eqs. (19) show the conditions (17) for static equilibrium are equivalent to

$$\alpha_i = 0 \qquad (i = 1, 2 \ldots m)$$

They require that p should contain no inertial component from the free modes. Hence, starting from an arbitrary p, the vector

$$p - \sum_1^m \alpha_i Mu_{(i)} = \left(E - \sum_1^m \frac{Mu_{(i)}u_{(i)}'}{u_{(i)}'Mu_{(i)}}\right) p = A'p \tag{21}$$

is the most general one fulfilling the requirements. This result is susceptible for a well-known dynamic interpretation.[2] On the assumption that the system is completely rigidified, the displacements are restricted to free modes and the equations of motion to Eqs. (7a). Now, from Eq. (21)

$$A'p = p - \sum_1^m \frac{u_{(i)}'p}{u_{(i)}'Mu_{(i)}} Mu_{(i)}$$

and in view of Eqs. (7a)

$$A'p = p - \sum_1^m \ddot{\eta}_i Mu_{(i)}$$

The fact that $A'p$ satisfies the rigid body equilibrium conditions then clearly results from an application of d'Alembert's principle to the rigidified structure; or, to borrow a statement from reference 3, the application of $A'p$ is equivalent to "consider the unrestrained system to be in equilibrium such that the external load is balanced by the inertia forces corresponding to the acceleration of the system as a rigid body due to the applied load." This new property of the matrix A may be used to advantage when setting up the vibration problem with the complementary energy method.[11] The same displacement functions are assumed as for the Rayleigh-Ritz method, and the structure is loaded by the resulting inertia forces, whose amplitudes for harmonic motion have the vectorial representation

$$p = \omega^2 Mx$$

We multiply to the left by A'

$$A'p = \omega^2 A'Mx = \omega^2 M(Ax)$$

and conclude that, whatever be x, the amplitude vector Ax is such that the resulting inertia force distribution is in equilibrium. Hence taking for x in succession all the unit vectors, what amounts to take for Ax the successive columns of A, we explore all the possibilities for equilibrium. Now, from its construction, A has exactly the degeneracy m and its n non-zero columns yield exactly n independent inertia force distributions satisfying rigid body equilibrium. For each of those the corresponding stress distribution should be computed by integration of the internal equilibrium and compatibility equations. From the stresses the maximum strain energy follows, whose expression for an arbitrary initial amplitude vector x will be of the form

$$(1/2)\omega^4 x' C^* x$$

The element common to the ith row and jth column of the elastance matrix C^* is $e_{(i)}'C^*e_{(j)}$, and belongs by definition to the interference energy (or energy itself if $i = j$) between the stress distributions due to the displacement amplitudes $Ae_{(i)}$ and $Ae_{(j)}$. Observing that the first m stress distributions vanish identically together with the m first columns of A

$$Ae_{(i)} = 0 \qquad (i = 1, 2, \ldots m)$$

the first m rows and columns of C^* must also vanish and

$$C^* = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & K^* \end{array}\right|$$

where K^* is a symmetrical, nonsingular (n, n) matrix.

For the maximum kinetic energy we have

$$(1/2)\omega^2(Ax)'M(Ax) = (1/2)\omega^2 x'A'MAx = (1/2)\omega^2 x'MA^2x$$

However, as a result of its definition and selective properties, a projection matrix enjoys the property of idempotence $A^2 = A$, whence the maximum kinetic energy is also $(1/2)\omega^2x'MAx$.

We apply now that form of Hamilton's principle which asserts that the difference between maximum strain energy and maximum kinetic energy is stationary with respect to arbitrary variations in the displacement amplitudes and obtain the vibration problem in the form

$$(\omega^2 C^* - MA)x = 0$$

In view of the structures of the matrices involved the problem is again carried over in the subspace of the elastic modes:

$$(\omega^2 K^* - N)z = 0$$

In order to obtain the low frequency modes, iteration should be conducted with the matrix $N^{-1}K^*$ and deflation set up with the aid of the elastance matrix K^*. It is known from general theory that the complementary energy method yields better eigenvalues than the corresponding Rayleigh-Ritz procedure.[12]

(5) Extended Influence Coefficients

From the considerations developed in section (4) it appears that the static problem

$$Cq = A'p \tag{22}$$

has always a solution for arbitrary p. The general solution is

$$q = P_0 p + \sum_1^m \gamma_i u_{(i)} \tag{23}$$

where the first term represents any particular solution. The arbitrary constants γ_i may be represented by the scalar product of p with arbitrary vectors g_i

$$\gamma_i = g_i'p$$

so that the general solution takes the form

$$q = \left(P_0 + \sum_1^m u_i g_i'\right)p = Pp \tag{24}$$

P is the general matrix of extended influence coefficients.

