Reprinted from the Proceedings
of the XV[th] International Astronautical Congress
Warszawa 1964, vol. I

OPTIMIZATION OF MULTIPLE IMPULSE ORBITAL TRANSFERS BY THE MAXIMUM PRINCIPLE

by

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Abstract

The transfer problem in a central gravitational field with minimum propellant expenditure is analyzed by Pontrjagin's maximum principle. Only the energy and angular momentum of the terminal orbits are given and no restrictions are imposed on the transfer duration nor on the transfer polar angle.

Under those conditions it is shown that there are no intermediate-thrust arcs.

When the available thrust is unbounded and the transfer achieved by means of impulses and coasting arcs a complete analytical solution can be obtained and the coasting arcs are necessarily of the Hohmann type.

All the multiple impulse transfers satisfying the maximum principle can be obtained by use of an impulse diagram. However further research is needed to eliminate transfers of a stationary and not minimal character.

1. Equations of Motion

The planar motion of a rocket of instantaneous mass M in a central gravitational field is described by the following differential equations

$$M\left(\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right)=-g_0M\frac{R^2}{r^2}+P\sin\psi$$

$$M\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=P\cos\psi \qquad (1)$$

$$P=V_e\left(-\frac{dM}{dt}\right)$$

V_e is the effective ejection velocity, g_0 the acceleration of the gravity field at the distance R of the center. The other notations are illustrated on Fig. 1.

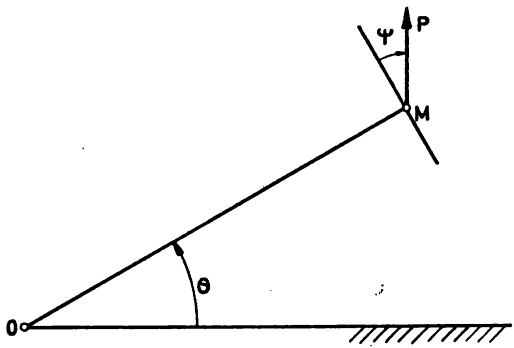

Fig. 1

2. Reduced Variables and Parameters

With R as the natural unit of length and g_0 as unit of acceleration, the natural unit of velocity is $\sqrt{g_0 R}$ (the orbital velocity at distance R) and the natural unit of time $\sqrt{R/g_0}$.

The following reduced variables are used

$$h = R/r; \qquad \tau = t\sqrt{g_0/R} \tag{2.1}$$

and, dots denoting differentiation with respect to the reduced time,

$$u = -\dot{h}/h; \qquad v = \dot{\theta}. \tag{2.2}$$

Finally, M_0 denoting any convenient reference value of the mass,

$$\mu = M_0/M. \tag{2.3}$$

There are two reduced parameters in the problem

$$c = V_e/\sqrt{Rg_0} \quad \text{the reduced ejection velocity}$$

$$a = \frac{P}{g_0 M_0} \quad \text{the acceleration factor.}$$

3. Canonical Equations of Motion and Adjoint System

In terms of the reduced variables and parameters the equations of motion (1) are rewritten in the form

$$\dot{h} = -uh \tag{3.1}$$

$$\dot{\theta} = v \tag{3.2}$$

$$\dot{u} = v^2 - u^2 - h^3 + a\xi h\mu \sin\psi \tag{3.3}$$

$$\dot{v} = -2uv + a\xi h\mu \cos\psi \tag{3.4}$$

$$\dot{\mu} = \frac{1}{c} a\mu^2 \xi \tag{3.5}$$

The control variables are: ψ which orients the thrust with respect to the local horizon, ξ which controls the amplitude of the thrust. If the thrust of the engine can be regulated, we may take ξ to represent any value between zero and unity. If we are only able to cutoff or relight, ξ can only take the values zero or unity. If the optimal trajectory requires an intermediate value of ξ along some subarc, it can be achieved in the second case by chattering, that is by a fast sequence of cutoffs and relights. We need therefore consider only the first case, the only difference in the second being that the optimum implying chattering would be, mathematically speaking, inaccessible.

From the right-hand sides of Eqs. (3.1–5) we build Pontrjagin's Hamiltonian

$$H = -uh\lambda_1 + v\lambda_2 + (v^2 - u^2 - h^3)\lambda_3 - 2uv\lambda_4 + a\xi\mu\left(\lambda_3 h \sin\psi + \lambda_4 h \cos\psi + \frac{1}{c}\mu\lambda_5\right) \tag{3.6}$$

and derive the adjoint system

$$\dot{\lambda}_1 = u\lambda_1 + 3h^2\lambda_3 - a\xi\mu(\lambda_3 \sin\psi + \lambda_4 \cos\psi) \tag{3.7}$$

$$\dot{\lambda}_2 = 0 \tag{3.8}$$

$$\dot{\lambda}_3 = h\lambda_1 + 2u\lambda_3 + 2v\lambda_4 \tag{3.9}$$

$$\dot{\lambda}_4 = -\lambda_2 - 2v\lambda_3 + 2u\lambda_4 \tag{3.10}$$

$$\dot{\lambda}_5 = -a\xi\left(\lambda_3 h \sin\psi + \lambda_4 h \cos\psi + \frac{2}{c}\mu\lambda_5\right) \tag{3.11}$$

Moreover, since the system is autonomous,

$$\dot{H} = \frac{\partial H}{\partial t} = 0 \quad \text{and} \quad H = \text{const.} \tag{3.12}$$

4. Boundary and Transversality Conditions

In a parametric description of a trajectory:

$$h = h(x); \quad \theta = \theta(x); \quad u = u(x); \quad v = v(x); \quad \mu = \mu(x)$$

the describing parameter x can be taken to vary between the fixed limits

$x = a$ and $x = b > a$. The geometry of the orbit of departure is described by its total energy $q(a)$ and angular momentum $p(a)$

$$q(a) = \frac{u^2(a)+v^2(a)}{2h^2(a)} - h(a) \tag{4.1}$$

$$p(a) = \frac{v(a)}{h^2(a)}. \tag{4.2}$$

The actual point of departure on this orbit is left free to be optimized. Similarly we only specify the energy and momentum of the terminal orbit

$$q(b) = \frac{u^2(b)+v^2(b)}{2h^2(b)} - h(b) \tag{4.3}$$

$$p(b) = \frac{v(b)}{h^2(b)} \tag{4.4}$$

leaving its angular orientation with respect to the departure orbit free together with the actual point of arrival on it.

Our problem consists in programming the controls so as to transfer between the two orbits with minimum fuel expenditure, that is, if the initial mass is specified

$$\mu(a) = 1 \quad \text{say} \quad (M_0 = \text{initial mass}) \tag{4.5}$$

we wish $\mu(b)$ to be minimum.

Adding to $\mu(b)$ the constraints (4.1 to 5) with constant multipliers α_i we set up the function

$$J = \mu(b)+\alpha_1[\mu(a)-1]+\alpha_2[u^2(a)+v^2(a)-2h^3(a)-2h^2(a)q(a)]+\alpha_3[v(a) -h^2(a)p(a)]+\alpha_4[u^2(b)+v^2(b)-2h^3(b)-2h^2(b)q(b)]+\alpha_5[v(b)-h^2(b)p(b)] \tag{4.6}$$

and derive the transversality conditions

$$\lambda_1(a) = \frac{\partial J}{\partial h(a)} = -6\alpha_2 h^2(a) - 4\alpha_2 h(a)q(a) - 2\alpha_3 h(a)p(a)$$

$$\lambda_2(a) = \frac{\partial J}{\partial \theta(a)} = 0$$

$$\lambda_3(a) = \frac{\partial J}{\partial u(a)} = 2\alpha_2 u(a)$$

$$\lambda_4(a) = \frac{\partial J}{\partial v(a)} = 2\alpha_2 v(a)$$

$$\lambda_5(a) = \frac{\partial J}{\partial \mu(a)} = \alpha_1$$

$$H(a) = -\frac{\partial J}{\partial \tau(a)} = 0$$

$$\lambda_1(b) = -\frac{\partial J}{\partial h(b)} = 6\alpha_4 h^2(b) + 4\alpha_4 h(b) q(b) + 2\alpha_5 h(b) p(b)$$

$$\lambda_2(b) = -\frac{\partial J}{\partial \theta(b)} = 0$$

$$\lambda_3(b) = -\frac{\partial J}{\partial u(b)} = -2\alpha_4 u(b)$$

$$\lambda_4(b) = -\frac{\partial J}{\partial v(b)} = -2\alpha_4 v(b) - \alpha_5$$

$$\lambda_5(b) = -\frac{\partial J}{\partial \mu(b)} = -1$$

$$H(b) = \frac{\partial J}{\partial \tau(b)} = 0.$$

The end values obtained for λ_2, together with Eq. (3.8) and the requirement of continuity of the multipliers, indicate that

$$\lambda_2 = 0 \text{ throughout the trajectory.} \tag{4.7}$$

This is a consequence of the absence of any specification on the amplitude of the transfer angle. It also indicates that Eq. (3.2) needs not be considered to solve the problem.

