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A VARIATIONAL APPROACH TO PURE MODE EXCITATION BASED ON CHARACTERISTIC PHASE LAG THEORY

by

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SUMMARY

Characteristic phase lag theory is applied to the derivation of an energy principle for pure mode excitation at a resonant frequency. The principle is not restricted to small damping nor to a specific variation of damping coefficients with frequency.

Preliminary remarks on structural damping are followed by a discussion of the simplifying assumption which makes the damping matrix proportional to the stiffness matrix. This procedure allows a gradual presentation of the general analysis based on the concept of characteristic phase lag. Both the torsional and the flexural response of a beam to single point excitation are presented as examples for the influence of internal damping on continuous structures.

SOMMAIRE

Pour obtenir expérimentalement des modes de vibration propres d'une grande pureté il est proposé de baser le réglage d'un groupe d'excitateurs sur un principe énergétique.

Ce principe est valable quelle que soit la nature et la distribution de l'amortissement interne pourvu que l'énergie dissipée par cycle demeure proportionnelle au carré de l'amplitude.

Quelques remarques préliminaires sur la représentation de l'amortissement interne sont suivies d'une discussion du cas simplifié où l'amortissement et l'élasticité de la structure sont dans un rapport constant. Le concept de déphasage caractéristique qui permet de se libérer de cette hypothèse est alors introduit comme base de démonstration du principe énergétique. Celui-ci affirme que l'énergie réactive fournie par cycle est stationnaire pour un réglage des excitateurs donnant un mode pur avec résonance de phase.

A titre d'exemple la réponse d'une poutre continue libre avec amortissement interne à une excitation localisée est analysée pour les trois premières résonances de phase en torsion et en flexion.

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NOTATION

q	column vector of generalized displacements
p	column vector of generalized external disturbances
x	column vector of frequency response amplitudes = a - ib
y	column vector of sinusoidal input amplitudes
x_r	column vector of elastic undamped mode
u_i	column vector of rigid or free mode
r_k	column vector of frequency response mode associated with a characteristic phase lag
y_k, x_i	column vector of excitation mode
M	symmetrical positive definite matrix of inertia coefficients
K	symmetrical semi-definite matrix of elasticity coefficients
C	symmetrical matrix of operational damping terms
G	symmetrical positive definite matrix of structural damping coefficients
ω	circular frequency
t	time
s	Laplace transform variable
ϕ_r	characteristic phase lag angle
γ	proportionality coefficient between K and G
μ	effective inertia coefficient for an undamped mode
E_a	active energy dissipated per cycle = $\pi y'b$
E_r	reactive energy = $\pi y'a$

Note: The row vector obtained by transposition of a column vector is denoted by the same symbol with a 'prime' (e.g. u_i'). In a similar fashion an asterisk* indicates transposition and replacement of the elements by their complex conjugates.

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1. INTRODUCTION

From general thermodynamical considerations structural damping may be represented by operational stress-strain relations. The correlative variations of stiffness with frequency are shown on the example of a simple Maxwell model. These considerations are necessary to fix the general character of validity underlying later developments, though the usual engineering assumption is retained for simplicity of presentation.

The case of proportionality between damping and stiffness distributions is first discussed, not only because an important part of previous research work rests on such a foundation, but also because it allows the general theory to be presented later as a natural extension of its energy properties.

The frequency response of continuous structures to single point excitation are worked out for the uniform free beam with uniform structural damping in torsion and in flexure. These examples illustrate the relative importance of the total vibration in phase, the total vibration in quadrature and the resonant term of an expansion in natural modes.

Characteristic phase lag theory is next applied to the case of arbitrary (and even large) damping distribution. It yields an expansion of the frequency response which retains the principal features of the expansion in normal modes, to which it reduces under the assumption of proportional damping.

Finally an energy principle is derived, which states that the total reactive energy input per cycle is stationary with respect to small variations in the distribution and intensity of the exciting forces provided there is simultaneously resonance and pure mode excitation. The indications of total reactive energy are consequently useful to check the optimum testing conditions. They may also provide a method for gradual adjustment of the individual shaker forces although the procedure suggested needs further investigation to establish its convergent character.

2. STRUCTURAL DAMPING

The assumption of viscous damping allows a simple mathematical formulation of damped motion problems. The single degree of freedom equation

$$m_1 \frac{d^2 q_1}{dt^2} + c \frac{dq_1}{dt} + k_1 q_1 = Q_1(t) \qquad (1)$$

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is typical. It governs the small displacement $q_1(t)$ about the equilibrium position under the influence of an external disturbance $Q_1(t)$.

This formulation does not represent adequately the effects of structural damping.

Simple mechanical models have been imagined to represent the anelastic behaviour of structural materials. They usually retain the simple viscous damper or dashpot as the type of damping, but they produce the required feature of heredity (or dependence on the past history of motion) by means of 'hidden' coordinates. The simplest example is the model of Fig. 1, which consists of a spring and a Maxwell unit in parallel and is actually a viscous damped system with two degrees of freedom. Its behaviour is fully described by the system of equations

$$m \frac{d^2 q_1}{dt^2} + k_1 q_1 + c \frac{d}{dt} (q_1 - q_2) = Q_1(t) \tag{2}$$

$$k_2 q_2 - c \frac{d}{dt} (q_1 - q_2) = 0 \tag{3}$$

Advantage may be taken of the fact that the second equation is free of external disturbance to carry out the elimination of q_2 which then becomes the 'hidden' coordinate. Assuming the system to be at rest in the equilibrium position at $t = 0$ one obtains for q_1, the single integro - differential equation[1]

$$m \frac{d^2 q_1}{dt^2} + k_1 q_1 + \int_0^t \phi(t - \tau) \frac{dq_1}{d\tau} d\tau = Q_1(t) \tag{4}$$

where, in this particular case, the heredity function is

$$\phi(t) = k_2 e^{-\frac{k_2}{c} t} \tag{5}$$

The ordinary type of eq. (1) must be recovered when k_2 tends to infinity.

The limiting form of the heredity function is there $c\delta(t)$, where $\delta(t)$ is Dirac's impulse function. Eq. (1) then follows from eq. (4) on account of the sifting property of the Dirac function.

Very general forms of heredity functions may be obtained by connecting several Maxwell units in parallel and multiplying the number of hidden coordinates.

A simple method to deal with eq. (4) is to take its Laplace transform. If $q_1(s)$, $\phi(s)$ and $Q_1(s)$ denote rebpectively the Laplace transforms of $q_1(t)$, $\phi(t)$ and $Q_1(t)$, one obtains instead of eqs. (4) and (5)

$$[ms^2 + k_1 + s\phi(s)]\, q_1(s) = Q_1(s) \tag{4'}$$

$$\phi(s) = c \frac{k_2}{k_2 + cs} \tag{5'}$$

The limiting form of $\phi(s)$ when k_2 tends to infinity is simply the constant c, which agrees with the fact that the Laplace transform of the Dirac impulse function is unity.

In the case of several Maxwell units in parallel the operational structure of the heredity function would be

$$\phi(s) = \Sigma \frac{\beta_i}{\alpha_i + s}$$

If the spring and the Maxwell units are thought of as a model for the anelastic properties of a single spring, one is led to represent the modulus of the spring by an operational term of the form

$$k_1 + s\phi(s) = \psi(s)$$

Secondary creep may be accounted for by suitable modifications in the model (essentially the chain of reaction forces would have to pass somewhere through a single dashpot), then

$$\psi(s) \to o \quad \text{when} \quad s \to o$$

since the static deflection is then unbounded.

In a similar manner stress-strain relations of continuous media with anelastic behaviour are obtained when the elastic constants are replaced by suitable operational expressions. Biot[2] has recently justified such relations on the basis of ONSAGER'S Thermodynamical reciprocity law for coupled irreversible phenomena associated with the use of hidden variables.

It will be sufficient for our purpose to discuss systems with a finite number of degrees of freedom. This procedure has the advantage of allowing a general formulation in matrix notation, the results of which are easily extended to continuous media. Accordingly we define

q a column vector of generalized displacement functions

p a column vector of generalized external disturbances

and replace eq. (1) by the matrix equation

$$M \frac{d^2}{dt^2} q + C \frac{d}{dt} q + K q = p \qquad (6)$$

where

M is a symmetrical positive definite matrix of inertia coefficients

K a semi-definite positive and symmetrical matrix of stiffness coefficients

C a symmetrical matrix of damping coefficients.

To account for structural damping the second term should be understood as

$$C * \frac{d}{dt} q$$

The symbol * meaning that each product of a damping term in C and a component velocity is a convolution integral involving some heredity function. Again it is simpler, from an algebraic standpoint, to consider the Laplace transform of eq. (6) which, under the assumption that there is equilibrium at t = o, would be

$$[s^2M + K + sC(s)]q(s) = p(s) \tag{7}$$

The elements of C are now operational expressions, this fact being indicated by the notation C(s).

Our chief interest lies in the frequency response of the system. Putting in (eq. 6)

$$p = ye^{i\omega t} \qquad q = xe^{i\omega t}$$

the relationship between the imput vector y and the forced output vector x is obtained by the substitution of $i\omega$ in the place of s

$$[K - \omega^2 M + i\omega C(i\omega)]\, x = y \tag{8}$$

Now the elements of $C(i\omega)$ are functions of $i\omega$ with real coefficients, hence

$$C(i\omega) = R(\omega^2) + i\omega I(\omega^2) \tag{9}$$

Since a change of sign in ω is equivalent to changing i into -i, this shows that in general the effective stiffness matrix is

$$K - \omega^2 I(\omega^2)$$

which is a function of frequency; and the matrix of equivalent viscous damping coefficients is

$$R(\omega^2)$$

also a function of frequency.

