Reprint from Vol. 11, 1965, No. 4

ASTRONAUTICA ACTA

Editor-in-Chief: MARTIN SUMMERFIELD. Princeton University, Princeton. N. J., U. S. A.

Springer-Verlag / Wien · New York

Canonical Transformations and the Thrust-Coast-Thrust Optimal Transfer Problem

By

B. Fraeijs de Veubeke

Université de Liège, Laboratoire de Techniques Aéronautiques et Spatiales, Liège, Belgium

With 2 Figures

(Received September 27, 1965)

Abstract — Résumé — Zusammenfassung

Canonical Transformations and the Thrust-Coast-Thrust Optimal Transfer Problem. Various formulations of the optimal steering and cutoff-relight programs for minimum fuel expenditure of a limited-thrust rocket in a central gravitational field are investigated. They are derived from a basic formulation in polar coordinates by canonical transformations of the PONTRYAGIN hamiltonian. The so-called (h, u, v) formulation is convenient for eliminating multipliers and obtaining explicit differential equations for the steering program and the switching function. The proof that there are no optimal intermediate-thrust arcs, if neither the time taken to describe the trajectory nor the polar angle it subtends are specified, is also carried out in this formulation. A canonical transformation by which the optimal steering control becomes a state variable is used to investigate the general case of intermediate-thrust extremals. The fundamental algebraic relations governing such extremals and first discovered by LAWDEN [2], are obtained by repeated differentiation of the chattering condition on the thrust switching function. Moreover LAWDEN's integration constant A can be identified as the isoperimetrical constant associated with a specification of the polar angle subtended by the trajectory. The so-called orbitally-linear formulation is convenient for complete integration of the adjoint differential equations during a coasting phase. It is used to establish in the simplest case (subtended polar angle and trajectory duration unspecified) rules for jumping over a coasting phase. To this purpose the values of all variables and multipliers at engine relight are given as algebraic functions of their values at the preceeding engine cutoff. Any result obtained in one formulation can be carried over into another by use of the appropriate canonical transformation.

Transformations canoniques dans le problème du transfert optimal avec extinctions et réallumages. Diverses formulations du contrôle optimal à exercer sur l'orientation de la poussée et sur le programme d'extinction et réallumage sont examinées en vue de minimiser la consommation d'ergols d'une fusée à poussée limitée dans un champ de gravitation central. Elles découlent toutes d'une formulation de base en coordonnées polaires par transformations canoniques du hamiltonien de PONTRJAGIN. La formulation (h, u, v) est pratique pour éliminer les multiplicateurs et obtenir les équations différentielles explicites gouvernant les programmes d'orientation et d'extinction et réallumage. Elle permet aussi de prouver l'absence d'extrémales à poussée intermédiaire, quand ni la durée de la trajectoire, ni l'angle polaire qu'elle sous-tend, ne sont imposés. La recherche d'arcs à poussée intermédiaire, dans le cas général, est menée à partir d'une transformation canonique qui fait du contrôle optimal d'orientation une variable d'état. Alors, les relations algébriques fondamentales le long de tels arcs, découvertes antérieurement par LAWDEN [2], s'obtiennent directement par différentiations répétées de la condition de réticence sur l'extinction et le réallumage. De plus la constante d'intégration A, introduite par LAWDEN, peut être identifiée avec la constante isopérimétrique de l'angle polaire sous-tendu par la trajectoire. La formulation „orbitale linéaire" est commode pour une intégration complète du systéme adjoint durant une phase non-propulsée. Elle permet ainsi dans le cas le plus simple (durée et angle polaire non spécifiés) d'établir des règles de transfert pour de telles phases. Dans ce but les valeurs des variables d'état et des variables adjointes au moment du réallumage sont fournies explicitement en fonction des valeurs des mêmes variables à l'extinction précédente. Tout résultat obtenu dans une formulation se laisse transposer dans une autre à l'aide des transformations canoniques.