The substitution of Eq. (24) in Eq. (22) leads, since p is arbitrary, to the following fundamental equation, satisfied by extended influence coefficients

$$CP = A' \qquad \text{or} \qquad P'C = A \tag{25}$$

It should be compared with the equations

$$CC^{-1} = C^{-1}C = E$$

holding for restrained systems.

A new formulation of the vibration problem, extending to unrestrained systems the properties of Eq. (8), is obtained from Eq. (14) by left-multiplying with P' and using Eq. (25)

$$(E - \omega^2 P'M)Ax = 0$$

or, in a form adapted to iteration

$$(P'M - \lambda E)y = 0 \qquad \text{where } y = Ax \tag{26}$$

In view of Eqs. (13), the elastic modes are retained with their original characteristic values. The original free modes however degenerate because of Eq. (12) in the trivial solution $y = 0$. It may be suspected, and examples do confirm, that the place of free modes is generally taken by parasitic modes. It is most important that the parasitic modes be eliminated in the iteration process and this may be achieved by a careful choice of the g_i vectors in relation to the particular matrix P_0 adopted. For example, we may require that a set of m parasitic modes $w_{(i)}$ be eigensolutions of Eq. (26) with zero characteristic value, and, therefore, obtain as conditions

$$P'Mw_{(j)} = P_0'Mw_{(j)} + \sum_{i=1}^m [u_{(i)}'Mw_{(j)}]g_{(i)} = 0 \qquad (j = 1, 2 \ldots m)$$

So long as the $w_{(j)}$ and $x_{(i)}$ are linearly independent this yields n nonsingular linear systems of the mth order for the $m \times n$ components of the g_i vectors.

(6) The Symmetrical Extended Influence Coefficients

As a particular choice, the parasitic modes of zero characteristic value may be the free modes themselves, in which case we obtain, on account of Eqs. (6a),

$$g_i = -\frac{P_0'Mu_{(i)}}{u_{(i)}'Mu_{(i)}}$$

The resulting matrix of extended influence coefficients is

$$G = \left(E - \sum_1^m \frac{u_{(i)}u_{(i)}'M}{u_{(i)}'Mu_{(i)}}\right)P_0 = AP_0$$

It must be independent of the particular choice P_0.

Indeed, we have the more general result

$$AP = AP_0 = G \tag{27}$$

which follows immediately from the definition of P included in Eq. (24) and from Eqs. (12).

It will now be proven that G is symmetrical. As a particular P matrix it satisfies both Eqs. (25) and (27)

$$G'C = A \qquad AG = G$$

and therefore

$$G'CG = G$$

Now the left side is obviously symmetrical, which proves the proposition.

When symmetrical extended influence coefficients are employed, Eq. (26) specializes in

$$(GM - \lambda E)y = 0 \tag{28}$$

This is the most natural extension to unrestrained systems of the formulation Eq. (8) an the associated iteration procedures successfully applied to restrained systems. Compared with the original Eq. (1), its eigensolutions and eigenvalues are unchanged except for the free modes whose frequency has been shifted from zero (infinite characteristic value) to infinity (zero characteristic value).

An explicit form of the extended influence coefficients may be found in the case of the Rayleigh-Ritz procedure described in section (3). Partitioning the vectors

$$q = \left[\begin{array}{c} a \\ \hline b \end{array}\right] \qquad p = \left[\begin{array}{c} r \\ \hline s \end{array}\right]$$

to match the partitioned forms of the matrices C and A, Eq. (22) reduces to

$$Kb = R'r + B's$$

The general solution for q is consequently

$$\begin{cases} a \text{ arbitrary} \\ b = K^{-1}R'r + K^{-1}B's \end{cases}$$

and a particular extended influence coefficients matrix is

$$P_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline K^{-1}R' & K^{-1}B' \end{array}\right]$$

From Eq. (27) follows then

$$G = \left[\begin{array}{c|c} RK^{-1}R' & RK^{-1}B' \\ \hline BK^{-1}R' & BK^{-1}B' \end{array}\right] \tag{29}$$

Nothing much is gained in this case. Actually the iteration with $K^{-1}N$ is more economical, both as an initial investment and because it proceeds in the subspace of elastic modes only. The use of extended influence coefficients becomes a practical proposition when it is combined in the integral equation approach with collocation and the use of an isostatic reference frame.