Similarly, the end values obtained for the Hamiltonian and Eq. (3.12) indicate that, in view of the required continuity of the Hamiltonian,

$$H = 0 \text{ throughout the trajectory.} \tag{4.8}$$

This consequence of the absence of specification concerning the duration of the transfer will be used in the form

$$K = a\xi\mu L \tag{4.9}$$

where

$$K = uh\lambda_1 + (u^2 - v^2 + h^3)\lambda_3 + 2uv\lambda_4 \tag{4.10}$$

$$L = h(\lambda_3 \sin\psi + \lambda_4 \cos\psi) + \frac{1}{c}\mu\lambda_5. \tag{4.11}$$

Elimination of α_2 and α_3 between the transversality conditions produces in view of Eqs. (4.1 and 4.2)

$$K(a) = 0. \tag{4.12}$$

Similarly, the elimination of α_4 and α_5

$$K(b) = 0. \tag{4.13}$$

Then, except for the end value

$$\lambda_5(b) = -1 \tag{4.14}$$

the other consequences of the transversality conditions merely define the α_i multipliers and are not of direct interest.

Circular orbits of departure or arrival are limiting cases for which

$$q = -\tfrac{1}{2}h; \quad p = h^{-\frac{1}{2}}; \quad u = 0; \quad v = h^{\frac{3}{2}} \tag{4.15}$$

so that the boundary values of h, u and v are all three specified.

5. Optimal Controls

In the Hamiltonian

$$H = -K + a\xi\mu L$$

only L is a function of the angle ψ. L reaches its maximum

$$L_{max} = N = h\sqrt{\lambda_3^2+\lambda_4^2} + \frac{1}{c}\mu\lambda_5 \tag{5.1}$$

for the choice

$$\sin\psi = \frac{\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}; \qquad \cos\psi = \frac{\lambda_4}{\sqrt{\lambda_3^2+\lambda_4^2}}. \tag{5.2}$$

The Hamiltonian itself will then reach its maximum for

$$\begin{aligned} \xi &= 1 \quad \text{if} \quad N > 0 \\ \xi &= 0 \quad \text{if} \quad N < 0. \end{aligned} \tag{5.3}$$

The indeterminate case where N remains zero on a subarc of the trajectory and the thrust might turn out to be intermediate or undetermined will be dismissed in the next section.

6. Impossibility of an Intermediate or Undetermined Thrust Arc

With the optimal choice (5.2) Eq. (3.11) becomes

$$\dot{\lambda}_5 = -a\xi\left[h\sqrt{\lambda_3^2+\lambda_4^2} + \frac{2}{c}\mu\lambda_5\right].$$

Along a subarc for which N remains zero this reduces to

$$\dot{\lambda}_5 = -\frac{1}{c}a\xi\mu\lambda_5.$$

This equation, together with Eq. (3.5), shows that

$$\lambda_5\mu = -ck \text{ a constant.} \tag{6.1}$$

Hence the equation $N = 0$ becomes equivalent to

$$h\sqrt{\lambda_3^2+\lambda_4^2} = k. \tag{6.2}$$

This result is equivalent to the "constancy of the primer" discovered by LAWDEN.[1] It is important to observe that the constant k is different from zero. For otherwise both λ_3 and λ_4 must remain zero along the subarc, Eq. (3.9) shows that the same must be true for λ_1 and Eq. (6.1) that it must be true for λ_5. If all the multipliers are zero it follows from the homogeneity of the adjoint system and the continuity requirement of the multipliers that they will remain zero along the whole trajectory. This would make it impossible to meet the condition (4.14). Hence λ_3 and λ_4 cannot vanish together and the optimum thrust orientation defined by Eqs. (5.2) is never indeterminate.

Since Eq. (6.2) holds along the subarc $N = 0$ it can be differentiated. The differential coefficients can be substituted from Eqs. (3.1), (3.9) and (3.10), whereby, in view of Eq. (4.7)

$$uk^2+h^3\lambda_1\lambda_3 = 0. \tag{6.3}$$

This is again differentiated, using the same equations as before and in addition (3.3) and (5.2) to produce

$$k^2(v^2-u^2-h^3)+3h^5\lambda_3^2+h^4\lambda_1^2+2vh^3\lambda_1\lambda_4 = 0. \tag{6.4}$$

Finally from Eq. (4.9) we can conclude that

$$K = a\xi\mu L_{\max} = a\xi\mu N = 0 \quad \text{or} \quad uh\lambda_1+(u^2-v^2+h^3)\lambda_3+2uv\lambda_4 = 0. \tag{6.5}$$

When this last equation is multiplied by $\lambda_1 h^3$ and the terms in $\lambda_1\lambda_3$ and $\lambda_1\lambda_4$ eliminated by use of Eqs. (6.3) and (6.4) it turns out that we must satisfy the condition

$$3uh^5\lambda_3^2 = 0. \tag{6.6}$$

There are two possibilities to satisfy (6.6) and (6.3) simultaneously

(a) by taking $u = 0$ and $\lambda_1 = 0$. Then, from Eqs. (6.4) and (6.5)

$$k^2(v^2-h^3)+3h^5\lambda_3^2 = 0; \quad \lambda_3(v^2-h^3) = 0$$

enforcing $v^2 = h^3$ and $\lambda_3 = 0$. From this last result and (3.9) also follows $\lambda_4 = 0$, in conflict with $k \neq 0$.

(b) by taking $u = 0$ and $\lambda_3 = 0$. Then $\sin\psi = 0$ and (3.3) requires tath $v^2 = h^3$. Equations (3.9) and (6.4) are satisfied by taking $h\lambda_1+2v\lambda_4 = 0$ and (6.5) is also satisfied. This case however turns out to be a simple circular orbital case, for, since v remains constant with h and $\cos\psi = \pm 1$, Eq. (3.4) requires $\xi = 0$.

The spiral arcs discovered by LAWDEN,[1] which are of the intermediate thrust type, do only occur when either the total transfer angle or the transfer duration are predetermined. In particular it can be shown[4] that Lawden's constant A is the isoperimetric constant associated with the polar angle and equivalent to our constant.