Kanonische Transformationen und das Problem des optimalen Bahnübergangs mit Schubunterbrechung. Es wurden verschiedene Formulierungen über optimale Lenkung und Brennschluß-Wiederzündungsprogramme für minimalen Treibstoffverbrauch einer schubbegrenzten Rakete in einem zentralen Gravitationsfeld untersucht. Sie wurden von einem Grundausdruck in Polarkoordinaten mit kanonischen Transformationen von der PONTRJAGIN-HAMILTON-Transformation hergeleitet. Der sogenannte (h, u, v)-Ausdruck ist zur Eliminierung von Multiplikatoren verwendbar und man erhält explizite Differentialgleichungen für das Lenkprogramm und die Schaltfunktionen. Der Beweis, daß hier keine optimalen Kurven für mittleren Schub existieren, wenn weder die Zeit für den Übergang noch der dazugehörige Polarwinkel angegeben sind, wird hier erbracht. Eine kanonische Transformation, bei welcher die optimale Lenkungssteuerung variiert wird, wird für die Untersuchung des allgemeinen Falles der Extrema für mittleren Schub verwendet. Die fundamentalen algebraischen Beziehungen, die solche Extrema bestimmen und zuerst von LAWDEN [2] entdeckt wurden, werden durch wiederholte Differentiation der Schaltbedingungen der Schubschaltfunktion erhalten. Außerdem kann LAWDENS Integrationskonstante A als die isoperimetrische Konstante, verbunden mit der Angabe des Polarwinkels des Überganges identifiziert werden. Der sogenannte orbitally-linear-Ausdruck wird zur gesamten Integration der damit verbundenen Differentialgleichungen während der Freiflugphase verwendet. Es wird im einfachsten Fall (Polarwinkel und Übergangsdauer nicht angegeben) zur Festsetzung der Regel zum Überspringen einer Freiflug-

phase verwendet. Zu diesem Zweck sind die Werte aller Variablen und Faktoren bei Triebwerkszündung als algebraische Funktionen der Werte des vorhergegangenen Brennschlusses gegeben. Jedes Resultat, das in einem Ausdruck erhalten wurde, kann in einen anderen mit Hilfe der passenden kanonischen Transformation verwandelt werden.

1. Canonical Transformations

The equations for optimal trajectories are derived from the variational formulation

$$\delta\left[J + \int_a^b \left(\sum_1^n \lambda_i \, dq_i - H \, dq_0\right)\right] = 0 \qquad (1.1)$$

where q_0 is the independant variable,

$q = (q_1, q_2, \ldots, q_n)$ the vector of state variables,

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the vector of adjoint multipliers.

$H = \sum_1^n \lambda_i g_i (q, q_0; \theta)$, the hamiltonian of the system, depends on a set of free controls $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$.

In a parametric representation of the trajectory:

$q = q(x)$, $q_0 = q_0(x)$, $\lambda = \lambda(x)$, $\theta = \theta(x)$, $a \leqslant x \leqslant b$

the function
$$J = I + \sum_1^p \mu_y U_y \qquad (1.2)$$

contains a function of terminal phases to be minimized

$$I = I(q(a), q_0(a); q(b), q_0(b)) \text{ min.} \qquad (1.3)$$

and the terminal constraints of the problem

$$U_y = U_y(q(a), q_0(a); q(b), q_0(b)) = 0 \qquad (1.4)$$
$$y = 1, 2 \ldots p$$

The vanishing of the first variation produces the following differential equations and transversality conditions:

$$dq_i = \frac{\partial H}{\partial \lambda_i} dq_0 \qquad d\lambda_i = -\frac{\partial H}{\partial q_i} dq_0$$
$$(i = 1, 2 \ldots n) \qquad (1.5)$$

$$dH = \frac{\partial H}{\partial q_0} dq_0 \qquad (1.6)$$

$$\lambda_i(a) = \frac{\partial J}{\partial q_i(a)} \qquad \lambda_i(b) = -\frac{\partial J}{\partial q_i(b)}$$
$$(i = 1, 2 \ldots n) \qquad (1.7)$$

$$H(a) = -\frac{\partial J}{\partial q_0(a)} \qquad H(b) = \frac{\partial J}{\partial q_0(b)} \qquad (1.8)$$

Moreover, for a continuous behaviour of state variables and multipliers, the principle requires continuity of the hamiltonian at points where discontinuities occur in the controls. The optimality conditions resulting from the vanishing of the first variation

$$\frac{\partial H}{\partial \theta_r} = 0 \qquad r = 1, 2 \ldots m \qquad (1.9)$$

are superseded by the condition

$$\theta = \arg \sup H \qquad (1.10)$$

whereby at each point the controls must be chosen so as to give the hamiltonian its maximum value. In the problem under consideration, guidance can be expressed in terms of free controls and (1.10) is justified by the strong variation criterion. This maximum principle has been extended by PONTRYAGIN to the case of bounded controls.

