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ROCKET PROPULSION

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CHAPTER 11

Elementary Problems of Overall Rocket Performance

11.1. Introduction

Problems of stability on the flight path are generally considered in the realm of exterior ballistics and fall outside the scope of the present book. The analysis of rocket performance in flight will therefore be restricted to the simple dynamics of a point mass. Even in this restricted field many problems arise in the choice of design parameters or in the programmation of guidance variables when a given performance is to be optimized. Some of these problems are of an elementary nature and will be dealt with in the present chapter. Other problems require the use of more advanced mathematical tools especially the calculus of variations. In Chapter 12 a general method of analysis of such problems will be given, together with some examples.

11.2. The Ideal Velocity Gain

The simplest performance equation is for a rocket flying in a vacuum without being subjected to any gravitational force. Then, in the absence of drag and gravitational field, and supposing the thrust to be tangential to the path

$$M\frac{\mathrm{d}V}{\mathrm{d}t}=F \tag{1}$$

where M denotes the instantaneous mass of the rocket and V the modulus of the flight velocity. The thrust modulus may be written as

$$F = mc \tag{2}$$

and c the effective exhaust velocity. Elimination of F and m gives

$$\mathrm{d}V = -c\frac{\mathrm{d}M}{M} \tag{4}$$

and on further integration

$$\Delta V = V - V_i = c \ln \frac{M_i}{M} \tag{5}$$

This is the well-known law of "ideal velocity increase", which connects the initial variables (V_i, M_i) and the instantaneous variables (V, M).

It is important to observe that the law holds true even for a variable mass flow $m(t)$, provided such variations do not affect the value of c. In particular, there is no gain or loss of velocity performance for a given mass ratio M_i/M, if propellant is burnt fast in a large rocket or very slowly in a micro-engine. This is not true, however, if the velocity performances are compared on the basis of the same propellant consumption $M_i - M = M_p$, for then we may write

$$\Delta V = c \ln \frac{1}{1 - M_p/M_i}$$

and the smaller M_i the larger the ideal velocity increase. A small engine, giving a minimum initial mass M_i, is thus advantageous unless flight time becomes a dominant consideration.

11.3. Gravitational Losses

What has been said above is still valid in the presence of a gravity field if the rocket is steered along a path lying in an equipotential surface of this field. If the gravitational acceleration has a retarding component g_s along the tangent to the flight path, eqn. (1) is modified to:

$$M\frac{\mathrm{d}V}{\mathrm{d}t} = F - Mg_s \tag{1'}$$

and eqn. (4) to:

$$\mathrm{d}V = -c\frac{\mathrm{d}M}{M} - g_s\mathrm{d}t \tag{4'}$$

Hence

This equation shows the importance of short burning times to minimize the gravitational loss represented by the last term. The requirement of short burning times conflicts with our previous considerations of maximum performance for a given propellant consumption. The existence of a best compromise regarding the engine size may be inferred, and further thought will be given to this problem at a later stage. Velocity, however, is not the only useful form into which chemical energy is converted. There is also potential energy gained against the gravity field. The total energy per unit mass of rocket is

$$E = \tfrac{1}{2} V^2 + \int_0^s g_s \mathrm{d}s \tag{6}$$

Then, multiplying eqn. (4′) by V, and noting that $V\mathrm{d}t = \mathrm{d}s$, we have:

$$\mathrm{d}E = V\mathrm{d}V + g_s \mathrm{d}s = -cV\frac{\mathrm{d}M}{M}$$

The ideal increase in total specific energy is obtained by substituting V from eqn. (5) and the real increase by substituting V from eqn. (5′). Hence, the difference between ideal and real gain, the gravitational energy loss, is, in differential form

$$\mathrm{d}G = -c\left(\int_0^t g_s \mathrm{d}t\right)\frac{\mathrm{d}M}{M} = -c\,\mathrm{d}\left(\ln\frac{M}{M_i}\int_0^t g_s \mathrm{d}t\right) + c\ln\frac{M}{M_i} g_s \mathrm{d}t$$

Integration between $(M_i, 0)$ and (M_b, t_b) yields:

$$G = -c\ln\frac{M_b}{M_i}\int_0^{t_b} g_s \mathrm{d}t + c\int_0^{t_b}\ln\frac{M}{M_i} g_s \mathrm{d}t = c\int_0^{t_b}\ln\frac{M}{M_b} g_s \mathrm{d}t \tag{7}$$

This condensed and general form of the gravitational energy loss shows, since the logarithm is positive in the whole range of integration, that a gain in potential energy ($g_s > 0$) is necessarily accompanied by a loss in total energy. It shows furthermore that this loss is reduced by short burning times. An equivalent formulation consists in substituting $\mathrm{d}t$ from eqn. (3)

$$G = -c\int_{M_i}^{M_b}\frac{g_s}{m}\ln\frac{M}{M_b}\,\mathrm{d}M \tag{8}$$

For constant g_s/m this leads to the closed form:

$$G = \frac{cg_s M_b}{m}\left[1 + \frac{M_t}{M_b}\left(\ln\frac{M_t}{M_b} - 1\right)\right], \left(\frac{g_s}{m} = \text{const.}\right) \tag{9}$$

If the mass flow is such that the acceleration $\dot{V} = a$ is constant, eqns. (4′) and (3) yield:

$$m = \frac{M}{c}(a + g_s) \tag{10}$$

and substitution of this in eqn. (8) gives, for a constant g_s:

$$G = \frac{1}{2}\,\frac{g_s}{a + g_s}\left(c\ln\frac{M_t}{M_b}\right)^2 \tag{11}$$

In the cases represented by eqns. (9) and (11), the reduction of energy loss by increase in the mass flow or in the acceleration of the rocket, both of which correspond to a reduction in burning time, is clearly indicated. The energy loss and the loss in velocity disappear entirely when the burning time tends to zero. In this limiting case of "impulsive burning", eqn. (5) remains valid.

11.4. A Classification of Orbits about an Inverse Square Law Attracting Centre

Before applying the performance equations to specific problems, it seems appropriate to recall some properties of the orbits described by satellites or ballistic missiles in the gravitational field of a planet. If the planet is an ideal isolated sphere of radius R and if g_0 denotes the acceleration of gravity at the surface, the radial acceleration is $g_0(R/r)^2$ at a distance r from the centre. From the radial and tangential components of the acceleration in polar coordinates, one gets as equations of motion for a point mass subjected to the gravitational field alone:

$$\ddot{r} - r\dot{\theta}^2 = g_0\left(\frac{R}{r}\right)^2$$

and

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

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First integrals of these equations are:

$$\tfrac{1}{2}\left[\dot{r}^2 + (r\dot{\theta})^2\right] - g_0 \frac{R^2}{r} = E \tag{12}$$

and

$$r^2\dot{\theta} = A \tag{13}$$

The constant E is the total specific energy, the potential energy being so defined that it vanishes when the point mass is at infinite distance:

$$-\int_r^{\infty} g_0 \frac{R^2}{r^2}\,\mathrm{d}r = -g_0 \frac{R^2}{r}$$

The constant A is twice the "areal velocity". Eliminating $\dot{\theta}$ between the two first integrals and replacing

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\,\dot{\theta} = \frac{A}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}\theta}$$

one obtains the differential equation of the orbit in polar coordinates:

$$\left(\frac{A}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{A^2}{r^2} - 2g\frac{R^2}{r} = 2E \tag{14}$$

Changing to $1/r$ as independent variable and differentiating with respect to θ,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right)\left\{\frac{d^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} - g_0\frac{R^2}{A^2}\right\} = 0$$

Apart from the special solution $r =$ constant, the general solution is

$$\frac{1}{r} = \frac{1}{p} - \frac{e}{p}\cos(\theta - \theta_0) \tag{15}$$

where e is the eccentricity and

$$\cancel{p}A^2 = g_0 R^2\, p \tag{16}$$

and substitution of this solution in eqn. (14) gives

$$(e^2 - 1)\left(\frac{g_0 R^2}{A}\right)^2 = 2E \tag{17}$$

Eqns. (16) and (17) connect the parameters p and e of the orbit with the energy and areal velocity constants. From eqn. (15), it appears that the perigee r_b, or shortest distance from the orbit to the attracting centre, is

$$r_b = \frac{p}{1+e}$$

whilst the apogee, which only exists if $e < 1$, is:

$$r_a = \frac{p}{1-e}$$

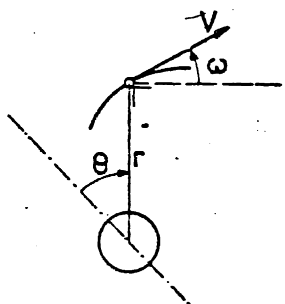
In order to classify the orbits, it will be convenient to introduce dimensionless quantities by using R and g_0 as units. We accordingly define

$$\varrho = r/R, \quad \alpha = r_a/R, \quad \beta = r_b/R$$

$$\gamma = \frac{V}{\sqrt{g_0 R}}$$

where V is the velocity modulus,

$$h = \frac{E}{g_0 R}, \quad a = \frac{A}{R\sqrt{g_0 R}}, \quad f = \frac{p}{R}$$



$e = \dfrac{\alpha-\beta}{\alpha+\beta}$

$a = \sqrt{\dfrac{2\alpha\beta}{\alpha+\beta}}$

$h = -\dfrac{1}{\alpha+\beta}$

Fig. 11.1. Definition of the angle ω.

and introduce the angle ω (Fig. 11.1) such that $V\cos\omega = r\dot{\theta}$. Eqns. (12) and (13) are then rewritten:

$$h = \tfrac{1}{2}\gamma^2 - \frac{1}{\varrho} \tag{12'}$$

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$$a = \gamma \varrho \cos \omega \tag{13'}$$

Eqns. (16) and (17):

$$a^2 = f \tag{16'}$$

$$e^2 - 1 = 2ha^2 \tag{17'}$$

and finally

$$\alpha(1 - e) = \beta(1 + e) = f \tag{18}$$

From this, we find that

$$\alpha\beta = -\frac{a^2}{2h} \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{2}{a^2} \tag{19}$$

so that α and β are the roots of the algebraic equation

$$2hx^2 + 2x - a^2 = 0 \tag{20}$$

This suggests representing an orbit as a point in a Cartesian diagram (a^2,h) (Fig. 11.2). In such a diagram, the curves "x = constant" are

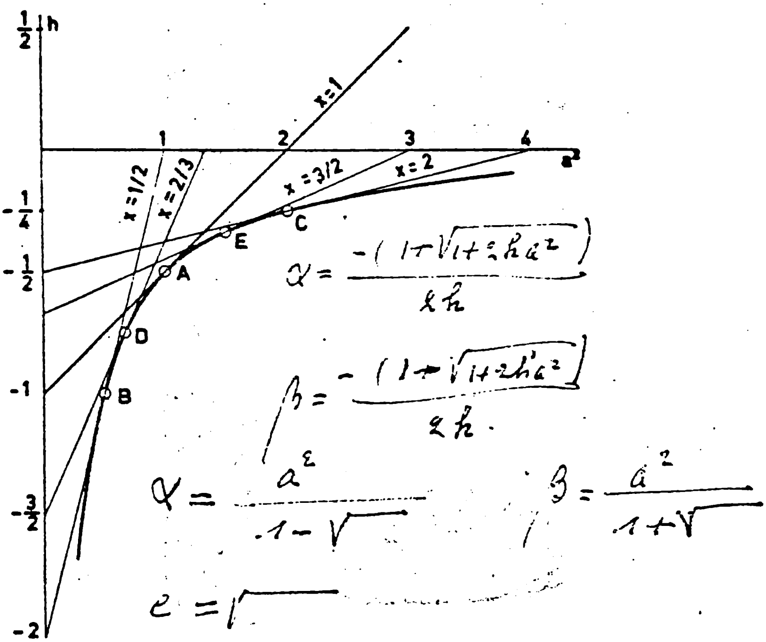

Fig. 11.2. Orbits about an inverse square law attracting centre characterized by their reduced energy h and reduced areal velocity a. Representative points of orbits with same apogee or perigee lie on a straight line. Envelope of straight-line family is hyperbolic locus BDAEC of circular orbits.

straight lines according to eqn. (20). Through each point of the region $h < 0$ pass two straight lines of this one parameter family, and the two values of the parameter x are the ratios of the perigee and apogee distances to the radius R. In the region $h > 0$, there passes a single line through a given point, corresponding to the perigee distance. Indeed, from eqn. (17′) e is then greater than 1 and the orbits of this region are hyperbolas. The envelope of the one parameter family is found in the usual way by eliminating x between eqn. (20) and its derivative with respect to x; it is the hyperbola

$$a^2h = -\tfrac{1}{2}$$

A circular orbit is a limiting case. For a given energy it has the highest kinetic momentum.

Along this curve, eqn. (20) has a double root and each point of it represents a circular orbit; in particular, point A is the circular orbit tangent to the surface.

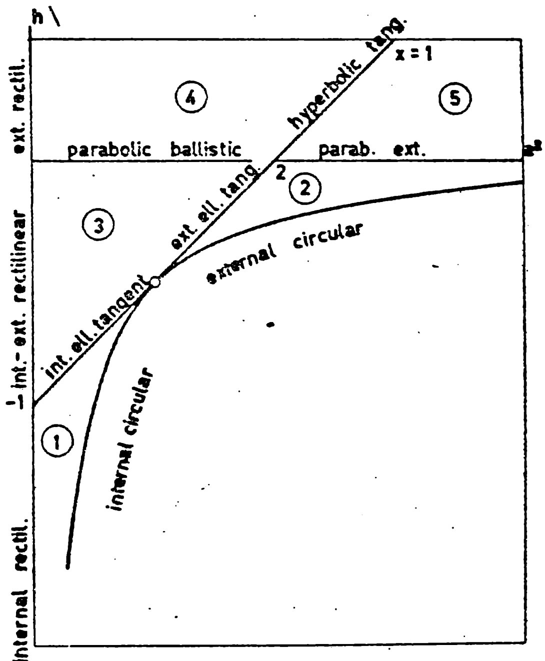

Fig. 11.3. Classification of orbits in five regions. Boundaries of regions: zero energy (parabolic) orbits, orbits touching the surface ($x = 1$) and circular orbits (hyperbolic locus).

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The boundaries constituted by this curve and the straight lines $x = 1$; $h = 0$ and $a^2 = 0$ delimit five regions corresponding to different types of orbit (Fig. 11.3).

Region 1 is a region of (virtual) internal elliptical orbits, both α and β being smaller than unity.
Region 2 is one of external elliptical orbits, or a satellite region, for which α and β are larger than one.
Region 3 is one of external–internal elliptical orbits ($\alpha > 1$, $\beta < 1$). Each of these orbits cuts the surface of the planet twice. It may be called the region of ballistic orbits.
Region 4 contains hyperbolic orbits cutting the surface of the planet and may be called the region of escape orbits.
Region 5 contains external hyperbolic orbits; it is the comet region.

11.5. The Elementary Ballistic Range Problem

Let us consider the problem of maximizing the range on a curved Earth for a given propellant consumption (Fig. 11.4). To obtain a

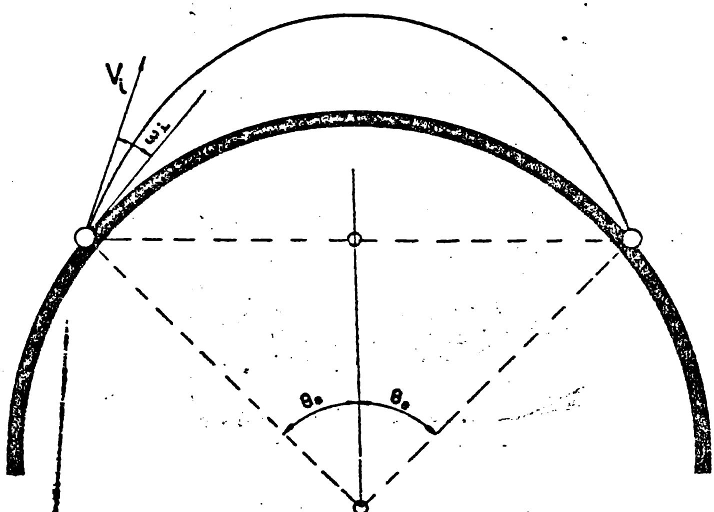

Fig. 11.4. Elliptic orbit of maximum range for given initial velocity. Second focus on the line joining launching and impact sites.

simple first approximation, the earth rotation and aerodynamic drag are neglected. It is also assumed that the burning time is so short that the burnout velocity is given by the ideal velocity

$$V_i = c \ln \frac{M_i}{M_b}$$

and may be regarded as an initial velocity impulsively communicated to a missile. The only problem remaining is the search for the best elevation angle ω_i to maximize range. The trajectory after burnout is an orbit of the type described by eqn. (15). If we take $r = R$ and $\theta = 0$ at the point of departure, r returns to R when $\theta = 2\theta_0$. Maximizing the range $2R\theta_0$ measured along the surface is equivalent to minimizing

$$\cos \theta_0 = \frac{1}{e}\left(1 - \frac{p}{R}\right) = \frac{1}{e}(1 - f)$$

The energy of the orbit is known from the initial values

$$h = \tfrac{1}{2}\gamma_i^2 - 1 \quad \text{where} \quad \gamma_i = V_i/\sqrt{g_0 R}$$

so that, eliminating a^2 between (16′) and (17′), we obtain a relation between the orbit parameters e and f:

$$e^2 - 1 = 2hf$$

We substitute for f to obtain:

$$\cos \theta_0 = \frac{1}{e}\left(1 - \frac{e^2 - 1}{2h}\right)$$

This may now be differentiated with respect to e to find the minimum. One finds

$$e^2 = -(1 + 2h) = 1 - \gamma_i^2$$

$$f = \gamma_i^2/(2 - \gamma_i^2) \tag{21}$$

and the maximum range is

$$2R\theta_0 = 2R \cos^{-1}\left(\frac{2\sqrt{1 - \gamma_i^2}}{2 - \gamma_i^2}\right) \tag{22}$$

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To obtain the optimum elevation angle ω_i, we equate the areal velocity as give by eqn. (13′) in the initial condition

$$a = \gamma_i \cos \omega_i$$

to its value taken from eqn. (16′) in which eqn. (21) is substituted and obtain

$$\cos \omega_i = \left(2 - \gamma_i^2\right)^{-\frac{1}{2}}$$

From this, we may compute

$$\sin 2\omega_i = 2\,\frac{\sqrt{1 - \gamma_i^2}}{2 - \gamma_i^2} = \cos \theta_0$$

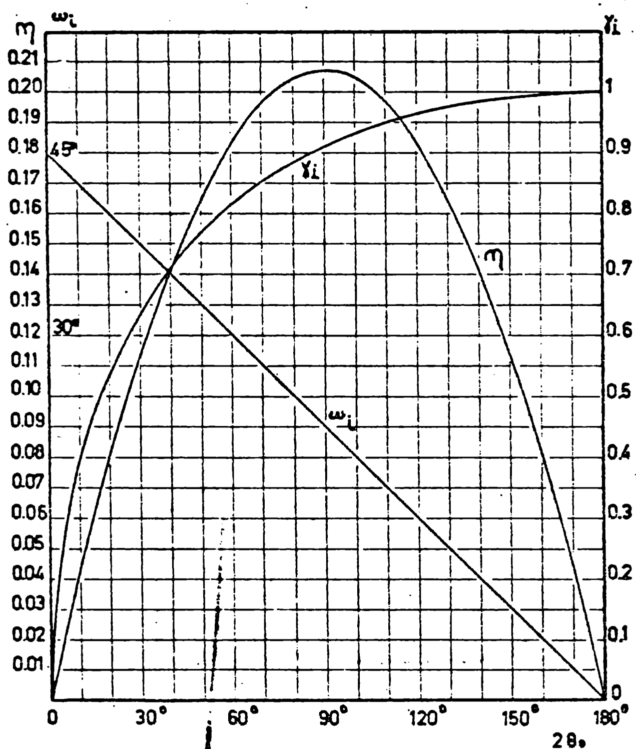

Fig. 11.5. Characteristics of ballistic orbits of maximum range. ω_i optimum launching angle, initial velocity required $V_i = \gamma_i \sqrt{g_0 R}$, apex altitude, $y = \eta R$.

and find the very simple result:

$$\omega_t = \frac{\pi}{4} - \tfrac{1}{2}\,\theta_o \tag{23}$$

Note that for very small ranges, we find the well-known flat Earth result: $\omega_t = 45°$. Other characteristics of interest are the maximum altitude reached along an orbit of maximum range:

$$\frac{p}{1-e} - R = R\,\frac{\sqrt{1-\gamma_t^2}\left(1-\sqrt{1-\gamma_t^2}\right)}{2-\gamma_t^2} \tag{24}$$

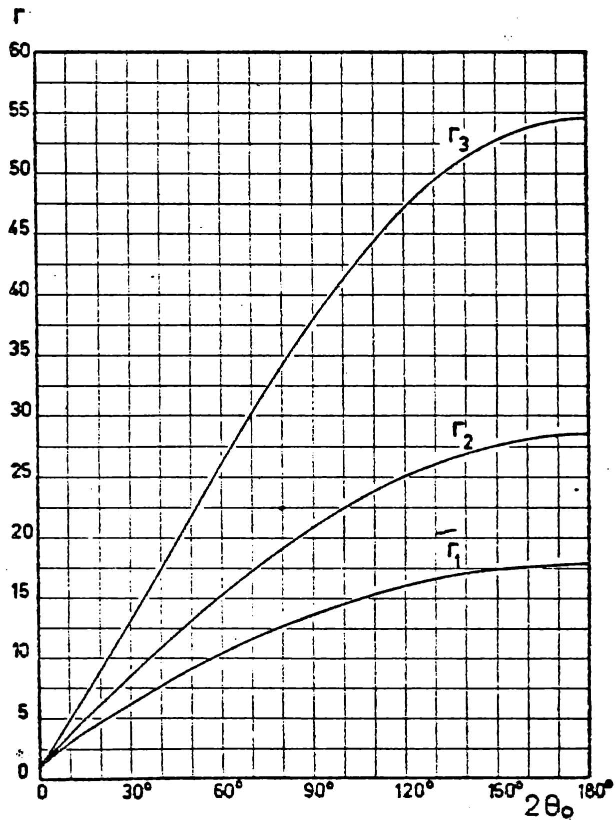

Fig. 11.6. Mass ratios required for ballistic range (impulsive burning). The curves correspond to effective exhaust velocities $c = b\sqrt{g_o R}$ with $b_1 = 0.35$, $b_2 = 0.30$, $b_3 = 0.25$. For $g_o = 9.81$ m/sec and $R = 6370$ km the corresponding specific impulses are approximately 280, 240 and 200 sec.

and the distance between foci which is found to correspond to:

$$\frac{p}{1-e} - \frac{p}{1+e} = R \cos \theta_0$$

Consequently, while the centre of the earth is always one focus of an orbit, the second focus of a ballistic orbit of maximum range lies on the line joining the launching site and the impact site. Figs. 11.5 and 11.6 illustrate the results obtained from eqns. (22), (23) and (24) and the ideal relationship between range and mass-ratio. These values are to be corrected for gravitational losses during the propulsion phase and aerodynamic drag losses. The latter are relatively unimportant for the long-range, large-sized missiles.

11.6. The Elementary Satellite Launching Problems

The problem is again idealized by neglecting drag, Earth rotation and burning time. A first impulse, communicated at launching, places the missile in a transfer orbit. Unless the energy of this orbit is high enough to cause the missile to escape from the attraction of the planet ($h \geqslant 0$), it will return to it. In both cases, it is necessary to modify the velocity vector by a second impulse at some transfer point so as to place the missile in the desired external elliptical orbit it should follow as a satellite. The programme would thus consist of:

1° an impulse at launching, boosting the velocity from $\gamma_1 = 0$ to some value γ_2. The characteristics of the transfer orbit which results are its reduced energy

$$h_2 = \tfrac{1}{2} \gamma_2^2 - 1 \tag{25}$$

and a reduced areal velocity $0 \leqslant a_2 = \gamma_2 \cos \omega_2 \leqslant \gamma_2$, depending on the orientation of the impulse.