(7) Collocation and Isostatic Reference Frames

Imagine any group of constraints applied to structure so as to make it isostatically restrained with respect to a rigid reference frame. The relative displacements of the structure are then given by

$$q = G_{\text{iso.}}p$$

where $G_{\text{iso.}}$ is a symmetrical matrix of generalized influence coefficients in the usual sense of relative elongations in the generalized coordinates due to unit generalized loads. We know that if p is replaced by $A'p$ the applied load fulfills the requirements of rigid body equilibrium. Consequently, the constraining loads with respect to the isostatic reference frame must vanish. Hence

$$q = G_{\text{iso.}}A'p$$

is a possible displacement vector of the unrestrained structure. The role played by the reference frame is thereby reduced to determine unique values γ_i, depending on the nature of the constraints, for the otherwise arbitrary free-mode components of the displacement [cf. the general solution Eq. (23)]. It may be concluded therefore that

$$P_0 = G_{\text{iso.}}A' \tag{30}$$

is a particular matrix of extended influence coefficients and

$$G = AG_{\text{iso.}}A' \tag{31}$$

the obviously symmetrical one. These expressions furnish us with practical means for computing the extended coefficients. From them we obtain the formulation

$$(AG_{\text{iso.}}A'M - \lambda E)y = 0 \tag{32}$$

derived from Eq. (28), and the simpler one

$$(AG_{\text{iso.}}M - \lambda E)y = 0 \tag{33}$$

deriving from Eqs. (26) and (30).

In the last case the parasitic modes must be shown to be harmless. They are, because it happens that their characteristic value is zero. To prove this it suffices to show that the equation

$$AG_{\text{iso.}}My = 0$$

possesses m linearly independent solutions for y; because of the existence of n elastic modes in Eq. (33) it cannot have more than m. Now, from a classical theorem of algebra,[10] the adjoint equation

$$MG_{\text{iso.}}A'y = 0$$

should then also have m independent solutions; but this is obviously true since

$$A'y = 0$$

has m solutions corresponding to the m solutions $u_{(i)}$ of the adjoint equation

$$Ay = 0$$

Examples of lumped mass systems treated in a manner equivalent to Eq. (33) are given by S. Levy[13] and by Scanlan and Rosenbaum.[14] In the notations of this paper their mathematical treatment is along the following lines. The equation

$$x = \omega^2 G_{iso.} Mx + \sum_1^m \alpha_i u_{(i)} \quad (34)$$

expresses that the absolute displacement amplitude due to the application of the vibratory inertia loads differs from the amplitude in the reference frame by a set of free-mode components. The intensities α_i of these components are determined from the conditions

$$\omega^2 u_{(j)}' Mx = 0 \qquad (j = 1, 2 \ldots m)$$

of rigid body equilibrium for the applied loads. They yield

$$-\alpha_j = \omega^2 \frac{u_{(j)}' MG_{iso.} Mu_{(j)}}{u_{(j)}' Mu_{(j)}}$$

and, when substituted in Eq. (34)

$$x = \omega^2 \left[E - \sum_1^m \frac{u_{(j)} u_{(j)}' M}{u_{(j)}' Mu_{(j)}} \right] G_{iso.} Mx$$

which brings us back to Eq. (33).