For a canonical change of variables from (q, q_0, λ, H) to (Q, Q_0, Λ, K)

$$\sum_1^n \lambda_i \, dq_i - H \, dq_0 = \sum_1^n \Lambda_j \, dQ_j - K \, dQ_0 + dF \qquad (1.11)$$

Only canonical transformations of MATHIEU type will be considered; they are such that F is identically zero. A particular case of such transformations are those generated by a function $V(q, q_0, \Lambda, K)$

$$\lambda_i = \frac{\partial V}{\partial q_i} \qquad H = -\frac{\partial V}{\partial q_0} \qquad Q_j = \frac{\partial V}{\partial \Lambda_j} \qquad Q_0 = -\frac{\partial V}{\partial K} \qquad (1.12)$$

we find

$$F = V - \left(K \frac{\partial V}{\partial K} + \sum_1^n \Lambda_j \frac{\partial V}{\partial \Lambda_j}\right) \qquad (1.13)$$

and F vanishes identically provided V is linear and homogeneous in the adjoint variables Λ_j and K. In that case the new coordinates (Q, Q_0) are only functions of the old (q, q_0) and the canonical change furnishes the linear transformations from the old multipliers and hamiltonian to the new. Similarly, for a generating function $V(Q, Q_0, \lambda, H)$

$$q_i = \frac{\partial V}{\partial \lambda_i} \qquad q_0 = -\frac{\partial V}{\partial H} \qquad \Lambda_j = \frac{\partial V}{\partial Q_j} \qquad K = -\frac{\partial V}{\partial Q_0}$$

$$F = \sum_1^n \lambda_i \frac{\partial V}{\partial \lambda_i} + H \frac{\partial V}{\partial H} - V \qquad (1.14)$$

vanishes identically if V is linear and homogeneous in the variables λ_i and H. In the application to the thrust-coast-thrust problem the canonical transformations are established directly from (1.11).

2. Dimensionless Variables and Parameters

All variables and parameters of the problem are made non-dimensional by reference to the following units:

r^*	conventional unit of distance from the attracting center,
g^*	acceleration of the gravitational field at the distance r^*,
$\sqrt{r^* g^*}$	orbital velocity at the distance r^*,
$\sqrt{r^*/g^*}$	unit of time; 2π units equal the period required to describe the circular orbit of unit radius,
M^*	conventional unit of mass of the vehicle; in most applications it is the mass at departure.

The notations for *dimensionless* quantities are:

r	distance from attracting center,
θ	polar angle measured from fixed reference,
t	time,
u_r	radial velocity,
u_θ	tangential velocity,
$\mu = M^*/M$	reciprocal of instantaneous mass,
ψ	angle of thrust orientation above local horizon, a control variable,
$a = \dfrac{F}{g^* M^*}$	acceleration factor, ratio of maximum thrust F to reference weight of vehicle, a parameter,
$c = V_e/\sqrt{r^* g^*}$	ratio of effective ejection velocity V_e to reference velocity, a parameter,
ξ	a control variable for thrust intensity.

For a fully throttlable engine the thrust ξF can be adjusted anywhere between zero and full thrust; hence ξ can take any real value between zero and one. If the engine can only be cutoff or relighted, ξ can only take the discrete values 0 and 1.

However the two cases need not be distinguished from the theoretical point of view for, in the limit, a fast sequence of cutoffs and relightings (Chattering) is equivalent to thrust modulation. In any optimal solution where intermediate thrust values are involved the alternative between chattering and a throttlable engine is a technical decision.

3. Formulation in Polar Coordinates

In the classical polar coordinates, the hamiltonian of the problem is

$$H = H_0 + a\,\xi\,\mu\,H_1 \tag{3.1}$$

where H_0 is independant of the control variables

$$H_0 = \lambda_r u_r + \lambda_\theta u_\theta/r + \lambda_{u_r}\left(\frac{u_\theta^2}{r} - \frac{1}{r^2}\right) - \lambda_{u_\theta}\frac{u_r u_\theta}{r} \tag{3.2}$$

and H_1 depends only on the control ψ

$$H_1 = \lambda_{u_r}\sin\psi + \lambda_{u_\theta}\cos\psi + \frac{1}{c}\lambda_\mu\,\mu \tag{3.3}$$