If for example, we return to the case of the model of Fig. 1, which has a frequency response

$$\left(k_1 - \omega^2 m + i\omega \frac{ck_2}{k_2 + i\omega c}\right) x_1 = y_1$$

the effective spring stiffness is

$$k_{eff} = k_1 + k_2 \frac{\omega^2 c}{k_2^2 + \omega^2 c^2} \tag{10}$$

and the equivalent coefficient of viscous damping

$$c_{eq} = c \frac{k_2^2}{k_2^2 + \omega^2 c^2} \tag{11}$$

The increase in spring stiffness with frequency is in qualitative agreement with the observed difference between isothermal and adiabatic moduli. To account for experimental evidence the simple engineering law

$$c_{eq} = g \frac{k_1}{\omega} \tag{12}$$

has been proposed for structural damping and extensively used in flutter calculations[3, 4]. Though it is not possible for a Maxwell model to follow this law exactly, quantitative agreement can be obtained over a wide range of testing frequencies. Let us examine this in more detail.

Put

$$H = k_2/k_1 \qquad \omega_0 = k_1/c \qquad \mu = \omega/\omega_0$$

then eqs. (10), (11) and (12) become respectively

$$k_{eff}/k_1 = 1 + \frac{H\mu^2}{H^2 + \mu^2} \tag{10'}$$

$$c_{eq} \cdot \frac{\omega_0}{k_1} = \frac{1}{1 + \mu^2 H^2} \tag{11'}$$

$$c_{eq} \cdot \frac{\omega_0}{k_1} = g/\mu \tag{12'}$$

Equating (11') and (12') at some value μ^* of the reduced frequency

$$g\mu^{*2} - H^2\mu^* + gH^2 = 0$$

The condition for a double root in μ^* yields

$$2g = H \qquad \mu^* = H \tag{13}$$

and a second order contact between the curves (11') and (12').

If for instance $g = 0.02$, which is on the lower side of observed values, we should have $H = 0.04$, which means that the asymptotic value of the effective modulus of the spring would be 4% higher than its static value.

This seems more than the observed difference but at the frequency of agreement

$$\mu^* = 0.04$$

the difference is only 2%.

Both damping laws and the effective spring stiffness ratio (10′) are plotted in Fig.3. There is a wide frequency range of agreement for damping.

The dashpot coefficient c may be chosen to shift the actual frequency at the contact point where necessary. Of course ω_o is really a physical constant of the material, whose value would result from careful measurements if the Maxwell model is found to be acceptable. Evidently by increasing the complexity of the model almost any experimental damping curve is reproducible in a large frequency range. The problem is to increase our knowledge of the damping curve before fitting a Maxwell model to duplicate it.

For subsequent discussion of the frequency response in n degrees of freedom, the following assumptions will be made:-

(a) The change in stiffness $-\omega^2 I(\omega^2)$ will be ignored,

(b) The equivalent viscous damping matrix will be approximated in the testing frequency range by the engineering law

$$R(\omega^2) = \frac{1}{\omega} G$$

where G is a symmetrical positive definite matrix of constants.

Hence our fundamental eq. (8) becomes

$$(K - \omega^2 M + iG)\, x = y \qquad (14)$$

The second assumption is really unnecessary but is used for the sake of simplicity. The main results will remain true for any damping distribution so long as $R(\omega^2)$ remains positive definite, which is necessarily the case since energy can only be dissipated by the system. Any additional type of damping source may be present such as slip in riveted joints and (even large) artificial damping in control surface deflections.

However, the energy dissipated must remain proportional to the square of the amplitude.

Concerning the first assumption the results will also remain true if stiffness increases remain proportional to K, that is if

$$K - \omega^2 I(\omega^2) = [1 + f(\omega^2)]K$$

It would then be possible to define the undamped natural modes in a rational manner by the eigenvalue problem

$$\left\{[1 + f(\omega^2)]K - \omega^2 M\right\} x = 0 \qquad (15)$$

and retain the benefits of orthogonality relations.

However both for the sake of simplicity and because the case just outlined seems a too special one, we retain our first assumption and define the undamped natural modes by the eigenvalue problem

$$(K - \omega^2 M)\, x = o \tag{16}$$

3. UNDAMPED NATURAL MODES

Eq. (16) is obtained as a special case of eq. (8) without external disturbance, by letting all dahspot coefficients vanish in the structural model. If the model includes secondary creep the dashpots responsible for unbounded static deflections must be frozen by letting their damping coefficient tend to infinity.

The undamped modes are then related to the quasi-static spring stiffnesses. In the frequency response of the damped system the resonant modes will be associated with somewhat higher dynamic moduli. In view of the small increases in stiffness with frequency the difference between a resonant mode and its quasi-static definition may be expected to be small. A measure of magnitude may be found by investigating the undamped modes associated with the asymptotic values of the moduli, all dashpots being frozen.

If the restraints necessary for ground resonance testing are flexible enough to simulate the conditions of free flight, we must allow for the existence of m independent free modes u_i which satisfy the equations

$$Ku_i = o \qquad (i = 1, 2 \ldots m) \tag{17}$$

and involve no elastic deformation energy. For that reason K was assumed to be only semidefinitive positive.

The elastic undamped modes x_r satisfy the equations

$$Kx_r = \omega_r^2 M x_r \qquad (r = 1, 2 \ldots n) \tag{18}$$

We recall the existence of orthogonality relations[5] which in our notations are in the form

$$u'_i\, Mu_j = o \qquad i \neq j \tag{19a}$$

$$u'_i\, Mx_r = o \tag{19b}$$

$$x'_r\, Mx_s = o \qquad r \neq s \tag{19c}$$

They likewise hold with K instead of M but, since those involving free modes are already implicit in the stronger statement eq. (17), we need only add the relation

$$x'_r\, Kx_s = o \qquad r \neq s \tag{19d}$$

It may be necessary to apply an orthogonalization process to each set of modes which happen to belong to the same eigenvalue in order that eqs. (19) hold without restriction. This is the case of free modes, which all belong to $\omega = o$. It may exceptionally be the case for elastic modes.

4. FREQUENCY RESPONSE UNDER RAYLEIGH ASSUMPTION

Returning to the frequency response Equation (14), we introduce the assumption of proportionality $G = \gamma K$, then

$$[(1 + i\gamma)\ K - \omega^2 M]x \ = \ y \qquad (20)$$

where $(1 + i\gamma)K$ represents the complex stiffness matrix.

This assumption goes back to Lord Rayleigh[6], who investigated its consequences in the case of viscous damping. It was revived by LEWIS and WRISLEY[7] as the basis of a method for exciting pure modes with a multiplicity of shakers. It is a decisive simplifying step since only two quadratic forms are involved which may simultaneously be reduced to a sum of squares.

In other words, the equations governing the normal coordinates are not only free of inertial and elastic coupling terms but even of coupling terms due to damping. The system is effectively split into a sum of independent, sub systems with one degree of freedom. With the object of removing later the need for the Rayleigh assumption, we present the detailed treatment of this problem in a slightly different way.

Instead of seeking the response amplitude x to a given excitation y we enquire about the excitation necessary to keep a normal undamped mode vibrating at a given amplitude u_i or x_r, although the forcing frequency differs from the natural frequency ω_r and damping is present. For an elastic natural mode we find

$$y_r \ = \ (1 + i\gamma)\ Kx_r - \omega^2 Mx_r$$

or, in view of eq. (18)

$$y_r \ = \ [(1 + i\gamma)\omega_r^2 - \omega^2]\ Mx_r \qquad (21)$$

and, for a free mode, in view of eq. (17)

$$z_i \ = \ -\omega^2 M u_i \qquad (22)$$

We call (21) and (22) 'normal excitation modes'. Their shape is here invariable, always being proportional to the inertia forces of the natural vibration.

Their amplitude and phase behave exactly like a damped system in one degree of freedom. If one puts

$$\tan \phi_r \ = \ \frac{\gamma}{1 - \omega^2/\omega_r^2} \qquad (23)$$

eq. (21) may be cast in the form

$$y_r = \frac{\omega_r^2 \gamma}{\sin \phi_r} e^{i\phi_r} . Mx_r \qquad (21')$$

$$\text{The ratio } \frac{\text{inertia forces}}{\text{exciting forces}} = \frac{\omega^2}{\omega_r^2} \cdot \frac{\sin \phi_r}{\gamma} . e^{-i\phi_r}$$

presents, as a function of frequency through eq. (23), the well known diagrammatic form[8] of Fig. 2.

The normal excitation modes have simple energy properties. From eqs. (19)

$$\left.\begin{aligned} x'_s y_r &= 0 \qquad r \neq s \qquad & u'_i y_r &= 0 \\ x'_r z_i &= 0 & u'_j z_i &= 0 \qquad j \neq i \end{aligned}\right\} \qquad (24)$$

Hence it appears that a normal excitation mode develops no energy input in a normal vibration mode except the one it keeps vibrating.

With this new interpretation of the orthogonality eqs. (19) it is easy to construct the response of the system to a general input y. Put

$$y = \sum_1^m a_i z_i + \sum_1^n \beta_r y_r \qquad (25)$$

so that, from the linearity of the equations

$$x = \sum_1^m a_i u_i + \sum_1^n \beta_r x_r \qquad (26)$$

The coefficients of the expansion are found by multiplying eq. (25) to the left by the u'_i and x'_r and using eqs. (24), there follows

$$a_i = \frac{u'_i y}{u'_i z_i} \qquad \beta_r = \frac{x'_r y}{x'_r y_r}$$

Substituting this result in eq. (26) and replacing the normal excitation modes by their values (22) and (21')

$$x = -\frac{1}{\omega^2 \mu} \sum_1^m [u'_i y]\, u_i + \sum_1^n \frac{[x'_r y] \sin \phi_r e^{-i\phi_r}}{\mu \dot{y} \omega_r^2} x_r \qquad (27)$$

The normal undamped modes have been supposed to be normalized in such a way that they all have the same effective inertia coefficient

$$\mu = u'_i M u_i = x'_r M x_r$$

The quantities between square brackets are a measure of the energy input of the excitation in each mode.

The expansion (27) is very useful when discussing the behaviour of the response.