The formulations Eqs. (32) and (33) are specially interesting when used in conjunction with collocation. Whatever be the kernel considered for the integral equation of the problem (usual Green's function in the isostatic reference frame or extended Green's function for the unrestrained system), any type of numerical integration, using the values of the integrand at several division points, and collocation at the same points, yields an equivalent lumped mass model with a diagonal mass matrix. The elements of $G_{iso.}$ are then local values of the usual Green's function in the reference frame, whose definition may be chosen in order to obtain simple checks against experimental data. The elements of $G_{iso.}A'$ or G are local values of the corresponding extended Green's functions. As a final practical observation, the invaluable Rayleigh quotient

$$\omega^2 = \frac{x' Mx}{x' MGMx}$$

is seen to be identical in view of Eqs. (31) and (13) and of the symmetry of MA to the simpler one

$$\omega^2 = \frac{x' Mx}{(Ax)' MG_{iso.} M(Ax)} = \frac{x' Mx}{x' MG_{iso.} Mx} \quad (35)$$

This makes the computation of G unnecessary when Eq. (33) is employed for iteration.

(8) Numerical Example

The example here concerns the normal vibration modes of a free elastic beam of constant cross section carrying a concentrated mass at the left end, equal to the total mass of the beam [see Fig. 1 (a)].

From the differential equation and boundary conditions the exact frequency equation is found to be

$$\mu(\sinh \mu \cos \mu - \cosh \mu \sin \mu) = 1 - \cosh \mu \cos \mu$$

in which $\mu^4 = \omega^2 ma^3/EI$. The mode shapes are given by

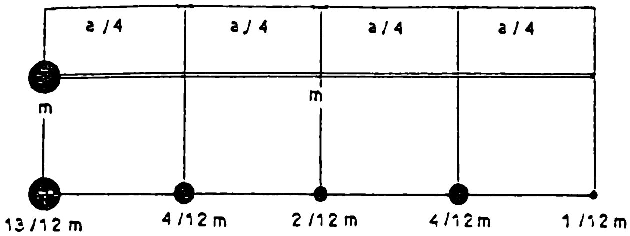

FIG. 1. (a) Elastic beam with concentrated mass, and (b) lumped mass model obtained from Simpson's rule and collocation.

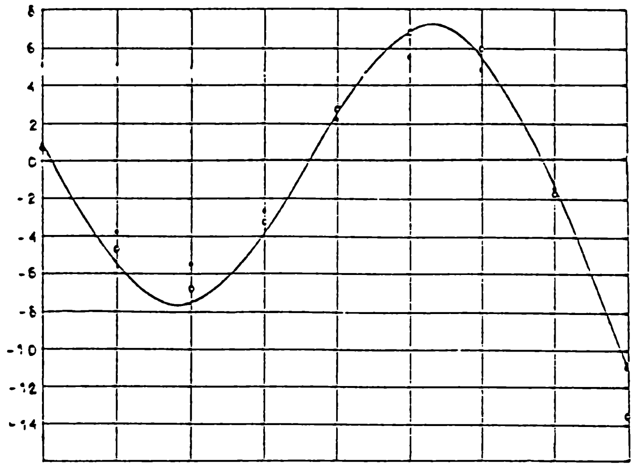

FIG. 2. Comparison of second elastic mode shapes on the basis of same effective mass:

—— = exact mode
● = lumped mass model
○ = lumped mass model after recomputation of the effective mass.

$$y(\zeta) = B(\cosh \mu\zeta + \cos \zeta) - A(\sinh \mu\zeta + \sin \mu\zeta)$$

where ζ is a dimensionless coordinate running from -1 at the left to zero at the right, and

$$A(\sinh \mu - \sin \mu) = B(\cosh \mu - \cos \mu)$$

The effective mass of a mode may be computed from

$$[y^2(-1) + \int_{-1}^{0} y^2(\zeta) d\zeta] m = \eta m$$

with $\qquad \eta = B^2 + (1/4) y^2(-1)$

The two first elastic modes have the following characteristics from which the approximate results found below may be evaluated:

First mode

$\mu = 4.041833 \qquad \mu^4 = 266.878 \qquad \eta = 40.96494$
$A = 6.290534 \qquad B = 6.321783$
normalized to have $y(-1) = 2$

Second mode

$\mu = 7.1340 \qquad \mu^4 = 2590 \qquad \eta = 29.43225$
$-A = 5.402871 \qquad -B = 5.402060$
normalized to have $y(-1) = 1$

This mode is represented by the continuous line of Fig. 2.