2° a coasting phase ending at some transfer point 3, where the reduced distance ϱ_3 lies between the values α and β of the final orbit. At this point the reduced velocity γ_4 and the angle ω_4 required to describe the final orbit are known from its energy h and areal velocity a, as from eqns. (12′) and (13′)

$$\tfrac{1}{2}\,\gamma_4^2 - \frac{1}{\varrho_3} = h \tag{26}$$

$$\gamma_4 \varrho_3 \cos \omega_4 = a \tag{27}$$

3° the velocity of the missile along the transfer orbit at the transfer point is projected orthogonally on the direction of γ_4 and a direction perpendicular to it. From these projections γ_3' and γ_3'', the energy and areal velocity of the transfer orbit may again be calculated:

$$\tfrac{1}{2}\,(\gamma_3'^2 + \gamma_3''^2) - \frac{1}{\varrho_3} = h_2 \tag{28}$$

$$(\gamma_3' \cos \omega_4 - \gamma_3'' \sin \omega_4)\varrho_3 = a_2 \tag{29}$$

The impulse necessary to pass from one velocity vector to the other is resolved in two components at right-angles; one must absorb the transverse component γ_3'', the other the difference $\gamma_4 - \gamma_3'$. Applying the impulsive thrust in the direction of the vectorial difference in velocities, the mass-ratio required will be given by:

$$\frac{c}{\sqrt{g_0 R}} \ln \frac{M_3}{M_4} = \sqrt{(\gamma_4 - \gamma_3')^2 + \gamma_3''^2}$$

and the mass ratio required for the initial impulse

$$\frac{c}{\sqrt{g_0 R}} \ln \frac{M_1}{M_2} = \gamma_2$$

Since the masses M_2 and M_3 at both ends of the transfer orbit are equal, we obtain, by summation, a relation involving the total mass ratio

$$T = \frac{c}{\sqrt{g_0 R}} \ln \frac{M_1}{M_4} = \gamma_2 + \left((\gamma_4 - \gamma_3')^2 + \gamma_3''^2\right)^{\frac{1}{2}} \tag{30}$$

This expression should be minimized with respect to the admissible variations, in programming. Since a and h are given, the parameters (γ_2, a_2, h_2, γ_3', γ_3'', γ_4, ϱ_3 and ω_4) are connected by the five eqns. (25) to (29). There are three degrees of freedom in the choice of programming. Let us first choose a fixed transfer point that is an admissible value of ϱ_3. Then ω_4 and γ_4 are also known and fixed by eqns. (26) and (27). We next choose a fixed value of γ_3' and consider γ_3'' as the third

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variable parameter. If then we transform the first term of eqn. (30) by eqns. (25) and (28):

$$T = \sqrt{2 - \frac{2}{\varrho_3} + \gamma_3'^2 + \gamma_3''^2} + \sqrt{(\gamma_4 - \gamma_3')^2 + \gamma_3''^2}$$

it appears, since γ_3'' varies independently in this expression, that a minimum occurs for $\gamma_3'' = 0$. The first rule will thus be that the transfer from one orbit to the other takes place tangentially. With $\gamma_3'' = 0$ and $\gamma_3' = \gamma_3$, we now have:

$$T = \sqrt{2 - \frac{2}{\varrho_3} + \gamma_3^2} + \gamma_4 - \gamma_3, \text{ provided } \gamma_4 \geqslant \gamma_3$$

The partial derivative of this expression with respect to γ_3 (ϱ_3 and hence γ_4 being kept constant) is always negative:

$$\frac{\partial T}{\partial \gamma_3} = \frac{\gamma_3}{\gamma_2} - 1 < 0$$

since from the energy eqns. (25) and (28)

$$\gamma_3^2 = \gamma_2^2 - 2\left(1 - \frac{1}{\varrho_3}\right) \tag{31}$$

and $\gamma_3 < \gamma_2$ because $\varrho_3 > 1$. The largest possible value of γ_3 will then make T a minimum for a given ϱ_3. The possibility that $\gamma_3 \geqslant \gamma_4$ must be ruled out as will now be shown. The fact that the satellite is on an orbit of the type contained in region 2 of Fig. 11.3 results in a set of inequalities

$$2h + 2 - a^2 < 0$$
$$h < 0$$
$$1 + 2a^2h > 0$$

They are most easily obtained by observing that the corresponding equalities represent the equations of the three boundaries of the region; respectively the straight line $x = 1$, the a^2 axis and the envelope or locus of circular orbits. The first inequality combined with eqns. (26) and (27) gives

$$\gamma_4^2(1-\varrho_3^2\cos^2\omega_4) + 2\left(1-\frac{1}{\varrho_3}\right) < 0 \tag{32}$$

while from eqns. (31) and (29)

$$\gamma_2^2 - a_2^2 = \gamma_2^2\sin^2\omega_2 = \gamma_3^2(1-\varrho_3^2\cos^2\omega_4) + 2\left(1-\frac{1}{\varrho_3}\right) \tag{33}$$

Substitution of ω_4 from this into the inequality gives

$$2\left(1-\frac{1}{\varrho_3}\right)\left(\frac{\gamma_4^2}{\gamma_3^2}-1\right) > \left(\frac{\gamma_4\gamma_2}{\gamma_3}\sin\omega_2\right)^2$$

and since $\varrho_3 > 1$ there follows $\gamma_3 \leqslant \gamma_4$. In fact the maximum value of γ_3 is for $a_2 = \gamma_2$ or $\omega_2 = 0$, *i.e.* for a tangential launching:

$$\gamma_3^2 = \frac{2\left(1-\dfrac{1}{\varrho_3}\right)}{\varrho_3^2\cos^2\omega_4 - 1} = \frac{\gamma_4^2(2+2h-\gamma_4^2)}{a^2-\gamma_4^2}$$

$$T = \gamma_3\varrho_3\cos\omega_4 + \gamma_4 - \gamma_3 = \gamma_4 + \sqrt{\frac{a-\gamma_4}{a+\gamma_4}}\sqrt{2+2h-\gamma_4^2}$$

and this is a minimum for the minimum of γ_4, which occurs at the apogee. It is convenient to express the resulting minimum in terms of reduced perigee and apogee distances. From eqn. (27), we get at the apogee where $\omega_4 = 0$.

$$\gamma_{4}a = a$$

and from eqn. (26)

$$\gamma_4^2 - \frac{2}{a} = 2h$$

whence

$$T = \frac{a}{a} + \sqrt{\frac{a-1}{a+1}}\cdot\sqrt{\frac{2(a-1)}{a}}$$

and finally, substituting a from the second of eqns. (19):

$$T = \sqrt{\frac{2\beta}{a(a+\beta)}} + (a-1)\sqrt{\frac{2}{a(a+1)}} \tag{34}$$

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This absolute minimum of T requires a tangential launching, a coasting period along one half of the elliptical transfer orbit, and finally a second small tangential impulse when touching the desired orbit at the apogee (Fig. 11.7).

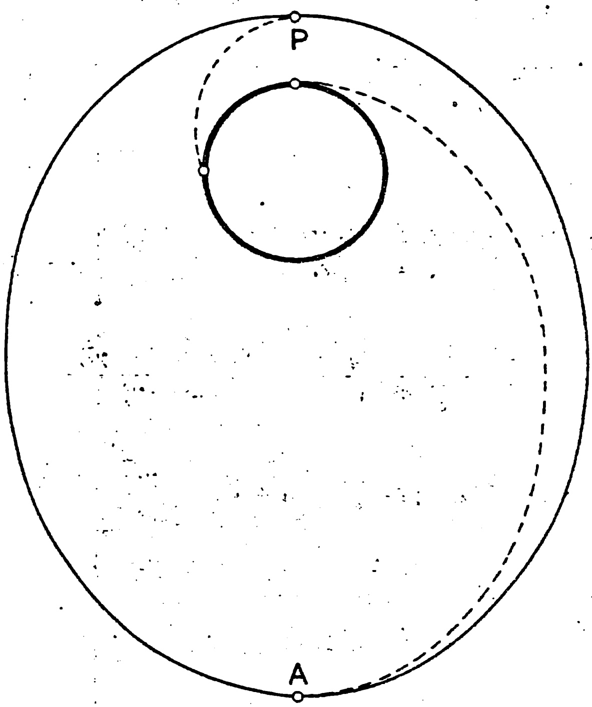

Fig. 11.7. Transfer orbits for launching an artificial satellite.

For circular satellite orbits ($\beta = \alpha$) the numerical value of T does not increase steadily with α, but, curiously enough, passes through a maximum for $\alpha = 7$, *i.e.* for an orbit of radius equal to seven Earth radii. This maximum value is $4/\sqrt{7}$, or about 1.5. It afterwards decreases again to an asymptotic value of $\sqrt{2}$. This last value corresponds to the escape along a parabolic orbit tangential to the Earth surface. It is of interest to note that both impulses are directed along equipotentials of the gravity field. Consequently, if we replace impulses by propulsive periods of finite time, *i.e.*, if we use engines of a normal size, the gravitational losses will remain small and the formula (34) is still a good approximation. However, while the use of a very small engine for the final impulse is fully justified, a finite burning time for the initial propulsive phase makes a tangential take-off difficult. It would require a launching ramp of great length and the achievement of orbital velocity at low altitude with enormous kinetic heating effects.

It is therefore logical to examine the penalty incurred for launching with a given initial angle ω_2.

The result $\gamma_3'' = 0$ still holds true, and from eqn. (30) we proceed to minimize

$$T = \gamma_2 + \gamma_4 - \gamma_3$$

We replace $a_2 = \gamma_2 \cos \omega_2$ in eqn. (29) and eliminate $\varrho_3 \cos \omega_4$ with eqn. (27)

$$\gamma_2\gamma_4 = \frac{a\gamma_3}{\cos \omega_2} \tag{35}$$

We also eliminate ϱ_3 and h_2 between eqns. (25), (26) and (28):

$$\gamma_2^2 + \gamma_4^2 = \gamma_3^2 + 2 + 2h \tag{36}$$

Combining the last two results:

$$(\gamma_2 + \gamma_4)^2 = \gamma_3^2 + 2 + 2h + \frac{2a}{\cos \omega_2}\gamma_3$$

and consequently

$$T = -\gamma_3 + \sqrt{\left(\gamma_3 + \frac{a}{\cos \omega_2}\right)^2 - \mu^2}$$

where

$$\mu^2 = \frac{a^2}{\cos^2 \omega_2} - 2 - 2h$$

According to the first inequality μ^2 is certainly a positive quantity; if T is now differentiated with respect to γ_3, the only remaining variable,

$$\frac{\mathrm{d}T}{\mathrm{d}\gamma_3} = \frac{\gamma_3 + \dfrac{a}{\cos \omega_2}}{\sqrt{\left(\gamma_3 + \dfrac{a}{\cos \omega_2}\right)^2 - \mu^2}} - 1 > 0$$

and the minimum of T, with given ω_2, occurs for the smallest value

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of γ_3. To find this value we eliminate γ_2 between eqns. (35) and (36):

$$\gamma_3^2 = \gamma_4^2 \frac{2 + 2h - \gamma_4^2}{a^2 - (\cos^2 \omega_2)\gamma_4^2} \cos^2 \omega_2$$

The analysis of this expression shows that when γ_4 varies from its lowest value, reached at the apogee, to its highest value reached at perigee, γ_3^2 increases and passes through a maximum to decrease again. Hence the minimum of γ_3 is reached when transfer is either at the apogee or at the perigee. For transfer at the apogee we obtain, putting $\varrho_3 = \alpha$ and using eqns. (19), p. 718.

$$\gamma_2 = \sqrt{\frac{2\alpha(\alpha - 1)}{\alpha^2 - \cos^2 \omega_2}}, \quad \gamma_3 = \frac{\cos \omega_2}{\alpha} \gamma_2, \quad \gamma_4 = \sqrt{\frac{2\beta}{\alpha(\alpha + \beta)}} \tag{37}$$

$$T = T(\alpha, \beta) = \sqrt{\frac{2\beta}{\alpha(\alpha + \beta)}} + (\alpha - \cos \omega_2) \sqrt{\frac{2(\alpha - 1)}{\alpha(\alpha^2 - \cos^2 \omega_2)}} \tag{38}$$

For transfer at the perigee we simply have to exchange α and β in these expressions. There may exist a critical angle ω_c for which $T(\alpha, \beta) = T(\beta, \alpha)$ and it is indifferent to transfer at $\varrho_3 = \alpha$ or β. This critical angle is easily obtained by equating the two minimum values of γ_3; this yields after reduction:

$$\cos^2 \omega_c = \alpha\beta - \alpha^2(\beta - 1) - \beta^2(\alpha - 1) \tag{39}$$

If ω_c exists, the absolute minimum T occurs for a transfer at the apogee if $\omega_2 < \omega_c$ and at perigee if $\omega_2 > \omega_c$. The formulae (37) and (38) are illustrated on Fig. 11.8 for the case $\alpha = 1.3$ and $\beta = 1.2$, where $\cos^2 \omega_c = 0.79$. To provide absolute figures for the velocities and mass ratio we have adopted the values

$$\sqrt{g_0 R} = 7.9 \text{ km/sec} \qquad \text{and} \qquad \frac{\sqrt{g_0 R}}{c} = 3.5$$

It is seen that up to launching angles of 30°, the penalty on the mass ratio is not unduly severe. The transfer orbit is shorter, terminates at the perigee, and therefore offers advantages for tracking and radio communications up to the end of the launching phase. Moreover, since the rocket will emerge sooner from the dense atmosphere and with a substantially lower velocity, the kinetic heating problems are simplified.

The main drawback of such a choice of transfer orbit is the gravitational loss incurred by finite burning time. Many problems of interplanetary flight have been studied elsewhere, and an interesting account of these by Lawden[10] also contains the principal references to the subject. Three-dimensional problems of this kind are dealt with by Bossart[11].

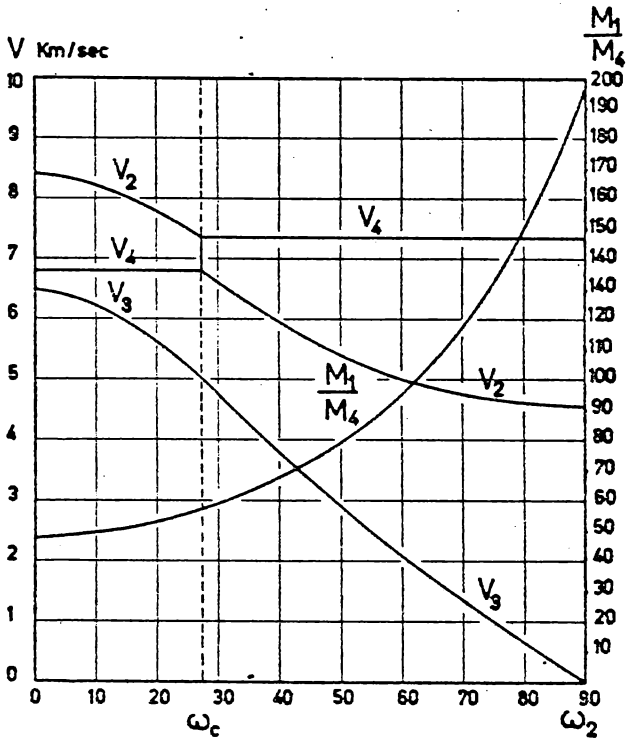

Fig. 11.8. Characteristics for optimum launching of a satellite as a function of the initial launching angle ω_i. Perigee and apogee of chosen orbit are respectively 1.2 R and 1.3 R. For launching angle smaller than critical ($\omega_c = 27.275°$) optimum transfer is at apogee, higher, at perigee. V_2 velocity of initial impulse, V_3 velocity at transfer point, $V_4 — V_3$ velocity boosting required at transfer point. Overall mass ratio computed for specific impulse of 240 sec.

11.7. Structural or Merit Indices

Before returning to the problem of optimum engine size, something must be said about the concept of structural index. The initial mass M_t of a single-stage rocket is conveniently split as follows

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$$M_t = M_b + M_p \tag{40}$$

where M_b is the mass after burnout and M_p the mass of the propellant. M_b itself is split into

$$M_b = M_s + M_e + M_u \tag{41}$$

with M_s the structural mass, M_e the rocket-engine mass, M_u the useful mass or payload. It is not always clear to which of these subdivisions a given component of the rocket belongs and the classification finally followed is never entirely free of conventional decisions. This, however, should not critically affect the value of preliminary optimization processes, now to be outlined.

In order to obtain a maximum performance of some specified type, various configurations must be investigated by modifying the relative sizes of the rocket components. In each configuration the structural weight and engine weight can be estimated by careful detail analysis. But, in order to reduce the number of configurations to be analysed in detail, a first approximation to the optimal design can generally be obtained by using structure and engine weight formulae of an elementary type. Such formulae, permitting the optimization to be carried out analytically, have been proposed by various authors. They are mostly based on elementary theoretical considerations though in the future they should preferably rely on a statistical analysis of actual designs.

A formula for combined structure and engine weight is provided by Vertregt's definition of a structural ratio[3,4]

$$s = \frac{M_s + M_e + M_p}{M_s + M_e} \tag{42}$$

When solved for $M_s + M_e$ this definition actually considers the sum of structure and engine mass to be proportional to the mass of propellant. The structural factors introduced by Engel[5] and Weisbord[8] imply the same type of proportionality and so are not fundamentally different. Separate formulae for structural weight and engine weight were suggested by Williams[7].

$$\zeta = \frac{M_s}{M_p} \tag{43}$$

$$\varepsilon = \frac{M_e}{M_t} \tag{44}$$

In the absence of actual data another reasonable assumption is to consider the structural weight to be proportional to the take-off weight as in ref.[9]

$$\sigma = \frac{M_s}{M_t} \tag{45}$$

In the same reference, the engine weight including the feed system, was taken to be proportional to the maximum thrust.

$$K = \frac{F_{max}}{g_0 M_e} \tag{46}$$

Optimizations carried out under this last assumption show that the values obtained for ε are far from being constant. The numerical values adopted for the structural factors s, ζ or σ and the engine factors ε or K are to be considered as "merit indices" indicative of the degree of perfection attained in technological execution.

11.8. Engine Size and Gravitational Losses

The necessity of a compromise regarding engine size became apparent as soon as gravitational losses showed the necessity of short burning times. Once a given size of engine is adopted, it is clear that in order to reduce the losses, it should develop its full thrust or mass flow throughout the powered flight (this statement may however have to be amended when drag is taken into account, see Chapter 12, p. 795). With m a constant maximum and assuming g_s to be either constant or replaced by some average value, eqns. (2), (3) and (5′), pp. 712, 713, give us the difference between the velocity at burnout (V_b) and the initial velocity (V_i):

$$V_b - V_i = c\left(\ln \frac{M_t}{M_b} - \frac{g_s}{F}(M_t - M_b)\right)$$

It is convenient to introduce the following notations

$$r = \frac{M_t}{M_b} \quad \text{the mass ratio}$$

$$a = \frac{F}{g_0 M_b} \text{ the gross acceleration factor at burnout}$$

and to rewrite the equation for the velocity gain as

$$\frac{\Delta V}{c} = \ln r - \frac{g_s}{g_0} \frac{r-1}{a} \tag{47}$$

The engine size will be determined by the value of ε. The best compromise depends on the kind of optimum performance which is required. One may for instance require the payload ratio

$$u = \frac{M_u}{M_t} \tag{48}$$

to be maximum for a given velocity performance ΔV. Assuming the validity of (46), using eqns. (40) and (41) and the definitions of r, ε and a we obtain

$$\varepsilon = \frac{a}{Kr} \tag{49}$$

and

$$u = \frac{\zeta + 1}{r} - \zeta - \frac{a}{Kr} \tag{50}$$

or

$$u = \frac{1}{r} - \sigma - \frac{a}{Kr} \tag{50'}$$

according to the definition adopted for the structural factor. The maximum of u with respect to the variables r and a, connected by eqn. (47), is obtained by elementary methods. In the case of eqn. (50) the optimum value of a turns out to be:

$$a = \frac{g_s}{2g_0}\left[1 + \sqrt{1 + 4\frac{g_0}{g_s}K(r-1)(\zeta+1)}\right] \tag{51}$$

A set of related optimum values is found by choosing a mass ratio, calculating a from eqn. (51), the performance ΔV from eqn. (47), ε from eqn. (49) and u from eqn. (50). Repeating this calculation for various values of r, one may finally establish the results with $\Delta V/c$ as the independent variable. In the case of eqn. (50′), the result eqn. (51) holds true with $\zeta = 0$, and the optimum value of a does not depend on the structural factor σ. This is otherwise obvious from eqn. (50′), where the maximum for u and the maximum for the sum $u + \sigma$ are seen to be identical problems. The calculations might still be performed in an elementary way under the more refined assumption that σ or ζ depend on the acceleration factor at burnout α.

Another and perhaps more logical requirement would be to make u a maximum for a given total energy gain. For the total energy gain

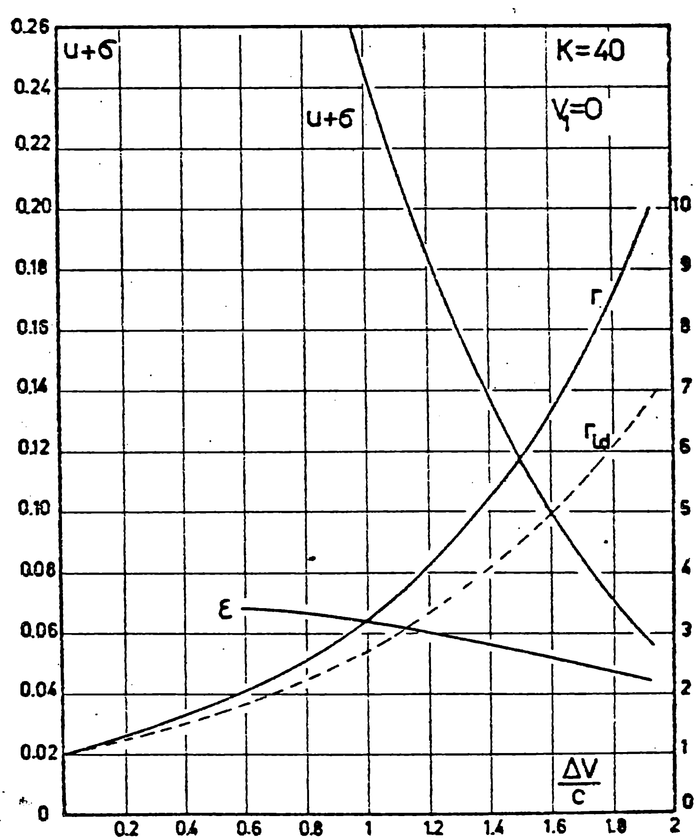

Fig. 11.9. Optimization of engine size for single-stage rocket in vertical flight with zero initial velocity. Horizontal scale: total specific energy at burnout converted to equivalent velocity ratio. Vertical scales: relative engine weight ε based on merit index $K = 40$; sum of relative payload and structural weight $u + \sigma$; mass ratio r; ideal mass ratio r_{id}.

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without gravitational losses one has:

$$\tfrac{1}{2}(V_b^2 - V_i^2) = \tfrac{1}{2}\,\Delta V^2 + V_i \cdot \Delta V = \frac{c^2}{2}\ln^2 r + V_i c \ln r$$

The real energy gain follows by subtracting from this the gravitational energy loss. This was calculated (eqn. 9, p. 715) when g_s/m is constant, as it is in the present problem, and the real energy gain is found to be

$$\frac{\Delta E}{c^2} = \frac{V_i}{c}\ln r + \tfrac{1}{2}\ln^2 r - \frac{g_s}{g_0}\,\frac{r\ln r - r + 1}{a} \tag{52}$$

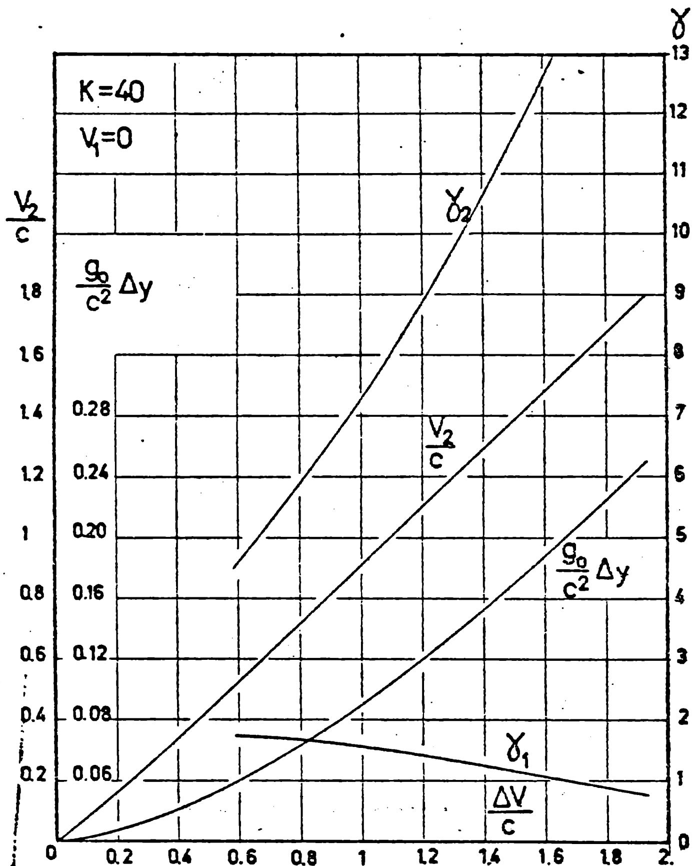

Fig. 11.10. Same horizontal scale as in Fig. 11.9. Vertical scales: net acceleration factor at take-off γ_1, at burnout γ_2; reduced velocity at burnout V_2/c; reduced altitude gain at burnout $\Delta y\, g_0/c^2$.