We assume all the elements of y to be real; all shakers are in phase (or 180° degrees out of phase) and one of them may be taken as marking the phase origin.

The response may then be split in a part which is in phase with the excitation and one which lags 90° behind, viz.

$$x = a - ib \quad \text{where}$$

$$a = -\frac{1}{\omega^2\mu}\sum_{1}^{m}[u'_i y]\,u_i + \sum_{1}^{n}\frac{[x'_r y]\,\sin\phi_r\cos\phi_r}{\mu\gamma\omega_r^2}x_r ,$$

$$b = \sum_{1}^{n}\frac{[x'_r y]\,\sin^2\phi_r}{\mu\gamma\omega_r^2}x_r$$

Let us disregard for the moment the relative importance of the input energy brackets. If damping is small, $\gamma \ll 1$, $\sin\phi_r$ will be small of order γ except in the neighbourhood of the resonant frequency $\omega = \omega_r$.

At resonance, there is in 'b' a resonant term which for $\sin\phi_r = 1$, is,

$$\frac{[x'_r y]}{\mu\gamma\omega_r^2}x_r$$

the corresponding term in 'a' disappears.

The other terms of 'b' are, compared to the resonant one, small of $O(\gamma^2)$, and the terms of 'a' small of $O(\gamma)$.

If a local amplitude of the resonant term is large, the addition of terms of $O(\gamma^2)$ in phase and of $O(\gamma)$ in quadrature results in a peak amplitude which is still correct to $O(\gamma^2)$. But in the vicinity of a node of the resonant term we are exposed to an error of $O(\gamma)$. It seems therefore important to discriminate, in the pick-up signals, the part in quadrature with the excitation 'b' and to reject the part in phase 'a'. It is shown on two relatively simple examples of continuous structures that with a single shaker good approximations to the resonant mode shape may be expected from the measurement of 'b'. Fig.4 (a,b,c) shows the first three resonance cases for torsion of a uniformly damped beam; Fig.5 (a,b,c) the first three cases for flexure. The value of γ is 0.04. In each case the total response 'b' is compared to the resonant term. The response 'a' is drawn to a magnification of ten. Its relative importance seems to grow systematically with the order of the resonance.

There are however situations which have been described at length[7,8] where the errors are necessarily larger:

(a) when the input energy bracket of the resonant term is vanishingly small (pseudo-resonance). This happens with a single shaker when it is located in the vicinity of a node.

b) when there are several modes grouped in a small range of frequency. Obviously there are then several sin ϕ terms which are not so small.

The only cure for these situations is to play on the input energy brackets by using a multiplicity of shakers.

The ideal would be to make $y = y_r$ for then, by definition, there is no other mode attacked than x_r. It would even become unnecessary to work at resonance, were it not advisable to minimize even small errors by using the principe of resonance. In practice this goal cannot be achieved because of the continuity of the structure but it may be reasonably approximated by a great number of concentrated shaking points.

The Rayleigh assumption is again significant with regard to the simplicity of the distribution law it yields for the shaking amplitudes.

If the mass alloted to each shaker may be determined, eq. (21) shows that the amplitude of the shaking force must be kept proportional to this mass and to the amplitude of the shaker's displacement. This is the basis of the method developed by Lewis and Wrisley[7].

5. FREQUENCY RESPONSE IN GENERAL : THE CHARACTERISTIC PHASE LAGS

It is difficult to decide whether the Rayleigh assumption is an over-simplification or of it is reasonably true. For the internal damping of an isotropic material one may question whether the complex moduli

$$E(1 + ig) \qquad G(1 + ig)$$

have the same value of g. If not, one must expect different damping intensities in bending and in torsion and failure of the Rayleigh assumption in cases where bending is coupled with torsion. Slip in riveted joints and other types of energy sinks may also be distributed in a manner quite different from the structure's flexibility.

Rayleigh has extended his treatment by what is essentially a perturbation method valid when damping is small[9]. It is however possible to present a general expansion formula for the frequency response, retaining most of the characteristics of eq. (27) and valid for any intensity and distribution of damping.

It is a simple transposition of a method originally devised to treat the case of viscous damping[10].

We now return to the general form presented by eq. (14) and concentrate on the physical ideas, leaving the detailed treatment for Appendix I.

With a view of extending the method of Section 4, one may first try to keep the old definition of an excitation mode. It appears however that the simple energy properties of eqs. (24) are no longer true. Because of the general form of the G matrix such an excitation mode would give energy inputs in all the undamped natural modes. If damping is small, a perturbation method similar to that of Rayleigh might be devised, but another approach is open, which consists in altering the definition of the excitation mode and the response mode so as to keep the truth of eqs. (24).

This happens if the definitions are based on the following typical relation between response and excitation. Observe on eqs. (22) and (21') that the displacements of a response mode are all in phase (the elements of u_i and x_r are real) and that the same is true for the elements of the corresponding excitation mode which all lead by the same phase angle ϕ_r.

To see whether such a property may hold in general we seek a solution of eq. (14) where

y is a vector with real elements

$x = e^{-i\phi_r}$ with r also having real elements.

This turns out to pose an eigen-value problem for the 'characteristic phase lag ϕ' and defines entirely a set of 'frequency response modes' and corresponding 'excitation modes' with the following properties : (proof of the statements may be found in Appendix I).

(a) For a given frequency there are as many real characteristic phase angles ϕ as there are degrees of freedom in the system.

(b) As functions of frequency they follow laws qualitatively similar to eq. (23). It may be shown that each ϕ_K is a growing function of ω^2, taking the value $\pi/2$ for one of the undamped natural frequencies ω_K and asymptotic to π when ω tends to infinity.

(c) To each characteristic phase lag angle ϕ_K there corresponds a frequency response mode r_K whose shape varies with frequency but which is, for $\omega = \omega_K$ identical to the corresponding undamped natural mode.

(d) To each characteristic phase lag angle ϕ_K there also corresponds an excitation mode

$$y_K = \frac{1}{\sin \phi_K} G\, r_K \tag{28}$$

(e) A given excitation mode develops no energy input in a frequency response mode except the one to which it corresponds. i.e. eqs. (24) are saved.

It follows that an expansion similar to eq. (27) exists. The form of the expansion given below assumes, for the sake of simplicity, that the free modes are undamped

$$Gu_i = o \qquad (i = 1, 2 \ldots m) \tag{29}$$

The free modes are then frequency response modes with a characteristic phase lag always equal to zero.

$$x = -\frac{1}{\omega^2 \mu} \sum_1^m [u'_i y]\, u_i + \sum_1^n \frac{[r'_K y] \sin \phi_K . e^{-i\phi_K}}{r'_K\, G\, r_K} . r_K \tag{30}$$

If we remember that

$$r_K \to x_K \qquad \phi_K \to \pi/2 \qquad \text{when } \omega \to \omega_K$$

nothing is changed in the discussion of the behaviour at resonance.

If the resonant frequencies are sufficiently separated and damping is not too large, good measurements of x_K result with few excitation points if the total response in quadrature is retained.

Otherwise we must again strive to make $y = Gx_K$ in order to excite the response mode $r_K = x_K$ only.

Since G is unknown, no direct method is available to this effect. Instead we may rely on an energy principle which is discussed in the last section.

6. STATIONARITY OF THE REACTIVE ENERGY IN PURE MODE EXCITATION

For a single degree of freedom the input $y \cos \omega t$ acting on the elementary displacement $rd.\cos(\omega t - \phi)$ produces per period an energy

$$yr \int_{\alpha=0}^{2\pi} \cos \alpha . d \cos(\alpha - \phi) = \pi yr \sin \phi$$

which we call the active energy (it is actually dissipated in the system during the period). The expression $\pi yr \cos \phi$ is the reactive energy. Active and reactive energy may be combined as the real and imaginary part of a single complex energy

$$\pi i yre^{-i\phi} = \pi iyx \quad \text{where} \quad x = re^{-i\phi}$$

Generalizing this for several degrees of freedom, a general (but real or 'in phase') input excitation y produces per period a complex energy $\pi i y'x$ where x is the complex output amplitude. If x is split in real and imaginary parts $x = a - ib$, we get

$$\text{the active energy} \qquad E_a = \pi y'b$$

$$\text{the reactive energy} \qquad E_r = \pi y'a$$

Expanding the excitation and response in their frequency response modes

$$y = \Sigma \beta_K y_K \qquad x = \Sigma \beta_K e^{-i\phi_K} . r_K$$

$$a = \Sigma \beta_K \cos \phi_K r_K \qquad b = \Sigma \beta_K \sin \phi_K r_K$$

the following quadratic forms are obtained

$$E_a = \pi \Sigma \beta_K^2 [y'_K r_K] \sin \phi_K = \pi \Sigma \beta_K^2 [r'_K G r_K] \tag{31}$$

$$E_r = \pi \Sigma \beta_K^2 [y'_K r_K] \cos \phi_K = \pi \Sigma \beta_K^2 [r'_K G r_K] \cot \phi_K \tag{32}$$

They both appear reduced to a sum of squares by virtue of the essential property e of Section 5 (See eqs. (A.11) of Appendix I). E_a is definite positive or at most semi-definite positive if there are undamped free modes, exactly as the G matrix. This was to be expected in view of the character of really dissipated energy of E_a. The sign of E_r on the contrary depends on the excitation frequency. Indeed, the factor cot ϕ_K, starting from a positive value at zero frequency, passes through zero at resonance and becomes more and more negative above resonance (See (b) of Section 5 and Appendix I). For a damped free mode this factor is zero at zero frequency and negative afterwards. The expansion term in eq. (32) for an undamped free mode is

$$- \pi \omega^2 \, [u'_i \, M \, u_i] \, \beta_i^2$$

which is always negative. It follows that the reactive energy takes any positive or negative value when the shape and intensity of the excitation is varied by changing its 'coordinates' β_K. The conditions for the reactive energy to be stationary with respect to variations of the coordinates are

$$\frac{\partial E_r}{\partial \beta_k} = 2\pi \beta_K \, [r'_K \, G \, r_K] \cot \phi_K = 0$$

Except for the trivial solution in which all the coordinates are zero, stationarity is satisfied only under the following circumstances:-

(a) All co-ordinates are zero except one, say β_j

(b) cot $\phi_j = 0$ that is $\omega = \omega_j$.