(a) *Rayleigh-Ritz*

All boundary conditions are for stresses, none for displacements. The assumed functions express

$$y = q_1 + q_2\zeta + q_3(2 - 3\zeta + \zeta^3)$$

features the translation free mode, the pitching free mode about the right end, and the elastic displacement due to a load applied at the concentrated mass, right end clamped. It satisfies but one of the stress boundary conditions: $y''(-1) = 0$.

The result is bad both for mode shape and frequency parameter

$$\mu^4 = 575.34$$

This serves to indicate that, although not required by the variational principle, good results with few assumed functions can only be obtained by satisfying also most of the stress boundary conditions. From other published results[15] this is a critical point to watch in all cases of unrestrained systems.

The next expansion tried,

$$y = q_1 + q_2\zeta + q_3(\zeta - 2\zeta^3 + \zeta^4)$$

satisfies both boundary conditions for flexural moments but not for shear loads. The elastic part is the deflection due to uniform loading of the beam on two end supports.

The main results are contained in Table 1. The error on the frequency parameter is about 5 per cent. The mode shape is here proportional to the last column of A. It was first chosen to be normalized like the first exact mode

$$y = 2 - 38\zeta + 100\zeta^3 - 50\zeta^4$$

It compares well and even slightly better if the comparison is made, as it should be, on the basis of the same effective mass. For this purpose it should be altered in the ratio

$$\left[\frac{40.96494}{43.01587}\right]^{1/2} = 0.9759$$

(b) *Complementary Energy*

The inertia loads due to the same expansions as in the Rayleigh-Ritz case were also investigated. Despite the bad mode shape the first expansion yields a fair approximation to the first frequency parameter $\mu^4 = 277.9$. Here are some more details for the second case.

The inertia loading, satisfying rigid body equilibrium, is found according to theory by using the last column of A in Table 1 as vector of displacement amplitude. The load consists of a distributed part

$$\omega^2(m/a)(-0.04 + 0.76\zeta - 2\zeta^3 + \zeta^4)$$

and a concentrated part $-\omega^2 m 0.04$ at the left end. Integration is carried out twice to find the bending moments

$$ma\left[-0.04\zeta - 0.02\zeta^2 + \frac{38}{300}\zeta^3 - 0.1\zeta^5 + \frac{1}{30}\zeta^6\right]$$

and as a check it is verified that the stress boundary conditions may be satisfied at both ends. From the strain energy

$$K^* = \frac{\omega^4 m^2 a^3}{EI}\left\|\frac{86\,887}{135\,135}\,10^{-4}\right\| \text{ and } \mu^4 = \frac{10.84}{630}\,\frac{135\,135}{86\,887} = 267.61$$

which is correct to about 0.27 per cent. Since only one elastic degree of freedom was retained, the mode shape and its effective mass are here the same as in the Rayleigh-Ritz case.

(c) *Collocation and Isostatic Reference Frame*

If one aims at several elastic modes this method is attractive on account of its low initial investment in numerical labor. Two elastic modes, besides the two free ones, were aimed at out of five degrees of freedom. The integral in the integral equation was approximated

TABLE 1
Numerical Results for First Elastic Mode from Rayleigh-Ritz Method

$$M = \frac{m}{630}\left\|\begin{matrix} 1260 & 315 & 126 \\ 315 & 210 & 63 \\ 126 & 63 & 31 \end{matrix}\right\| \qquad u'_{(1)} = (1 \quad 0 \quad 0) \qquad u'_{(2)} = (-1 \quad 4 \quad 0)$$

$$A = \frac{1}{25}\left\|\begin{matrix} 0 & 0 & -1 \\ 0 & 0 & -6 \\ 0 & 0 & 25 \end{matrix}\right\| \qquad N = \left\|\frac{271}{15750}\,m\right\| \qquad K = \left\|\frac{48}{10}\,EI\,a^{-3}\right\|$$

$$\mu^4 = 279.83 \qquad x' = (2 \quad 12 \quad -50) \qquad x'Mx = 43.0\,m$$

by Simpson's rule with four intervals and the subdivision points used for collocation. This is equivalent to the lumped mass model of Fig. 1 (b); the coordinates are the displacements of the concentrated masses.