Instead of being tied by eqn. (57), the variables r and α are now tied by eqn. (52), and the maximum of u, as given by eqn. (50), occurs for:

$$\alpha = \frac{g_s}{2g_0}\,\frac{r-1}{\dfrac{V_i}{c}+\ln r}\left[1+\sqrt{1+4\frac{g_0}{g_s}\,\frac{(r\ln r - r + 1)}{(r-1)^2}\left(\frac{V_i}{c}+\ln r\right)K(\zeta+1)}\right] \qquad (53)$$

Again, the corresponding result for eqn. (50′) is obtained by putting $\zeta = 0$. The initial velocity ratio V_i/c appears here as a new parameter influencing the engine size.

Figs. 11.9 and 11.10 illustrate this type of optimization in the case of vertical flight in a uniform gravity field ($g_s = g_0$) starting at zero velocity ($V_i = 0$) and for a merit index $K = 40$. The factor σ was used to characterize the structural efficiency and since $(u + \sigma)$ is optimized no numerical value was assigned to σ. In both diagrams the horizontal scale is given as $\sqrt{2\Delta E}/c$, which is a kind of reduced equivalent velocity. Fig. 11.9 shows the decrease in the optimum value of $(u + \sigma)$, the engine ratio ε and the mass ratio r necessary to obtain a given ΔE performance. The interrupted curve gives the mass ratios when gravitational losses are neglected. Fig. 11.10 shows the initial net acceleration

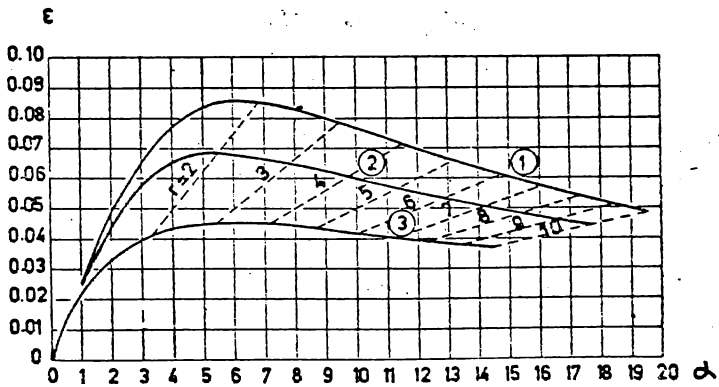

Fig. 11.11. Correlation between relative engine weight ε and gross acceleration factor at burnout α for three types of optimization of relative payload in vertical flight: 1° for velocity gain only, initial velocity unimportant; 2° for total energy gain, zero initial velocity; 3° for total energy gain, initial velocity equal to effective exhaust velocity. Engine merit index $K = 40$.

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factor $\gamma_1 = \alpha/r - 1$ and the same factor at burnout $\gamma_2 = \alpha - 1$. Also, the subdivision of the energy gain between the kinetic and potential forms may be estimated from the curves giving the reduced velocity at burnout and the reduced gain in altitude.

Fig. 11.11 illustrates the correlation between the engine ratio ε and α in the three different cases of vertical flight and for $K = 40$:

1° When optimizing $(u + \sigma)$ for a given velocity gain alone.

2° For a given energy gain with $V_t = 0$.

3° For a given energy gain with $V_t = c$.

The dashed lines passing through the origin correspond to constant mass ratios.

11.9. Optimal Mass Ratios for Multistaged Rockets

A substantial increase in payload ratio may be obtained by rocket staging. In this technique, masses which have become useless are abandoned step by step and the chemical energy that would have been spent to accelerate them up to the final velocity is recuperated. It is convenient to distinguish between a step-rocket and a sub-rocket and to index them in the order indicated by Fig. 11.12. Each sub-rocket is divided in the same manner as expressed by eqns. (40) and (41), p. 732

$$M_{tn} = M_{bn} + M_{pn} \tag{54}$$

$$M_{bn} = M_{sn} + M_{en} + M_{un} \tag{55}$$

It is further considered that the useful mass of a given sub-rocket is made of the next sub-rocket

$$M_{un} = M_{t(n+1)} \tag{56}$$

except of course for the last sub-rocket ($n = N$), which contains the real payload:

$$M_{uN} = M_{t(N+1)} = M_u \tag{57}$$

Denoting by u_n the partial payload ratio

$$u_n = \frac{M_{un}}{M_{tn}} = \frac{M_{t(n+1)}}{M_{tn}} \tag{58}$$

the overall payload ratio is

$$u = \frac{M_u}{M_{t1}} = \frac{M_u}{M_{tN}} \frac{M_{tN}}{M_{t(N-1)}} \cdots \frac{M_{t2}}{M_{t1}}$$

or

$$u = u_N \cdot u_{N-1} \ldots u_1 \tag{59}$$

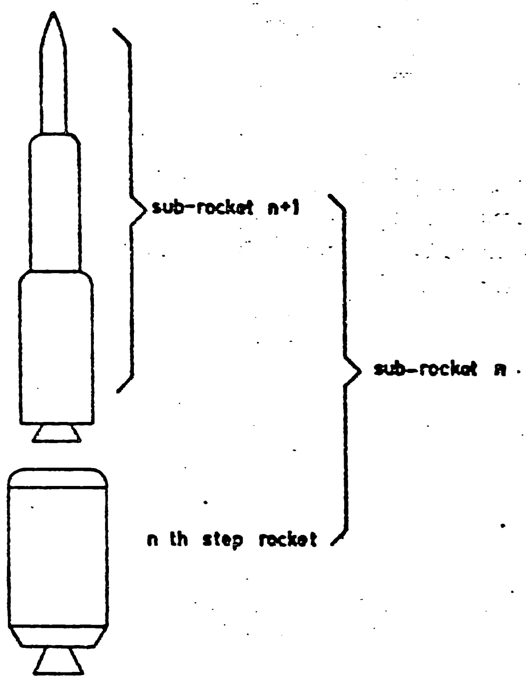

Fig. 11.12. Definition adopted for N-staged rockets in series.

An interesting problem consists in finding the optimum distribution of mass ratio between the successive sub-rockets in order to maximize the over-all payload ratio. Certain assumptions must be made regarding the use of merit indices for the structures and the engines and several will be tried and discussed in the next section.

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11.9.1. Rocket staging neglecting gravitational losses

It is assumed that for each sub-rocket, the partial boost in velocity follows the ideal law

$$(\Delta V)_n = c_n \ln r_n \tag{60}$$

where c_n denotes the effective exhaust velocity of the engine of the step-rocket and r_n the partial mass ratio

$$r_n = \frac{M_{in}}{M_{bn}} \tag{61}$$

The total velocity performance is

$$\Delta V = \sum_1^N c_n \ln r_n \tag{62}$$

To maximize u with respect to the distribution of mass ratios, we need to know how a partial payload ratio u_n is affected by the mass ratio r_n. If a structural factor s_n, as defined by eqn. (42), p. 732, is assumed for each sub-rocket, we find from eqns. (54), (55) and (58)

$$u_n = \frac{1}{r_n} \, \frac{s_n - r_n}{s_n - 1} \tag{63}$$

It is equivalent to maximize $\ln u$ instead of u itself, and from eqn. (59) this amounts to stating the problem in the form:

$$\sum_1^N \ln u_n \text{ maximum}$$

The variables are not independent but are connected by the performance condition eqn. (62). Following a procedure first applied by Vertregt, we use a Lagrangian multiplier v^{-1} to make the variables independent and obtain the modified problem:

$$U = \sum_1^N \ln u_n - \frac{1}{v} \left(\Delta V - \sum_1^N c_n \ln r_n\right) \text{ maximum} \tag{64}$$

We then substitute expressions (63) for the u_n and derive the conditions $\partial U / \partial r_n = 0$ for a maximum; they yield:

$$r_n = \left(1 - \frac{v}{c_n}\right) s_n \qquad n = 1,2, \ldots, N \tag{65}$$

so that the optimum mass ratios are determined once the value of the Lagrangian multiplier is known. It is sufficient for this purpose to substitute eqns. (65) back into the performance condition (62):

$$\Delta V = \sum_{1}^{N} c_n \left[\ln s_n + \ln\left(1 - \frac{v}{c_n}\right)\right] \tag{66}$$

where everything is known except v. In general, the value of v satisfying this equation can only be found by a method of successive approximations. There is however one case where the solution of eqn. (66) is elementary; it is that of equal effective exhaust velocities. If

$$c_n = c \qquad n = 1,2,\ldots,N$$

then

$$1 - \frac{v}{c} = \frac{1}{s}\exp\left(\frac{\Delta V}{Nc}\right)$$

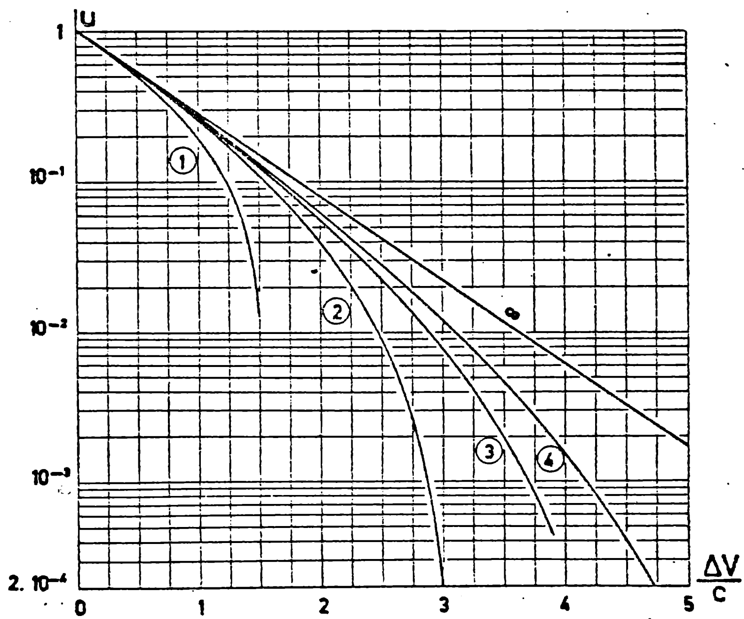

Fig. 11.13. Payload ratio *versus* velocity performance for multistaged rockets. Optimization neglecting gravitational losses and using same overall structural efficiency factor of Vertregt $s = 4.7$.

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where s is defined as the geometrical average of the s_n:

$$\ln s = \frac{1}{N} \sum_{1}^{N} \ln s_n$$

Then

$$r_n = \frac{s_n}{s} \exp\left(\frac{\Delta V}{Nc}\right) \qquad (67)$$

$$u = \frac{\left[s \exp\left(-\frac{\Delta V}{Nc}\right) - 1\right]^N}{(s_1 - 1)(s_2 - 1)\ldots(s_N - 1)}$$

As a further specialization, when all the structural factors are identical ($s_n = s$):

$$u = \left[\frac{s \exp\left(-\frac{\Delta V}{Nc}\right) - 1}{(s - 1)}\right]^N \qquad (68)$$

It is clear from eqn. (67) that in this last case all the partial mass ratios are identical, and then, from eqn. (60), that the velocity increments are equally divided between stages. Formula (68) is illustrated by Fig. 11.13 with a hypothetical value of $s = 4.7$. It will be observed that the increase in the number of stages is beneficial to the overall payload ratio, in the whole performance range. This naturally provokes consideration of the upper limit which could be obtained by subdivision into an infinite number of stages. Taking the logarithm of u in eqn. (68) and applying L'Hospital's rule with respect to the limit $N \to \infty$, we get

$$\ln u = \frac{-s}{s-1} \frac{\Delta V}{c} \qquad (69)$$

In the diagram Fig. 11.13 this equation is represented by the upper straight line. Inasmuch as the theory takes no account of the additional components required to effect the separation of stages, this last result is unrealistic but may serve for purposes of comparison with other theories. The significance of eqn. (69) is also clear when derived from considerations of continuity, which will prove useful in other cases where no definite limit is obtained by going to an infinite number of stages.

From eqn. (42), p. 732, there follows:

$$M_s + M_e = \frac{1}{s-1} M_p$$

Instead of considering M_p to be the total mass of propellant at take-off, we now assume this relation to hold at any time during the powered flight. It implies that the rocket is continuously losing engine weight and structural weight in proportion to the propellant consumed. The rocket mass at any time is then

$$M = M_u + M_p + \frac{1}{s-1} M_p = M_u + \frac{s}{s-1} M_p \tag{70}$$

and

$$\mathrm{d}M = \frac{s}{s-1} \mathrm{d}M_p$$

whilst only the mass flow of propellant is active in determining the engine thrust

$$m\mathrm{d}t = -\mathrm{d}M_p$$

The equation of motion in the absence of gravity is then:

$$M\mathrm{d}V = mc\mathrm{d}t = -c\mathrm{d}M_p = -c\frac{s-1}{s}\mathrm{d}M \tag{70'}$$

When all the propellant is consumed, the mass of the rocket reduces to M_u by virtue of eqn. (70), so that the integration of eqn. (70′) between $M = M_t$ and $M = M_u$ leads to :

$$\Delta V = c\frac{s-1}{s}\ln\frac{1}{u}$$

in accordance with eqn. (69).

Instead of using an overall structural factor of the type eqn. (42), p. 732, separate merit indices of type eqn. (44) and eqns. (43) or (45) may be used. However, as shown by the analysis of section 11.8, (p. 733) a definite value of ε_n is really determined only by the gravitational losses incurred for the production of a definite $(\Delta V)_n/c_n$, or a definite energy gain.

Thus the choice of specific ε_n values really implies that some consideration be given to gravitational losses. But then the gravitational losses should really also be considered in the overall performance equation tying together the variables r_n; in other words, the use of eqn. (62) is difficult to justify. The same considerations naturally apply to the theory based on the use of the s_n coefficients, in which the choice of engine size is contained implicitly.

Before trying to analyse the problem taking full account of gravity losses, we may try to restore some logic into the approximation neglecting these losses by introducing the idea that the gross acceleration factor reached at each burnout of a stage rocket is limited to a definite value for structural or other reasons (like integrity of the guidance or instrumentation equipment). This approach has the advantage of offering a better guarantee to the significance of the σ or ζ structural factors, should they depend appreciably on this acceleration limit. The acceleration factor at burnout is

$$a = \frac{F}{g_0 M_b} = \frac{F}{g_0 M_e} \frac{M_e}{M_i} \frac{M_i}{M_b}$$

Assuming again that the engine weight is proportional to the maximum thrust it can deliver (eqn. 46, p. 733), we obtain:

$$a = \varepsilon K r \quad \text{(this is naturally the same as eqn. 49, p. 734)}$$

so that for a fixed a, ε really becomes inversely proportional to the mass ratio. Such a relation is expected to hold for each sub-rocket:

$$\varepsilon_n = \left(\frac{M_e}{M_i}\right)_n = \frac{a_n}{K_n r_n} \tag{49'}$$

Let us now see how u_n depends on r_n, using the structural factors ζ_n as defined by eqn. (45), p. 733. We find for the initial mass (dropping subscript n)

$$M_i = M_u + \frac{a}{Kr} M_i + (1 + \zeta) M_p$$

replace M_p by $(M_i - M_b)$ and divide by M_i

$$\left(1 - \frac{a}{Kr}\right) = u + (1 + \zeta)\left(1 - \frac{1}{r}\right)$$

This is eqn. (50) again and it is assumed to hold for each sub-rocket

$$u_n = \frac{1 + \zeta_n - a_n/K_n}{r_n} - \zeta_n \tag{71}$$

We substitute this in eqn. (64) and find as optimum conditions

$$\frac{1}{u_n} \frac{du_n}{dr_n^{-1}} - \frac{1}{v} c_n r_n = 0$$

or after reduction,

$$\zeta_n r_n = \left(1 - \frac{v}{c_n}\right)\left(1 + \zeta_n - \frac{a_n}{K_n}\right) \tag{72}$$

Again v is known through back substitution in eqn. (62):

$$\Delta V = \sum_1^N c_n \left[\ln\left(1 + \zeta_n - \frac{a_n}{K_n}\right) - \ln \zeta_n + \ln\left(1 - \frac{v}{c_n}\right)\right] \tag{73}$$

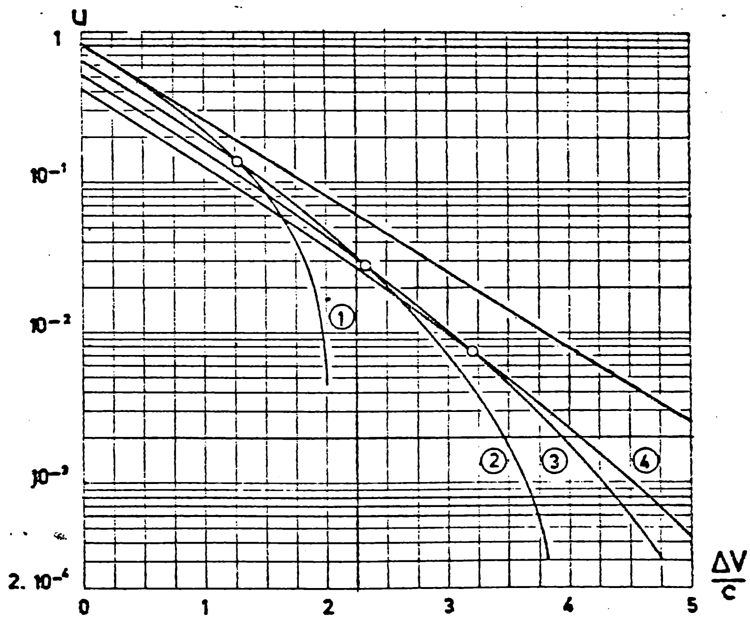

Fig. 11.14. Payload ratio *versus* velocity performance for multistaged rockets. Optimization neglecting gravitational losses and using same structural efficiency factor $\zeta = 0.12$, same engine merit index $K = 40$, and same limitation to gross-acceleration factor at burnout $a = 8$.

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This equation is again easily solved when all c_n are identical. The overall payload ratio, when all c_n, K_n, a_n and ζ_n are identical, is found to be

$$u = \left[\left(1 + \zeta - \frac{a}{K}\right) \exp\left(-\frac{\Delta V}{Nc}\right) - \zeta\right]^N \tag{74}$$

It is only logical to find that small values of α are beneficial to u. Indeed, since gravitational effects are ignored, there is not the slightest advantage of using the large thrust engines required to produce large α values. But if too low values of α_n are used in the presence of gravitational effects in the present theory, the values found for u will become much too optimistic.

The diagram Fig. 11.14 illustrating formula (74) shows that a given number of stages is now superior to any other number in some specific range of performance, and it is immediately seen from eqn. (74) that for $\Delta V = 0$ the value of u decreases with increasing N since $1 - a/K < 1$. One should not expect therefore to find any significant limit performance by going over to $N = \infty$ or to a continuous case. Similar calculations using a structural factor of the type σ defined by eqn. (45) would replace eqns. (71) to (74) by the following:

$$u_n = -\sigma_n + \left(1 - \frac{a_n}{K_n}\right) \frac{1}{r_n} \tag{71'}$$

$$\sigma_n r_n = \left(1 - \frac{v}{c_n}\right)\left(1 - \frac{a_n}{K_n}\right) \tag{72'}$$

$$\Delta V = \sum_1^N c_n \left[\ln\left(1 - \frac{a_n}{K_n}\right) - \ln \sigma_n + \ln\left(1 - \frac{v}{c_n}\right)\right] \tag{73'}$$

$$u = \left[\left(1 - \frac{a}{K}\right) \exp\left(-\frac{\Delta V}{Nc}\right) - \sigma\right]^N \tag{74'}$$

11.9.2. Rocket staging including gravitational losses

Let us consider the influence of the retarding component of the gravity field, denoting it by g_n for the portion of trajectory described during the operation of the nth stage. This value is either a constant or should be evaluated as a suitable average. Eqn. (60), p. 740, is now replaced by an equation similar to eqn. (47), p. 734:

$$(\Delta V)_n = c_n \ln r_n - c_n \frac{g_n}{g_0} \frac{r_n - 1}{a_n} \tag{75}$$

and the problem is modified from the form (64), p. 740. to the following form:

$$U = \sum_1^N \ln u_n - \frac{1}{v}\left(\Delta V - \sum_1^N c_n \ln r_n + \sum_1^N c_n \frac{g_n}{g_0} \frac{r_n - 1}{a_n}\right) \max \tag{76}$$

Again we use eqn. (71) to represent the partial payload ratio as a function of r_n and a_n. We may now take two standpoints. As in the previous section, we may consider the a_n to be specified constants and be content to derive, as optimal conditions, the equations:

$$\frac{\partial U}{\partial r_n} = 0, \qquad n = 1,2,\ldots, N \tag{77}$$

As one can verify, eqns. (77) are quadratics in r_n. The appropriate root is easily determined from the condition that it should reduce to eqn. (72) when g_n is made to vanish. By giving several values to the Lagrangian multiplier, the r_n are determined and the corresponding ΔV may be calculated. However, the new formulation offers the more interesting possibility of determining the best acceleration factors a_n themselves. It is sufficient for this purpose to add the set of optimal conditions

$$\frac{\partial U}{\partial a_n} = 0 \qquad n = 1,2,\ldots, N \tag{78}$$

Indeed one easily recognizes in this procedure the application to a staged rocket of the type of optimization developed for a single rocket in the first part of section 11.8 (p. 733).

From the conditions eqns. (77) and (78), one may conclude that eqn. (51), p. 734, remains valid for all sub-rockets and, if the r_n were known, would determine the a_n independently of the value of the Lagrangian multiplier. The equations that may be formed by elimination of the a_n are unfortunately of the third degree in the r_n and the best procedures to solve the set of optimal conditions again appear to be of the semi-inverse type. We may for instance rewrite the eqns. of type (51) in the form

$$r_n = 1 + \frac{a_n\left(a_n - \dfrac{g_n}{g_o}\right)}{\dfrac{g_n}{g_o} K_n(1 + \zeta_n)} \tag{79}$$

and derive another set to compute the multiplier

$$\frac{v}{c_n} = 1 - r_n + r_n \frac{a_n - \dfrac{g_n}{g_o}}{a_n(1 + \zeta_n)} \tag{80}$$

The procedure may then consist in assuming a_n, computing r_n from eqn. (79) and v from eqn. (80), repeating and interpolating until v has the same value from each stage. The corresponding ΔV is then determined by the addition of eqns. (75). For a required ΔV, the whole process must be repeated with different values of v, until a safe interpolation can be made. Account may be taken of a possible dependence of ζ_n on a_n by incorporating the differential coefficients $d\zeta_n/da_n$ in conditions (78). Corrections may also have to be made for the assumed g_n values from examination of the trajectories obtained.