7. Coasting Arcs

In the absence of thrust the vehicle coasts along an arc of Keplerian orbit, with the well known integrals

$$\frac{u^2+v^2-2h^3}{2h^2} = q \quad \text{const. of total energy} \tag{7.1}$$

$$v/h^2 = p \quad \text{const. of angular momentum} \tag{7.2}$$

$$\mu = \text{const.} \tag{7.3}$$

The adjoint system posseses the integrals (A, B, C constants)

$$\lambda_1 = \frac{2A}{h} - 3B \tag{7.4}$$

$$\lambda_3 = Bu/h^2 \tag{7.5}$$

$$\lambda_4 = -\frac{A+Bq}{ph^2} + Bp \tag{7.6}$$

$$\lambda_5 = C. \tag{7.7}$$

These integrals satisfy the condition

$$K = 0 \quad \text{along a coasting arc} \tag{7.8}$$

to which (4.9) reduces for $\xi = 0$. There is a third integral of the third order differential system in $(\lambda_1, \lambda_3, \lambda_4)$. For non circular orbits it gives to K and consequently to the Hamiltonian a non-zero constant value and would only come into play for transfers of specified duration or time-optimal transfers. As can readily be verified the constants (A, B) are uniquely determined from a local set of values of $(\lambda_1, \lambda_3, \lambda_4)$ satisfying $K = 0$ except when u and $\dot{u} = v^2 - u^2 - h^3$ vanish simultaneously. This is the circular orbital case which warrants a special study. Putting $u = 0$ and noting that h and v are constants, the complete integration of the adjoint differential system in $(\lambda_1, \lambda_3, \lambda_4)$ yields

$$\begin{aligned} \lambda_1 &= \frac{3v}{h} P \sin\theta - \frac{2v}{h} Q \\ \lambda_3 &= P \cos\theta \\ \lambda_4 &= -2P \sin\theta + Q \end{aligned} \tag{7.9}$$

where $\theta = vt - \theta_0$ and θ_0, P and Q are the constants of integration. The condition $K = 0$ is always satisfied, which leads to the observation that a circular arc of this nature will never be part of an optimal transfer of specified duration.

Owing to the special nature of the behaviour of the adjoint system along a circular orbit, the discussions will be somewhat lengthened if one wishes to understand thoroughly and control the limiting forms assumed by certain results established for elliptical orbits by means of Eqs. (7.4 to 6).

8. Impulse Extremals

The other type of extremal corresponds to $\xi = 1$, that is maximum thrust. Only the limiting case of unbounded acceleration parameter a will be considered. It can be analyzed without recourse to numerical integration and the results can provide insight into the difficult boundary value problem associated with limited thrust. The orbital transfer is then accomplished by a succession of impulses and coasting arcs, a problem whose optimization has been widely investigated by more conventional techniques.

The approach through the maximum principle has the advantage of not limiting in advance the number of impulses. Such an approach has been followed by BREAKWELL using an extremely ingenious set of variables. The present version, in addition to the elimination of the intermediate thrust case, has permitted the analytical discussion to proceed very far towards the synthesis of the optimal trajectories. In fact very simple iterative tests can now be applied to establish the optimal trajectory for any specific numerical case.

Impulses can be treated as regular extremals by the simple device of the parametric description. If primed variables denote differential coefficients with respect to the description parameter x, the basic equations (3.1 to 5) can be rewritten as

$$\begin{aligned} h' &= -uh\tau' \\ \theta' &= v\tau' \\ u' &= (v^2-u^2-h^3)\tau'+a\tau'\xi\mu h z_3 \\ v' &= -2uv\tau'+a\tau'\xi\mu h z_4 \\ \mu' &= a\tau'\xi\frac{1}{c}\mu^2. \end{aligned} \tag{8.1}$$

The direction cosines of the thrust (z_3, z_4) have been given their optimal values (5.2)

$$z_3 = \frac{\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}; \quad z_4 = \frac{\lambda_4}{\sqrt{\lambda_4^2+\lambda_3^2}}. \tag{8.2}$$

Putting $\tau' = 1/a$, $\xi = 1$, and letting a go to infinity, the evolution of the system is governed by

$$\tau' = 0; \quad h' = 0; \quad \theta' = 0 \tag{8.3}$$

$$u' = \mu h z_3; \quad v' = \mu h z_4 \tag{8.4}$$

$$\mu' = \frac{1}{c}\mu^2. \tag{8.5}$$

Along this "impulse extremal" both time and position of the vehicle stand still, while velocity and mass are modified. A similar treatment of the adjoint system yields

$$\lambda_1' = -\mu\sqrt{\lambda_3^2+\lambda_4^2} \tag{8.6}$$

$$\lambda_3' = 0; \quad \lambda_4' = 0 \tag{8.7}$$

$$\lambda_5' = -\left[R\sqrt{\lambda_3^2+\lambda_4^2}+\frac{2}{c}\mu\lambda_5\right]. \tag{8.8}$$

From (8.7) follows that z_3 and z_4 remain constant, they are the direction cosines of the impulse. The magnitude of the impulse is conveniently measured by the total increase in characteristic velocity

$$\varphi = c \ln \mu \tag{8.9}$$

Since, from (8.9) and (8.5)

$$\varphi' = c\frac{\mu'}{\mu} = \mu \tag{8.10}$$

integration of Eqs. (8.4) yields for the increases in velocity components

$$\Delta u = hz_3\Delta\varphi; \quad \Delta v = hz_4\Delta\varphi. \tag{8.11}$$

Also integration of Eq. (8.6) gives

$$\Delta\lambda_1 = -\Delta\varphi\sqrt{\lambda_3^2+\lambda_4^2}. \tag{8.12}$$

Equation (8.8) will be used to find the change occurring in the quantity N, defined by (5.1)

$$N' = \frac{1}{c}(\mu\lambda_5)' = -\frac{1}{c}\mu N$$

or again, in view of (8.10)

$$\frac{dN}{d\varphi} = -\frac{1}{c}N. \tag{8.13}$$

Thus, if N is zero at some point along the impulse extremal, it will remain zero. This is always the case for, according to equation (4.9)

$$K = a\xi\mu L_{max} = a\xi\mu N$$

and, letting a go to infinity with $\xi = 1$, N must vanish to keep K finite. The result

$$N = 0 \quad \text{along an impulse extremal} \tag{8.14}$$

will be later exploited in the form

$$h\sqrt{\lambda_3^2+\lambda_4^2} = -\frac{1}{c}\mu\lambda_5. \tag{8.15}$$

9. Properties Deduced from the Weierstrass–Erdmann Corner Rules

As a result of the elimination of the intermediate thrust possibility, any optimal trajectory will consist of combinations of impulses and coasting arcs.

The synthesis of the trajectory is dominated by the requirement that phase variables and multipliers remain continuous. As a consequence the same requirements apply to the quantities K and N.

Because K vanishes along a coasting arc and also at both ends of the trajectory, K must vanish at both ends of any impulse extremal. This requirement suggests that we evaluate K' along an impulse extremal. Using Eqs. (8.3 to 7) and (8.10)

$$\frac{dK}{d\varphi} = \mu^{-1}K' = n \tag{9.1}$$

where

$$n = uh\sqrt{\lambda_3^2+\lambda_4^2}+h^2\frac{\lambda_1\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}. \tag{9.2}$$

With the help of the same equations it turns out that

$$n' = 0. \tag{9.3}$$

Thus, since n is a constant, K cannot vanish at both ends unless

$$n = 0 \quad \text{along an impulse extremal.} \tag{9.4}$$

Consequently K itself vanishes along this extremal and finally

$$K = 0 \quad \text{along the whole trajectory.} \tag{9.5}$$

Referring back to Eq. (8.15) it appears that, should λ_3 and λ_4 vanish simultaneously, so would λ_5, *and conversely*. It would follow from the continuity of λ_5 and the fact that it remains constant along a coasting arc that λ_5 would stay zero up to the beginning of the next impulse extremal, where (8.15) again applies. Thus λ_3 and λ_4 would again vanish on the next impulse. This situation would repeat itself up to the end of the trajectory, making it impossible to satisfy the end value (4.14). Ruling this case out, there follows

(a) the direction cosines of an impulse are never undetermined,

(b) the quantity $-\frac{1}{c}\mu\lambda_5$ is a positive constant along an impulse,

(c) λ_5 itself is negative and decreasing.

Moreover, since both λ_5 and μ are constant on a coasting arc it follows from the continuity requirements that $-\frac{1}{c}\mu\lambda_5$ is a trajectory constant. There is no loss in generality in scaling the multipliers so that

$$-\frac{1}{c}\mu\lambda_5 = 1 \quad \text{along the whole trajectory.} \tag{9.6}$$

This reduces the quantity N to

$$\left.\begin{array}{c} N = h\sqrt{\lambda_3^2+\lambda_4^2}-1 \\ \text{continuous and negative along a coasting arc} \\ \text{zero along an impulse extremal} \end{array}\right\} \tag{9.7}$$

To this property we add (9.4) which, by virtue of (9.7) becomes

$$u+h^3\lambda_1\lambda_3 = 0 \quad \text{whenever} \quad N = 0. \tag{9.8}$$

10. Conditions for a Transfer Orbit

Because of the special behaviour of the multipliers along a circular orbit, the discussion of this case will be dissociated and non-circular orbits investigated first.