In order to apply PONTRYAGIN's maximum principle, H_1 is first maximized with respect to the control ψ:

$$\sin\psi = \lambda_{u_r}/\lambda \quad \cos\psi = \lambda_{u_\theta}/\lambda \quad \lambda = \sqrt{\lambda_{u_r}^2 + \lambda_{u_\theta}^2} \tag{3.4}$$

giving

$$(H_1)_{\max} = \overline{H}_1 = \lambda + \frac{1}{c}\mu\,\lambda_\mu \tag{3.5}$$

The choice of ξ for maximum H is then

$$\begin{aligned} \xi &= 0 \quad \text{if } \overline{H}_1 < 0 \\ \xi &= 1 \quad \text{if } \overline{H}_1 > 0 \end{aligned} \tag{3.6}$$

The case $\overline{H}_1 = 0$, if instantaneous, means only commutation from zero to full thrust or conversely. If this case can persist for some time, ξ is as yet undetermined, and possible intermediate thrust arcs must be investigated. This investigation will be carried out with another set of variables in section 6.

The equations of motion are obtained from

$$\begin{gathered} \frac{dr}{dt} = \frac{\partial H}{\partial \lambda_r} \quad \frac{d\theta}{dt} = \frac{\partial H}{d\lambda_\theta} \\ \frac{du_r}{dt} = \frac{\partial H}{\partial \lambda_{u_r}} \quad \frac{du_\theta}{dt} = \frac{\partial H}{\partial \lambda_{u_\theta}} \quad \frac{d\mu}{dt} = \frac{\partial H}{\partial \lambda_\mu} \end{gathered} \tag{3.7}$$

The adjoint system, governing the multipliers, is

$$\begin{gathered} \frac{d\lambda_r}{dt} = -\frac{\partial H}{\partial r} \qquad \frac{d\lambda_\theta}{dt} = -\frac{\partial H}{\partial \theta} = 0 \\ \frac{d}{dt}\lambda_{u_r} = -\frac{\partial H}{\partial u_r} \qquad \frac{d}{dt}\lambda_{u_\theta} = -\frac{\partial H}{\partial u_\theta} \\ \frac{d}{dt}\lambda_\mu = -\frac{\partial H}{\partial \mu} \end{gathered} \tag{3.8}$$

to which can be added as a consequence

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \tag{3.9}$$

Because the hamiltonian does not depend explicitly on θ and t, the adjoint variables λ_θ and H are isoperimetrical constants. Their values depend on the constraints imposed respectively on the polar angle subtended by the complete flight path and on its duration.

The following is an example of a set of constraints specifying completely a problem and its transversality conditions:

(1) Departure from an orbit of known energy $q(a)$ and known angular momentum $k(a)$:

$$c_1 = \frac{1}{2}\left[u_r^2(a) + u_\theta^2(a)\right] - \frac{1}{r(a)} - q(a) = 0 \tag{3.10}$$

$$c_2 = r(a)\,u_\theta(a) - k(a) = 0 \tag{3.11}$$

Equivalent constraints are the specification of apocenter $\alpha(a)$ and pericenter $\beta(a)$ of the orbit since these determine the energy and angular momentum through the formulas

$$q(a) = -\frac{1}{\alpha(a) + \beta(a)} \tag{3.12}$$

$$k^2(a) = \frac{2\,\alpha(a)\,\beta(a)}{\alpha(a) + \beta(a)} \tag{3.13}$$

(2) Arrival on an orbit of specified energy and angular momentum

$$c_3 = \frac{1}{2}\left[(u_r^2(b) + u_\theta^2(b)\right] - \frac{1}{r(b)} - q(b) = 0 \tag{3.14}$$

$$c_4 = r(b)\,u_\theta(b) - k(b) = 0 \tag{3.15}$$

with analogous relations for apocenter and pericenter values $\alpha(b)$ and $\beta(b)$.