As a numerical example take a three-stage rocket in vertical flight ($g_n = g_o$) with the following characteristics:

$c_1 = 2.3$ km/sec	$K_1 = 30$	$\zeta_1 = 0.10$
$c_2 = 2.5$ km/sec	$K_2 = 40$	$\zeta_2 = 0.12$
$c_3 = 2.1$ km/sec	$K_3 = 25$	$\zeta_3 = 0.15$

Fig. 11.15 shows the three values of a_n related to r_n through eqn. (79). Fig. 11.16 shows the relations between the a_n and the Lagrangian multiplier v derived from eqns. (80). From this diagram an optimum set of a_n is obtained for each common value of v; the corresponding set of r_n is then given by Fig. 11.15 and the corresponding ΔV may be calculated. The result is a relation between ΔV and v as illustrated by Fig. 11.17. It should be noted that from conditions (77) and (78), we also have

$$\frac{v}{c_n} = \frac{a_n - g_n/g_o}{a_n(1 + \zeta_n)} r_n u_n$$

so that the partial payload ratios vanish for $v = 0$, this last value giving the highest ΔV performance of the staged rocket, which corresponds

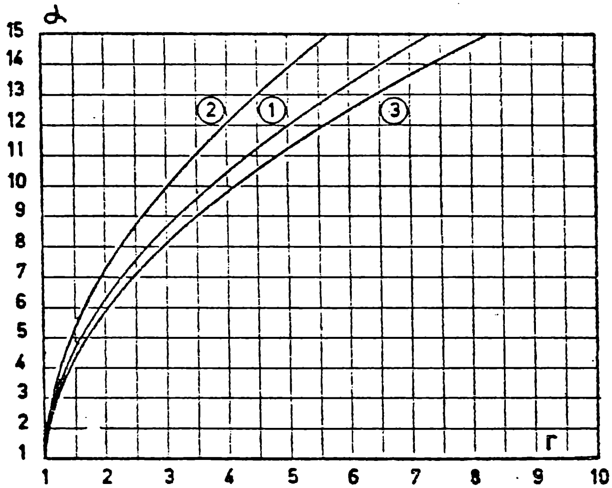

Fig. 11.15. Optimization of three-stage rocket with following characteristics:

$c_1 = 2.3$ km/sec	$K_1 = 30$	$\zeta_1 = 0.10$
$c_2 = 2.5$ km/sec	$K_2 = 40$	$\zeta_2 = 0.12$
$c_3 = 2.1$ km/sec	$K_3 = 25$	$\zeta_3 = 0.15$

for total velocity gain, reduced by gravity action in vertical flight. Optimum mass ratio of each stage as function of gross-acceleration factor at burnout.

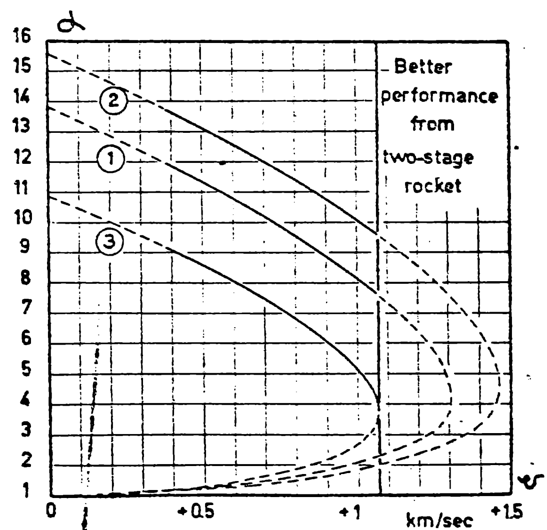

Fig. 11.16. Same conditions as in Fig. 11.15. Optimum gross-acceleration factors at burnout against value of the Lagrangian multiplier.

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to zero payload. For positive values of v, there is a positive payload. For instance for $v = 0.38$, we obtain

$r_1 = 5.00$	$\alpha_1 = 12$	$u_1 = 0.0400$	$(\Delta V)_1 = 2.935$ km/sec
$r_2 = 4.94$	$\alpha_2 = 13.8$	$u_2 = 0.0369$	$(\Delta V)_2 = 3.280$ km/sec
$r_3 = 3.65$	$\alpha_3 = 9.2$	$u_3 = 0.0642$	$(\Delta V)_3 = 2.114$ km/sec

and finally $u = u_1 u_2 u_3 = 0.9476 \cdot 10^{-4}$ for $\Delta V = 8.329$ km/sec.

We also observe that no optimum set exists for values of v larger than the maximum on the α_3 curve of Fig. 11.16. This value is about 1.075 km/sec and corresponds to

$\alpha_1 = 7.6$	$r_1 = 2.527$	$(\Delta V)_1 = 1.670$ km/sec	$u_1 = 0.235048$
$\alpha_2 = 9.6$	$r_2 = 2.848$	$(\Delta V)_2 = 2.135$ km/sec	$u_2 = 0.189000$
$\alpha_3 = 3.8$	$r_3 = 1.375$	$(\Delta V)_3 = 0.462$ km/sec	$u_3 = 0.575818$

or $u = 0.02558$ for $\Delta V = 4.267$ km/sec.

The reason is presumably that, for such a low performance, better overall payload ratios are obtainable from a two-stage rocket. Indeed, discarding the third, and worst, stage and taking $v = 0.96$ km/sec, we find:

$\alpha_1 = 8.5$	$r_1 = 2.939404$	$(\Delta V)_1 = 1.955$ km/sec	$u_1 = 0.177834$
$\alpha_2 = 10.5$	$r_2 = 3.232160$	$(\Delta V)_2 = 2.401$ km/sec	$u_2 = 0.145302$

or $\quad u = 0.02584$ for $\Delta V = 4.356$ km/sec.

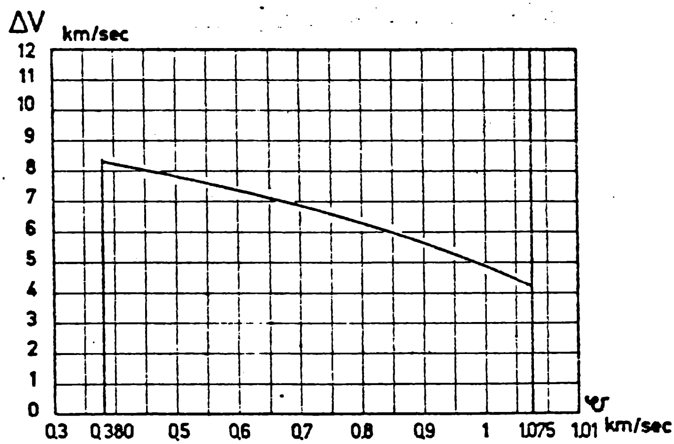

Fig. 11.17. Same conditions as in Figs. 11.15 and 11.16. Velocity gain as function of the Lagrangian multiplier.

Consequently the useful part of the α_n *versus* v curves is limited as indicated.

For equal values of $g_n = g_s$, $K_n = K$, $c_n = c$ and $\zeta_n = \zeta$, the multiplier plays no part. All the α_n evidently become equal and the r_n likewise. For an N-stage rocket

$$\frac{\Delta V}{c} = N\left(\ln r - \frac{g_s}{g_0}\,\frac{r-1}{\alpha}\right)$$

and

$$u = \left(\frac{1+\zeta-\alpha/K}{r} - \zeta\right)^N$$

These are parametric equations for the u *versus* $\Delta V/c$ curve, the connection between α and r being given by eqn. (79). The case of vertical flight ($g_s = g_0$) with $K = 40$ and $\zeta = 0.12$, is illustrated in Fig. 11.18 for $N = 1$, 2, 3 and 4. Again each number of steps has a superior per-

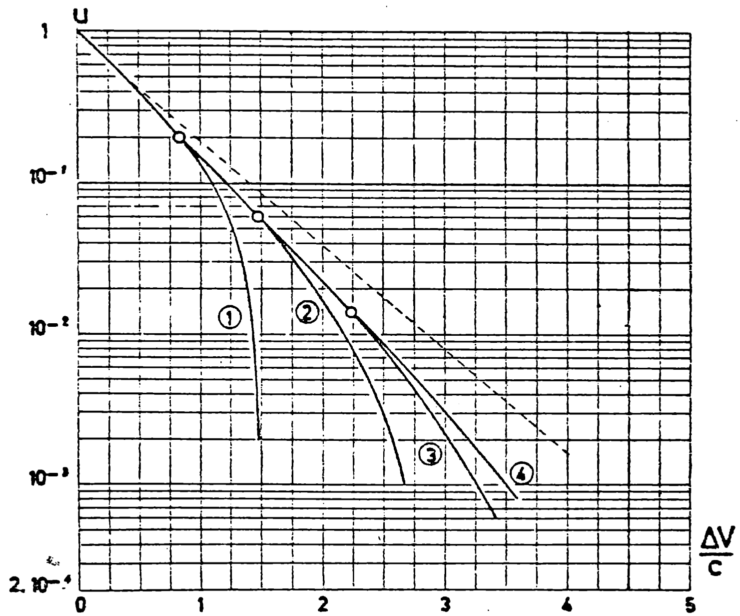

Fig. 11.18. Payload ratio *versus* velocity performance for multistaged rockets. Optimization for imposed velocity gain as reduced by gravity in vertical flight. Same structural efficiency $\zeta = 0.12$ and engine merit index $K = 40$ for all stages. Dashed line is absolute limit obtained from analysis of continuously staged rocket (in parallel) with same efficiency factors.

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formance in a given range, and no limit performance is to be found by assuming an infinite number of steps or a continuous case. On the other hand, as will be shown in section 11.9.4 (p. 757) such a limit does exist when the rocket staging is done in parallel instead of in series.

Up to now, we have tried to optimize rocket staging for a given increase in velocity alone. We may try, more correctly, to optimize for a given increase in total specific energy, as was done in section 11.8 (p. 733), to determine the best engine size in a single stage. Though the equations to be solved are easily derived, the amount of numerical work clearly becomes quite considerable. The difficulties arise mainly from the fact that the partial energy gain from a single stage depends on the initial velocity at which this stage begins to operate. Indeed, referring back to eqn. (52), p. 736:

$$(\Delta E)_n = c_n V_{in} \ln r_n + \tfrac{1}{2} c_n^2 \ln^2 r_n - c_n^2 \frac{g_n}{g_o} \frac{r_n \ln r_n - r_n + 1}{a_n} \quad (n = 1,2,\ldots,N) \quad (81)$$

Clearly, given r_n and a_n, $(\Delta E)_n$ will increase with V_{in}, and allowing velocity reached at burnout of a stage to drop by waiting some time before firing the next stage would be detrimental to the performance of the latter. The best initial velocity of a stage is thus given by the burnout of the previous stage and, according to eqn. (75), we shall assume

$$V_{i(n+1)} = V_{in} + c_n \ln r_n - c_n \frac{g_n}{g_o} \frac{r_n - 1}{a_n} \qquad (n = 1,2,\ldots,N) \quad (82)$$

the initial velocity V_{i1} being known (and equal, say, to zero). To take into account the performance condition

$$\Delta E = \sum_1^N (\Delta E)_n$$

we use a Lagrangian multiplier λ, and for conditions eqns. (82), a set of $(N - 1)$ multipliers $\lambda\beta_n$, and formulate the problem as one of making maximum the quantity

$$U = \sum_1^N \ln u_n + \lambda \sum_1^N (\Delta E)_n + \lambda \sum_1^{N-1} \beta_n \left(- V_{i(n+1)} + V_{in} + c_n \ln r_n - c_n \frac{g_n}{g_o} \frac{r_n - 1}{a_n} \right)$$

with respect to arbitrary values of the $2N$ unknown (r_n, a_n) and the $N - 1$ unknowns V_{in} $(n = 2, 3, \ldots N)$. In the expression of U, the u_n

are replaced, for instance, by eqns. (71), p. 745 and the $(\Delta E)_n$ are given by eqns. (81). The conditions $\partial U/\partial V_{in} = 0$ are simple; they yield

$$\beta_{n-1} = \beta_n + c_n \ln r_n \qquad (n = 2,3,\ldots,N) \tag{83}$$

The other conditions $\partial U/\partial r_n = 0$ and $\partial U/\partial \alpha_n = 0$ are:

$$\frac{1}{u_n}\frac{\partial u_n}{\partial r_n} + \lambda c_n \frac{g_n}{g_0}\frac{1}{a_n^2}\cdot[c_n(r_n \ln r_n - r_n + 1) + \beta_n(r_n - 1)] = 0 \tag{84}$$

$$\frac{1}{u_n}\frac{\partial u_n}{\partial a_n} + \lambda c_n \frac{V_{in}}{r_n} + \lambda c_n(c_n \ln r_n + \beta_n)\left(\frac{1}{r_n} - \frac{g_n}{g_0}\frac{1}{a_n}\right) = 0 \tag{85}$$

The $2N$ equations, together with the $(2N-2)$ eqns. (83) and (82) and

$$\Delta E = \sum_1^N (\Delta E)_n \tag{86}$$

form a set of $(4N-1)$ eqns. for the $(4N-1)$ unknowns (r_n, a_n, V_{in}, β_n and λ). We may observe that for the last stage, since $\beta_N = 0$, the pair eqns. (84) and (85) yield, on elimination of λ, a quadratic in a_N whose appropriate root is that given by eqn. (53), but such a type of equation may not be used for the other stages. There is no straightforward solution to the system and any method involves some amount of trial and successive approximations. One such method might run along the following lines. First assume a plausible set of optimal (r_n) values. Compute the β_n in succession from eqns. (83) starting from the last. Take the pair of eqns. (84) and (85) for $n = 1$ and, since V_{i1} is known, eliminate λ, and solve the quadratic in a_1 to obtain this value. Substitute this in eqn. (82) for $n = 1$ to calculate V_{i2}. This procedure may then be repeated to calculate a_2 and V_{i3} from eqns. (84), (85) and (82) for $n = 2$, $n = 3$, etc ... Once these calculations, which require no trial and error procedure, are done, take eqns. (84) or (85) or any convenient combination of them to examine the values obtained for λ. If all those values, which we will denote by λ_n, are the same, the set is optimal for the resulting value of $\Delta E = (\Delta E)_0$, which may however be different from the required ΔE. If the values are not the same, apply a perturbation procedure, which will essentially consist in giving to each r_n in succession a small increment δr_n, and run through the calculation again to find the $\delta\lambda_n$ and $\delta\Delta E$. We may thus compute the approximate differential coefficients

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$$\frac{\delta \lambda_n}{\delta r_m} = \frac{\partial \lambda_n}{\partial r_m} \quad \text{and} \quad \frac{\delta \Delta E}{\delta r_m} = \frac{\partial \Delta E}{\partial r_m}$$

If the corrections needed are small enough, we shall have:

$$\lambda_n + \sum_1^N \frac{\partial \lambda_n}{\partial r_m} \delta r_m = \lambda \qquad (n = 1, 2, \ldots, N)$$

$$(\Delta E)_0 + \sum_1^N \frac{\partial \Delta E}{\partial r_m} \delta r_m = \Delta E$$

and these are $(N + 1)$ linear equations for the n corrections δr_m and the common λ value.

11.9.3. Step rockets in parallel

Another possibility for increasing the payload would consist in a subdivision of the rocket in elements arranged in parallel instead of in series. The main advantage of such an arrangement is a reduction in gravitational losses owing to the fact that all the engines may be used to give their full thrust from the beginning, instead of being fired in succession. The rocket is, however, likely to be bulky and the drag penalty may be severe except for very large rockets, where the energy loss due to drag is insignificant compared to the gravitational losses. Another drawback is the reduction in nozzle efficiency of the engines which are used for propulsion after dropping the first stage. If these engines are used from the start, their nozzle-expansion ratio is limited by the high-density atmosphere at low altitude; in the series arrangement their expansion ratio may be adjusted to the reduced density prevailing at the altitude where they are fired. The analysis of the optimum staging of the parallel scheme is no more complicated than the series scheme if all the effective exhaust velocities of the engines are assumed to be identical, an assumption which is likely to be verified because the nozzle expansion ratios will be similar. Let F_1 denote the total thrust available at take-off, F_2 the thrust remaining after dropping the first step (which may consist of several tanks and engines wrapped around a central body) and in general by F_n the thrust remaining after dropping the $(n - 1)$th step. As before, let a_n and r_n be defined by

$$a_n = \frac{F_n}{g_0 M_{bn}}, \qquad r_n = \frac{M_{in}}{M_{bn}} \tag{87}$$

M_{in} being the mass just after dropping the $(n - 1)$th step and M_{bn}

the same mass after consumption of the propellant in the nth step tanks (all the engines working at the time are assumed to be fed by these tanks alone). Then

$$M_{tn} = M_{bn} + M_{pn}$$

and, when the nth step is dropped

$$M_{t(n+1)} = M_{bn} - \zeta_n M_{pn} - M_{en}$$

the second term being the mass of the tanks which are dropped. If K_n is the merit index of the engines dropped,

$$M_{en} = \frac{F_n - F_{n+1}}{K_n g_o}$$

The nth partial payload is defined as

$$u_n = \frac{M_{t(n+1)}}{M_{tn}}$$

and, from the preceding relations, we find

$$u_n = \left(\frac{1 + \zeta_n - a_n/K_n}{r_n} - \zeta_n\right)\left(1 - \frac{a_{n+1}}{K_n r_{(n+1)}}\right)^{-1} \quad (n = 1,2,\ldots, N - 1) \tag{88}$$

For $n = N$ we have exceptionally, because $F_{(N+1)} = 0$ and $M_{t(N+1)} = M_u$ the true mass of payload,

$$u_N = \frac{M_u}{M_{t,\,N-1}} = \frac{1 + \zeta_N - a_N/K_N}{r_N} - \zeta_N\,. \tag{89}$$

Moreover, on the assumption that the effective exhaust velocities of all the engines are the same, the velocity increment given by each step keeps the simple form

$$(\varDelta V)_n = c\left(\ln r_n - \frac{g_n}{g_o}\,\frac{r_n - 1}{a_n}\right) \qquad (n = 1,2,\ldots, N) \tag{90}$$

where gravitational losses are included. Comparing eqn. (88) and eqn. (71), p. 745, it is immediately apparent that, all other things being equal, the $(N - 1)$ first partial payload ratios are higher for the parallel arrangement. Since as before

$$u = \frac{M_u}{M_{t1}} = u_1 u_2 \ldots u_N$$

the problem of maximizing u for a given total velocity gain may be set up as follows:

$$U = \sum_1^N \ln\left(\frac{1 + \zeta_n - a_n/K_n}{r_n} - \zeta_n\right) - \sum_1^{N-1} \ln\left(1 - \frac{a_{n+1}}{K_n r_{n+1}}\right) +$$

$$+ \frac{c}{v} \sum_1^N \left(\ln r_n - \frac{g_n}{g_o} \frac{r_n - 1}{a_n}\right) \max$$

The conditions for a maximum

$$\frac{\partial U}{\partial r_n} = 0, \qquad \frac{\partial U}{\partial a_n} = 0 \qquad (r = 1,2,\ldots,N)$$

reveal that for $n = 1$, eqn. (79), p. 748, remains valid:

$$r_1 = 1 + \frac{a_1(a_1 - g_1/g_o)}{\frac{g_1}{g_o} K_1(1 + \zeta_1)} \tag{91}$$

but for the remaining values of n, we have (elimination of v)

$$\left(K_{n-1} + \zeta_n K_n - \frac{K_n(1 + \zeta_n)}{r_n}\right) a_n^2 - \frac{g_n}{g_o} (K_{n-1} - K_n)\, a_n -$$

$$\frac{g_n}{g_o} K_n K_{n-1}(1 + \zeta_n)(r_n - 1) = 0 \qquad (n = 2,3,\ldots,N) \tag{92}$$

There is only one positive (appropriate) root for this quadratic in a_n, and we may note that if $K_{n-1} = K_n$ it becomes remarkably simpler:

$$a_n^2 = \frac{g_n}{g_o} K_n r_n \tag{92'}$$

Other combinations of the optimal conditions yield relations of type eqns. (80), p. 748, valid for all n:

$$\frac{v}{c} = \left(1 - \frac{g_n}{g_o a_n}\right)\left(1 - \frac{\frac{a_n}{K_n} + \zeta_n r_n}{1 + \zeta_n}\right) \qquad (n = 1,2,\ldots,N) \tag{93}$$

The procedure for solving the eqns. (91), (92), and (93) is exactly like that for the series arrangement. Numerical example:

Vertical flight of a two-step rocket with $K_1 = K_2 = 40$ and $\zeta_1 = \zeta_2 = 0.12$. A common value of $v/c = 0.1348$ is found to correspond to

$a_1 = 14$	$r_1 = 5.062$	$(\Delta V)_1 = 1.33165\, c$	$u_1 = 0.03454$
$a_2 = 14.2$	$r_2 = 5.041$	$(\Delta V)_2 = 1.33333\, c$	$u_2 = 0.03175$

giving finally $u = 1.0968 \cdot 10^{-3}$ for $\Delta V/c = 2.665$.

11.9.4. Step rockets in parallel — Continuous case

For any given ΔV performance, the payload ratio increases with the number of step rockets in parallel and there is a definite upper limit reached by letting this number tend to infinity or by passing to a continuous case, which is equivalent. This is in contrast to staging in series, where the payload ratio passes through a maximum for a certain number of stages and decreases again for a higher number. It is not without interest to know the absolute upper limit of u given by the continuous case of rocket staging in parallel and this we now proceed to evaluate. The total mass of the rocket at any time will be analysed as follows

$$M = M_u + (1 + \zeta)\, M_p + \frac{F}{Kg_0} \tag{94}$$

The second term represents the mass of propellants and tanks at the instant considered; the tanks are expended continuously at the rate of propellant consumption. The last term is the instantaneous engine mass, which is kept proportional to the thrust delivered. Since no inactive engine weight is involved (all the engines still carried by the rocket are active) the equation can be considered effectively as a limiting continuous case of the arrangement in *parallel*.

The optimum continuous variation in engine thrust will apparently consist of two phases. In the first, the total thrust will remain constant and only propellant and tank masses will be continuously expended until the acceleration reaches a sufficiently high value. From then on, during a second phase, engine thrust will decrease and consequently engine mass will also be continuously discarded. Let F_1 be the engine thrust of the first phase from values M_1 and V_1 to M_2 and V_2. Then the equation of motion:

$$\mathrm{d}V = \left(\frac{F_1}{M} - g_s\right) \mathrm{d}t$$

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the equation of thrust

$$F_1 = -c\frac{dM_p}{dt}$$

and the differential of eqn. (94)

$$dM = (1 + \zeta)\, dM_p$$

yield, after elimination of dt and dM_p and integration:

$$\frac{1+\zeta}{c}(V_2 - V_1) = \ln r_1 - \frac{g_{12}}{g_o a_1}(r_1 - 1) \tag{95}$$

where

$$r_1 = \frac{M_1}{M_2}, \qquad a_1 = \frac{F_1}{g_o M_2} \tag{96}$$

and g_{12} an average of g_s for this phase. For the second phase, of variable thrust, we introduce the instantaneous gross-acceleration factor

$$\beta = \frac{F}{g_c M} \tag{97}$$

transforming eqn. (94) into

$$\left(1 - \frac{\beta}{K}\right) M = M_u + (1 + \zeta)\, M_p \tag{98}$$

Between the two phases, we allow for a finite reduction in thrust from F_1 to F_2, implying a change in mass due to engine loss

$$M_2 - M_2^* = \frac{F_1 - F_2}{K g_o}$$

This equation, by virtue of definitions (96) and (97), may be written as

$$\left(1 - \frac{\beta_2}{K}\right)\frac{M_2^*}{M_2} = 1 - \frac{a_1}{K} \tag{99}$$

where β_2 is the initial value of β in the second phase:

$$\beta_2 = \frac{F_2}{g_o M_2^*}$$

Using the thrust equation

$$F = -c\frac{dM_p}{dt}$$

to eliminate $\mathrm{d}t$ and eqn. (98) to eliminate M_p, the equation of motion may be obtained in the form:

$$\mathrm{d}\ln\left[\left(1-\frac{\beta}{K}\right)\frac{M}{M_2^*}\right] = -\frac{1+\zeta}{c} f(\beta)\,\mathrm{d}V$$

where

$$f(\beta) = \frac{\beta}{(\beta - g_s/g_o)\,(1-\beta/K)} \tag{100}$$

Now let the required velocity performance be $\Delta V = V_3 - V_1$ and integrate between V_2 and V_3 (where $M_p = 0$):

$$\ln\frac{M_u}{M_2^*\left(1-\dfrac{\beta_2}{K}\right)} = -\frac{1+\zeta}{c}\int_{V_2}^{V_3} f(\beta)\,\mathrm{d}V$$

From this result and eqn. (99), we finally get

$$\ln u = \ln\frac{M_u}{M_1} = \ln\left(1-\frac{\alpha_1}{K}\right) - \ln r_1 - \frac{1+\zeta}{c}\int_{V_2}^{V_3} f(\beta)\,\mathrm{d}V \tag{101}$$

The payload ratio depends on the parameters α_1, r_1 and V_2, which are connected by eqn. (95), and on the unknown β (here considered as a function of the velocity) in the interval $V_2 \leqslant V \leqslant V_3$. Given r_1 and α_1, and consequently V_2, the maximum of $\ln u$, as a function of β, exists and is simply given by local minimum values of $f(\beta)$. Those minima are reached for

$$\beta^2 = \frac{g_s}{g_o}K \quad \text{where} \quad f_{\min} = \left(1-\sqrt{\frac{g_s}{Kg_o}}\right)^{-2} \tag{102}$$

and depend on the local slope of the trajectory with respect to equipotential surfaces of the gravitational field. Because g_{12} and g_2 (the value of g_s at the beginning of the second phase) depend on r_1 and α_1, it is a more delicate matter to examine the maximum of $\ln u$ with respect to those values. By the assumption that the first phase occurs in vertical flight, we avoid the necessity of analysing the trajectory and find as maximum conditions with respect to variations in the parameters α_1,

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r_1 and V_2 as connected by eqn. (95) with $g_{12} = g_o$:

$$\left.\begin{aligned} r_1 &= 1 + \frac{\alpha_1(\alpha_1 - 1)}{K} \\ \alpha_1 &= \beta_2 \end{aligned}\right\} \tag{103}$$

Since $g_2 = g_o$ we also have

$$\beta_2^2 = K \tag{104}$$

Consequently there is no discontinuous reduction in thrust when passing from the first to the second phase. If the second-phase trajectory remains vertical, the thrust is continuously reduced to keep the acceleration factor constant and equal to $\sqrt{K}$. If, however, the trajectory is curved, the thrust is still further decreased to allow for the reduction in g_s. For $K = 40$ we have

$$\alpha_1 = \beta = 6.324555 \qquad r_1 = 1.841886$$

and if

$$V_1 = 0, \qquad V_2 = 0.477676 \frac{c}{1+\zeta}$$

The altitude reached at the beginning of the second phase is found

$$z_2 - z_1 = \left(\frac{c}{1+\zeta}\right)^2 \frac{1}{g_o}\left[\frac{r_1 - 1}{\alpha_1} - \frac{1}{2}\left(\frac{r_1 - 1}{\alpha_1}\right)^2 - \frac{\ln r_1}{\alpha_1}\right] = \left(\frac{c}{1+\zeta}\right)^2 \frac{1}{g_o} 0.02768$$

If the performance required is equal to V_2, the payload ratio would be

$$\ln u = \ln\left(1 - \frac{\alpha_1}{K}\right) - \ln r_1$$

or

$$u = \frac{1}{r_1}\left(1 - \frac{\alpha_1}{K}\right) = 0.457078$$

For $\zeta = 0.12$ this performance point in the diagram of Fig. 11.18 (p. 751), lies very slightly above the curve of the single-stage rocket. For higher performances required in vertical flight, since

$$\frac{\mathrm{d} \ln u}{\mathrm{d} V_3} = -\frac{1+\zeta}{c}\left(1-\frac{1}{\sqrt{K}}\right)^{-2}$$

the performance points lie on a straight line (dashed line of Fig. 11.18). For instance $u = 1.612 \cdot 10^{-3}$ corresponds to $\Delta V/c = 4$.