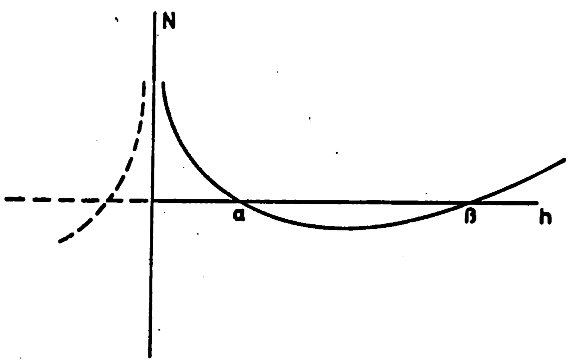

Fig. 2

The values of the multipliers can be taken from Eqs. (7.4 to 7). Then, in view of Eqs. (7.1 and 2)

$$N = \sqrt{\left(\frac{A+Bq}{p}\right)^2 \frac{1}{h^2} - 2AB + 2B^2h} - 1 \tag{10.1}$$

while condition (9.8) becomes

$$u(1+2AB-3B^2h) = 0 \quad \text{whenever} \quad N = 0. \tag{10.2}$$

10.1. Provided $(A+Bq)$ does not vanish, the function $N(h)$ given by (10.1) has the shape illustrated on Fig. 2.

There can exist a positive interval $\alpha < h < \beta$ along which the coasting condition $N < 0$ is satisfied. In $h = \alpha$, where $N = 0$

$$\left(\frac{A+Bq}{p}\right)^2 \frac{1}{\alpha^2} - 2AB + 2B^2\alpha = 1. \tag{10.3}$$

We try first to satisfy (10.2) in $h = \alpha$ by taking

$$1 + 2AB - 3B^2\alpha = 0. \tag{10.4}$$

Combining this with (10.3)

$$\left(\frac{A+Bq}{p}\right)^2 = B^2\alpha^3. \tag{10.5}$$

This turns out to be the condition for $N = 0$ to have a double root ($\alpha = \beta$). Indeed, substitution of the left hand side of (10.5) and of $2AB$ from (10.4) into (10.1) gives

$$N = \sqrt{\frac{B^2}{h^2}(h-\alpha)^2(2h+\alpha)+1} - 1$$

and the N curve is tangent to the axis in $h = \alpha$. This case can therefore be dismissed as that of a circular transfer orbit along which N would remain zero.

It can be concluded that for non circular orbits (10.2) must be satisfied *at both ends* of the transfer orbit by taking

$$u = 0 \quad \text{for} \quad h = \alpha \quad \text{and} \quad h = \beta. \tag{10.6}$$

(Should we go from $h = \alpha$ (or β) and return to α (or β) without reaching β (or α) we would have accomplished a complete circuit of an elliptical orbit. This is not a real transfer since, except for an increase in time and polar angle, all variables and multipliers return to their departure values).

From the result (10.6) follows that a transfer takes place from peri-apsis to apo-apsis or conversely along an elliptical orbit. In other words the transfer orbits are necessarily of the Hohmann type. The values of h at the apsides are roots of equation

$$p^2h^2 - 2h - 2q = 0 \tag{10.7}$$

deduced from (7.1 and 2) by elimination of v and insertion of $u = 0$. Hence

$$\alpha+\beta = 2/p^2; \quad \alpha\beta = -\frac{2q}{p^2} \tag{10.8}$$

$$p = \sqrt{\frac{2}{\alpha+\beta}}; \quad q = -\frac{\alpha\beta}{\alpha+\beta}. \tag{10.9}$$

From (7.5) and (10.6) follows

$$\lambda_3 = 0 \quad \text{or} \quad \sin\psi = 0 \quad \text{in} \quad h = \alpha \quad \text{and} \quad h = \beta. \tag{10.10}$$

The impulses adjacent to the transfer orbit are tangential. It also follows that the condition $N = 0$ at the apsides reduces to

$$|\alpha\lambda_4(\alpha)| = 1; \quad |\beta\lambda_4(\beta)| = 1. \tag{10.11}$$

It will be convenient to introduce the notation

$$\Lambda = h\lambda_4 \tag{10.12}$$

and rewrite (10.11) as

$$\Lambda(\alpha) = \pm 1; \quad \Lambda(\beta) = \pm 1 \tag{10.13}$$

$\Lambda = +1$ ($\cos\psi = +1$) corresponds to a positive impulse signal or acceleration,

$\Lambda = -1$ ($\cos\psi = -1$) to a negative impulse signal or deceleration.

Two types of Hohmann transfers will be distinguished

(a) Transitive transfers, which preserve the sign of the impulse. They are characterized by

$$\Lambda(\alpha) = \Lambda(\beta). \tag{10.14}$$

Calculating B and $A+Bq$ from equations

$$\lambda_4(\alpha) = \frac{\Lambda(\alpha)}{\alpha} = -\frac{A+Bq}{p\alpha^2} + \beta p$$

$$\lambda_4(\beta) = \frac{\Lambda(\beta)}{\beta} = -\frac{A+Bq}{p\beta^2} + \beta p$$

and replacing into Eq. (7.4), there comes in view of (10.9) and (10.14)

$$\lambda_1(\alpha) = -\frac{\Lambda(\alpha)}{\sqrt{2(\alpha+\beta)}} \frac{3\alpha+5\beta}{\alpha+\beta} \tag{10.15}$$

$$\lambda_1(\beta) = -\frac{\Lambda(\beta)}{\sqrt{2(\alpha+\beta)}} \frac{3\beta+5\alpha}{\alpha+\beta}. \tag{10.16}$$

Hence, in a transitive transfer the sign of λ_1 is also preserved; it is always opposite to the sign of the impulse.

(b) Reflexive transfers, changing the sign of the impulse

$$\Lambda(\beta) = -\Lambda(\alpha). \tag{10.17}$$

By a similar calculation

$$\lambda_1(\alpha) = \frac{-3\Lambda(\alpha)}{\sqrt{2(\alpha+\beta)}} \tag{10.18}$$

$$\lambda_1(\beta) = \frac{-3\Lambda(\beta)}{\sqrt{2(\alpha+\beta)}}. \tag{10.19}$$

In a reflexive transfer the value of λ_1 is simply changed in sign, opposite to that of the adjacent impulse.

10.2 The case $A+Bq = 0$ will be taken up later as it proves to be important only for orbits of departure or arrival.

10.3 Circular transfer orbits.

The multipliers are taken from Eqs. (7.9) with the result

$$N = h\sqrt{P^2\cos^2\theta+4P^2\sin^2\theta-4PQ\sin\theta+Q^2} - 1 \tag{10.20}$$

P can be assumed to be a modulus, that is to be a positive quantity by proper determination of the phase angle θ_0. To find the extremal values of the quantity under the radical sign, the derivative is set equal to zero

$$\cos\theta(6P^2\sin\theta-4PQ) = 0.$$

The solution $\sin\theta = (2Q)/(3P)$, implying $|Q| < (3P)/2$, gives a value $P^2-\frac{1}{3}Q^2$ always smaller than the value $(2P+|Q|)^2$ reached for $\sin\theta = 1$ if Q is negative or $\sin\theta = -1$ if Q is positive. Consequently $N = 0$ is reached at most in one position with $N < 0$ elsewhere around the circle. This cannot be a transfer case. If however

$$Q = 0, \quad N = Ph\sqrt{\frac{5-3\cos 2\theta}{2}} - 1$$

and $N = 0$ can be reached in $\theta = \pi/2$ and $\theta = 3\pi/2$ with $N < 0$ elsewhere. Moreover, since u is identically zero and λ_3 vanishes with $\cos\theta$ where $N = 0$, condition (9.8) is also satisfied. This is a semi-circle transfer case with tangent adjacent impulses. Because $\sin\theta$ changes sign between the transfer ends, it is of the reflexive type. It is also seen to verify Eqs. (10.18 and 19) for $h = \alpha = \beta$, which proves the validity of these equations in this limiting case.