(3) Specification of the vehicle mass at departure. This is simply formulated by defining M^* to be the departure mass and gives the constraint

$$c_5 = \mu(a) - 1 = 0 \tag{3.16}$$

(4) The orbital transfer should be optimized with respect to propellant expenditure:

$$\mu(b) \text{ minimum} \tag{3.17}$$

The transversality conditions yield the initial and final values of the adjoint variables. They are derived from the function of terminal values to be minimized, (3.17) in the present case, to which the constraints on terminal values are added by means of unknown constant multipliers μ_i

$$J = \mu(b) + \sum_1^5 \mu_i c_i$$

we find in succession

$$\lambda_\theta(a) = \frac{\partial J}{\partial \theta(a)} = 0 \qquad \lambda_\theta(b) = -\frac{\partial J}{\partial \theta(b)} = 0 \quad (3.18)$$

and as a consequence the isoperimetrical constant

$$\lambda_\theta \equiv 0 \quad (3.19)$$

Also

$$H(a) = -\frac{\partial J}{\partial t(a)} = 0 \qquad H(b) = \frac{\partial J}{\partial t(b)} = 0 \quad (3.20)$$

and as a consequence the isoperimetrical constant

$$H \equiv 0 \quad (3.21)$$

Next we consider

$$\begin{aligned} \lambda_r(a) &= \frac{\partial J}{\partial r(a)} = \frac{\mu_1}{r^2(a)} + \mu_2 u_\theta(a) \\ \lambda_{u_r}(a) &= \frac{\partial J}{\partial u_r(a)} = \mu_1 u_r(a) \\ \lambda_{u\theta}(a) &= \frac{\partial J}{\partial u_\theta(a)} = \mu_1 u_\theta(a) + \mu_2 r(a) \end{aligned} \quad (3.22)$$

and eliminate the unknown multipliers μ_1 and μ_2 between them. The result, combined with (3.18), turns out to be expressible in the simple form

$$H_0(a) = 0 \quad (3.23)$$

In the same manner there comes the result

$$H_0(b) = 0 \quad (3.24)$$

after elimination of μ_3 and μ_4 between equations

$$\lambda_r(b) = -\frac{\partial J}{\partial r(b)} \qquad \lambda_{u_r}(b) = -\frac{\partial J}{\partial u_r(b)}$$

$$\lambda_{u\theta}(b) = -\frac{\partial J}{\partial u_\theta(b)}$$

and consideration of (3.18).

Finally

$$\lambda_\mu(a) = \frac{\partial J}{\partial \mu(a)} = \mu_5$$

which gives no real information, and

$$\lambda_\mu(b) = -\frac{\partial J}{\partial \mu(b)} = -1 \quad (3.25)$$

This last result is very important, even if for convenience the λ variables are scaled differently than suggested by eq. (3.25), the end value of λ_μ must be negative.

Equation (3.19) is a consequence of the absence of specification concerning the polar transfer angle and provides a considerable simplification of the analysis. If the polar angle were specified to be θ, we would have to add the constraint

$$c_6 = \theta(b) - \theta(a) - \theta = 0 \quad (3.26)$$

and we would obtain the transversality conditions

$$\lambda_\theta(a) = \lambda_\theta(b) = -\mu_6$$

The actual value of the isoperimetrical constant λ_θ would have to be determined by satisfaction of the constraint (3.26).

Also, instead of (3.23) and (3.24), we would have

$$H_0(a) - \lambda_\theta \frac{u_\theta(a)}{r(a)} = H_0(b) - \lambda_\theta \frac{u_\theta(b)}{r(b)} = 0 \quad (3.27)$$

Equation (3.21) is a consequence of the absence of specification concerning the transfer duration. Should this duration be required to be T, the following constraint would have to be added:

$$c_7 = t(b) - t(a) - T = 0 \quad (3.28)$$

and would yield the transversality conditions

$$H(a) = H(b) = \mu_7$$

The actual value of H as an isoperimetrical constant would have to be determined from the constraint (3.28) itself.

As we shall see in section 6, the existence of intermediate thrust extremal arcs depends entirely on the isoperimetrical constants.