11.10. Integration of Vertical Trajectories

More exact estimations of rocket performance require the inclusion of terms previously neglected to facilitate the analytical treatment of certain design optimizations. A more correct equation of motion in vertical flight is the following:

$$M\dot{V} = F - Mg - D \tag{105}$$

The behaviour of the variable mass M, variable thrust F and drag term D will now be briefly considered. Under conditions of constant mass flow of the propellants, the thrust is still increasing with altitude because of the reduction in ambient pressure (see section 2.4.4, p. 93). At the altitude y,

$$F(y) = F(0) \cdot \left[1 + k_F\left(1 - \frac{p_a(Y)}{p_a(0)}\right)\right] \tag{106}$$

where the coefficient of pressure thrust k_F is theoretically equal to:

$$\frac{1}{k_F} = \frac{2\gamma}{\gamma-1}\left[\left(\frac{p_c}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\frac{p_e}{p_a(0)} + \frac{p_e}{p_a(0)} - 1 \tag{107}$$

with $p_a(Y)$ ambient pressure at altitude y
$p_a(0)$ ambient pressure at reference altitude $y = 0$
p_c chamber pressure
p_e exit pressure
γ specific heat ratio of exhaust gases.

If $c(0)$ denotes the effective exhaust velocity at zero altitude:

$$F(0) = -c(0)\dot{M} \tag{108}$$

an equation from which the instantaneous mass of the rocket is easily derived. The drag is essentially a function of velocity and altitude

$D(y, V)$ and for flights to very high altitudes even the acceleration of gravity should be regarded as a function of altitude

$$g(y) = g_0 \left(\frac{R}{R+y}\right)^2 \qquad (R \text{ the Earth's radius})$$

The numerical integration of eqns. (105) and (108) must then necessarily proceed together with the equation for the altitude gain

$$\dot{y} = V \tag{109}$$

To simplify the integration, the time may be eliminated and replaced by the altitude as the independent variable. Denoting by an apostrophe the differential coefficient with respect to altitude, we then obtain the system

$$V' = \frac{1}{V}\left[\frac{F(y) - D(y, V)}{M(y)} - g(y)\right]$$

$$M' = -\frac{1}{V}\,\frac{F(0)}{c(0)}$$

The rocket mass may also be taken as the independent variable and this choice is particularly interesting during a take-off phase because it removes a singularity at $V = 0$ and because, in the beginning, the variations in mass are more important than the variations in velocity.

11.11. Zero-Lift Trajectories

In the relatively dense atmosphere prevailing at low altitudes, a curved flight path is preferably obtained by steering the rocket along a zero-lift trajectory. The rocket then has no angle of attack, which means its axis of revolution remains tangential to the flight path. Except for a winged missile, an angle of attack would involve a large amount of induced drag for a small amount of lift. If the thrust vector remains aligned with the rocket axis the curvature of the flight path is produced by the gravitational field and depends on the initial conditions. The rocket will then describe a perfect trailing turn. If γ denotes the angle of the tangent to the flight path with the horizontal, the intrinsic equations of motion of a zero-lift trajectory in a uniform gravity field will be

$$
\begin{aligned}
M\dot{V} &= F - D - Mg_{\circ} \sin \gamma \\
V\dot{\gamma} &= -g_{\circ} \cos \gamma
\end{aligned}
\tag{110}
$$

By considering the rocket as a point mass, we evidently neglect the inertia forces involved in the continuous change of orientation of the rocket axis. In practice these forces must be balanced either by aerodynamic forces or by the action of vernier engines or a disalignment of the thrust vector (gimballed rocket engine).

In the next section it will be shown that the corrections involved in eqns. (110) are generally small enough to be neglected. In the integration of eqns. (110) and (108) a difficulty is encountered at the start, where $V = 0$ and γ is necessarily equal to $\pi/2$.

To investigate the nature of the singularity at this point, and to obtain a convenient method of integration, we begin with a special case of integrability in closed form. Introducing the gross-acceleration factor β defined by eqn. (97), p. 758, and neglecting drag, eqns. (110) are rewritten:

$$
\begin{aligned}
\mathrm{d}V &= g_{\circ} (\beta - \sin \gamma)\, \mathrm{d}t \\
V\mathrm{d}\gamma &= -g_{\circ} \cos \gamma\, \mathrm{d}t
\end{aligned}
$$

and the case of a constant β is considered. With the change of variable

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \Gamma$$

$$\cos\gamma = \frac{2\Gamma}{1+\Gamma^2}, \qquad \sin\gamma = \frac{1-\Gamma^2}{1+\Gamma^2}, \qquad \mathrm{d}\gamma = -\frac{2\mathrm{d}\Gamma}{1+\Gamma^2} \tag{111}$$

we obtain

$$
\begin{aligned}
\mathrm{d}V &= g_{\circ}\left(\beta - \frac{1-\Gamma^2}{1+\Gamma^2}\right)\mathrm{d}t \\
V\mathrm{d}\Gamma &= g_{\circ}\Gamma\mathrm{d}t
\end{aligned}
$$

Eliminating $\mathrm{d}t$ between the two and integrating, we find

$$V = g_{\circ} H \Gamma^{\beta-1}(1+\Gamma^2) \tag{112}$$

and a second integration gives for the flight time

$$\Delta t = H\left(\frac{\Gamma^{\beta-1}}{\beta-1} + \frac{\Gamma^{\beta+1}}{\beta+1}\right) \tag{113}$$

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H is an arbitrary constant of integration, whose significance is associated with the value of the velocity $2g_0H$, when the flight path reaches its apex ($\Gamma = 1$). Despite the fact that for $V = 0$ the flight path always starts vertically ($\Gamma = 0$ and $\gamma = \pi/2$) there is one parameter family of curved trajectories such that the thrust vector remains tangential to the flight path. It is interesting to observe that the rate of turn of the path (or of the rocket axis)

$$\frac{d\gamma}{dt} = -\frac{2}{H} \Gamma^{2-\beta} (1 + \Gamma^2)^{-2} \tag{114}$$

is infinite at the origin if $\beta > 2$, zero if $\beta < 2$, finite and equal to $-2/H$ if $\beta = 2$. This indicates that because of the inertia of the rocket about a transverse axis it will be difficult to steer it along a perfect trailing turn if its initial gross-acceleration factor is higher than two. The altitude gained by the rocket and the horizontal range covered are also easily found to be respectively

$$\Delta y = gH^2 \left(\frac{\Gamma^{2\beta-2}}{2\beta - 2} - \frac{\Gamma^{2\beta+2}}{2\beta + 2} \right) \tag{115}$$

$$\Delta x = 2gH^2 \left(\frac{\Gamma^{2\beta-2}}{2\beta - 2} + \frac{\Gamma^{2\beta+2}}{2\beta + 2} \right) \tag{116}$$

To integrate a zero-lift trajectory in general we may now re-define

$$\beta = \frac{F - D}{M} \tag{117}$$

and subdivide the interval of variation of Γ into portions along which β will be assumed to remain constant. Along each subinterval from Γ_n to Γ_{n+1} the formulae (112) to (116) are applicable with suitable values of $\beta = \beta_n$ and $H = H_n$. The value of β_n is estimated from (117) at the origin of the subinterval. Then, applying (112) at the origin

$$V_n = g_0 H_n \Gamma_n^{\beta_n - 1} (1 + \Gamma_n^2)$$

and, since V_n is known, this equation determines H_n. Alternatively, from the same equation applied at the end of the previous interval,

$$V_n = g_0 H_{n-1} \Gamma_n^{\beta_{n-1} - 1} (1 + \Gamma_n^2)$$

and by comparison

$$H_n = H_{n-1}\, \Gamma_n^{\beta_{n-1}-\beta_n}$$

Next, β_n and H_n being determined, eqns. (112) to (116) applied for $\Gamma = \Gamma_{n+1}$, yield the values $V_{(n+1)}$, $t_{(n+1)}$, $y_{(n+1)}$ and $x_{(n+1)}$. The value of M_{n+1} is then obtained from eqn. (108) and all the initial data for the next subinterval are known. The value H_1, chosen arbitrarily for the first subinterval, is the parameter for the family of zero-lift trajectories.

11.12. Effects of the Rotational Inertia During Steady-State Gravity Turns

Allowing for a deflection angle δ of the thrust vector with respect to the rocket axis, the equations of a perfect (zero-lift) gravity turn are:

$$M\dot{V} = F\cos\delta - D - Mg_0 \sin\gamma$$

$$MV\dot{\gamma} = F\sin\delta - Mg_0\cos\gamma$$

$$I\ddot{\gamma} = -Fp\sin\delta$$

The last equation expresses the equilibrium between the rotational inertia effects during the turn and the moment of the thrust with respect to the transversal axis through the centre of gravity. The turning moment of aerodynamic origin is neglected as being of a smaller order of magnitude. Eliminating sin δ between the last two equations

$$\frac{I}{pM}\ddot{\gamma} + V\dot{\gamma} + g_0\cos\gamma = 0 \qquad (118)$$

Moreover, since δ is small (as verified *a posteriori*), the first equation can be written

$$M\dot{V} = F - D - Mg_0\sin\gamma \qquad (119)$$

Except for the first term in eqn. (118), we have the same system as described by eqns. (110). If the rotational inertia term is really negligible, the rate of turn will be given as before by:

$$\dot{\gamma} = -\frac{g_0}{V}\cos\gamma \qquad (120)$$

Differentiating,

$$\ddot{\gamma} = \frac{g_0}{V^2}\cos\gamma\,\dot{V} + \frac{g_0}{V}\dot{\gamma}\sin\gamma$$

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and resubstituting for the approximate rate of turn and for the tangential acceleration

$$\ddot{\gamma} = \frac{g_o \cos\gamma}{V^2}\left[\frac{F-D}{M} - 2g_o \sin\gamma\right] \tag{121}$$

We may now check on the relative importance of the rotational inertia term by taking the ratio of this term to the last (gravity) term in eqn. (118). This ratio appears to be:

$$\left(\frac{\varrho}{p}\right)\frac{\varrho g_o}{V^2}\left[\frac{F-D}{Mg_o} - 2\sin\gamma\right]$$

where ϱ is the radius of gyration as defined by:

$$I = M\varrho^2$$

The ratio $\varrho g_o/V^2$ governs the order of magnitude of this expression, which is generally very much smaller than unity, except for very large rockets turning at very low velocities. Finally, the deflection angle of the thrust is obtained from the equation of rotational equilibrium:

$$\sin\delta = -\left(\frac{\varrho}{p}\right)\frac{\varrho g_o}{V^2}\left(\frac{F-D}{Mg_o} - 2\sin\gamma\right)\frac{Mg_o\cos\gamma}{F} \tag{122}$$

As may be verified by numerical examples, this angle is generally very small, its order of magnitude being also governed by the same characteristic ratio. It is also observed to change sign according to whether the gross-acceleration factor β, defined by eqn. (117) is smaller or larger than 2 sin γ.

The equations (120) and (122) may be considered as the characteristics of the steady-state perfect gravity turn. It is usual for a rocket to be steered along a purely vertical segment of trajectory at take-off before being deflected on its true course. The original differential equations of the perfect gravity turn show that from the initial conditions $V_i \neq 0$, $\gamma_i = \pi/2$, $\dot{\gamma}_i = 0$, prevailing at the end of the vertical segment, the unique solution is $\gamma = \pi/2$. Thus, in order to initiate a perfect gravity turn, we must allow a transient phase with an aerodynamic angle of attack. This phase is critical and the programming of the thrust deflection to minimize the angle of attack during the transient phase is an interesting problem in itself, but it falls outside the limited scope of this Chapter.

11.13. The Equation of Motion on Rotating Earth

In the study of more general three-dimensional trajectories, the following simplifying assumptions are made:

1. The mass of the rocket is negligible in comparison with the mass of the Earth.
2. Axes passing through the centre of the Earth and pointing to fixed stars form a Galilean system.
3. The atmosphere is entrained with the rotation of the Earth.
4. The Earth is a perfect spherical body.

Let r be the distance of the rocket to the centre of the Earth, θ the longitude and Φ the latitude. The square of the absolute velocity of the rocket is then

$$2a = \dot{r}^2 + r^2 \cos^2\Phi(\dot{\theta} + \Omega)^2 + r^2\dot{\Phi}^2$$

where Ω is the angular velocity of the Earth with respect to the fixed stars. The components of the absolute acceleration in polar co-ordinates are then:

$$a_r = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial a}{\partial \dot{r}}\right) - \frac{\partial a}{\partial r} = \ddot{r} - r\cos^2\Phi(\dot{\theta} + \Omega)^2 - r\dot{\Phi}^2$$

$$a_\theta = \frac{1}{r\cos\Phi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial a}{\partial \dot{\theta}}\right) = 2\dot{r}\cos\Phi(\dot{\theta} + \Omega) - 2r\sin\Phi(\dot{\theta} + \Omega)\,\dot{\Phi} + r\cos\Phi\ddot{\theta} \quad (123)$$

$$a_\Phi = \frac{1}{r}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial a}{\partial \dot{\Phi}}\right) - \frac{\partial a}{\partial \Phi}\right] = 2\dot{r}\dot{\Phi} + r\ddot{\Phi} + r\sin\Phi\cos\Phi(\dot{\theta} + \Omega)^2$$

Draw an axis Ox_1 from the centre O of the Earth to the instantaneous position of the rocket and let Oz_1 be perpendicular to Ox_1 in the meridian plane. The angle between Oz_1 and the polar axis Oz is then equal to the instantaneous latitude Φ. Consider the plane through O and normal to Oz_1 and rotate it about Ox_1 through an angle α (positive if clockwise looking from O) until it contains the velocity vector V of the rocket *relative to the Earth*. The rotated plane will be referred to as the instantaneous *relative* orbital plane and the normal to it makes with the polar axis an angle ν related to α and the latitude as follows:

$$\cos \nu = \cos \alpha \cos \Phi \tag{124}$$

In the instantaneous orbital plane let $\pi/2 - \beta$ be the angle between Ox_1 and the relative velocity vector. Then

$$\dot{r} = V \sin \beta \tag{125}$$

$$\dot{\theta} r \cos \Phi = V \cos \beta \cos \alpha \tag{126}$$

$$\dot{\Phi} r = V \cos \beta \sin \alpha \tag{127}$$

The time derivatives of r, θ and Φ may now be substituted from these expressions in the formulae (123). Moreover, the acceleration components are taken:

1. In the direction of the relative velocity

$$a_V = a_r \sin \beta + \cos \beta (a_\theta \cos \alpha + a_\Phi \sin \alpha)$$

2. Normal to the relative velocity in the instantaneous orbital plane

$$a_n = a_r \cos \beta - \sin \beta (a_\theta \cos \alpha + a_\Phi \sin \alpha)$$

3. Normal to the instantaneous orbital plane

$$a_\nu = - a_\theta \sin \alpha + a_\Phi \cos \alpha$$

After performing the calculations and reductions, we get:

$$a_V = \dot{V} + \Omega^2 r \cos \Phi (\sin \Phi \sin \alpha \cos \beta - \cos \Phi \sin \beta)$$

$$a_n = V\dot{\beta} - \frac{V^2}{r} \cos \beta - \Omega^2 r \cos \Phi (\cos \Phi \cos \beta + \sin \Phi \sin \beta \sin \alpha) - 2\,\Omega V \cos \Phi \cos \alpha \tag{128}$$

$$a_\nu = V\dot{\alpha} \cos \beta + \frac{V^2}{r} \tan \Phi \cos^2 \beta \cos \alpha + \Omega^2 r \sin \Phi \cos \Phi \cos \alpha$$
$$- 2\Omega V (\sin \Phi \cos \beta - \sin \beta \cos \Phi \sin \alpha)$$

Taking now the components of the radial gravitational force and noting that the drag $D(V, r)$ is opposed to the relative velocity vector, whilst the thrust is assumed to act in an independent direction, the equations of motion may be rewritten

$$a_V + g_0 \frac{R^2}{r^2} \sin \beta = \frac{1}{M} (F \cos \varepsilon \cos \zeta - D) \tag{129}$$

$$a_n + g_0 \frac{R^2}{r^2} \cos \beta = \frac{1}{M} F \sin \varepsilon \cos \zeta \tag{130}$$

$$a_\nu = \frac{1}{M} F \sin \zeta \tag{131}$$

The modulus of the thrust and the angles ε and ζ are steering variables. The variables α, β, V, r and Φ are governed by the differential system (125), (127), (129), (130) and (131). Equation (126) is not coupled with the others and serves by a simple quadrature to determine the longitude. We may note that the first two terms of a_ν may be written in the form

$$-\frac{V \cos \beta}{\sin \alpha \cos \Phi} \frac{\mathrm{d}}{\mathrm{d}t} (\cos \alpha \cos \Phi) \tag{132}$$

and disappear if the angle between the polar axis and the instantaneous orbital plane is kept constant. This is the case if the ζ angle is so chosen that

$$F \sin \zeta = M\Omega^2 r \sin \Phi \cos \nu - 2\Omega VM (\sin \Phi \cos \beta - \sin \beta \cos \Phi \sin \alpha)$$

and the orbital plane remains fixed *relative* to the earth. The integration may then proceed with eqns. (125), (127), (129) and (130) only, the angle α being related to Φ by the auxiliary equation

$$\cos \alpha \cos \Phi = \cos \nu = \text{const.} \tag{133}$$

In the particular case of an equatorial orbit:

$$\nu = 0, \quad \Phi = 0, \quad \alpha = 0, \quad \zeta = 0$$

and the expressions for the acceleration components are considerably simplified. At altitudes where drag has a negligible influence on certain characteristics of the motion the use of variables connected with relative motion is unnecessarily complicated. To change the description to absolute motion, it suffices to put $\Omega = 0$ in eqns. (128) and to re-define V as the magnitude of the *absolute* velocity, $\pi/2 - \beta$ as the angle between the absolute velocity vector and the axis Ox_1 and $\pi/2 - \alpha$ as the angle between the meridian plane and the instantaneous *absolute* orbital plane. The acceleration components are then defined in the corresponding absolute directions. In particular, a_ν is now the acceleration component normal to the instantaneous orbital plane and its

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expression reduces to (132). Consequently if $\zeta = 0$, eqn. (133) is valid and the absolute orbital plane remains fixed.

In computing the long-time effects of drag on the life of artificial satellites, the assumption is generally made that the drag force is tangential to the absolute path. This conflicts with our second assumption. If we keep the assumption of the atmosphere entrained by the Earth's rotation, the drag terms affect all three of the equations of motion in the absolute description. Except for equatorial orbits, drag will cause a slow rotation of the absolute orbital plane. The same phenomenon and other related ones are caused by the Earth's oblateness[12]. It is not difficult to extend our description of the equations of motion to include the effects of the Earth's oblateness since the gravitational potential is most easily expressed in terms of the distance and the latitude. In general, if $U(r, \Phi)$ is the specific potential energy of the gravitational field, the first terms of the eqns. of motion (129), (130) and (131) are respectively

$$a_V + \frac{\partial U}{\partial r} \sin \beta + \frac{1}{r} \frac{\partial U}{\partial \Phi} \sin \alpha \cos \beta$$

$$a_n + \frac{\partial U}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial U}{\partial \Phi} \sin \alpha \sin \beta$$

$$a_r + \frac{1}{r} \frac{\partial U}{\partial \Phi} \cos \alpha$$

In first approximation for the Earth[12]

$$U = -g_0 R \left[\frac{R}{r} + J \frac{R^3}{r^3} \left(\frac{1}{3} - \sin^2 \Phi \right) + \frac{D}{35} \frac{R^5}{r^5} (35 \sin^4 \Phi - 30 \sin^2 \Phi + 3) \right]$$

where R is the equatorial radius (3441.69 nautical miles) and

$$J = 1.637 \cdot 10^{-3}, \qquad D = 10.6 \cdot 10^{-6}$$

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CHAPTER 12

Variational Methods in Optimizing Rocket Performance

12.1. Guidance Variables and Their Programming

In most problems the parameters describing the behaviour of a given system are connected by differential equations or non-holonomic constraints and the motion of the system is obtained by integrating these equations with a comprehensive set of initial conditions. In many cases the number of independent variables is larger than the number of constraints and the motion is not properly determined unless some "program" is assigned to a group of variables representing the extra degrees of freedom. Such variables are appropriately called "guidance variables". In most cases they can be identified by the fact that their differential coefficients, or time derivatives, if time is taken as the independent variable, do not appear in the equations of constraint. In aeronautical engineering the guidance variables are precisely those subject to pilot action; in guided missile engineering, those monitored by the command link.

The arbitrariness in the programming of guidance variables vanishes when additional requirements are introduced such as the stipulation of maximum performance of some sort, coupled with prescribed initial and terminal conditions. The quest for a programming to achieve the optimal performance belongs to the realm of the calculus of variations. In this quest two different approaches may be used.

In the first, the quantity to be maximized, or possibly a related quantity which is simultaneously maximized, is expressed as a functional to which the elementary rules of the calculus of variations are applicable. This method requires eliminations and changes of variables, generally inspired by physical intuition, and is intimately connected with the reduction of a Pfaffian system to an appropriate canonical form. When it succeeds, it generally provides the answer in the simplest form and also facilitates direct proofs of the maximum character of the solution[9]. Some extremal arcs, which are part of the complete solution, may be lost in the elimination process but are generally easily retrieved as expressing physical limitations in the guidance variables. To take examples: if the thrust of a variable thrust rocket engine is limited by the lower value zero and some maximum upper value, the maximal arc may comprise, in addition to a programmed thrust arc indicated by the solution of the functional, a zero thrust arc and a maximum thrust arc. If the rocket has a lifting surface with an angle of attack subject to programming, physical limitations on the angle of attack, imposed by stalling, may result in the embodiment of extremal arc flows with maximum positive or negative angles of attack. A similar situation arises when the limitation is in the transversal acceleration that the missile structure can take. Orientation of thrust with respect to the instantaneous velocity vector is also a form of guidance variable; it can be subject to similar limitations requiring the existence of special extremals.