Thus the fact that the reactive energy is insensitive with respect to variations in the shape and intensity of the excitation is an indication both of resonance and of pure mode excitation since then $y = \beta_j y_j$ and the response is $x = \beta_j x_j$. In such a situation the stationary value of the reactive energy is actually zero. However the converse is not generally true; for many sets of values of the coordinates the reactive energy may be zero without being stationary. There is also a stationarity property for

$$\cot \phi = E_r / E_a \qquad (33)$$

which is a sort of average value for the phase lag of response over excitation. It does not require resonance but only pure mode excitation. There is resonance when the stationary value of this ratio is zero. The behaviour of the total reactive energy, which seems already to provide valuable control of the optimum testing conditions, would still be more valuable if it provided a method for adjusting the individual shaking forces. The following remarks are pertinent to this aspect. The adjustment of a single shaker force amplitude s_m makes all coordinates vary according to linear laws $\beta_K = \beta_{KO}^{(m)} + \gamma_K^{(m)} . s_m$. The reactive energy is then a second degree equation in the variable s_m. Simple examples show that, according to circumstances, this equation may or may not have real roots. Its extremal value is also either a maximum or a minimum. It is suggested that, according to a repeated cycle, each shaker be adjusted to make the reactive energy extremal. Periodic control of the frequency may be made in the early stages by observing peak amplitudes, later, by adjusting the stabilized extremal value of the reactive energy to zero. When no free modes are present and the first resonant mode is sought, the stationary value is an absolute minimum. So are then the extremal values in each adjusting step and the procedure obviously converges. Further research is needed to establish convergence in the general case.

ACKNOWLEDGMENTS

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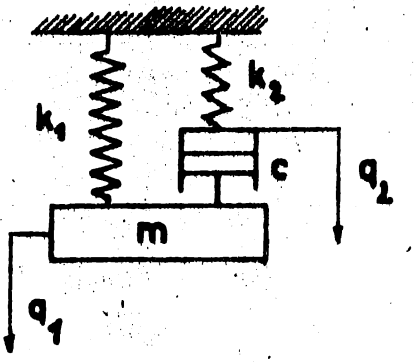

Fig.1 Maxwell Model for Structural Damping

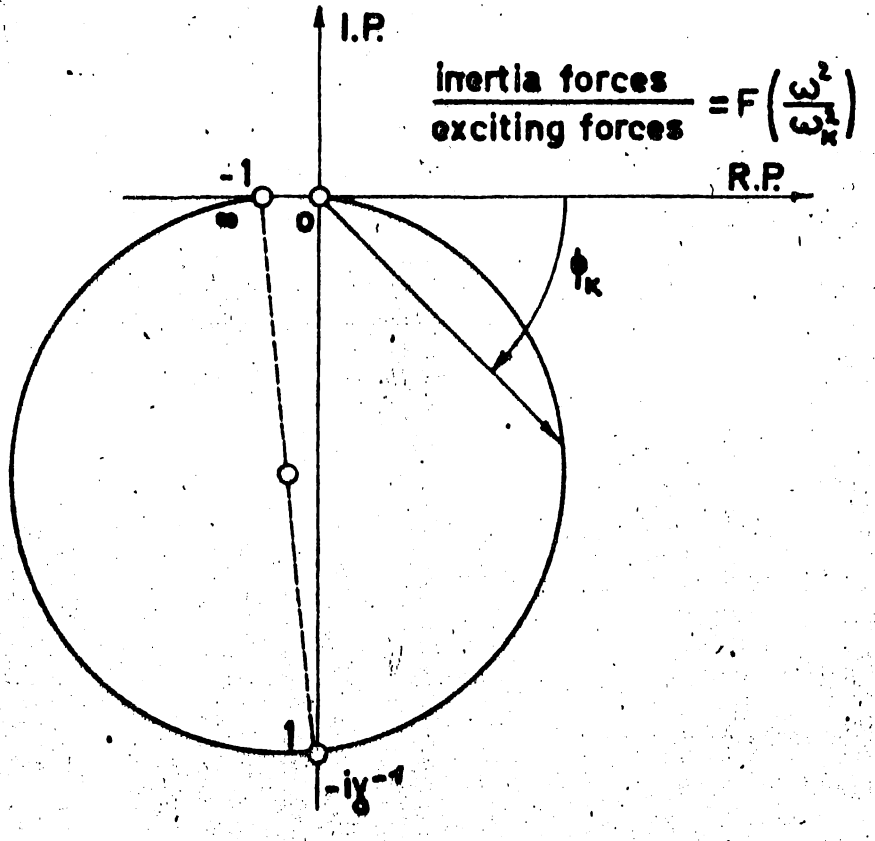

Fig.2 Circle Diagram in Structural Damping

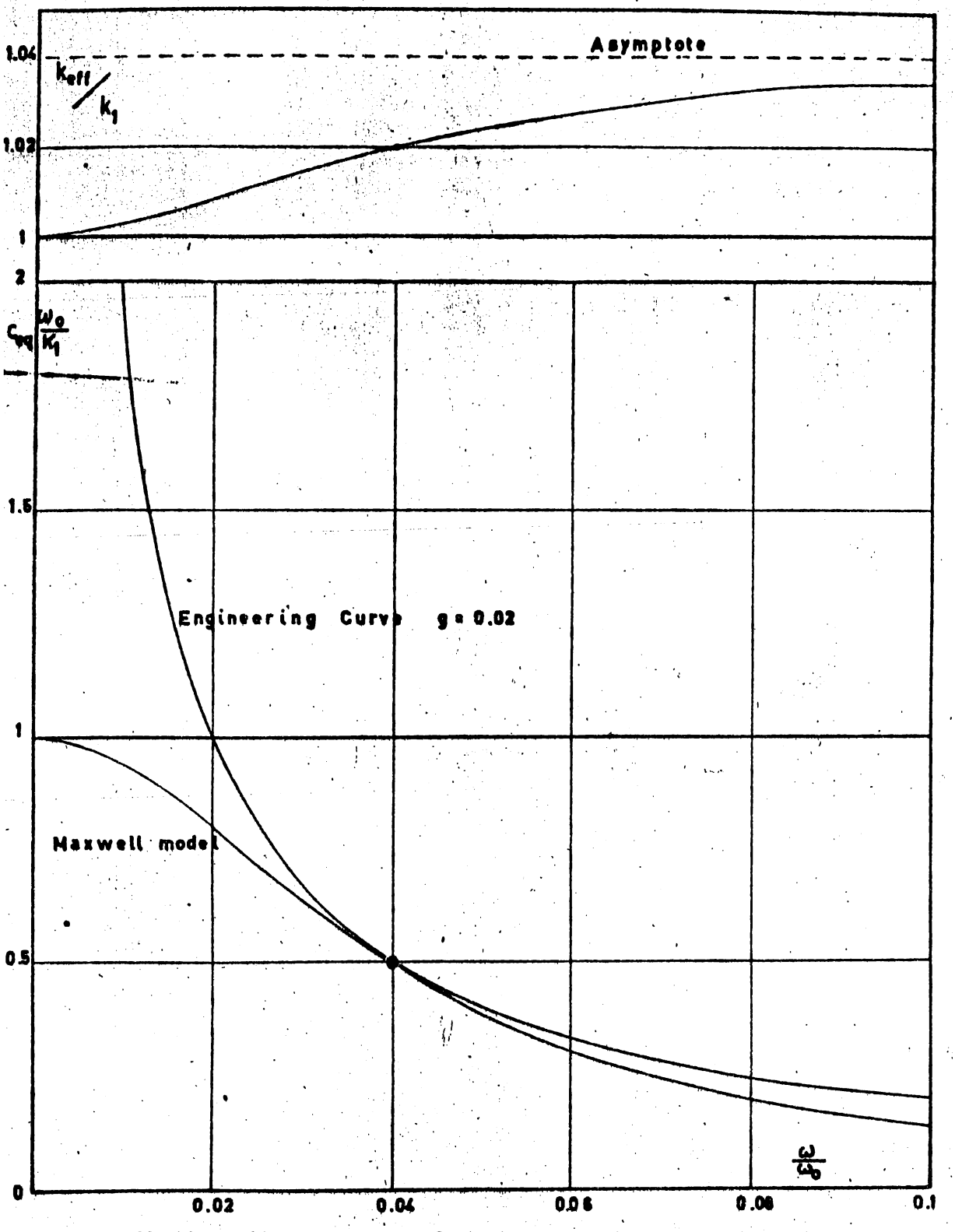

Fig.3 Effective Stiffness and Equivalent Viscous Damping as Functions of Frequency