A final important remark is that λ_{u_r} and $\lambda_{u\theta}$ cannot vanish simultaneously for a finite period of time. Hence the optimum angle ψ is never but locally indeterminate. The proof of this statement relies on the examination of the adjoint differential equations (3.8). If λ_{u_r} and $\lambda_{u\theta}$ remain zero for some finite time, we conclude from

$$\frac{d\lambda_{u_r}}{dt} = -\lambda_r + \lambda_{u_\theta} u_\theta / r$$

that λ_r also remains zero. Then from

$$\frac{d\lambda_r}{dt} = \lambda_\theta \frac{u_\theta}{r^2} + \lambda_{u_r}\left(\frac{u_\theta^2}{r^2} - \frac{2}{r^3}\right) - \lambda_{u_\theta} \frac{u_r u_\theta}{r^2}$$

that λ_θ has to vanish. With these results

$$\frac{d\lambda_{u\theta}}{dt} = -\lambda_\theta \frac{1}{r} - 2\lambda_{u_r} \frac{u_\theta}{r} + \lambda_{u_\theta} \frac{u_r}{r}$$

is identically satisfied. But the differential system for $(\lambda_r, \lambda_\theta, \lambda_{u_r}, \lambda_{u\theta})$ is homogeneous in these variables. Hence, by the requirement of continuity of the multipliers, these adjoint variables will be zero throughout the trajectory and so will be H_0. Finally the differential equation for λ_μ will reduce to

$$\frac{d\lambda_\mu}{dt} = -\frac{2}{c} a \xi \mu \lambda_\mu \qquad \text{with } \bar{H}_1 = \frac{1}{c} \mu \lambda_\mu$$

and we can verify that $\mu^2 \lambda_\mu =$ constant. Hence either λ_μ remains positive, we have a single full powered flight, but it is impossible to satisfy (3.25); or λ_μ remains negative and we have a single coasting flight, in which case the problem is of course meaningless.

4. Canonical Transformation to the (h, u, v) Variables

In [3] a convenient set of state variables was used to simplify analytical results for impulsive thrust.

The change from (r, u_r, u_θ) to (h, u, v) is defined by

$$h = \frac{1}{r} \quad u = \frac{u_r}{r} \quad v = \frac{u_\theta}{r} \tag{4.1}$$

The corresponding change in adjoint variables is readily given by the canonical transformation

$$\lambda_r\, dr + \lambda_{u_r}\, du_r + \lambda_{u\theta}\, du_\theta = \lambda_h\, dh + \lambda_u\, du + \lambda_v\, dv \tag{4.2}$$

Replacing in the right hand side the differentials by their values taken from (4.1) and identifying with the left-hand side, there comes

$$\begin{aligned} \lambda_r &= -\frac{1}{r^2}\lambda_h - \frac{u_r}{r^2}\lambda_u - \frac{u_\theta}{r^2}\lambda_v \\ \lambda_{u_r} &= \frac{1}{r}\lambda_u \\ \lambda_{u\theta} &= \frac{1}{r}\lambda_v \end{aligned} \tag{4.3}$$

with the help of (4.1) and (4.3) the hamiltonian (3.1—3) can be expressed in the new variables:

$$H_0 = -u\, h\, \lambda_h + v\, \lambda_\theta + (v^2 - u^2 - h^3)\, \lambda_u - 2\, u\, v\, \lambda_v \tag{4.4}$$

$$H_1 = h\, (\lambda_u \sin\psi + \lambda_v \cos\psi) + \frac{1}{c}\mu\, \lambda_\mu \tag{4.5}$$

The optimal choice of thrust orientation can be obtained from

$$\sin\psi = \lambda_u/\Lambda \qquad \cos\psi = \lambda_v/\Lambda \tag{4.6}$$

$$\Lambda = \sqrt{\lambda_u^2 + \lambda_v^2} = r\, \lambda \tag{4.7}$$

5. Differential Equations for Control Variables

From eqs. (4.6) we can write

$$\frac{d}{dt}(\Lambda \sin\psi) = \frac{d\lambda_u}{dt} = -\frac{\partial H}{\partial u}$$

$$\frac{d}{dt}(\Lambda \cos\psi) = \frac{d\lambda_v}{dt} = -\frac{\partial H}{\partial v}$$

or, explicitly

$$\sin\psi \frac{d\Lambda}{dt} + \Lambda \cos\psi \frac{d\psi}{dt} = h\, \lambda_h + 2\, u\, \lambda_u + 2\, v\, \lambda_v$$

$$\cos\psi \frac{d\Lambda}{dt} - \Lambda \sin\psi \frac{d\psi}{dt} = -\lambda_\theta - 2\, v\, \lambda_u + 2\, u\, \lambda_v$$

combinations of those equations yield, in view of (4.6)

$$\frac{d\Lambda}{dt} = h\, \lambda_h \sin\psi - \lambda_\theta \cos\psi + 2\, u\, \Lambda \tag{5.1}$$

$$\Lambda \frac{d\psi}{dt} = h\, \lambda_h \cos\psi + \lambda_\theta \sin\psi + 2\, v\, \Lambda \tag{5.2}$$