A second approach consists in using a set of Lagrangian parameters and treating the problems as one of Mayer's type. This method furnishes immediately a set of complementary differential equations and end conditions. The connections between extremals of different types are generally indicated by the so-called Weierstrass-Erdmann corner conditions and if some ambiguity still remains concerning the proper synthesis of the optimal path it is generally removed, as the examples will show, by an appeal to the Weierstrass criterion of strong variation. Generally, the true maximum or minimum character of the performance investigated is sufficiently clear from physical intuitions to make this additional, and difficult, proof unnecessary. In particular cases, investigated by Miele[4], this proof is of elementary character.

The method of Mayer[1], whose applicability to aeronautical and astronautical performance problems was first systematically pointed out by Cicala[2], has the advantage of providing a standardized approach, whereby the skill of the operator is reduced to setting up a correct statement of the problem, including all the side conditions. It is therefore unnecessary, and it would indeed be difficult in the limited space available, to review all the optimal performance problems considered or even satisfactorily solved in the extensive literature of today.

The examples which have been selected are intended to supplement existing knowledge and are thought to be sufficiently representative of the possible situations encountered to provide the reader with the techniques to use in new problems. Some unification of the field and

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more symmetry is achieved by applying Mayer's method in parametric form[3]. Time is then rejected as being necessarily the best independent variable and discontinuous solutions are thereby included as regular extremals.

12.2. Vertical Flight of a Continuously Staged Rocket for Maximum Payload Ratio

This problem was solved by elementary methods in Chapter 11. A comparison with the present type should prove to be instructive.

The variables $V(\sigma)$, $M(\sigma)$, $t(\sigma)$, $F(\sigma)$ will be considered to depend on the parameter σ. As soon as they are known as functions of σ, we shall have a parametric description of the optimal path. The differential coefficients will be denoted as follows:

$$\frac{\mathrm{d}V}{\mathrm{d}\sigma} = V^{\circ} \qquad \frac{\mathrm{d}M}{\mathrm{d}\sigma} = M^{\circ} \qquad \text{etc.}$$

The parameter will be monotonically increasing from σ_1 to σ_2 and since time cannot flow backwards, we shall have

$$t^{\circ} \geqslant 0$$

The constraints between dependent variables will be put in the form of homogeneous relations of the first degree between differential coefficients. For example the equation of vertical motion will be written

$$[G]_1 = M(V^{\circ} + gt^{\circ}) - Ft^{\circ} = 0 \tag{1}$$

To obtain all the extremals it will be essential to express in analytical form that the thrust can only decrease, as engines fall away from the rocket, or remain constant. This is achieved by writing

$$[G]_3 = F^{\circ} + A^2 t^{\circ} = 0 \tag{2}$$

where the auxiliary variable A can take any real value. The thrust and fuel consumption are related by

$$Ft^{\circ} = -cM_p^{\circ}$$

The instantaneous mass of the rocket is

$$M = M_u + (1 + \zeta)M_p + \frac{F}{Kg} \tag{3}$$

where as before M_u stands for the payload mass, ζM_p for the mass of fuel tanks and the last term represents the instantaneous mass of the rocket engines assumed to be proportional to the instantaneous thrust. Since eqn. (3) is not a differential constraint, let us eliminate M_p from the problem by differentiation. We obtain

$$[G]_2 = M^\circ + \frac{1+\zeta}{c} Ft^\circ + \frac{A^2}{Kg} t^\circ = 0 \tag{4}$$

The original system of constraints consists of eqns. (1), (2) and (4). If one of the variables is considered as the independent one, we have, in all, four dependent variables connected by three constraints. The guidance variable, whose differential coefficient does not appear in the constraints, is $A(\sigma)$. With the help of three Lagrangian parameters λ_1, λ_2 and λ_3, we set up the function

$$G = \lambda_1[G]_1 + \lambda_2[G]_2 + \lambda_3[G]_3$$

For any path along which the constraints are satisfied

$$I = \int_{\sigma_1}^{\sigma_2} G\mathrm{d}\sigma = 0$$

If we compare only such paths, the first variation

$$\delta I = 0$$

Working out this first variation we obtain

$$[\lambda_1 M\delta V + \lambda_2\delta M + \lambda_3\delta F + U\delta t]_{\sigma_1}^{\sigma_2}$$
$$+ \int_{\sigma_1}^{\sigma_2} \{[G]_M\delta M + [G]_V\delta V + [G]_F\delta F + [G]_t\delta t + [G]_A\delta A\}\, \mathrm{d}\sigma = 0 \tag{5}$$

where, if q is any one of the variables,

$$[G]_q = -\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial G}{\partial q^\circ}\right) + \frac{\partial G}{\partial q}$$

The Euler equations of this variational formulation are

$$[G]_q = 0$$

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and we assume them to be satisfied:

$$[G]_M = \lambda_1(V^\circ + gt^\circ) - \lambda_2^\circ = 0 \tag{6}$$

$$[G]_V = -[\lambda_1 M]^\circ = 0 \tag{7}$$

$$[G]_F = -\lambda_1 t^\circ + \lambda_2 \frac{1+\zeta}{c} t^\circ - \lambda_3^\circ = 0 \tag{8}$$

$$[G]_t = -U^\circ = 0 \tag{9}$$

with

$$U = \lambda_1(Mg - F) + \lambda_2\left(\frac{1+\zeta}{c}F + \frac{A^2}{Kg}\right) + \lambda_3 A^2 \tag{10}$$

Finally

$$[G]_A = 2At^\circ\left[\frac{1}{Kg}\lambda_2 + \lambda_3\right] = 0 \tag{11}$$

Eqn. (5) is now reduced to the "transversality condition"

$$[\lambda_1 M\delta V + \lambda_2\delta M + \lambda_3\delta F + U\delta t]_{\sigma_1}^{\sigma_2} = 0 \tag{12}$$

We observe that there are two "isoperimetrical constants" involved in the problem because G does not depend on V or t but only on their differential coefficient. As a consequence, the corresponding Euler equations are immediately integrable

$$\lambda_1 M = \text{constant} \qquad U = \text{constant}$$

and the transversality condition may be written as follows:

$$\lambda_1 M\delta[V(\sigma_2) - V(\sigma_1)] + U\delta[t(\sigma_2) - t(\sigma_1)]$$
$$+ \lambda_2(\sigma_2)\delta M(\sigma_2) - \lambda_2(\sigma_1)\delta M(\sigma_1) + \lambda_3(\sigma_2)\delta F(\sigma_2) - \lambda_3(\sigma_1)\delta F(\sigma_1) = 0$$

We now state our problem as that of minimizing the take-off mass $M(\sigma_1)$ for a given payload M_u and a given velocity gain $V(\sigma_2) - V(\sigma_1)$. It follows then that

$$\delta[V(\sigma_2) - V(\sigma_1)] = 0$$

and since for $\sigma = \sigma_2$ the fuel is completely burned and M_u is prescribed

$$\delta M(\sigma_2) = \frac{1}{Kg}\delta F(\sigma_2)$$

We can transform the transversality condition into

$$U\delta[t(\sigma_2) - t(\sigma_1)] + \left[\frac{\lambda_2(\sigma_2)}{Kg} + \lambda_3(\sigma_2)\right]\delta F(\sigma_2) - \lambda_3(\sigma_1)\delta F(\sigma_1) - \lambda_2(\sigma_1)\delta M(\sigma_1) = 0$$

Since the quantities $t(\sigma_2) - t(\sigma_1)$, $F(\sigma_2)$ and $F(\sigma_1)$ are not prescribed, the transversality condition will enforce the necessary requirement of a stationary value, which is

$$\delta M(\sigma_1) = 0 \tag{13}$$

if, and only if,

$$U = 0 \tag{14}$$

$$\lambda_2(\sigma_2) + Kg\lambda_3(\sigma_2) = 0 \tag{15}$$

$$\lambda_3(\sigma_1) = 0 \tag{16}$$

$$\lambda_2(\sigma_1) \neq 0 \tag{17}$$

12.2.1. The identity between Eulerian equations

From the general theory of the Mayer problem in parametric form[3] an identity is known to exist between the Eulerian equations:

$$\lambda_1^\circ[G]_1 + \lambda_2^\circ[G]_2 + \lambda_3^\circ[G]_3 + M^\circ[G]_M + V^\circ[G]_V + F^\circ[G]_F + t^\circ[G]_t + A^\circ[G]_A \equiv 0 \tag{18}$$

As a consequence it is generally true that when all equations are satisfied except one, the last one is automatically satisfied also.

12.2.2. The Weierstrass-Erdmann corner conditions[5]

The corner conditions require that the quantities $\partial G/\partial q^\circ$ remain continuous at any corner or junction between extremals. In our case these quantities are λ_1, M, λ_2, λ_3 and U. The first condition will be satisfied if we give to eqn. (7), p 776, the same integration constant throughout the minimal path. Moreover, since the equations governing the multipliers are homogeneous in the multipliers, there is no loss in generality in giving one of the multipliers an arbitrary value at some point. In consequence we choose to integrate eqn. (7) in the form

$$\lambda_1 M = M(\sigma_1) \tag{19}$$

which assigns to $\lambda_1(\sigma_1)$ the value of 1. The last of the corner conditions

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will be satisfied by keeping $U = 0$ throughout in accordance with eqn. (14). It follows then from eqn. (10), p. 776, that

$$\lambda_1(Mg - F) + \frac{1+\zeta}{c}\lambda_2 F + \frac{A^2}{Kg}(\lambda_2 + Kg\lambda_3) = 0 \tag{20}$$

Finally, the remaining corner conditions will be satisfied by ensuring the continuity of λ_2 and λ_3.

12.2.3. The constant thrust arc

Examination of eqn. (11) reveals three possible solutions, corresponding to three possible types of extremal arcs. We may first satisfy eqn. (11) by setting $A = 0$. From eqn. (2), p. 774, this is seen to correspond to a constant thrust. Along the constant thrust arc the behaviour of λ_1 will be governed by eqn. (19), and the behaviour of λ_2 by eqn. (20), which, in view of $A = 0$, reduces to

$$\lambda_2 \frac{1+\zeta}{c} = -\lambda_1\left(\frac{Mg}{F} - 1\right) \tag{21}$$

or, if the acceleration factor

$$\beta = \frac{F}{Mg} \tag{22}$$

is introduced, by

$$\lambda_2 \frac{1+\zeta}{c} = \frac{M(\sigma_1)}{M}\left(1 - \frac{1}{\beta}\right) = \frac{M(\sigma_1)}{M} - \frac{gM(\sigma_1)}{F} \tag{23}$$

The behaviour of λ_3 follows from eqn. (8), which in view of eqns. (19) and (22) reduces to

$$\lambda_3^\circ = -\frac{gM(\sigma_1)}{F}t^\circ \tag{24}$$

Then, in view of the identity eqn. (18) and the fact that $M^\circ \neq 0$ along the constant thrust arc, the last of the Euler equations, namely eqn. (6), p. 776, is automatically satisfied.

12.2.4. The constant acceleration arc

We also satisfy eqn. (11) by putting

$$\lambda_2 + Kg\lambda_3 = 0 \tag{25}$$

In this case the multiplier λ_1 is still governed by eqn. (19) and λ_2 by eqn. (23), since eqn. (20) again reduces to eqn. (21) by virtue of eqn. (25). Eqn. (24) also remains valid, but the thrust law along this arc must be such that eqn. (25) remains satisfied. This thrust law is immediately found by differentiating eqn. (25)

$$\lambda_2^{\circ} + Kg\lambda_3^{\circ} = 0$$

and substituting λ_3° from eqn. (24) and λ_2° from eqn. (6), p. 776

$$\lambda_1(V^{\circ} + gt^{\circ}) - Kg\frac{gM(\sigma_1)}{F}t^{\circ} = 0$$

Making use of eqns. (1), p. 774, and (19) this reduces to

$$gt^{\circ}M(\sigma_1)\left[\frac{1}{M}\frac{F}{Mg} - K\frac{g}{F}\right] = 0$$

This equation can be satisfied continuously ($t^{\circ} \neq 0$) only by the constant acceleration law

$$\left(\frac{F}{Mg}\right)^2 = \beta^2 = K \tag{26}$$

We may note that eqn. (6), used in deriving this law, is automatically satisfied by virtue of identity (18) $M^{\circ} \neq 0$.

12.2.5. The discontinuous solutions

The last possibility to satisfy eqn. (11) by putting

$$t^{\circ} = 0 \tag{27}$$

should not be overlooked. This equation states that the time does not vary along this type of extremal, whereas the parameter σ does, and it corresponds to a discontinuity whose nature will be examined. The possibility of including discontinuities of technical significance as regular extremals is another specific advantage of the parametric representation.

If we keep the guidance variable A finite along this extremal, the original constraints eqns. (1), (2) and (4), pp. 774-775, readily show

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that V, F and M remain constant and no physical discontinuities occur. The discontinuous solution is then meaningless. But if we let A tend to infinity in such a manner that

$$\lim \quad A^2 t^\circ = a^2 \quad \text{finite}$$

then we shall still have

$$V^\circ = 0$$

i.e. no discontinuity in velocity, but now

$$F^\circ = -a^2 \qquad M^\circ = -\frac{a^2}{Kg}$$

Consequently, during the interval $\Delta\sigma$ of this type of extremal

$$\Delta F = -a^2 \Delta\sigma \qquad \Delta M = -\frac{a^2}{Kg} \Delta\sigma$$

and, if we refer back to eqn. (3), p. 775,

$$\Delta M_p = 0$$

There is no fuel consumption during the extremal and the variation in total mass is due entirely to loss in engine mass. Consequently, the discontinuous solution represents the possibility for a finite amount of engine mass to separate from the rocket at a given time. While λ_1 is still given by eqn. (19), eqns. (6) and (8), p. 776, reduce to

$$\lambda_2^\circ = 0 \qquad \lambda_3^\circ = 0 \tag{28}$$

and eqn. (20) must now be understood in the sense that

$$\lim A^2 \left[\frac{\lambda_2}{Kg} + \lambda_3 \right] = -\lambda_1 (Mg - F) - \frac{1 + \zeta}{c} \lambda_2 F \tag{29}$$

which implies that eqn. (25) holds true also along the discontinuous solution.

Besides the fact that it will in general be ruled out by the initial conditions, there is no sense in beginning the path by a discontinuous solution. However, such a solution may be appropriate at the end of a constant thrust extremal or of a constant acceleration extremal. In both cases eqn. (21), p. 778, will be satisfied at the beginning of the discontinuous arc and the right-hand side of eqn. (29) will be zero. It will not remain zero. The value it takes after the discontinuity is obtained by noting that λ_2 will retain the value given by eqn. (23) at the start of the discontinuity and using eqn. (19). This value is:

$$\frac{gM(\sigma_1)\,\Delta M}{\beta_i(M_i + \Delta M)}\left[\beta_i^2 - K + K\frac{\Delta M}{M_i}(\beta_i - 1)\right]$$

where the subscript i indicates the start of the discontinuity. Hence, unless

$$\frac{\Delta M}{M_i} = -\frac{\beta_i^2 - K}{K(\beta_i - 1)} < 0 \tag{30}$$

the right-hand side of eqn. (29) will differ from zero at the end of the discontinuity and eqn. (21), p. 778, will no longer be satisfied. This result is very important because it shows that after a discontinuity it becomes impossible to prolong the path by a new constant thrust or constant acceleration arc. The exceptional case represented by eqn. (30) requires that $\beta_i > \sqrt{K}$ and will be ruled out by the strong variation criterion of Weierstrass.

12.2.6. Stationary paths

The constant thrust arc is the only one for which the initial conditions (16) and (17) can be simultaneously satisfied. Consequently, any stationary path of our problem will begin with a constant thrust solution. Let σ^* denote the value of the parameter at which that initial arc possibly branches in one of another type. Then, taking $t(\sigma_1) = 0$ without loss of generality, eqns. (19), p. 777, (23), and (24), p. 778, give

$$\lambda_1(\sigma^*) = \frac{M(\sigma_1)}{M(\sigma^*)} \tag{31}$$

$$\frac{1+\zeta}{c}\lambda_2(\sigma^*) = -g\frac{M(\sigma_1)}{F} + \frac{M(\sigma_1)}{M(\sigma^*)} \tag{32}$$

$$\lambda_3(\sigma^*) = -g\frac{M(\sigma_1)}{F}t(\sigma^*) \tag{33}$$

While integration of eqn. (4), p. 775, with $A = 0$ and constant F yields

$$M(\sigma^*) = M(\sigma_1) - \frac{1+\zeta}{c} Ft(\sigma^*) \tag{34}$$

To branch off in any of the other extremal arcs and since λ_2 and λ_3 must remain continuous, condition (25) must be satisfied at the end of the constant thrust arc. With the help of eqns. (32) and (33) this condition is transformed into

$$\beta(\sigma^*) = 1 + Kx(\sigma^*) \tag{35}$$

where x stands for the reduced time variable

$$x = g\frac{1+\zeta}{c}t \tag{36}$$

With the same notation eqn. (34) is transformed into

$$\beta(\sigma_1) = \frac{\beta(\sigma^*)}{1 + x(\sigma^*)\,\beta(\sigma^*)} = \frac{1 + Kx(\sigma^*)}{1 + x(\sigma^*)\,[1 + Kx(\sigma^*)]} \tag{37}$$

A plot of eqns. (35) and (37) is given on Fig. 12.1. The initial acceleration factor passes through a maximum

$$\beta(\sigma_1) = \frac{K}{2\sqrt{K} - 1} \tag{38}$$

for the value

$$x(\sigma^*) = \frac{\sqrt{K} - 1}{K} \tag{39}$$

at which

$$\beta(\sigma^*) = \sqrt{K} \tag{40}$$

It then decreases again, returning to unity for

$$x(\sigma^*) = \frac{K-1}{K} \quad \text{at which} \quad \beta(\sigma^*) = K \tag{41}$$

At the maximum point $\beta(\sigma^*)$ has exactly the value required by eqn. (26) and all conditions to branch over to a constant acceleration arc are satisfied. Moreover, since eqn. (25) holds true along the constant acceleration arc, we may extend it until the final velocity gain is obtained and meet the end condition (15). Hence one of the possible minimal paths is established. We shall now proceed to show that no other solution is obtained by a further branching.

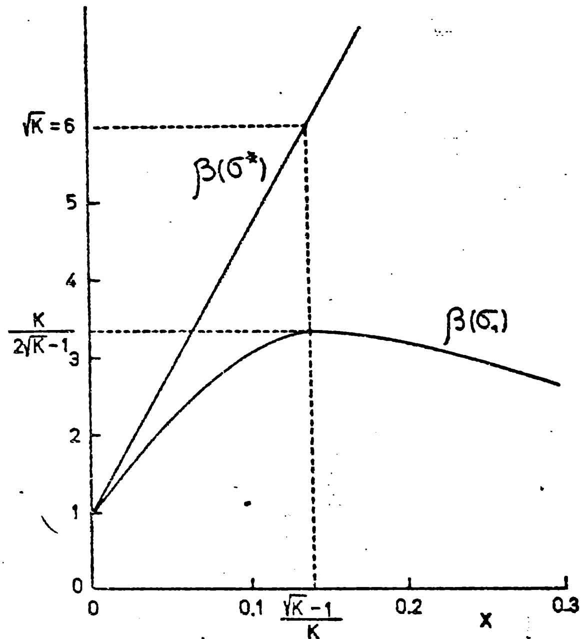

Fig. 12.1.

At any point of the constant acceleration arc it appears permissible to branch over again to a constant thrust arc since eqns. (17), (23) and (24) are common to both types of arc and the multipliers may be kept continuous. (In fact, since F remains continuous, even λ_3° will remain

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continuous). However, eqn. (25) will not be satisfied after that, a circumstance which prevents any further branching and also prevents the end condition (15) from being satisfied. Consequently, this type of further branching is ruled out.

At any point of the constant acceleration arc it is also permissible to branch over to a discontinuous solution, since it shares eqn. (25) with the constant acceleration arc and eqns. (28) merely require λ_2 and λ_3 to remain constant. We have seen, however, that a further branching is then ruled out and the exceptional case does not apply since at the end of the constant acceleration arc we have $\beta_t = \sqrt{K}$. It must then be concluded that the path cannot be extended beyond the discontinuity. It may well terminate with the discontinuity because eqn. (25) is verified at any point of it and this meets the terminal requirement (15). This new possibility of a constant thrust–constant acceleration–discontinuity path does not differ in reality from the previous solution where there is no discontinuity at the end. For no velocity is gained along the discontinuity, no fuel is burned, and the only difference is that the engines still remaining when the prescribed velocity is acquired are separated in part or *in toto* from the payload.

The last possibility is to branch over from the initial constant thrust arc to a discontinuity. It appears possible at any value of the reduced time $x(\sigma^*)$ provided eqn. (25) be satisfied, and this occurs, according to our previous calculations, when the initial acceleration factor is related to $x(\sigma^*)$ by eqn. (37). Further branching is possible if condition (30) can be satisfied. However β would therefore have to exceed $\sqrt{K}$ and this will be shown to violate the strong variation criterion of Weierstrass.

Consequently, this type of path terminates either on the constant thrust arc itself or after a subsequent discontinuity, the distinction between the two possibilities being once more irrelevant.

After this somewhat lengthy discussion we are faced with only two distinct stationary paths: the constant thrust + constant acceleration (+ discontinuity) type and the simpler constant thrust (+ discontinuity) type. One of the two will be the minimal path with respect to take-off mass. The choice may be decided by direct comparison only because of the possibility to calculate directly the take-off mass per unit payload. This will first be done in order to furnish tangible results.

It will later be shown that the same choice is dictated by the Weierstrass strong variation criterion.

12.2.7. Integration along the stationary paths

Integration of eqns. (1) and (4), pp. 774-775, along the constant thrust arc up to the value σ^*, produces the general results

$$\frac{M(\sigma_1)}{M(\sigma^*)} = 1 + x(\sigma^*)\,\beta(\sigma^*) = \frac{1}{1 - \beta(\sigma_1)\,x(\sigma^*)} \tag{42}$$

$$\frac{1+\zeta}{c}[V(\sigma^*) - V(\sigma_1)] = \ln\frac{1}{1 - \beta(\sigma_1)\,x(\sigma^*)} - x(\sigma^*) \tag{43}$$

Along a constant acceleration arc we have, because of the constant acceleration $g(\sqrt{K} - 1)$

$$\frac{1+\zeta}{c}[V(\sigma_2) - V(\sigma^*)] = (\sqrt{K} - 1)\,[x(\sigma_2) - x(\sigma^*)] \tag{44}$$

On the other hand, if we differentiate eqn. (26), p. 779

$$F^\circ = g\sqrt{K}\,M^\circ$$

and substitute in eqn. (2), p. 774, to find A^2, then introduce in eqn. (4), p. 775, with $F = \sqrt{K}\,Mg$, we obtain

$$\frac{\mathrm{d}M}{M} = -\frac{K}{\sqrt{K}-1}\,\mathrm{d}x$$

This becomes upon integration

$$\frac{M(\sigma^*)}{M(\sigma_2)} = \exp\left\{\frac{K}{\sqrt{K}-1}[(x(\sigma_2) - x(\sigma^*)]\right\}$$

$$= \exp\left\{\frac{K}{(\sqrt{K}-1)^2}\,\frac{1+\zeta}{c}[V(\sigma_2) - V(\sigma^*)]\right\} \tag{45}$$

Finally, from eqn. (3) at $t(\sigma_2)$ when the fuel is burned out

$$\frac{M(\sigma_2)}{M_u} = \frac{K}{K - \beta(\sigma_2)} \tag{46}$$

From the same eqn. at $t(\sigma_1) = 0$

$$(1+\zeta)\frac{M_p(\sigma_1)}{M_u} = \frac{M(\sigma_1)}{M_u}\left(1 - \frac{\beta(\sigma_1)}{K}\right) - 1 \tag{47}$$

12.2.8. Results for simple constant thrust path

In this case we have to take $\sigma^* = \sigma_2$. Since condition (15), p. 777, is to be satisfied in the end, $\beta(\sigma_1)$ is given by eqn. (37), p. 782,

$$\beta(\sigma_1) = \frac{1 + Kx(\sigma_2)}{1 + x(\sigma_2)\,[1 + Kx(\sigma_2)]}$$

Substitution in eqn. (43) yields

$$\frac{1+\zeta}{c}[V(\sigma_2) - V(\sigma_1)] = \ln\{1 + x(\sigma_2)\,[1 + Kx(\sigma_2)]\} - x(\sigma_2) \tag{48}$$

whereby $x(\sigma_2)$ is determined by the required velocity gain. From eqn. (42)

$$\frac{M(\sigma_1)}{M(\sigma_2)} = 1 + x(\sigma_2)\,[1 + Kx(\sigma_2)]$$

From eqn. (35), p. 782,

$$\beta(\sigma_2) = 1 + Kx(\sigma_2)$$

and from eqn. (46)

$$\frac{M(\sigma_2)}{M_u} = \frac{K}{K - 1 - Kx(\sigma_2)}$$

It follows therefore that

$$\frac{M(\sigma_1)}{M_u} = K\frac{1 + x(\sigma_2)\,[1 + Kx(\sigma_2)]}{K - 1 - Kx(\sigma_2)} \tag{49}$$

The payload vanishes for $\beta(\sigma_2) = K$ or $x(\sigma_2) = 1 - 1/K$, the value for which $\beta(\sigma_1)$ reduces to unity.