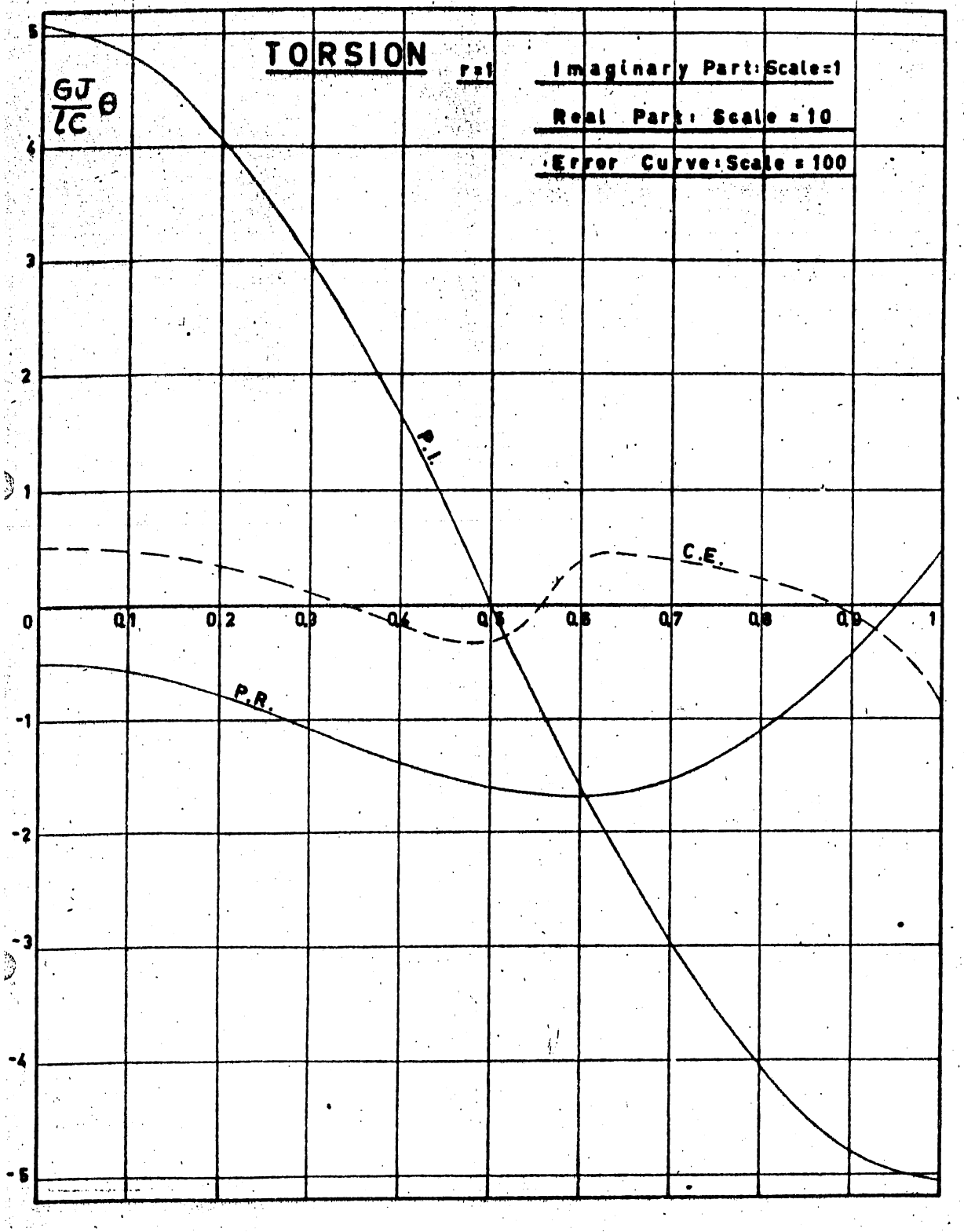

Fig. 4a Torsional Vibrations of a Free Uniform Beam under Concentrated Harmonic Torque : First Resonance

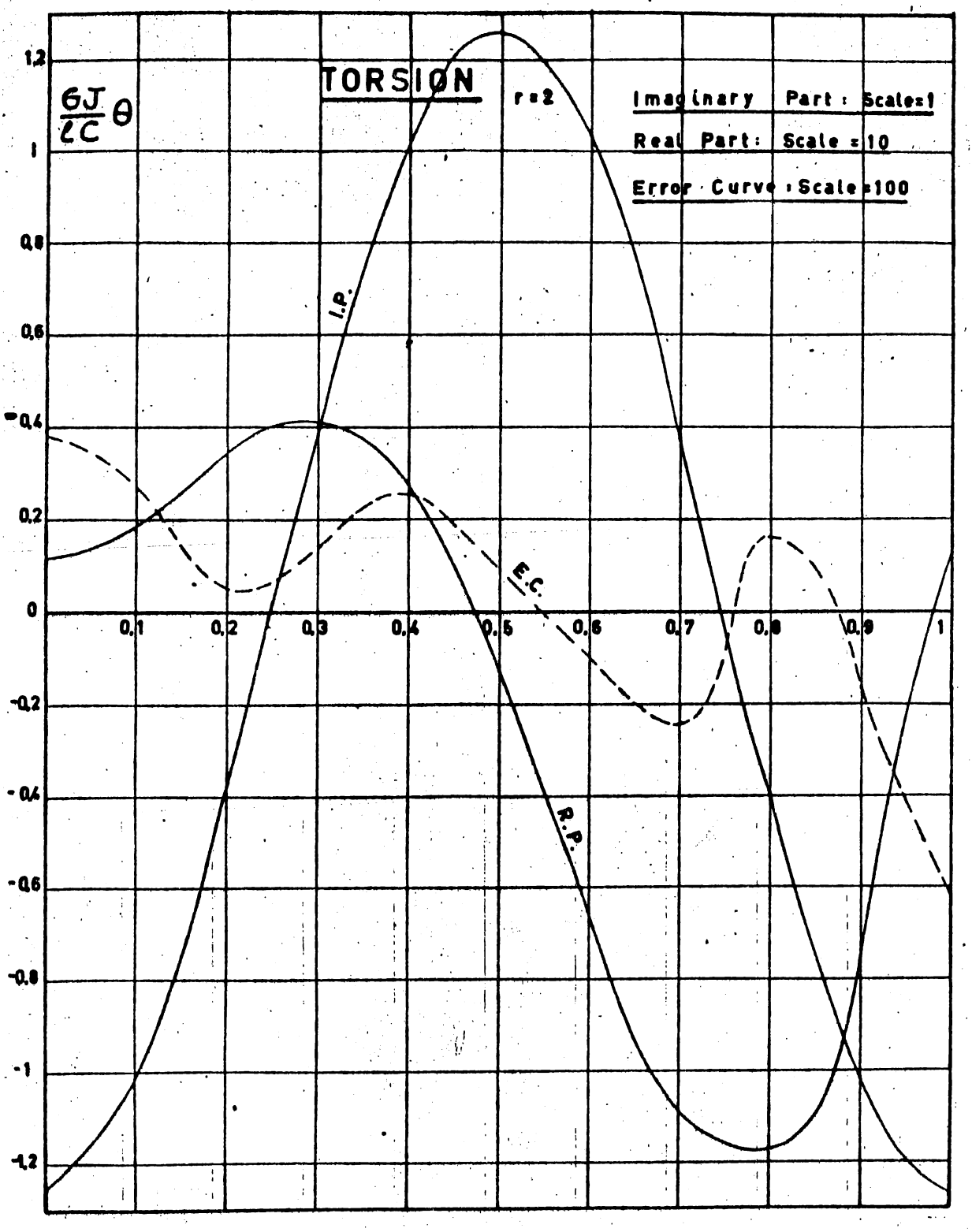

Fig.4b Torsional Vibrations of a Free Uniform Beam under Concentrated Harmonic Torque : Second Resonance

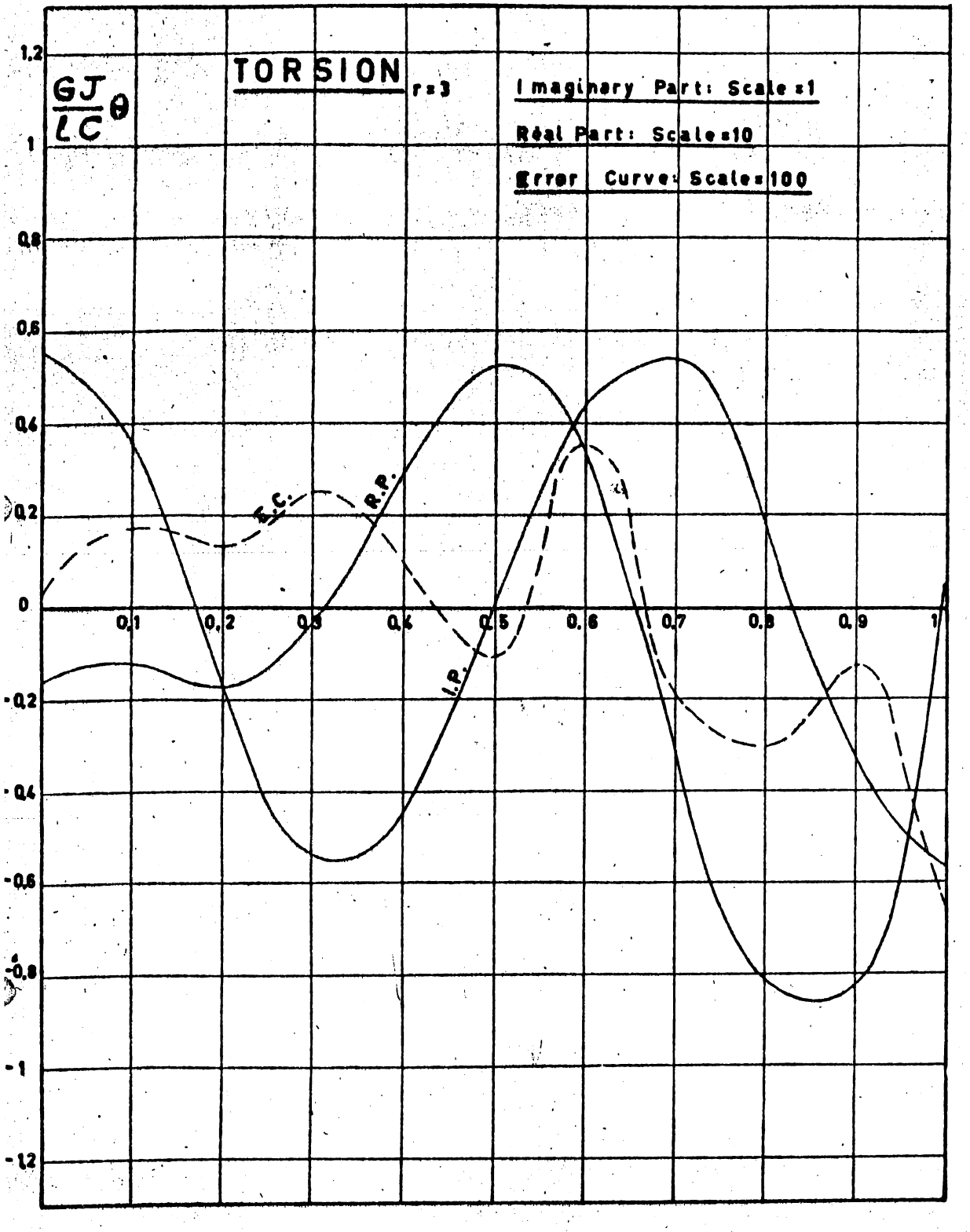

Fig.4c Torsional Vibrations of a Free Uniform Beam under Concentrated Harmonic Torque : Third Resonance