Iteration was performed according to Eq. (33) and Table 2 gives the main results. The $G_{iso.}$ matrix was computed from the deflections of the beam on two end supports. This causes the first and last columns of the iteration matrix to vanish and allows the temporary discarding of the first and last coordinates during the iteration process. Besides initial investment this is another advantage of Eq. (33) over Eq. (32). In that respect, clamping the beam at one end would have proven slightly less advantageous.

The frequency parameter is correct to about 0.6 per cent; the mode shape (of same effective mass) within plotting accuracy of the exact first mode.

The results of a subsequent deflation and iteration for the second elastic mode are contained in Table 3. The mode shape (black dots in Fig. 2) compares reasonably with the exact one, which suggests that the large discrepancy in the frequency parameter might be due to the crude estimate of the effective mass in the lumped mass model. Indeed, the following correction, which is simple enough to apply, provides substantial improvement. On the assumption (correct for constant cross sections) that the elastic curve between two stations is a cubic, the displacement at mid-station is given by the simple formula

$$y[x + (h/2)] = (1/2)\{y(x) + y(x + h) + (h/4)[y'(x) - y'(x + h)]\}$$

The slopes may be computed in the isostatic reference frame since their differences only are involved. From ordinary beam theory we have for the slopes at the stations due to the concentrated inertia loads

$$\left\|\frac{y'h}{4}\right\| = \omega^2 \frac{a}{16}\frac{a^2}{96EI}\left\|\begin{matrix} 0 & 21 & 24 & 15 & 0 \\ 0 & 12 & 18 & 12 & 0 \\ 0 & -3 & 0 & 3 & 0 \\ 0 & -12 & -18 & -12 & 0 \\ 0 & -15 & -24 & -21 & 0 \end{matrix}\right\| Mx_{(2)} = \left\|\begin{matrix} -\ 4.175918 \\ +\ 2.484788 \\ +\ 7.150795 \\ -\ 2.484788 \\ -10.125672 \end{matrix}\right\|$$

The application of the formula gives the mode values at mid-stations:

$$-8.962658 \quad -6.280293 \quad 13.013875 \quad -3.514210$$

The effective mass is then recomputed from its exact expression evaluating the integral by Simpson's formula with eight intervals. The new value is found to be $106.825m$. The mode shape then compares better (white circles in Fig. 2) with the exact one and the frequency parameter is overcorrected

$$\mu^4 = 1808.74\,\frac{162.558}{106.825} = 2752.4$$

but lies within 6.26 per cent of the exact value.

(9) Conclusions

It is suggested that normal modes of complete aircraft structures in free flight be computed on the basis of lumped inertia models. The results would appear to be reliable provided (1), the models be obtained as a result of some formula for numerical integration combined with collocation applied to the integral equation of the problem; (2), the constraints used to evaluate $G_{iso.}$ conform with those of the static rig for comparison against experimental data; and (3), the higher frequency modes be corrected for effective mass and frequency by techniques similar to the one described.

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TABLE 2
Numerical Results for First Elastic Mode from Lumped Mass Model

$$M = \frac{m}{12}\left\|\begin{matrix} 13 & & & & \\ & 4 & & & \\ & & 2 & & \\ & & & 4 & \\ & & & & 1 \end{matrix}\right\| \qquad G_{iso} = \frac{a^3}{768\ EI}\left\|\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 11 & 7 & 0 \\ 0 & 11 & 16 & 11 & 0 \\ 0 & 7 & 11 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right\|$$

$$u'_{(1)} = (\ 1 \quad 1 \quad 1 \quad 1 \quad 1\ ) \qquad u'_{(2)} = (\ -1 \quad 0 \quad 1 \quad 2 \quad 3\ )$$

$$A = \frac{1}{120}\left\|\begin{matrix} 16 & -20 & -4 & 4 & 4 \\ -65 & 100 & -10 & -20 & -5 \\ -26 & -20 & 104 & -44 & -14 \\ 13 & -20 & -22 & 52 & -23 \\ 52 & -20 & -28 & -92 & 88 \end{matrix}\right\| \qquad x_{(1)} = \left\|\begin{matrix} 428.058 \\ -1369.394 \\ -1540.722 \\ 84.150 \\ 2657.665 \end{matrix}\right\|$$