12.2.9. Results for constant thrust–constant acceleration path

The transition is given by the values eqns. (39) and (40), p. 782, from which

$$\frac{M(\sigma_1)}{M(\sigma^*)} = \frac{2\sqrt{K}-1}{\sqrt{K}} \tag{50}$$

Consequently

$$\frac{1+\zeta}{c}[V(\sigma^*) - V(\sigma_1)] = \ln\frac{2\sqrt{K}-1}{\sqrt{K}} - \frac{\sqrt{K}-1}{K} \tag{51}$$

This last result establishes the value of the transition velocity. The right-hand side of eqn. (45) is then known and since $\beta(\sigma_2) = \beta(\sigma^*) = \sqrt{K}$, eqn. (46) yields

$$\frac{M(\sigma_2)}{M_u} = \frac{\sqrt{K}}{\sqrt{K}-1} \tag{52}$$

Finally, combining eqns. (50), (45) and (52) we get

$$\frac{M(\sigma_1)}{M_u} = \frac{2\sqrt{K}-1}{\sqrt{K}-1}\exp\left\{\frac{K}{(\sqrt{K}-1)^2}\frac{1+\zeta}{c}[V(\sigma_2) - V(\sigma^*)]\right\} \tag{53}$$

Naturally, in this solution $V(\sigma_2)$ should be higher than $V(\sigma^*)$ and then the value of eqn. (53) is always smaller than that of eqn. (49) and is the absolute minimum we look for. For instance, when $K = 36$ and $V(\sigma_1) = 0$ we find from eqns. (39), p. 782, and (51)

$$\frac{1+\zeta}{c}V(\sigma^*) = 0.46724$$

$$x(\sigma^*) = 0.13889$$

$$\beta(\sigma_1) = 3.273$$

The velocity requirement for which the payload vanishes in the constant thrust path is given by eqn. (48) with

$$x(\sigma_2) = 35/36$$

or

$$\frac{1+\zeta}{c}V(\sigma_2) = 2.61134$$

For the same velocity requirement the minimal path gives

$$M_u = M(\sigma_1)\,\frac{5}{11}\exp\left(-\frac{36}{25}(2.61134 - 0.46724)\right) = 0.02084\, M(\sigma_1)$$

The ratio of initial thrust to payload is given by

$$\frac{F(\sigma_1)}{gM_u} = \beta(\sigma_1)\,\frac{M(\sigma_1)}{M_u} = \frac{K}{\sqrt{K}-1}\exp\left(\frac{K}{(\sqrt{K}-1)^2}\,\frac{1+\zeta}{c}\,[V(\sigma_2) - V(\sigma_1)]\right) \quad (54)$$

which in this example takes on the value

$$\frac{F(\sigma_1)}{gM_u} = \frac{36}{5}\,e^{3.76} = 309.2$$

Finally, the ratio of fuel weight to payload is from eqn. (47)

$$(1+\zeta)\,\frac{M_p(\sigma_1)}{M_u} - 1 = 2\exp\left(\frac{K}{(\sqrt{K}-1)^2}\,\frac{1+\zeta}{c}\,[V(\sigma_2) - V(\sigma_1)]\right) - 1 \quad (55)$$

In this particular case

$$2e^{3.76} - 1 = 84.9$$

An interesting variant of the problem consists in dropping the assumption that the tanks are expended at the rate fuel is being burned. Eqn. (3), p. 775, is then transformed into

$$M = M_u + M_p + \zeta M_p(\sigma_1) + \frac{F}{Kg} \quad (3')$$

and ζ is put equal to zero in eqn. (4) and also in all the Euler equations of the problem. Eliminating $M_p(\sigma_1)$ between eqn. (3′) for $\sigma = \sigma_1$ and for $\sigma = \sigma_2$ we get

$$M(\sigma_2) = \frac{\zeta}{1+\zeta}\,M(\sigma_1) + \frac{1}{1+\zeta}\,M_u + \frac{1}{Kg}\,F(\sigma_2) - \frac{\zeta}{1+\zeta}\,\frac{1}{Kg}\,F(\sigma_1) \quad (3'')$$

Hence the substitution of $\delta M(\sigma_2)$ in the transversality condition produces the modified end conditions:

$$\frac{\zeta}{1+\zeta}\,\lambda_2(\sigma_2) + Kg\lambda_3(\sigma_1) = 0 \quad (16')$$

$$\frac{\zeta}{1+\zeta}\,\lambda_2(\sigma_2) - \lambda_2(\sigma_1) \neq 0 \quad (17')$$

while eqn. (15), p. 777, remains valid. We note that, since from eqns. (16′) and (17′) $\lambda_2(\sigma_1) + Kg\lambda_3(\sigma_1) \neq 0$, the minimal path must again begin with a constant thrust solution. The transition conditions to the constant acceleration arc must, however, be worked out differently with the help of eqn. (16′). Also, the relationship between the overall payload ratio and the overall mass ratio is different; it follows from eqn. (3″) if it is divided by $M(\sigma_1)$

$$\frac{M(\sigma_2)}{M(\sigma_1)}\left(1-\frac{1}{\sqrt{K}}\right) = \frac{\zeta}{1+\zeta}\left(1-\frac{1}{K\beta(\sigma_1)}\right) + \frac{1}{1+\zeta}\,\frac{M_u}{M(\sigma_1)}$$

The detailed solution involves only algebraic manipulations.

12.3. Vertical Flight of a Continuously Staged Rocket with other Minimal Requirements

Instead of minimizing the total take-off mass for a given payload and velocity gain we may require a minimum of fuel consumption. Thus we should try to enforce the condition

$$\delta M_p(\sigma_1) = 0$$

which, by virtue of eqn. (3) is equivalent to

$$\delta F(\sigma_1) = Kg\delta M(\sigma_1)$$

Through substitution of this in the transversality condition we obtain the condition

$$\lambda_2(\sigma_1) + Kg\lambda_3(\sigma_1) = 0 \tag{16″}$$

as replacing our former condition eqn. (16), p. 777. This is satisfied by a constant acceleration arc. A discussion similar to the one given shows that a single constant acceleration path satisfies all the requirements of a minimal path. From eqns. (45) and (46), p. 785, with

$$\sigma^* = \sigma_1 \qquad x(\sigma_1) = 0 \qquad \beta(\sigma_1) = \beta(\sigma_2) = \sqrt{K}$$

we obtain

$$\frac{M(\sigma_1)}{M_u} = \frac{\sqrt{K}}{\sqrt{K}-1}\exp\left(\frac{K}{(\sqrt{K}-1)^2}[V(\sigma_2) - V(\sigma_1)]\right) \tag{53′}$$

References p. 812

and from eqn. (47) we get

$$(1+\zeta)\frac{M_p(\sigma_1)}{M_u} = \exp\left(\frac{K}{(\sqrt{K}-1)^2}[V(\sigma_2)-V(\sigma_1)]\right)-1 \tag{51'}$$

Finally

$$\frac{F(\sigma_1)}{gM_u} = \frac{K}{\sqrt{K}-1}\exp\left(\frac{K}{(\sqrt{K}-1)^2}[V(\sigma_2)-V(\sigma_1)]\right) \tag{54'}$$

In view of the difficulties of production of large engines and their cost another significant minimal requirement should be that of minimum initial thrust. When we enforce the stationary condition

$$\delta F(\sigma_1) = 0$$

in the transversality condition, we obtain the initial conditions

$$\lambda_2(\sigma_1) = 0 \qquad \lambda_3(\sigma_1) \neq 0$$

which replace eqns. (16) and (17), p. 777. The only possibility for the initial arc is a constant thrust one with the additional conditions

$$\left(\frac{F}{Mg}\right)_{\sigma_1} = \beta(\sigma_1) = 1$$

The initial value of λ_3, which remains undetermined, can always be adjusted so as to satisfy the branching requirements on a constant acceleration path, when $\beta = \sqrt{K}$, and it can again be shown that this type of constant thrust–constant acceleration path is the minimal one. From eqn. (42), p. 785,

$$\frac{M(\sigma_1)}{M(\sigma^*)} = 1+\sqrt{K}\,x(\sigma^*) = \frac{1}{1-x(\sigma^*)}$$

from which it follows that

$$\frac{M(\sigma_1)}{M(\sigma^*)} = \sqrt{K} \quad \text{and} \quad x(\sigma^*) = \frac{\sqrt{K}-1}{\sqrt{K}} \tag{56}$$

From eqn. (39), p. 782,

$$\frac{1+\zeta}{c}[V(\sigma^*)-V(\sigma_1)] = \ln\sqrt{K} - \frac{\sqrt{K}-1}{\sqrt{K}} \tag{51''}$$

Eqns. (45) and (52) are valid; if they are combined with eqn. (56) they yield

$$\frac{M(\sigma_1)}{M_u} = \frac{K}{\sqrt{K}-1} \exp\left(\frac{K}{(\sqrt{K}-1)^2}[V(\sigma_2) - V(\sigma^*)]\right) \tag{53''}$$

Since $\beta(\sigma_1) = 1$, this is also the value of $F(\sigma_1)/gM_u$. Finally

$$(1+\zeta)\frac{M_p(\sigma_1)}{M_u} = \frac{K-1}{\sqrt{K}-1} \exp\left(\frac{K}{(\sqrt{K}-1)^2}[V(\sigma_2) - V(\sigma^*)]\right) - 1 \tag{55''}$$

It is interesting to compare the three minimal problems in a given situation. Take $K = 36$ and suppose the velocity performance to be given by $V(\sigma_1) = 0$

$$\frac{1+\zeta}{c} V(\sigma_2) = 5.15$$

(for $\zeta = 0.1$ and $c = 2350$ m/sec this would approximately yield the escape velocity of 11 km/sec.) The application of the formulas yields the following table of results, where the first column corresponds to minimum take-off weight, the second to minimum fuel consumption, and the third to minimum initial thrust.

TABLE 1

	I	II	III
$\frac{1+\zeta}{c} V(\sigma^*)$	0.46724		0.95843
$\frac{M(\sigma_1)}{M_u}$	1867	1995	3011
$(1+\zeta)\frac{M_p(\sigma_1)}{M_u}$	1696	1661	2926
$\frac{F(\sigma_1)}{gM_u}$	6110	11969	3011

From this table it appears that the minimum take-off weight is a good compromise. The reduction in fuel consumption in case II is slight and is obtained at the expense of a large increase in initial thrust, whereas the minimum initial thrust requires a tremendous increase in fuel.

References p. 812

12.4. Application of the Weierstrass Strong Variation Test

Consider the Weierstrassian excess function defined by

$$E = G(\bar{q},\bar{q}^\circ) - G(q,q^\circ) - \Sigma\ (\bar{q}^\circ - q^\circ)\frac{\partial}{\partial q^\circ}G(q,q^\circ)$$

where $G(\bar{q},\bar{q}^\circ)$ is obtained by a finite change of the guidance variable A into $\bar{A}$, the other variables undergoing associated changes compatible with the original constraints. Thus M, F and V do no change, but M° changes into $\bar{M}^\circ$, F° into $\bar{F}^\circ$, V° into $\bar{V}^\circ$ and t° into $\bar{t}^\circ$ according to the laws

$$M(\bar{V}^\circ + g\bar{t}^\circ) - F\bar{t}^\circ = 0$$

$$\bar{M}^\circ + \frac{1+\zeta}{c}F\bar{t}^\circ + \frac{\bar{A}^2}{Kg}\bar{t}^\circ = 0$$

$$\bar{F}^\circ + \bar{A}^2\bar{t}^\circ = 0$$

derived from eqns. (1), (2), and (4), pp. 774-775. $G(\bar{q},\bar{q}^\circ)$ is then obtained by multiplying the left-hand side of these equations by λ_1, λ_2 and λ_3 and adding the results. Because of the homogeneity of $G(q,q^\circ)$ with respect to the q°, we may write

$$G(q,q^\circ) = \Sigma\ q^\circ\frac{\partial}{\partial q^\circ}G(q,q^\circ)$$

and the excess function can be simplified into

$$E = G(\bar{q},\bar{q}^\circ) - \Sigma\ \bar{q}^\circ\frac{\partial}{\partial q^\circ}G(q,q^\circ)$$

After performing all calculations and reductions we find

$$E = \bar{t}^\circ(\bar{A}^2 - A^2)\left(\frac{\lambda_2}{Kg} - \lambda_3\right) \tag{57}$$

To apply the test to a minimum problem we must also consider the coefficient of the variation of the quantity to be minimized, as it appears in the transversality condition. A necessary condition for a path to be minimal is then that at all of its points the excess function should vanish or should have a sign opposed to that of this coefficient. In the case of a maximum the sign should be equal. If we look into the problem of minimizing $M(\sigma_1)$ the coefficient of its variation in the transversality condition is

$$-\lambda_2(\sigma_1)$$

For the stationary paths discovered, the first arc is of the constant thrust variety. From eqn. (23), p. 778,

$$-\lambda_2(\sigma_1) = \frac{c}{1+\zeta}\left(1 - \frac{1}{\beta(\sigma_1)}\right)$$

Since $\beta(\sigma_1) \geqslant 1$ this coefficient is negative and the test to apply is

$$E \geqslant 0$$

There is no difficulty along a constant acceleration arc, for which, by virtue of eqn. (25), p. 778, $E = 0$. Along the initial constant thrust arc $A = 0$. Since $\bar{l}^\circ > 0$, the test reduces to

$$\lambda_2 + Kg\lambda_3 \geqslant 0$$

From eqns. (23) and (24), p. 778,

$$\frac{1+\zeta}{c}(\lambda_2 + Kg\lambda_3) = \frac{M(\sigma_1)}{M}\frac{1}{\beta(\sigma_1)} - \frac{1}{\beta(\sigma_1)}\frac{1+\zeta}{c}gt$$

Since further

$$M = M(\sigma_1) - \frac{1+\zeta}{c}Ft$$

we find, using the reduced time defined by eqn. (36), p. 782,

$$\frac{1+\zeta}{c}(\lambda_2 + Kg\lambda_3) = \frac{1}{1 - x\beta(\sigma_1)} - \frac{1+Kx}{\beta(\sigma_1)} \geqslant 0$$

to be satisfied for any x from zero to the end of the constant thrust arc. The equality sign defines a curve $\beta(\sigma_1) = f(x)$, which is of course the same as the lower curve of Fig. 12.1 and is reproduced for convenience in Fig. 12.2. This curve is entirely to the left of the curve

$$\beta(\sigma_1) = x^{-1}$$

and so in the region for which

$$1 - x\beta(\sigma_1) > 0$$

In this region the condition can then be put in a form analogous to eqn. (37), p. 782,

$$\beta(\sigma_1) \geqslant \frac{1 + Kx}{1 + x[1 + Kx]}$$

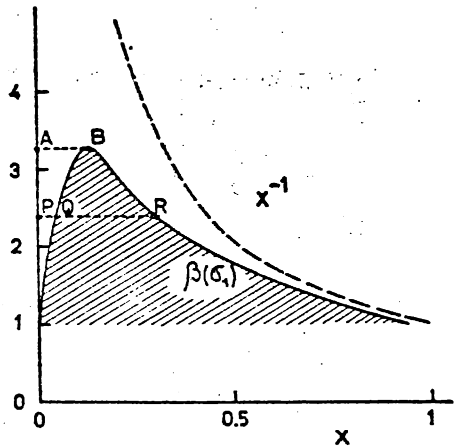

Fig. 12.2

It should be clear that the reverse inequality holds in the shaded region of Fig. 12.2, which is then a forbidden region. For a constant thrust arc with a given $\beta(\sigma_1)$ the representative point moves, as x increases, along a horizontal line. The end of the arc is on the curve, either in B, where transition is possible to the constant acceleration path, or on some other point in order to satisfy the end condition (15), p. 777. The path AB satisfies the Weierstrass criterion and corresponds to the constant thrust–constant acceleration solution. A path PQ also satisfies the criterion and corresponds to the simple constant thrust path with a velocity requirement smaller than that which would already be reached in B. A path PR, however, violates the criterion by entering in the forbidden region. Hence the alternative solution of a simple constant thrust path giving the same final velocity as the constant thrust–constant acceleration path must be rejected. This was already confirmed by direct calculation.

12.5. The Problem of Thrust Programming along a Curved Path

The consideration of aerodynamic drag, which may be important in denser layers of the atmosphere, indicates the desirability of programming the engine thrust to strike the best compromise between gravitational losses and aerodynamic losses. In this problem we consider the flight of a rocket in a vertical plane and uniform gravitational field. The engine mass is not subject to continuous variation as in the former problem (*i.e.* the rocket is not continuously staged) but the rocket engine propelling the stage considered is capable of yielding a variable thrust.

If γ denotes the angle of slope of the path with respect to the horizontal direction and the thrust acts tangentially to the path, the equation of motion along the tangent is

$$[G]_1 = M(V^\circ + gt^\circ \sin\gamma) - Ft^\circ + Dt^\circ = 0 \tag{58}$$

where the drag is assumed to be a function of velocity and altitude

$$D(z, V)$$

The equation of motion normal to the path in the absence of lift is

$$[G]_2 = V\gamma^\circ + gt^\circ \cos\gamma = 0 \tag{59}$$

The engine thrust is taken to be

$$F = mc$$

where c is the effective ejection velocity of gases and m their rate of mass flow. Since the variation in rocket mass is solely due to m, we also have

$$M^\circ = -mt^\circ \tag{60}$$

so that the equation of conservation of mass can be written

$$[G]_3 = M^\circ + \frac{F}{c}t^\circ = 0 \tag{61}$$

References p. 812

Since the altitude enters in the problem through D, we must also consider the equation of altitude gain

$$[G]_4 = z^\circ - Vt^\circ \sin\gamma = 0 \tag{62}$$

Since the thrust is variable, we consider F to be a function of a parameter α. Explicitly we may take[3]

$$F = F_{\max} \frac{1 + \tanh\alpha}{2} \tag{63}$$

Hence F varies continuously from zero, for $\alpha = -\infty$, to $F_{\max}$ the maximum thrust the engine can deliver, for $\alpha = +\infty$. In this case α is the actual guidance variable. If we consider again the function

$$G = \lambda_1[G]_1 + \lambda_2[G]_2 + \lambda_3[G]_3 + \lambda_4[G]_4$$

built up with the help of Lagrangian parameters, the Euler equations of the problem are

$$[G]_V = -[\lambda_1 M]^\circ + \lambda_1 t^\circ D_V + \lambda_2\gamma^\circ - \gamma_4 t^\circ \sin\gamma = 0 \tag{64}$$

$$[G]_\gamma = -[\lambda_2 V]^\circ + \lambda_1 Mgt^\circ \cos\gamma - \lambda_2 gt^\circ \sin\gamma - \lambda_4 Vt^\circ \cos\gamma = 0 \tag{65}$$

$$[G]_M = -\lambda_3^\circ + \lambda_1(V^\circ + gt^\circ \sin\gamma) = 0 \tag{66}$$

$$[G]_z = -\lambda_4^\circ + \lambda_1 t^\circ D_z \tag{67}$$

$$[G]_t = U^\circ = 0 \tag{68}$$

$$[G]_\alpha = \left[-\lambda_1 + \frac{1}{c}\lambda_3\right] t^\circ \frac{dF}{d\alpha} = 0 \tag{69}$$

where

$$U = \lambda_1(Mg\sin\gamma + D) + \lambda_2 g\cos\gamma - \lambda_4 V\sin\gamma + F\left(-\lambda_1 + \frac{1}{c}\lambda_3\right) \tag{70}$$

and

$$D_V = \frac{\partial D}{\partial V} \qquad D_z = \frac{\partial D}{\partial z}$$

The transversality condition is

$$\left[\lambda_1 M\delta V + \lambda_2 V\delta\gamma + \lambda_3\delta M + \lambda_4\delta z + U\delta t\right]_{\sigma_1}^{\sigma_2} = 0 \tag{71}$$

12.5.1. The sustaining phase

The sustaining phase is the extremal arc along which there is a continuous thrust programming. Its characteristics are obtained when we satisfy eqn. (69) by taking

$$c\lambda_1 - \lambda_3 = 0 \tag{72}$$

No restriction is imposed on the duration of flight; then, since U is constant by virtue of eqn. (68), the arbitrary variation of the flight duration in the transversality condition requires $U = 0$. Hence from eqns. (70) and (72)

$$\lambda_1(Mg \sin\gamma + D) + \lambda_2 g \cos\gamma - \lambda_4 V \sin\gamma = 0 \tag{73}$$

A third homogeneous relation between multipliers is obtained by differentiation of eqn. (72) and substitution of all derivatives with the help of the Euler equations and the original constraints. This furnishes

$$-\lambda_1(\omega D + VD_V) + \lambda_2 g \cos\gamma + \lambda_4 V \sin\gamma = 0 \tag{74}$$

where $\omega = V/c$. This relation does not involve the thrust (or the guidance variable) nor have eqns. (65) or (67) been used. It is therefore advisable to repeat the differentiation procedure on eqn. (74). After simplifications the result may be put in the form

$$-C\lambda_1 \frac{F-D}{Mg} + \lambda_1(A-B)\sin\gamma + \lambda_1 Mg\cos^2\gamma - 2\lambda_4 V\cos^2\gamma = 0 \tag{75}$$

where

$$A = 2\omega D + (2+\omega)VD_V + V^2 D_{VV} - \frac{V^2}{g}[(\omega - 1)D_z + VD_{Vz}] - Mg\sin\gamma$$

$$B = (\omega + 1)D + VD_V$$

$$C = (\omega^2 + \omega - 1)D + (1 + 2\omega)VD_V + V^2 D_{VV} - Mg\sin\gamma$$

Eqns. (73), (74) and (75) are homogeneous in λ_1, λ_2, and λ_4 and their compatibility condition furnishes an algebraic relation to be satisfied between the original variables:

$$\frac{F-D}{Mg} = \frac{1}{C}(A\sin\gamma - B\,\mathrm{cosec}\,\gamma) \tag{76}$$

References p. 812

This expression actually yields the thrust programming along the sustaining phase and allows the numerical integration of the equations of motion to be carried out. If we consider drag to be given by the usual law

$$D = K\varrho(z)V^2 \tag{77}$$

with a constant drag coefficient K, we have

$$VD_V = V^2 D_{VV} = 2D \qquad D_z = -\frac{g}{V_a^2} D$$

where

$$V_a = \sqrt{-\varrho g \frac{\mathrm{d}z}{\mathrm{d}\varrho}} \tag{78}$$

is a velocity characterizing the gradient of the atmospheric density. In standard atmosphere this velocity is slightly less than the velocity of sound. The quantities A, B and C are then given by the much simpler expressions:

$$\begin{aligned} A &= \left[6 + 4\omega + \left(\frac{V}{V_a}\right)^2 (\omega + 1)\right] D - Mg \sin\gamma \\ B &= (3 + \omega)D \\ C &= (\omega^2 + 4\omega + 2)D - Mg \sin\gamma \end{aligned} \tag{79}$$

Another important consequence of the system of eqns. (72) to (75) is that the multipliers are entirely determined along the sustaining phase, except of course for a scale factor.

12.5.2. The maximum thrust arc

Another way of satisfying equation (69), p. 796, is to take $\alpha = \infty$, a value of the guidance variable for which

$$\frac{\mathrm{d}F}{\mathrm{d}\alpha} = F_{\max} \frac{1}{2\cosh^2\alpha}$$

is equal to zero. The thrust remains equal to its maximum value and the system of original constraints is to be integrated under that assumption. Eqns. (64) to (67), p. 796, may be used to integrate the multipliers, starting from a set of known values. If the algebraic relation $U = 0$ is satisfied by the set of initial values at the start of the integration, it will remain satisfied afterwards. This is an immediate consequence of the identity between Euler equations. Since $\alpha^\circ = 0$ along the arc and provided the original constraints are satisfied, this identity reduces to

$$V^\circ[G]_V + \gamma^\circ[G]_\gamma + M^\circ[G]_M + z^\circ[G]_z + t^\circ U^\circ = 0$$

Once eqns. (64) to (67) are used to calculate the multipliers, $U^\circ = 0$ follows and consequently U remains equal to its initial value. An alternative procedure consists in using three of the equations (64) to (67) and determining the fourth multiplier from $U = 0$.