The corresponding limiting case of a circular transitive transfer is obtained by setting $P = 0$, whereby N and n remain zero around the circle and the length of arc of this transfer is left undetermined. In fact all variables and multipliers remain constant, except time and polar angle. The only physical interest of such a transfer would be to delay the arrival or to rotate at will and without fuel expenditure the axis' of the terminal orbit.

11. The Impulse Invariant

It is already known that h, λ_3 and λ_4 remain constant during an impulse extremal. If furthermore the impulse is tangential at peri-apsis or apo-apsis

$$u = 0; \quad \lambda_3 = 0; \quad \sqrt{\lambda_3^2+\lambda_4^2} = |\lambda_4| = \frac{1}{h}$$

and

$$z_4 = h\lambda_4 = \Lambda = \pm 1 .$$

This reduces (8.12) to

$$\Delta\lambda_1 = \lambda_{1+} - \lambda_{1-} = -\frac{1}{h}\Delta\varphi \tag{11.1}$$

and (8.11) to

$$\Delta v = \Lambda h \Delta\varphi . \tag{11.2}$$

But also, from (7.2)

$$\Delta p = p_+ - p_- = \frac{1}{h^2}\Delta v - \Lambda\frac{1}{h}\Delta\varphi. \tag{11.3}$$

Eliminating $\Delta\varphi$ between (11.1) and (11.3) results in

$$p_+ + \Lambda\lambda_{1+} = p_- + \Lambda\lambda_{1-}. \tag{11.4}$$

The quantity $p+\Lambda\lambda_1$ is an impulse invariant.

12. Recurrence Formulas in a Succession of Transfers

Because the multipliers are exactly known along a transfer orbit and must remain continuous, there are rules to be followed for more than a single transfer.

Let $h = \alpha_n$ be the inverse distance where an impulse is applied between a first transfer from α_{n-1} to α_n and a next one from α_n to α_{n+1}. Recurrence formulas will be established between the ratios α_n/α_{n-1} and α_{n+1}/α_n or, more precisely, between

$$x = \frac{2}{1+\alpha_n/\alpha_{n-1}} = \frac{2r_n}{r_n+r_{n-1}} \tag{12.1}$$

and

$$y = \frac{2}{1+\alpha_{n+1}/\alpha_n} = \frac{2r_{n+1}}{r_n+r_{n+1}}. \tag{12.2}$$

The impulse $\Delta\varphi$ will be represented by a point in an (x, y) diagram. If the impulse is an acceleration

$$\alpha_{n+1} < \alpha_{n-1} \quad \text{or} \quad (\alpha_{n+1}/\alpha_n)\cdot(\alpha_n/\alpha_{n-1}) < 1$$

which is equivalent to $x+y > 2$. Similarly if the impulse is a braking one $\alpha_{n+1} > \alpha_{n-1}$ and $x+y < 2$. Hence the straight line $x+y = 2$ divides the (x, y) diagram in two domains: an upper right acceleration domain (A) and a lower left braking domain (B).

12.1. Succession of two transitive transfers

$$\Lambda(\alpha_{n-1}) = \Lambda(\alpha_{n+1}) = \Lambda(\alpha_n) = \Lambda.$$

From (10.16) where $\beta = \alpha_n$ and $\alpha = \alpha_{n-1}$

$$\Lambda\lambda_{1-} = -\frac{1}{\sqrt{2}}\frac{3\alpha_n+5\alpha_{n-1}}{(\alpha_n+\alpha_{n-1})^{\frac{3}{2}}} \tag{12.3}$$

and from (10.15) with $\beta = \alpha_{n+1}$ and $\alpha = \alpha_n$

$$\Lambda\lambda_{1+} = -\frac{1}{\sqrt{2}}\frac{3\alpha_n+5\alpha_{n+1}}{(\alpha_n+\alpha_{n+1})^{\frac{3}{2}}}. \tag{12.4}$$

Noting that

$$p_- = \sqrt{\frac{2}{\alpha_n+\alpha_{n-1}}}; \quad p_+ = \sqrt{\frac{2}{\alpha_n+\alpha_{n+1}}}.$$

and substituting into the invariance relation (11.4) produces the condition

$$\frac{\alpha_n+3\alpha_{n+1}}{(\alpha_n+\alpha_{n+1})^{\frac{3}{2}}} = \frac{\alpha_n+3\alpha_{n-1}}{(\alpha_n+\alpha_{n-1})^{\frac{3}{2}}} \tag{12.5}$$

that, in terms of the variables x and y, becomes

$$y(3-y)^2 = (2-x)(1+x)^2 \tag{12.6}$$

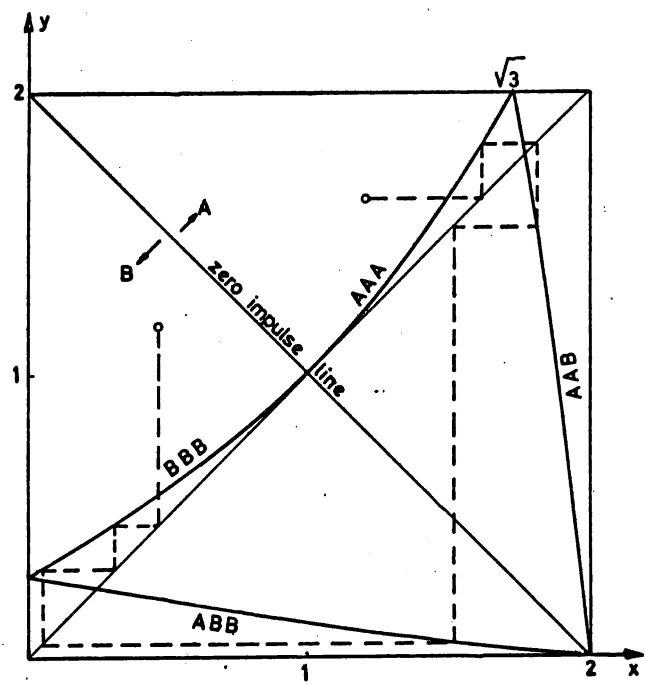

Fig. 3. The recurrent impulse diagram

Given x, this equation has three roots for y. One is trivial $y = 2-x$, corresponding to the zero impulse case. The other two are then roots of the quadratic

$$y^2 - y(4+x) + (1+x)^2 = 0.$$

Finally, since y cannot become larger than 2, the only significant value is

$$y = \tfrac{1}{2}\left\{4+x-\sqrt{3(4-x^2)}\right\} = R(x). \qquad (12.7)$$

This curve of the (x, y) diagram (Fig. 3) lies partly in the (B) domain, partly in the (A) domain. The part in the (B) domain has been labelled *BBB* since the impulse in α_n is a braking one, the previous one in α_{n-1} must also have been a braking one because the transfer was assumed transitive and similarly next one in α_{n+1} will have to be.

Correspondingly the part in the (A) domain is labelled *AAA*. The function $y = R(x)$ is tabulated in Table 1.

Table 1

x	$y = R(x)$
0	0.267949
0.2	0.376631
0.4	0.502944
0.6	0.647729
0.8	0.812549
1.0	1
1.2	1.214359
1.4	1.463068
1.6	1.760769
1.732051	2

The strength of the central impulse can be calculated from (11.3), or equivalently

$$\Delta\varphi = \sqrt{\alpha_n}\left|\sqrt{y}-\sqrt{2-x}\right|. \qquad (12.8)$$

12.2. Transitive transfer followed by a reflexive one

$$\Lambda(\alpha_{n-1}) = \Lambda(\alpha_n) = -\Lambda(\alpha_{n+1}) = \Lambda.$$

While (12.3) remains valid, (12.4) must be replaced by

$$\Lambda\lambda_{1+} = \frac{-3}{\sqrt{2(\alpha_n+\alpha_{n+1})}} \qquad (12.9)$$

and (12.5) becomes

$$\frac{1}{(\alpha_n+\alpha_{n+1})^{\frac{1}{2}}} = \frac{\alpha_n+3\alpha_{n-1}}{(\alpha_n+\alpha_{n-1})^{\frac{3}{2}}} \qquad (12.10)$$

or

$$y = (2-x)(1+x)^2 = S(x). \tag{12.11}$$

This curve lies entirely in the (A) domain and must be labelled AAB. The function is tabulated in Table 2.