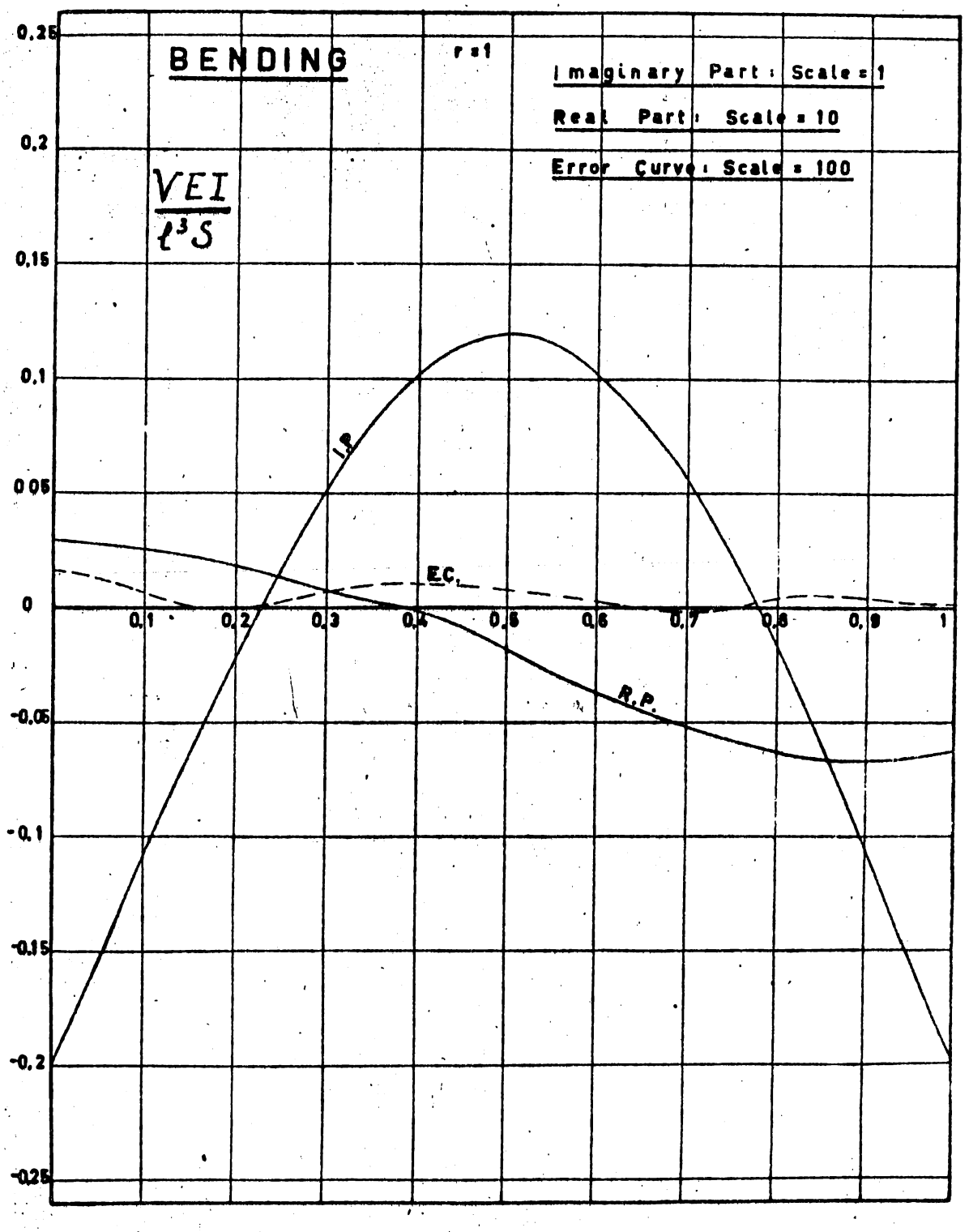

Fig. 5a Bending Vibrations of a Free Uniform Beam due to Harmonic Shear Load : First Resonance

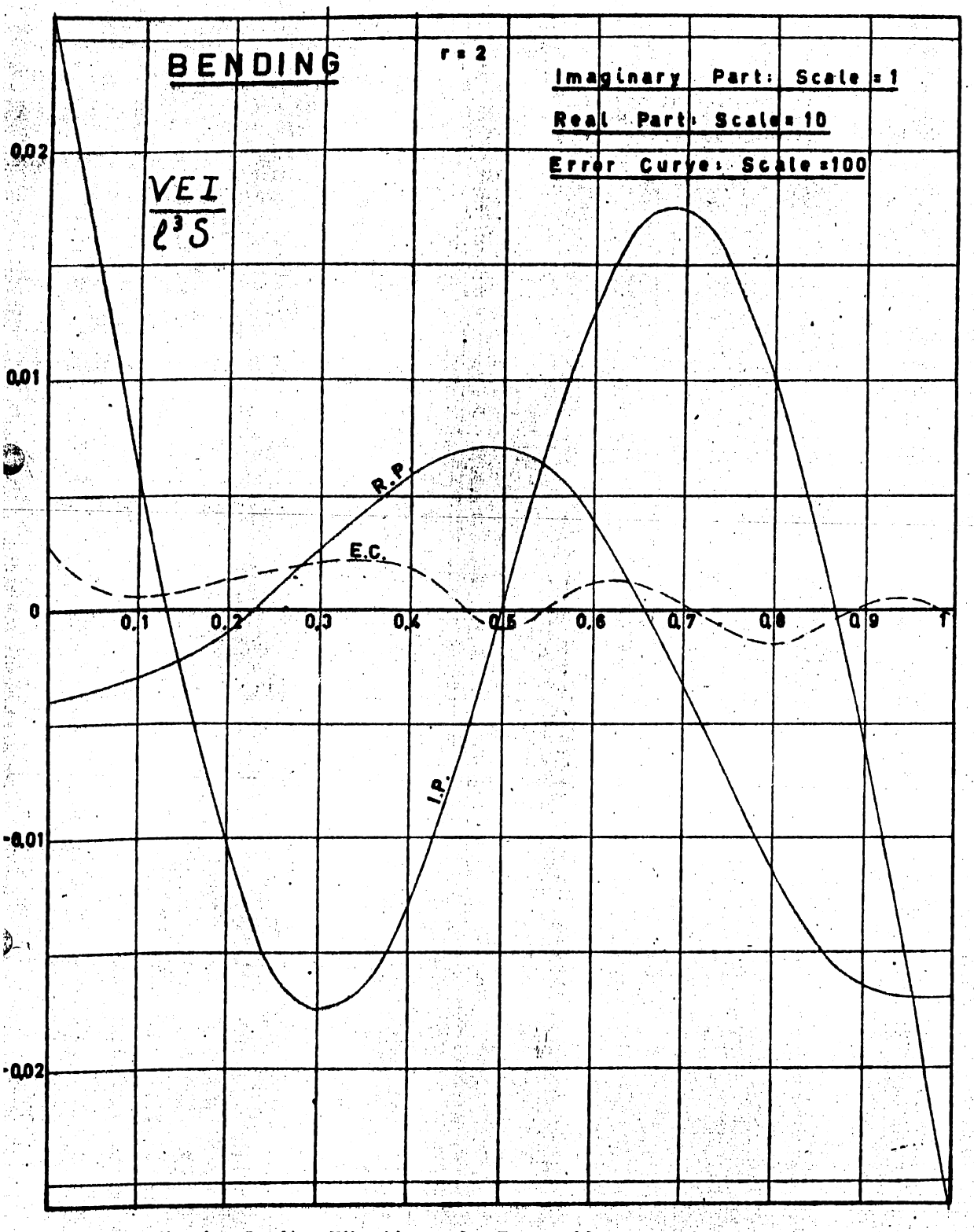

Fig.5b Bending Vibrations of a Free Uniform Beam due to Harmonic Shear Load : Second Resonance