$$AG_{iso}M = \frac{m\ a^3}{1\ 105\ 920\ EI}\left\|\begin{matrix} 0 & -784 & -480 & -592 & 0 \\ 0 & 2600 & 1440 & 1640 & 0 \\ 0 & 2624 & 1920 & 2432 & 0 \\ 0 & -232 & 0 & 344 & 0 \\ 0 & -4528 & -3360 & -5104 & 0 \end{matrix}\right\|$$

$$\mu_1^4 = 268.466 \qquad x'_{(1)}M\,x_{(1)} = 1\ 810\ 180\ m\ .$$

TABLE 3
Numerical Results for Second Elastic Mode from Lumped Mass Model and Deflation of the First Mode

$$AG_{iso}MA_1 = \frac{m\ a^3}{1105920EI}\left\|\begin{matrix} -451.729 & -339.348 & -229.858 & -619.324 & -215.741 \\ 1445.119 & 1177.520 & 639.775 & 1727.412 & 690.173 \\ 1625.921 & 1023.551 & 1019.657 & 2530.348 & 776.522 \\ -88.804 & -144.588 & 49.174 & 338.628 & -42.412 \\ -2804.629 & -1767.308 & -1806.955 & -5273.646 & -1339.460 \end{matrix}\right\|$$

$$x'_{(2)} = (\ 1.170\ 186 \quad -13.004\ 797 \quad 5.110\ 218 \quad 11.281\ 949 \quad -25.951\ 253\ )$$

$$\mu_2^4 = 1\ 808.74 \qquad x'_{(2)}M\,x_{(2)} = 162.558\ m\ .$$

Appendix

Symmetrical Extended Coefficients as a Limiting Case of Dynamic Flexibilities

Harmonic excitation $p = e^{i\omega t}y$ produces a forced response $q = e^{i\omega t}x$ whose amplitude must satisfy $(C - \omega^2 M)x = y$. To the expansion of x in normal modes

$$x = \sum_1^m \alpha_i u_{(i)} + \sum_1^n \beta_r x_{(r)} \tag{36}$$

there corresponds an expansion for y, that in view of Eqs. (4) and (5) may be written in the form

$$y = -\omega^2 \sum_1^n \beta_i M u_{(i)} + \sum_1^n \beta_r(\omega_r^2 - \omega^2) M x_{(r)} \tag{37}$$

So long as the forcing frequency is neither zero nor equal to one of the natural frequencies, the expansion (37), similar to Eq. (18) is capable of representing any forcing amplitude. The coefficients must then have the values

$$-\alpha_i = \frac{u_{(i)}'y}{\omega^2 u_{(i)}' M u_{(i)}} \qquad \beta_r = \frac{x_{(r)}'y}{(\omega_r^2 - \omega^2) x_{(r)}' M x_{(r)}}$$

which, when substituted in Eq. (36), yield an expansion formula for the dynamic flexibilities matrix

$$x = (C - \omega^2 M)^{-1} y$$

$$(C - \omega^2 M)^{-1} = -\frac{1}{\omega^2} \sum_1^m \frac{u_{(i)} u_{(i)}'}{u_{(i)}' M u_{(i)}} + \sum_1^n \frac{x_{(r)} x_{(r)}'}{(\omega_r^2 - \omega^2) x_{(r)}' M x_{(r)}}$$

For restrained systems the influence coefficients matrix is obtained as a limiting case by letting the forcing frequency vanish

$$C^{-1} = \sum_1^n \frac{x_{(r)} x_{(r)}'}{\omega_r^2 x_{(r)}' M x_{(r)}} \tag{38}$$

It may be shown that the same expression applies to the matrix G for unrestrained systems. In other words, this matrix obtains as a limiting case of dynamic flexibilities by removing the free mode components of the expansion which have a simple pole at $\omega^2 = 0$, before passing to the limit.

To prove it, denote by G the expansion (38) and observe that it already is symmetrical from structure and that it satisfies the relation

$$AG = G$$

as a result of Eqs. (13). There remains to show that it satisfies Eq. (25). To this effect multiply to the left by C and use Eq. (4)

$$CG = \sum_1^m \frac{M x_{(r)} x_{(r)}'}{x_{(r)}' M x_{(r)}}$$

Now, on account of the expansion (20), there follows

$$CG = E - \sum_1^m \frac{M u_{(i)} u_{(i)}'}{u_{(i)}' M u_{(i)}} = A'$$

Q.E.D.