Table 2

x	$y = S(x)$
1.732051	2
1.75	1.890625
1.80	1.568000
1.85	1.218373
1.90	0.841000
1.95	0.435125
2	0

Formula (12.8) assumes here the special form

$$\Delta\varphi = \sqrt{\alpha_n}\, x\sqrt{2-x}. \tag{12.12}$$

12.3. Reflexive transfer followed by a transitive one

$$-\Lambda(\alpha_{n-1}) = \Lambda(\alpha_n) = \Lambda(\alpha_{n+1}) = \Lambda.$$

Equation (12.4) holds true but (12.3) must be replaced by
and (12.5) becomes

$$\Lambda\lambda_{1-} = \frac{-3}{\sqrt{2(\alpha_n+\alpha_{n-1})}} \tag{12.13}$$

$$\frac{\alpha_n+3\alpha_{n+1}}{(\alpha_n+\alpha_{n+1})^{\frac{3}{2}}} = \frac{1}{(\alpha_n+\alpha_{n-1})^{\frac{1}{2}}} \tag{12.14}$$

or

$$x = 2-y(3-y)^2 = T^{-1}(y). \tag{12.15}$$

Table 3

y	$x = T^{-1}(y)$
0	2
0.05	1.564875
0.10	1.159000
0.15	0.781625
0.20	0.432000
0.25	0.109375
0.267949	0

This curve, symmetrical of $y = S(x)$ with respect to the zero impulse line, lies entirely in the (B) domain and must be labelled ABB. The function

$T^{-1}(y)$ is tabulated in Table 3. Formula (12.8) assumes the special form

$$\Delta\varphi = \sqrt{\alpha_n}(2-y)\sqrt{y}. \tag{12.16}$$

12.4. Succession of two reflexive transfers

$$-\Lambda(\alpha_{n-1}) = \Lambda(\alpha_n) = -\Lambda(\alpha_{n+1}) = \Lambda .$$

Combination of (12.9) and (12.13) leads to

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{\alpha_{n-1}}{\alpha_n} \quad \text{or} \quad x+y = 2 .$$

Hence $\Delta\varphi = 0$ is the only possibility and this case can be ruled out.

13. The Recurrent Impulse Diagram

The (x, y) diagram of Fig. 3 contains virtually all the multiple-impulse transfer cases satisfying the necessary (but not sufficient) maximum principle condition for minimum fuel expenditure.

The use of this diagram is best explained on an example.

A knowledge of the apses of the departure orbit and a decision as to the one where the first impulse will be applied determine the x coordinate of the first point.

The y coordinate determines the sign and intensity of the first impulse. Suppose the point is chosen in the (A) domain: the first impulse is an acceleration.

The y coordinate is brought back as a x coordinate by reflexion in the "mirror line" $y = x$. This operation produces the transfer; it prepares the application of a second impulse at the other apsis of the orbit resulting from the first impulse. The choice of the second y coordinate determines the second impulse and, if the operation is ended, the characteristics of the final orbit.

This describes the simple Hohmann case.

If the game is to continue by at least a second transfer, the second y coordinate must be such that the second point falls on one of the curves AAA or AAB or ABB (the first impulse was an acceleration); suppose that it can be taken on the AAA curve. Then, after a second reflexion on the mirror, the third point must be taken in the (A) domain and we have the representation of a three acceleration impulse transfer.

Again if the game is to continue the third point will have to fall on either the AAA curve again, or the AAB curve. The general rule is of course that if the previous impulse lies on a curve, the next usable curve must have its two first letters coincident with the two last ones of the previous curve. Suppose we are now on the AAB curve. After reflexion the game can be ended by the use of a braking impulse or prolonged by use of the *ABB* curve.

The next prolongation makes necessarily use of the *BBB* curve on which a series of points can be taken in succession. A result of these rules, based on the considerations of Section 12, is that after switching from acceleration to braking is impossible to revert to acceleration.

14. Classes of Terminal Orbits

A terminal (departure or arrival) orbit will be said to be of class 1 if N remains negative on it except of course at the point of departure or arrival itself where it must vanish together with n.

Such orbits mark certainly complete stops for the trajectory. Orbits of class 2 are those on a portion of which at least appears a "firing signal" $N > 0$.

The frontier between the two classes consists of course of the transfer orbits where $N < 0$ but vanishes at both apses. The distinction is presumed to be significant for sufficiency conditions and will be shown to be associated with allowable domains in the (x, y) diagram.

14.1. Non-circular orbits with $A+Bq \neq 0$

This is the general case. It was established that if N and n vanish simultaneously for some h, a range of negative N values exists only if u also vanishes there. The same is true then of λ_3 so that departure or arrival is tangential at an apsis $h = \omega$ of the terminal orbit. For N to be negative elsewhere it is clearly necessary and sufficient that it should be negative at the other apsis $h = \omega^*$ where u and λ_3 are again zero. Thus one should have

$$N = |\omega^* \lambda_4(\omega^*)| - 1 = |\Lambda(\omega^*)| - 1 < 0. \tag{14.1}$$

In Fig. 2 this situation corresponds to either $\omega = \alpha$ or $\omega = \beta$ and $\alpha < \omega^* < \beta$.

Taking the multipliers from (7.4 to 6) and eliminating A and B between the expressions of $\lambda_4(\omega^*) = \dfrac{\Lambda(\omega^*)}{\omega^*}$

$$\lambda_4(\omega) = \frac{\Lambda(\omega)}{\omega} \quad \text{and} \quad \lambda_1(\omega)$$

there comes

$$\frac{\Lambda(\omega^*)}{\Lambda(\omega)} = -\left(4+3\frac{\omega}{\omega^*}\right) - \left(1+\frac{\omega}{\omega^*}\right)x \tag{14.2}$$

with

$$x = \frac{2\lambda_1(\omega)}{p\Lambda(\omega)}.$$

Noting that $\Lambda^2(\omega) = 1$, condition (14.1) is equivalent to

$$\left(\frac{(\Lambda(\omega^*)}{\Lambda(\omega)}\right)^2 < 1$$

and, using Eq. (14.2) takes the form

$$\left(1+\frac{\omega}{\omega^*}\right)x^2+2\left(4+3\frac{\omega}{\omega^*}\right)x+3\left(5+3\frac{\omega}{\omega^*}\right)<0. \qquad (14.3)$$

Whence, after evaluation of the roots,

$$-3 > x = \frac{2\lambda_1(\omega)}{p\Lambda(\omega)} > -\frac{3\omega+5\omega^*}{\omega+\omega^*}. \qquad (14.4)$$

Thus inequality (14.1) is translated into an authorized range for the first multiplier at departure or arrival. Replacing p in terms of the apsis value ω and ω^* and distinguishing between the possible signs of $\Lambda(\omega)$, these ranges are

$$-\frac{3}{\sqrt{2(\omega+\omega^*)}} > \lambda_1(\omega) > -\frac{3\omega+5\omega^*}{(\omega+\omega^*)\sqrt{2(\omega+\omega^*)}} \qquad \Lambda(\omega) = 1 \quad (14.5)$$

$$\frac{3}{\sqrt{2(\omega+\omega^*)}} < \lambda_1(\omega) < \frac{3\omega+5\omega^*}{(\omega+\omega^*)\sqrt{2(\omega+\omega^*)}} \qquad \Lambda(\omega) = -1 \quad (14.6)$$

Should $\lambda_1(\omega)$ reach a boundary value, the orbit becomes of the transfer type. Comparison with Eqs. (10.15 and 16) or (10.18 and 19) shows that the boundaries on the left are for reflexive transfers, on the right for transitive transfers. When $\lambda_1(\omega)$ lies within the authorized range the terminal orbit is of class 1. Otherwise it is of class 2 and has in the neighborhood of the opposite apsis ω^* a range of positive N values.