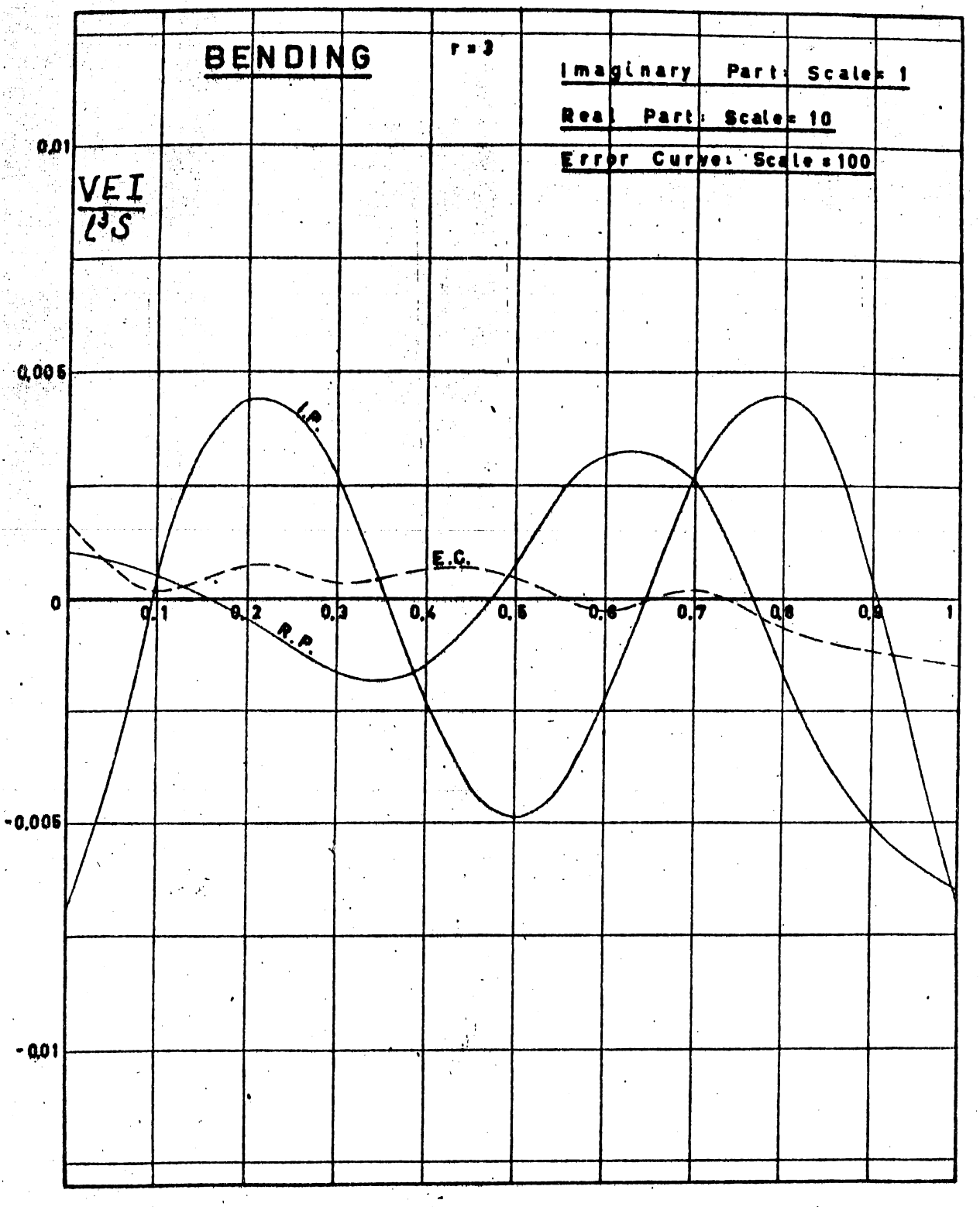

Fig. 5c Bending Vibrations of a Free Uniform Beam due to Harmonic Shear Load : Third Resonance

APPENDIX I

CHARACTERISTIC PHASE LAG THEORY

ST • Sciences Ai

Differentiate eq. (A.3) with respect to ω^2 and multiply the result to the left by r'_K

$$r'_K G v_K = \frac{d(\tan \phi_K)}{d\omega^2} [r'_K (K - \omega^2 M) r_K] + \tan \phi_K [r'_K (K - \omega^2 M) v_K] - \tan \phi_K [r'_K M r_K]$$

Transpose eq. (A.3), multiply it to the right by v_K and subtract from the previous result

$$\frac{d(\tan \phi_K)}{d\omega^2} [r'_K (K - \omega^2 M) r_K] - \tan \phi_K [r'_K M r_K] = 0$$

In view of eq. (A.4) where $r_K^* = r'_K$ we get the result

$$\frac{d(\tan \phi_K)}{d\omega^2} = \tan^2 \phi_K \frac{r'_K M r_K}{r'_K G r_K} > 0 \qquad (A.5)$$

Hence ϕ_K is a growing function of ω^2.

From eq. (A.2) a characteristic phase lag takes the value $\pi/2$ when

$$(K - \omega^2 M) r = 0$$

Thus, if ω is a natural undamped frequency and r the corresponding mode shape, from eq. (A.2) again $\omega^2 \tan \phi$ tends to a finite limit when ω tends to infinity. In this case

$$\tan \phi \simeq \psi/\omega^2 \text{ where } \psi \text{ is a root of } \det (\psi M + G) = 0$$

and is consequently negative, which agrees with eq. (A.5).

Since $\tan \phi$ cannot be zero except in the case of undamped free modes (where it was shown to remain zero) or when ω tends to infinity, it follows that each root ϕ is a growing function of ω^2 which passes only once through the value $\pi/2$ and reaches asymptotically the value π.

Consequently each of the characteristic phase lags belongs to a frequency response mode r_K which becomes identical to x_K when $\omega = \omega_K$ and $\phi_K = \pi/2$.

(c) The corresponding excitation mode is found from eq. (A.1) in the form

$$y_K = \cos \phi_K (K - \omega^2 M) r_K + \sin \phi_K G r_K \qquad (A.6)$$

and, if use is made of eq. (A.2) in one of the other useful forms

$$y_K = \frac{1}{\cos \phi_K} (K - \omega^2 M) r_K = \frac{1}{\sin \phi_K} G r_K \qquad (A.7)$$

In particular, for an undamped free mode,

$$y = z_i = - \omega^2 M u_i \qquad (A.8)$$

APPENDIX I

CHARACTERISTIC PHASE LAG THEORY

(a) In eq. (14) put $x = e^{-i\phi}r$ and, assuming y and r to have real elements, separate real and imaginary parts

$$\cos\phi\,(K - \omega^2 M)\,r + \sin\phi\,Gr = y \tag{A.1}$$

$$-\sin\phi\,(K - \omega^2 M)\,r + \cos\phi\,Gr = o \tag{A.2}$$

The last equation is homogeneous and poses an eigenvalue problem for ϕ

$$\det\,[G - \tan\phi\,(K - \omega^2 M)] = o$$

There are m + n roots for $\tan\phi$ satisfying

$$G\,r_K = \tan\phi_K\,(K - \omega^2 M)\,r_K \tag{A.3}$$

Should $\tan\phi_K$ and hence r_K be complex, multiply to the left by the transpose conjugate r_K^*

$$\tan\phi_K = \frac{r_K^*\,G\,r_K}{r_K^*\,(K - \omega^2 M)\,r_K} \tag{A.4}$$

hence ϕ_K and consequently r_K are real since $G = G^*$, $K = K^*$, $M = M^*$ and both numerator and denominator are Hermitian forms.

(b) For $\omega = o$ the initial values of the phase lag angles

$$\tan\phi_K\,(o) = \frac{r'_K\,G\,r_K}{r'_K K r_K} \geq o$$

since G and K are semidefinite positive. The only vectors which make the denominator zero are the u_i. A free mode u_i satisfies eq. (A.2) for $\omega = o$ with

$$\phi = \pi/2 \qquad \text{if } Gu_i \neq o$$

$$\phi \quad \text{arbitrary if } Gu_i = o.$$

In the last case (undamped free mode) $\phi = o$ and $r = u_i$ is a solution for any value of the frequency.

Next denote by v_K the vector whose elements are derivatives with respect to ω^2 of the elements of r_K, i.e. r_K

$$v_K = \frac{d(r_K)}{d\omega^2}$$

(d) From eq. (A.3), multiplying to the left by r'_j

$$r'_j \, G \, r_K = \tan \phi_K \, [r'_j \, (K - \omega^2 M) r_K]$$

Similarly, if eq. (A.3) is used for the mode of index j and then transposed and multiplied to the right by r_K

$$r'_j \, G \, r_K = \tan \phi_j \, [r'_j \, (K - \omega^2 M) r_K]$$

hence if $\tan \phi_j \neq \tan \phi_K$

$$r'_j \, G \, r_K = o \qquad \text{(A.9)}$$

$$r'_j \, (K - \omega^2 M) r_K = o \qquad \text{(A.10)}$$

These orthogonality relations still hold if a multiple root occurs for the characteristic phase lag by orthogonalizing the resulting independent modes. If undamped free modes are present

$$G \, u_i = o$$

replaces eq. (A.9) as a stronger statement, whilst eq. (A.10) becomes simply

$$u'_i \, M \, r_K = o$$

Finally the undamped free modes may be orthogonalized between themselves (eqs. 19a of Sect. 3). From all this there results

$$r'_y \, y_K = o \qquad j \neq K \qquad \text{(A.11)}$$

Thus each excitation mode has no other energy input than in its corresponding frequency response mode.

(e) The y_K form a complete linearly independent set, or base, of the vectorial space. For, should a relation of dependency exist

$$\Sigma \, \alpha_K \, y_K = o$$

then, on account of eqs. (A.11)

$$\alpha_K \, [r'_K \, y_K] = o \qquad \text{would follow.}$$

But, from eqs. (A.7)

$$\cos \phi_K \, [r'_K \, y_K] = r'_K \, (K - \omega^2 M) \, r_K$$

$$\sin \phi_K \, [r'_K \, y_K] = r'_K \, G \, r_K$$

and

$$[r'_K \, y_K]^2 = [r'_K \, (K - \omega^2 M) r_K]^2 + [r'_K \, G \, r_K]^2$$

This cannot be zero except in the case of an undamped free mode at $\omega = o$. Hence for a non-zero frequency of excitation one has essentially $\alpha_K = o$ and linear independence.

An expansion of $i\delta$ thus always possible and unique

$$y = \Sigma \beta_K y_K$$

where, in view of eqs. (A.11)

$$\beta_K = \frac{[r'_K\ y]}{[r'_K\ y_K]} = \frac{[r'_K\ y] \sin \phi_K}{[r'_K\ G\ r_K]}$$

and there follows the associated expansion

$$x = \Sigma \beta_K e^{-i\phi_K} r_K$$

For undamped free modes it has the particular form of eq. (30) of Section 5.