12.5.3. The zero thrust arc

$dF/d\alpha$ is also zero for $\alpha = -\infty$, which corresponds to zero thrust or coasting phase. The calculation of the multipliers follows the same procedure as in the case of the maximum thrust arc. This exhausts the possibilities of the differential system. For the solution $t^\circ = 0$ of eqn. (69), p. 796, does not involve any real discontinuity because F cannot become infinite.

12.5.4. The Weierstrassian excess function

After calculation the Weierstrassian excess function reduces to

$$E = t^\circ(\bar{F} - F)\left(\frac{1}{c}\lambda_3 - \lambda_1\right) \tag{80}$$

12.5.5. Minimal paths

Now that the nature of the extremals is determined we must state the precise problem in order to solve and analyse the composition of the minimal path. Among the various optimal performance problems associated with the original set of constraints let us choose the following one: Proceed from an altitude z_1 and velocity V_1 to an altitude z_2, velocity V_2 and inclination γ_2 with a minimum of fuel consumption.

References p. 812

the final burnout mass M_2 being known. The absence of restriction on flight duration already led us to use the result $U = 0$. We now have to put

$$\delta z(\sigma_1) = 0 \qquad \delta V(\sigma_1) = 0 \qquad \delta z(\sigma_2) = 0 \qquad \delta V(\sigma_2) = 0$$
$$\delta \gamma(\sigma_2) = 0 \qquad \delta M(\sigma_2) = 0$$

in the transversality condition (71), p. 796, reducing it to

$$-\lambda_2(\sigma_1) V(\sigma_1) \delta\gamma(\sigma_1) - \lambda_3(\sigma_1) \delta M(\sigma_1) = 0$$

We suppose $V(\sigma_1) = V_1 \neq 0$ so that the arbitrariness of $\gamma(\sigma_1)$ leads us to take

$$\lambda_2(\sigma_1) = 0 \tag{81}$$

and the stationary condition $\delta M(\sigma_1) = 0$ ($M(\sigma_1)$ should be a minimum for minimum fuel consumption) is enforced provided

$$\lambda_3(\sigma_1) \neq 0$$

We shall assume that the arbitrary factor affecting the multipliers is so chosen that $\lambda_3(\sigma_1)$ is positive, in which case the Weierstrass criterion is

$$E \geqslant 0 \tag{82}$$

Along a sustaining phase the criterion is automatically satisfied by virtue of eqn. (72), p. 797. Along a maximum thrust arc $\bar{F}$ can only be smaller than F and $l_0 > 0$ and the criterion reduces to

$$\frac{1}{c}\lambda_3 - \lambda_1 \leqslant 0 \qquad \text{maximum thrust} \tag{83}$$

Along a coasting arc, $F = 0$, $\bar{F}$ can only be positive and

$$\frac{1}{c}\lambda_3 - \lambda_1 \geqslant 0 \qquad \text{coasting} \tag{84}$$

The Weierstrass-Erdmann corner conditions require the continuity of the multipliers and of U. It follows from this that at any type of corner we have

$$\frac{1}{c}\lambda_3 - \lambda_1 = 0$$

If one of the arcs joined at the corner corresponds to a sustaining phase, this is obvious from eqn. (72), p. 797. If the corner is between a maximum thrust and a coasting arc, it expresses the condition under which U remains equal to zero without discontinuity in the multipliers, despite the discontinuity in thrust.

We now have all the elements necessary for discussing the synthesis of a minimal path. The discussion will serve the purpose of demonstrating the determinateness of the problem; it should not be necessarily considered as the best method of integration.

At the beginning of the sustaining phase an arbitrary value may be taken for λ_3 and a set of values for (z, V, γ, M) can be guessed. All variables, including multipliers, are then determined for the sustaining phase. But first we integrate backwards along a coasting arc or a maximum thrust arc, the type being dictated by the Weierstrass criterion. At the altitude z_1 the sign of $\lambda_3(\sigma_1)$ should be checked to determine whether the criterion was correctly applied (normally λ_3 should not change sign). The initially guessed values must be corrected until $V(\sigma_1) = V_1$ and $\lambda_2(\sigma_1) = 0$ so that we are left with two degrees of freedom only at the beginning of the sustaining phase. The following possibilities may arise:

1) The path terminates on a coasting arc. We then integrate along the sustaining phase up to when $M(\sigma_2) = M_2$, continue along a coasting arc up to $z(\sigma_2) = z_2$ and determine the two freedoms by the terminal conditions $V(\sigma_2) = V_2$ and $\gamma(\sigma_2) = \gamma_2$.

2) The path terminates on a full thrust arc following directly the sustaining phase. The end of the sustaining phase must then be guessed and this third freedom is compensated by the existence of three terminal requirements (M_2, V_2, γ_2) at z_2.

3) The sustaining phase is followed by a coasting arc and terminates on a full thrust arc. This may happen for large terminal velocities at low angles of inclination, such as required in satellite launching. If the end of the sustaining period is again unknown, the coasting phase has to end when $\lambda_3 = c\lambda_1$ and is determined.

References p. 812

We may note that the choice between the three possibilities is again dictated by the Weierstrass criterion.

Because of the large number of guesses and corrections required, the actual integration in a specific case is of course a lengthy process. If we restrict ourselves to the satisfaction of "natural boundary conditions" on the multipliers, in this case the sole condition $\lambda_2(\sigma_1) = 0$ "semi-inverse" solutions are easier to calculate. The trajectories obtained are minimal with respect to the end values obtained by the integration, as if they had been prescribed *a priori*. When a sufficient batch of such semi-inverse solutions is calculated it becomes possible to make reasonable assumptions for the specific problem and to correct rationally for them by a method of variation of the guessed parameters.

12.5.6. Boost phases[3]

A very large maximum thrust will tend to reduce the variations in altitude and inclination of the path during the full thrust phases of short duration. A considerable simplification of the analysis will follow if the assumption is made that the engine can deliver an infinitely large thrust. One then obtains a limiting case of the minimal path, which may provide useful indications for the real case of finite maximum thrust. By letting $F_{\max}$ tend to infinity and simultaneously t° to zero in such a manner that

$$\lim F_{\max} t^\circ = I \qquad \text{a finite positive quantity}$$

a discontinuous solution of the Euler equations is found. Equation (69), p. 796, is now satisfied by virtue of $t^\circ = 0$. The original constraints reduce to

$$\gamma^\circ = 0 \quad \text{and} \quad z^\circ = 0$$

expressing that neither inclination of the path nor altitude will vary, but also

$$MV^\circ - I = 0 \quad \text{and} \quad M^\circ + \frac{I}{c} = 0$$

After elimination of I

$$V^\circ + c\frac{M^\circ}{M} = 0$$

an equation which can be integrated along the arc to give

$$\Delta V = c \ln \frac{M_i}{M_f}$$

where M_i stands for the initial and M_f for the final mass along the arc. This is the familiar law of ideal velocity gain, extensively used in Chapter 11. The discontinuous solution thus represents a boost in velocity acquired by the instantaneous burning of a finite amount of fuel. The equation $U = 0$, which must be satisfied, is again understood in the sense

$$\lim F_{\max} \left(-\lambda_1 + \frac{1}{c} \lambda_3 \right) = -\lambda_1 (Mg \sin \gamma + D) - \lambda_2 g \cos \gamma + \lambda_4 V \sin \gamma$$

and requires that the condition (72), p. 797, now be satisfied along the boost phase as it is along the sustaining phase. Consequently, the boost phase automatically satisfies the Weierstrass criterion. All the equations between multipliers are compatible and integrable. A set of integrals is

$$\lambda_3 = c\lambda_1 = \frac{K_1}{M} \qquad \lambda_2 = \frac{K_2}{V} \qquad \lambda_4 = K_4$$

This solution contains three arbitrary constants determined by the corner conditions. The synthesis of a boost-sustain-boost type of minimal path presents new features of simplicity. For the initial condition $\lambda_2(\sigma_1) = 0$ can only be achieved by taking $K_2 = 0$ along the initial boost phase. This means that λ_2 will still be zero at the start of the sustaining phase; and this imposes a new compatibility condition in the homogeneous system eqns. (73), (74) and (75), p. 797. In fact eqns. (73) and (74) are compatible with $\lambda_2 = 0$ if, and only if,

$$\sin \gamma = \frac{(\omega - 1)D + VD_V}{Mg} \tag{85}$$

This determines the initial inclination of the path from the knowledge of mass and velocity at the start of the sustaining phase. Consequently, when starting the integration of the sustaining phase only two unknown parameters must be assumed, the take-off mass and the velocity jump of the initial boost. Carrying the integration through to final altitude and adding a second boost phase to meet the prescribed terminal velocity, we can determine the two unknowns from a knowledge of the terminal inclination of the path and the terminal mass. Backward integration is even simpler since it involves only one unknown parameter, the terminal velocity jump, to be adjusted by satisfying condition (85) at the appropriate point.

In the boost-sustain-coast-boost variety an additional unknown is the duration of the coasting phase; the additional condition is $\lambda_3 = c\,\lambda_1$, at the corner between coast and boost phases.

12.6. Thrust Programming along a Vertical Path

This is a particular case of the problem treated in section 12.5. The thrust programming of the sustaining phase is obtained from eqn. (76), p. 797, by inserting $\sin\gamma = \operatorname{cosec}\gamma = 1$. However, if eqns. (73), (74) and (75) are taken into consideration and $\cos\gamma = 0$, the two first equations already yield a compatibility condition

$$Mg = (\omega - 1)D + VD_V \tag{86}$$

similar to eqn. (85) and valid throughout the sustaining phase. On the other hand, eqn. (75) requires the vanishing of the coefficient of λ_1. This produces the simpler form

$$\frac{F - D}{Mg} = \frac{\omega VD_V + V^2 D_{VV} - \dfrac{V^2}{g}\left[(\omega - 1)D_z + VD_{Vz}\right]}{\omega^2 D + 2\omega VD_V + V^2 D_{VV}} \tag{87}$$

of the thrust law. It is readily seen to be identical with the previous one as modified by the insertion of result (86) in the coefficients A and C. Finally, the thrust can be expressed solely as a function of velocity and altitude. In the case of the drag law expressed by eqn. (77), p. 798, we find

$$\frac{F}{D} = 1 + \frac{(\omega + 1)^2}{\omega^2 + 4\omega + 2}\left[2 + \left(\frac{V}{V_a}\right)^2\right] \tag{88}$$

and

$$Mg = (\omega + 1)D \tag{89}$$

After substitution in the equation of motion (58), p. 795, and elimination of t^c from the equation of altitude gain (62), p. 796, we find

$$\frac{d\omega}{d\zeta} = \frac{\omega(\omega + 1)\beta^{-1} - (2 + \omega)}{\omega^2 + 4\omega + 2} \tag{90}$$

where ζ stands for the reduced altitude

$$\zeta = \frac{g}{c^2} z \qquad \text{and} \qquad \beta = \left(\frac{V_a}{c}\right)^2$$

In an exponential atmosphere, where V_a is independent of altitude, which may be an excellent approximation in the region where drag is important, eqn. (90) is integrable by separation of variables:

$$\frac{\zeta - \zeta_1}{\beta} = \omega - \omega_1 + \frac{\gamma}{2} \ln \left[\frac{2\omega + 1 - \beta - \gamma}{2\omega + 1 - \beta + \gamma} \times \frac{2\omega_1 + 1 - \beta + \gamma}{2\omega_1 + 1 - \beta - \gamma}\right]$$
$$+ \frac{3 + \beta}{2} \ln \frac{\omega^2 + (1 - \beta)\omega - 2\beta}{\omega_1^2 + (1 - \beta)\omega_1 - 2\beta} \tag{91}$$

with $\gamma = \sqrt{(1 + \beta)^2 + 4\beta}$, (ω_1, ζ_1) being any pair of values known along the sustaining phase. This result is essentially due to Tsien and Evans[6]. From eqn. (90) it will be observed that according to whether the initial value of ω is greater or smaller than

$$\omega^* = \frac{1}{2}(1 + \beta + \gamma) \tag{92}$$

the velocity will increase or decrease along the sustaining phase. The case when it remains stationary at ω^* corresponds to a constant thrust/drag ratio

$$\frac{F}{D} = 2 + \omega^* \tag{93}$$

In general, β is small enough to replace (92) in a good approximation by

$$\omega^* = 2\beta$$

From eqn. (89) we find then that the combination of weight, ejection velocity, atmospheric density at the altitude considered, and drag

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coefficient required to move at the constant reduced velocity ω^* along the sustaining phase is

$$\frac{gM}{4K\varrho V_a^2}\left(\frac{c}{V_a}\right)^2 = 1 + 2\beta \simeq 1 \tag{94}$$

The left-hand member represents what might be called a "performance parameter" of the rocket. It is usually much larger than one, especially for large-size rockets for which the drag coefficient increases roughly as the square of the diameter, while the weight increases as the third power. Such rockets will therefore require large accelerations along the sustaining phase: It may even happen that the drag to weight ratio required by eqn. (89) for a sustaining phase is never reached. The minimum consumption path is then simply made of a full thrust arc, possibly followed by coasting, and thus the desirability of a thrust programming is eliminated.

12.7. The Case of a Prescribed Path

We may consider the path to be prescribed if γ is given as a function of the altitude. In that case some transverse force will in general be needed to steer the rocket on that path. When the transverse force is produced by aerodynamic lift, the drag is affected by it and is generally expressible in the form

$$D = D_0(z, V) + L^2 D_1(z, V) \tag{95}$$

where L is the aerodynamic lift. The lift required is known from the equations of motion normal to the path

$$L = M\left(V\frac{\gamma^\circ}{t^\circ} + g\cos\gamma\right)$$

Since

$$\frac{\gamma^\circ}{t^\circ} = \gamma' V \sin\gamma \qquad \text{where} \qquad \gamma' = \mathrm{d}\gamma/\mathrm{d}z$$

$$L = M(\gamma' V^2 \sin\gamma + g\cos\gamma) \tag{96}$$

When eqns. (95) and (96) are substituted in eqn. (58), p. 795, for the motion tangential to the path, the equation normal to it is automatically satisfied and we may take $\lambda_2 \equiv 0$. The modified Euler equations are

$$[G]_V = -[\lambda_1 M]^\circ + \lambda_1 t^\circ [D_{oV} + L^2 D_{1V} + 4LD_1 MV\gamma' \sin\gamma] - \lambda_4 t^\circ \sin\gamma = 0 \quad (64')$$

$$[G]_M = -\lambda_3^\circ + \lambda_1 [V^\circ + gt^\circ \sin\gamma + 2LD_1(\gamma' V^2 \sin\gamma + g\cos\gamma)t^\circ] = 0 \qquad (66')$$

$$\begin{aligned}[G]_z = & -\lambda_4^\circ - \lambda_4 t^\circ \gamma' V \cos\gamma \\ & + \lambda_1 t^\circ [D_{oz} + L^2 D_{1z} + 2LMD_1(\gamma'' V^2 \sin\gamma + \gamma'^2 V^2 \cos\gamma - g\gamma' \sin\gamma)] = 0 \quad (67')\end{aligned}$$

while

$$U = \lambda_1(Mg\sin\gamma + D) - \lambda_4 V \sin\gamma + F\left[-\lambda_1 + \frac{1}{c}\lambda_3\right] = 0 \qquad (97)$$

and eqn. (69), p. 796, remains valid. Along the sustaining phase we shall have

$$\lambda_3 = c\lambda_1$$

Differentiation followed by substitution of the derivatives produces the following algebraic relation:

$$\begin{aligned}\lambda_1[D + cD_{oV} + cL^2 D_{1V} + 4LcD_1 MV\gamma' \sin\gamma - 2LD_1 M(\gamma' V^2 \sin\gamma + g\cos\gamma)] & \\ - \lambda_4 c \sin\gamma = 0 & \qquad (98)\end{aligned}$$

Eqns. (97) and (98) are now homogeneous in λ_1 and λ_4 and their condition of compatibility furnishes the algebraic relation without multipliers:

$$\begin{aligned}Mg(\sin\gamma + 2\omega LD_1 \cos\gamma) = & \\ (\omega - 1)D + VD_{oV} + L^2 VD_{1V} + 2(2-\omega)MV^2 LD_1 \gamma' \sin\gamma & \qquad (86')\end{aligned}$$

For $\gamma = \pi/2$ when $L = 0$, this equation reduces to eqn. (86), p. 804. When the transverse force is produced by other means than aerodynamic lift, we may put $D_1 \equiv 0$ and we obtain an extension of the validity of eqn. (85) to the whole sustaining phase. The addition of eqn. (86′) to the original set of constraints is sufficient to integrate the sustaining phase. We need only to eliminate F between eqns. (58) and (61), p. 795. The thrust is then obtained from eqn. (61) by differentiation of M. In principle, however, the explicit value of the thrust as an algebraic function of the other variables can be found by differentiating eqn. (86′) and substituting the derivatives of the original constraints.

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Since the expression is extremely complicated it will not be reproduced here.

The use of lifting surfaces to modify the natural curvature of the trajectory imposed by gravitation naturally suggests the investigation of performance problems where lift as well as thrust is programmed in some optimal way. The main purpose of aerodynamic lift is in general to provide an extended range. An investigation of the maximum range problem with the lift as guidance variable and the thrust as a known function of time will be found in ref.[7].

12.8. Thrust Programming for Maximum Range

Reverting to zero-lift, naturally curved trajectories, let us briefly investigate the modifications in thrust programming brought about by the inclusion of side conditions on the range. It will be necessary for this purpose to introduce the horizontal distance covered by means of the kinematic relation

$$[G]_5 = x^\circ - Vt^\circ \cos\gamma = 0 \tag{99}$$

and a fifth multiplier λ_5. The equation governing this multiplier is simply

$$[G]_x = -\lambda_5^\circ = 0$$

Hence λ_5 reduces to an isoperimetrical constant. A new term appears in the transversality condition

$$\lambda_5\delta[x(\sigma_2) - x(\sigma_1)] = \lambda_5\delta\Delta x$$

If no condition were prescribed for the range Δx, the arbitrariness of its variation would impose $\lambda_5 = 0$ and lead back to the earlier formulation. However, if the range is imposed or if it is required to be maximum, $\delta\Delta x = 0$ and λ_5 in both cases is a constant different from zero. A typical problem is one of maximizing the range for a given payload and fuel consumption, whereby $M(\sigma_1)$ and $M(\sigma_2)$ are known and

$$\delta M(\sigma_1) = 0 \qquad \delta M(\sigma_2) = 0$$

with a prescribed initial velocity $V(\sigma_1) = V_1$ and altitude $z(\sigma_1) = z_1$, from which

$$\delta V(\sigma_1) = 0 \qquad \delta z(\sigma_1) = 0$$

and a prescribed final altitude $z(\sigma_2) = z_2$ giving

$$\delta z(\sigma_2) = 0$$

The remaining arbitrary variations, namely $\delta[t(\sigma_2) - t(\sigma_1)]$, $\delta V(\sigma_2)$, $\delta\gamma(\sigma_1)$ and $\delta\gamma(\sigma_2)$, will then impose the conditions

$$\lambda_1(\sigma_2) = 0 \tag{100}$$

$$\lambda_2(\sigma_1) = 0 \tag{101}$$

$$\lambda_2(\sigma_2) = 0 \tag{102}$$

and

$$U = \lambda_1(Mg \sin\gamma + D) + \lambda_2 g \cos\gamma - \lambda_4 V \sin\gamma - \lambda_5 V \cos\gamma + F\left(-\lambda_1 + \frac{1}{c}\lambda_3\right) = 0 \tag{103}$$

The Euler equations (66), (67) and (69), p. 796, are not modified but (64) and (65) have an additional term

$$[G]_V = -[\lambda_1 M]^\circ + \lambda_1 t^\circ D_V + \lambda_2 \gamma^\circ - \lambda_4 t^\circ \sin\gamma - \lambda_5 t^\circ \cos\gamma = 0 \tag{64'}$$

$$[G]_\gamma = -[\lambda_2 V]^\circ + \lambda_1 Mg t^\circ \cos\gamma - \lambda_2 g t^\circ \sin\gamma - \lambda_4 V t^\circ \cos\gamma + \lambda_5 V t^\circ \sin\gamma = 0 \tag{65'}$$

The same types of extremal arcs exist: maximum thrust, coasting and sustaining phase, the latter being still characterized by

$$\lambda_3 = c\lambda_1 \tag{72}$$

Differentiation of this relation followed by substitution of the derivatives produces relation (74), p. 797, with an additional term

$$-\lambda_1(\omega D + V D_V) + \lambda_2 g \cos\gamma + \lambda_4 V \sin\gamma + \lambda_5 V \cos\gamma = 0 \tag{74'}$$

This may also be differentiated a second time to produce

$$\lambda_1\left[(A - B)\sin\gamma - C\frac{F - D}{Mg} + Mg\cos^2\gamma\right] - 2\lambda_4 V \cos^2\gamma + 2\lambda_5 V \sin\gamma\cos\gamma = 0 \tag{75'}$$

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where A, B and C have the same significance as before. However, eqns. (103) (simplified by 72), (74') and (75') contain the constant λ_5 and no longer yield a compatibility condition As a matter of fact, when the multipliers are eliminated, the additional relation between the original variables is a differential equation and not a simple algebraic one. A better procedure is then to avoid the elimination of multipliers but to integrate numerically with the aid of the algebraic relations obtained. For a drag law of type eqn. (77), p. 798, eqn. (67), p. 796, can be rewritten

$$\lambda_4^\circ + \frac{g}{V_a^2} t^\circ \lambda_1 D = 0$$

and from eqns. (58), p. 795, (66), p. 796, and (72), p. 797,

$$Mc\lambda_1^\circ = \lambda_1(F - D)t^\circ$$

The elimination of $\lambda_1 D$ between the two yields

$$\lambda_4^\circ - \frac{gc}{V_a^2}[\lambda_1 M]^\circ = 0$$

In the case of an exponential atmosphere, where V_a is constant, this relation is integrable:

$$\lambda_4 - \frac{gc}{V_a^2}\lambda_1 M = -k\lambda_5 \tag{104}$$

where k is a new unknown constant. The existence of this additional integral was discovered by Bryson and Ross[8]. When eqn. (104) is added to the set (103), (74') and (75') a compatibility condition may be written and an explicit form of the thrust program follows, which depends on the unknown constant k

$$\frac{F - D}{Mg} = \frac{1}{C}\left[A \sin\gamma + \frac{Mg\cos\gamma\left(1 - 2\dfrac{cV}{V_a^2}\right) + kB}{\cos\gamma - k\sin\gamma}\right] \tag{105}$$

In calculating the second member we may now use the expressions

(79), p. 798. Bryson and Ross have given an equivalent form of this result, which follows from the substitution

$$\frac{F-D}{Mg} = \frac{1}{g}\left(\frac{\mathrm{d}V}{\mathrm{d}t} + \sin\gamma\right) = \frac{1}{g}\left(V\cos\gamma\frac{\mathrm{d}V}{\mathrm{d}x} + \sin\gamma\right)$$

i.e. they use it to integrate the velocity profile along the sustaining phase with the horizontal distance as independent variable. The synthesis of the minimal path is very similar to the one discussed in section 12.5.5 (p. 799) except for the additional unknown represented by k.

We limit ourselves to the combination of a boost phase followed by a sustaining phase and then coasting. The initial condition $\lambda_2(\sigma_1) = 0$ again implies $\lambda_2 = 0$ at the end of the boost phase, and the compatibility of eqns. (103), (72), (74′) and (104) at this point requires the relation eqn. (85), p. 803. (Another possibility $k = \cot\gamma$ is ruled out by eqn. (105) as giving an infinite thrust to begin the sustaining phase.) If the velocity jump is taken as initial unknown, the initial angle γ is determined by eqn. (85). The sustaining phase is then integrated with a second unknown k until the fuel is burned out. The coasting period may be ended when $\lambda_2(\sigma_2) = 0$. The two unknowns are then determined by successive corrections to obtain the prescribed final altitude and $\lambda_1(\sigma_2) = 0$. If only k is corrected to produce the end condition $\lambda_1(\sigma_2) = 0$ the solution obtained is minimal with respect to the altitude obtained from the integration, if such an altitude had been prescribed *a priori*.

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