14.2. Non-circular orbits with $A+Bq = 0$

$$N = \sqrt{2B^2(h+q)}-1.$$

From Fig. 4, where this case is illustrated, it appears that the range of h for with $N < 0$ depends on the energy q of the orbit.

Since N vanishes for $h = \omega$

$$2B^2(q+\omega) = 1 \qquad (14.7)$$

and condition (9.8) becomes

$$u\left(1-\frac{3\omega+2q}{2(\omega+q)}\right) = 0. \qquad (14.8)$$

Of the two possibilities offered by (14.8): $\omega = 0$ or $u = 0$ only the last can be retained for orbits of class 1. Departure or arrival are still tangential at the largest value of h on the orbit, that is at peri-apsis. We must now distinguish between the elliptical and the hyperbolic case.

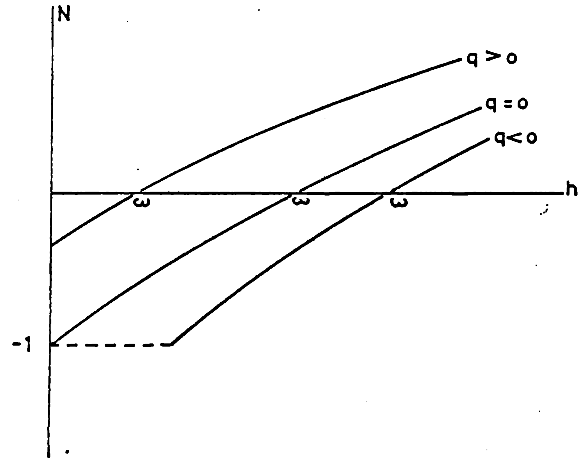

Fig. 4

Elliptical case $q < 0$. At the apo-apsis

$$h = \omega^* < \omega, \qquad \omega^* + q = \frac{(\omega^*)^2}{\omega + \omega^*} > 0,$$

and N is still real and negative.

$$\lambda_1(\omega) = -\frac{B}{\omega}(3\omega + 2q) = -B\frac{3\omega + \omega^*}{\omega + \omega^*}$$

$$\lambda_4(\omega) = \frac{\Lambda(\omega)}{\omega} = Bp = B\sqrt{\frac{2}{\omega + \omega^*}} = \lambda_4(\omega^*) = \frac{\Lambda(\omega^*)}{\omega^*}.$$

Hence the value authorized for $\lambda_1(\omega)$ is given by

$$\frac{2\lambda_1(\omega)}{p\Lambda(\omega)} = x = -\left(3 + \frac{\omega^*}{\omega}\right).$$

It verifies inequality (14.3) if due account is taken of $\omega^* < \omega$. It can therefore be concluded that it is the value of $\lambda_1(\omega)$ within the previously defined range for which $A + Bq = 0$ occurs as a particular but non singular case.

Hyperbolic case $q > 0$. While condition $A + Bq = 0$ yields nothing new in the elliptical case it is essential for hyperbolic terminal orbits of class 1.

Only then can N remain negative up to the point at infinity $h = 0$. The authorized $\lambda_1(\omega)$ value is unique. From

$$\lambda_1(\omega) = -B\left(3+\frac{2q}{\omega}\right)$$

$$\lambda_4(\omega) = Bp = \frac{B}{\omega}\sqrt{2(q+\omega)}.$$

this value is

$$\lambda_1(\omega) = -\left(3+\frac{2q}{\omega}\right)\frac{\Lambda(\omega)}{\sqrt{2(q+\omega)}}. \tag{14.9}$$

14.3. Circular orbits

The complete discussion should again be based on the values (7.10) of the multipliers. It was already observed in Section 10.3 that the general case can provide a class 1 orbit.

(a) $Q > 0$. $N = 0$ is an isolated maximum of N for $\sin\theta = -1$ provided $\omega(Q+2P)-1 = 0$. Hence $\Lambda(\omega) = \omega\lambda_4(\omega) = \omega(Q+2P) = 1$ and the adjacent impulse is an acceleration. Also

$$\lambda_1(\omega) = -\frac{3v}{\omega}P-\frac{2v}{\omega}Q = \frac{v}{\omega}P-\frac{2v}{\omega^2}.$$

Moreover P has an authorized range $0 < P < \dfrac{1}{2\omega}$ in order to satisfy the assumption that both P and Q are positive. The corresponding authorized range of the first multiplier is

$$-\frac{3v}{2\omega^2} > \lambda_1(\omega) > -\frac{2v}{\omega^2} \quad (\Lambda(\omega) = 1) \tag{14.10}$$

(b) $Q < 0$. $N = 0$ is again an isolated maximum of N for $\sin\theta = +1$ provided $\omega(2P-Q) = 1$ this gives $\Lambda(\omega) = -1$ a braking impulse. The authorized range of P is as before and then

$$\frac{3v}{2\omega^2} < \lambda_1(\omega) < \frac{2v}{\omega^2} \quad (\Lambda(\omega) = -1). \tag{14.11}$$

Noting that for a circular orbit $\dfrac{v}{\omega^2} = \dfrac{1}{\sqrt{\omega}}$ the ranges (14.10 and 11) for class 1 orbits are identical to those deduced from (14.5 and 6) in the limit $\omega^* = \omega$. In conclusion (14.5 and 6) are uniformly valid.

15. Allowable Domains for Class 1 Terminal Orbits

15.1. Departure orbits

If a departure orbit (α_0, α_1) is of class 1, the value of $\lambda_{1-}(\alpha_1)$ must satisfy inequalities (14.4), where $\omega = \alpha_1$ and $\omega^* = \alpha_0$. Corresponding inequalities are found for $\lambda_{1+}(\alpha_1)$ by use of the invariance relation (11.4) with

$$p_- = \sqrt{\frac{2}{\alpha_0+\alpha_1}}; \quad p_+ = \sqrt{\frac{2}{\alpha_1+\alpha_2}}.$$

They are

$$\frac{-1}{(\alpha_0+\alpha_1)^{\frac{1}{2}}} > \frac{\sqrt{2}}{\Lambda(\alpha_1)}\lambda_{1+}(\alpha_1) + \frac{2}{\sqrt{\alpha_1+\alpha_2}} > -\frac{\alpha_1+3\alpha_0}{(\alpha_0+\alpha_1)^{\frac{3}{2}}}. \tag{15.1}$$

If the transfer following the impulse is transitive, Eq. (10.15) gives the value of $\lambda_{1+}(\alpha_1)$ and (15.1) becomes

$$\frac{1}{(\alpha_0+\alpha_1)^{\frac{1}{2}}} < \frac{\alpha_1+3\alpha_2}{(\alpha_1+\alpha_2)^{\frac{3}{2}}} < \frac{\alpha_1+3\alpha_0}{(\alpha_0+\alpha_1)^{\frac{3}{2}}}$$