APPENDIX II

TORSIONAL VIBRATIONS OF A UNIFORM FREE BEAM WITH UNIFORM STRUCTURAL DAMPING

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TORSIONAL VIBRATIONS OF A UNIFORM FREE BEAM WITH UNIFORM STRUCTURAL DAMPING

The beam has constant values of torsional rigidity J, polar moment of inertia I_p and specific mass ρ. Its' structural damping is accounted for by the complex shear modulus $G(1 + ig)$.

If submitted to distributed external torques $c(z)e^{i\omega t}$ and concentrated end torques $C(o)e^{i\omega t}$ and $C(l)e^{i\omega t}$ its (complex) forced vibration amplitude $\theta(z)$ is governed by the following differential equation and boundary conditions

$$\left.\begin{aligned} GJ\,(1 + ig)\,\theta'' + \omega^2\rho I_p\,\theta &= -\,c(z) \\ GJ\,(1 + ig)\,\theta'(o) &= C(o) \\ GJ\,(1 + ig)\,\theta'(l) &= C(l) \end{aligned}\right\} \qquad (B.1)$$

The natural undamped modes are found by putting $c = o$, $C(o) = C(l) = o$. Hence with

$$k^2 = \frac{\omega^2\rho I_p}{GJ} \qquad (B.2)$$

$$\theta'' + k^2\theta = o \qquad \theta'(o) = \theta'(l) = o$$

The modes are

$$\left.\begin{aligned} \theta_o(z) &= 1 \quad \text{(free mode)} \quad \omega = o. \\ \theta_r(z) &= \cos(k_r z/l) \quad \text{with} \quad k_r = r\pi\ (r = 1, 2\ldots) \end{aligned}\right\} \qquad (B.3)$$

The natural frequencies result from eq. (B.2). For the damped case put

$$\lambda^2 = \frac{k^2}{1 + ig} \qquad (B.4)$$

The solution of eqs. (B.1) when the excitation consists uniquely in the concentrated torque to the right is easily found to be

$$\frac{GJ}{lC(l)}\,\theta(z) = -\,\frac{\lambda}{k^2 \sin\lambda}\cos(\lambda z/l) \qquad (B.5)$$

Since λ is complex a practical method consists to write $\lambda = \alpha(1 - i\beta)$ substitute in eq. (B.4) and equate real and imaginary parts. Whence

$$g = \frac{2\beta}{1 - \beta^2} \qquad (B.6) \qquad \alpha^2 = \frac{k^2}{(1 \mp \beta^2)(1 + g^2)} \qquad (B.7)$$

The real and imaginary parts of eq. (B.5) have been computed for $\beta = 0.02$ or $g = 0.040016$ in the three first resonant conditions. The corresponding values of α are given by eq. (B.7) with the appropriate values of k. On Figs. 4a, 4b and 4c it is observed that the complementary vibration in phase (real part) is small but grows in relative importance with the order of the resonance.

It is to be expected from theory that the vibration in quadrature (imaginary part) closely approximates the natural mode which comes into resonance. This is most clearly seen by expanding the vibration in frequency response modes and picking out the resonant term, whose shape is that of the resonant mode in question. The expansion is simple to obtain since we are in a case where the Rayleigh assumption holds.

It is readily verified on eqs. (B.1) that the natural modes (B.3) are kept in forced vibration solely by the following distributed excitation torques

$$\left.\begin{aligned} c_0(z) &= -\omega^2 \rho I_p \, \theta_0(z) \\ c_r(z) &= g\,GJ \frac{r^2\pi^2 e^{i\phi_r}}{l^2 \sin \phi_r} \theta_r(z) \end{aligned}\right\} \qquad (B.8)$$

where $\tan \phi_r = g/(1 - \omega^2/\omega_r^2)$

To find the coefficients of the expansion the concentrated end torque is represented as a distributed torque by means of Dirac's delta function

$$C(l).\delta(z - l) = \sum_0^\infty A_r \, c_r(z) \qquad (B.9)$$

where $\delta(z - l) = 0$ for $z < l$

$$\int_0^z \delta(z - l)dz = \begin{cases} 0 & z < l \\ 1 & z = l \end{cases}$$

Eq. (B.9) is treated as a formal generalized Fourier expansion for which the orthogonality relations (as expected from energy input consideration) hold

$$\int_0^l c_r(z)\, \theta_s(z)dz = 0 \qquad r \neq s$$

Or more simply here

$$\int_0^l \theta_r(z)\, \theta_s(z)dz = \begin{cases} 0 & r \neq s \\ l/2 & r = s \neq 0 \\ l & r = s = 0 \end{cases}$$

Thus, multiplying eq. (B.9) by $\theta_r(z)$, integrating between 0 and l and using the sifting property of Dirac's function

$$C(l) \cos r\pi = A_r\, g\, GJ \frac{r^2\pi^2 e^{i\phi_r}}{2l \sin \phi_r} \qquad (B.10)$$

With those values of the A_r the series (B.9) is divergent, but, it may be summed by Borel's method. The corresponding series expansion for the angular vibration is, however, convergent

$$\theta(z) = \sum_{0}^{\infty} A_r \theta_r(z) \qquad \text{(B.11)}$$

For a natural vibration frequency the resonant term of (B.11) is ($\phi = \pi/2$)

$$\frac{GJ}{lC(l)} \theta_{res}(z) = - \frac{i\, 2\cos r\pi}{g\, r^2\pi^2} \cos(r\pi z/l)$$

The difference between this term and the imaginary part of eq. (B.5) has been computed. It is very small and is displayed as an error curve with a magnification factor of 100. It tends to increase with the order of resonance.

APPENDIX III.

FLEXURAL VIBRATIONS OF A UNIFORM FREE BEAM WITH UNIFORM STRUCTURAL DAMPING

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FLEXURAL VIBRATIONS OF A UNIFORM FREE BEAM WITH UNIFORM STRUCTURAL DAMPING

Structural damping is represented here by the complex Young's modulus $E(1 + ig)$. There are distributed loads $p(z)e^{i\omega t}$ and concentrated end shear loads. The amplitude of forced deflexions $V(z)$ are governed by

$$\left.\begin{aligned} EI(1+ig)\,V'''' - \omega^2 m V &= p \qquad V''(o) = V''(l) = o \\ -EI(1+ig)\,V'''(o) &= S(o) \qquad -EI(1+ig)\,V'''(l) = S(l) \end{aligned}\right\} \qquad (C.1)$$

Natural undamped modes comprise two orthogonalized free modes and elastic modes

$$\left.\begin{aligned} U_1(z) &= 1 \qquad U_2(z) = 1 - 2z/l \\ V_r(z) &= c_2(k_r)\, s_1(k_r z/l) - s_2(k_r)\, c_1(k_r z/l) \end{aligned}\right\} \qquad (C.2)$$

The k_r's are roots of the transcendental equation

$$1 - \cos k \cosh k = o$$

with

$$EI\, k^4 = \omega^2 m l^4$$

$$s_1(x) = \sin x + \sinh x \qquad c_1 = s_1' \quad s_2 = c_1' \quad c_2 = s_2' \quad s_1 = c_2'$$

For the damped case put

$$\lambda^4 = \frac{k^4}{1 + ig}$$

Then, if only the right end shear load is acting, the response, based on eqs. (C.1), is found in closed form as

$$\frac{EI\,V(z)}{l^3\,S(l)} = \frac{\lambda}{k^4}\,\frac{s_2(\lambda)\,c_1(\lambda z/l) - c_2(\lambda)\,s_1(\lambda z/l)}{2(1 - \cos\lambda\cosh\lambda)} \qquad (C.3)$$

Putting $\lambda = \alpha(1 - i\beta)$ one finds

$$g = \frac{4\beta(1-\beta^2)}{1 - 6\beta^2 + \beta^4} \qquad \alpha^4 = \frac{k^4}{(1 - 6\beta^2 + \beta^4)(1 + g^2)}$$

The real and imaginary part of eq. (C.3) are shown on Figs. 5a, 5b and 5c for the three first resonant conditions and for the choice $\beta = 0.01$ or $g = 0.04$. The conclusions are similar to those of the torsion case.

The Rayleigh assumption holds. It is readily verified on eqs. (C.1) that the natural modes are kept vibrating by the following distributions of excitation loads only

$$q_1(z) = -\omega^2 m\, U_1(z) \qquad q_2(z) = -\omega^2 m\, U_2(z)$$

$$p_r(z) = \frac{EI\, k_r^4\, g\, e^{i\phi_r}}{l^3 \sin \phi_r}\, V_r(z)$$

When the right end shear load acts only we use the formal Fourier expansion in distributed loads

$$S(l)\, \delta(z - l) = A_1 q_1(z) + A_2 q_2(z) + \sum_1^\infty B_r\, p_r(z)$$

Whence, by using the orthogonality relations

$$\int_0^l U_0\, U_i\, dz = 0 \qquad \int_0^l U_i\, V_r\, dz = 0 \qquad \int_0^l V_r\, V_s\, dz = 0$$

$$\int_0^l U_0^2 dz = l \qquad \int_0^l U_1^2 dz = l/3 \qquad \int_0^l V_r^2 dz = l s_2^2(k_r)$$

and the sifting property of the Dirac function, the coefficients are obtained. In particular

$$S(l)[c_2(k_r)\, s_1(k_r) - s_2(k_r)\, c_1(k_r)] = B_r\, s_2^2(k_r)\, \frac{EIgk_r^4 e^{i\phi_r}}{l^3 \sin \phi_r}$$

The expansion of the response is convergent

$$V = A_1\, U_1 + A_2\, U_2 + \sum_1^\infty B_r\, V_r$$

The term which comes into resonance is worked out as

$$\frac{EI}{l^3\, S(l)}\, V_{res}(z) = i \cdot \frac{2(\cos k_r \sinh k_r - \sin k_r \cosh k_r)}{s_2^2(k_r)}\, V_r(z)$$

The difference between this and the imaginary part of eq. (C.3) is again extremely small as may be seen on the error curves of Figs. 5a, 5b and 5c.

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