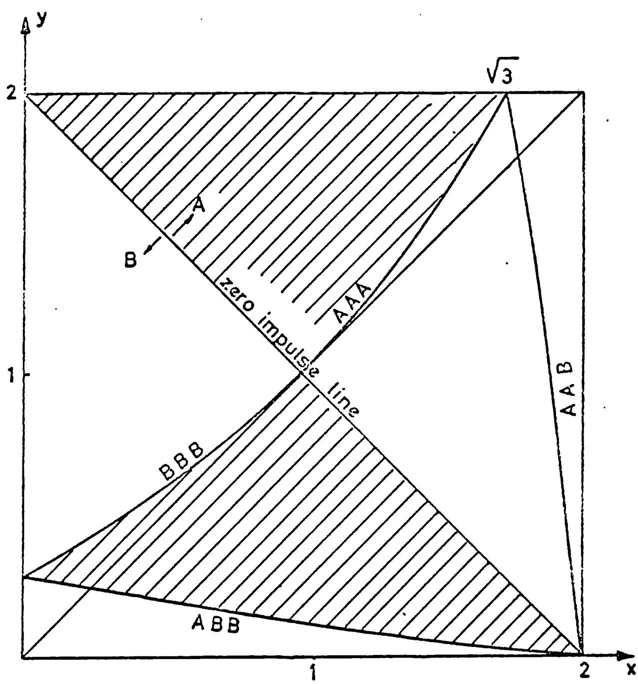

Fig. 5. For class 1 departure orbit, first impulse must lie in shaded area when second is of same sign

or, using definitions (12.1 and 2) for $n = 1$,

$$2-x < y(3-y)^2 < (2-x)(1+x)^2. \qquad (15.2)$$

These inequalities are satisfied when the departure point in the (x, y) diagram lies in the shaded areas of Fig. 5. Should the transfer following the impulse be reflexive, Eq. (10.18) is used for $\lambda_{1+}(\alpha_1)$ and yields

$$\frac{1}{(\alpha_0+\alpha_1)^{\frac{1}{2}}} < \frac{1}{(\alpha_1+\alpha_2)^{\frac{1}{2}}} < \frac{\alpha_1+3\alpha_0}{(\alpha_0+\alpha_1)^{\frac{3}{2}}}$$

or

$$2-x < y < (2-x)(1+x)^2. \qquad (15.3)$$

The departure point should lie in the shaded area of Fig. 6. It will be observed that in this case the first impulse is necessarily an acceleration.

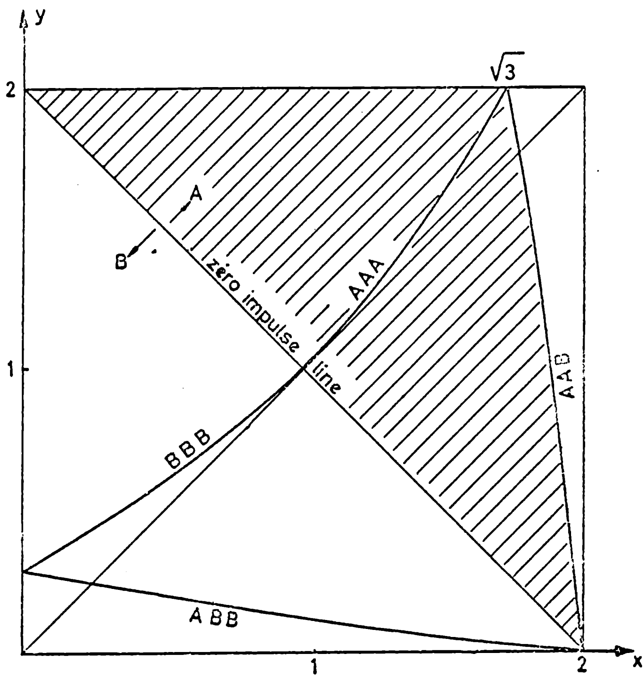

Fig. 6. For class 1 departure orbit, first impulse must lie in shaded area when second is of opposite sign

15.2. Orbits of arrival

If the final orbit (α_N, α_{N-1}) is of class 1, $\lambda_{1+}(\alpha_N)$ is submitted to the inequalities (14.4) with $\omega^* = \alpha_{N+1}$. They are modified into

$$\frac{-1}{(\alpha_N+\alpha_{N+1})^{\frac{1}{2}}} > \frac{\sqrt{2}}{\Lambda(\alpha_N)}\lambda_{1-}(\alpha_N)+\frac{2}{\sqrt{\alpha_N+\alpha_{N-1}}} > -\frac{\alpha_N+3\alpha_{N+1}}{(\alpha_N+\alpha_{N+1})^{\frac{3}{2}}} \qquad (15.4)$$

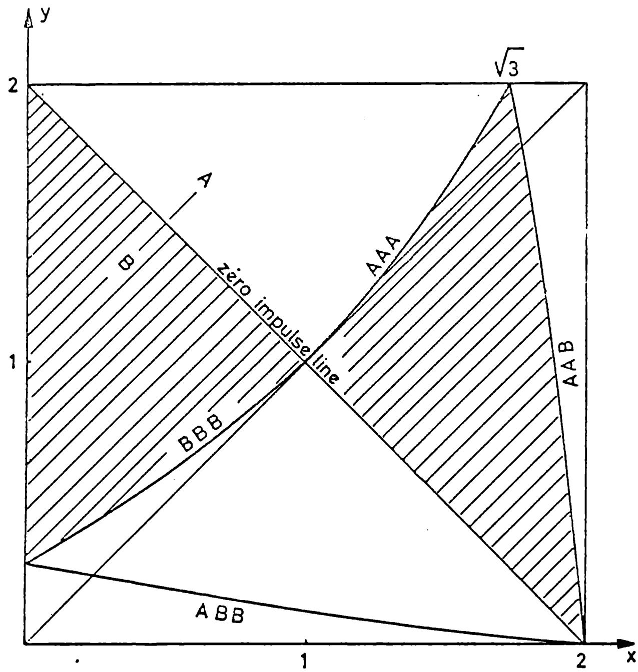

Fig. 7. For class 1 arrival orbit, last impulse must lie in shaded area when previous impulse is of same sign

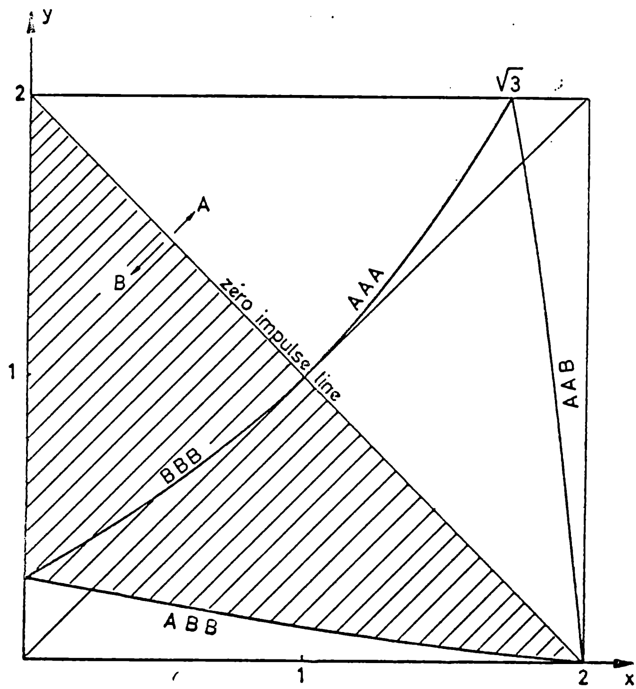

Fig. 8. For class 1 arrival orbit, last impulse must lie in shaded area when previous impulse is of opposite sign

by the invariance relation (11.4). Calculating $\lambda_{1-}(\alpha_N)$ by means of (10.16) if the preceding transfer is transitive

$$\frac{1}{(\alpha_N+\alpha_{N+1})^{\frac{1}{2}}} < \frac{\alpha_N+3\alpha_{N-1}}{(\alpha_N+\alpha_{N-1})^{\frac{3}{2}}} < \frac{\alpha_N+3\alpha_{N+1}}{(\alpha_N+\alpha_{N+1})^{\frac{3}{2}}}$$

or, using definitions (12.1 and 2) for $n = N$

$$y < (2-x)(1+x)^2 < y(3-y)^2. \tag{15.5}$$

The final impulse in the (x, y) diagram must lie in the shaded area of Fig. 7.

Should the last transfer be of the reflexive type, $\lambda_{1-}(\alpha_N)$ is calculated from (10.19) and (15.4) yields

$$\frac{1}{(\alpha_N+\alpha_{N+1})^{\frac{1}{2}}} < \frac{1}{(\alpha_N+\alpha_{N-1})^{\frac{1}{2}}} < \frac{\alpha_N+3\alpha_{N+1}}{(\alpha_N+\alpha_{N+1})^{\frac{3}{2}}}$$

or

$$y < 2-x < y(3-y)^2$$

and the final impulse should lie in the shaded area of Fig. 8.

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