Instead of the angular velocity of thrust orientation relative to the local horizon, let us introduce its absolute angular velocity

$$\omega = \frac{d\chi}{dt}$$

where (Fig. 1)

$$\chi = \theta + \frac{\pi}{2} - \psi$$

so that

$$\frac{d\psi}{dt} = \frac{d\theta}{dt} - \omega = v - \omega \tag{5.3}$$

Substitution into equation (5.2) gives

$$-\omega\, \Lambda = h\, \lambda_h \cos\psi + \lambda_\theta \sin\psi + v\, \Lambda \tag{5.4}$$

This equation can again be differentiated and the differential coefficients of the state variables and adjoint variables replaced by the corresponding

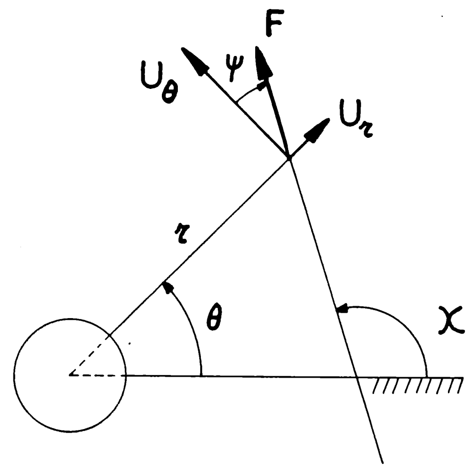

Fig. 1. Motion in a central gravitational field. $\theta + \frac{\pi}{2} - \psi = \chi$

canonical equations. For the derivatives of Λ and ψ, use can be made of eqs. (5.1—2). The end result turns out to be

$$-\Lambda \frac{d\omega}{dt} = 2\, \omega\, (h\, \lambda_h \sin\psi - \lambda_\theta \cos\psi + u\, \Lambda) + 3\, h^3\, \Lambda \sin\psi \cos\psi \tag{5.5}$$

Complete elimination of the adjoint variables is possible at this stage if the assumption is retained that the polar transfer angle is unspecified. Then we have $\lambda_\theta = 0$ and eqs. (5.4—5) are homogeneous in the adjoint variables Λ and λ_h. They yield a compatibility condition

$$\frac{d\omega}{dt} = 2\, \omega\, (\omega + v) \tan\psi - 2\, \omega\, u - 3\, h^3 \sin\psi\, \cos\psi \tag{5.6}$$

Equations (5.3) and (5.6) provide now a differential system to calculate ψ and ω together with the canonical equations of motion:

$$\frac{dh}{dt} = \frac{\partial H}{d\lambda_h} = -u\, h$$

$$\frac{d\theta}{dt} = \frac{dH}{d\lambda_\theta} = v$$

$$\frac{du}{dt} = \frac{\partial H}{\partial \lambda_u} = v^2 - u^2 - h^3 + a\, \xi\, \mu\, h \sin\psi \tag{5.7}$$

$$\frac{dv}{dt} = \frac{\partial H}{\partial \lambda_v} = -2\,u\,v + a\,\xi\,\mu\,h\cos\psi$$

$$\frac{d\mu}{dt} = \frac{\partial H}{\partial \lambda_\mu} = \frac{1}{c}\,a\,\xi\,\mu^2$$

In fact, if ω is eliminated between (5.3) and (5.5), one obtains a second order differential equation, which can be identified with the one discovered by FAULDERS [1]

$$\frac{d^2\psi}{dt^2} = -2\left(\frac{d\psi}{dt} - v\right)\left(\frac{d\psi}{dt} - 2\,v\right)\tan\psi - 2\,u\frac{d\psi}{dt} + 3\,h^3\sin\psi\,\cos\psi + a\,\xi\,\mu\,h\cos\psi \quad (5.8)$$

While FAULDERS' assumptions included constant vehicle mass and minimum transfer time, his result is seen to remain true for variable mass and any optimality criterion but essentially for free polar transit angle.

A useful addition to FAULDERS' equation, or better to the differential system (5.3—6), would be a differential equation to decide on cutoff and relight. This decision depends on the sign of $\bar{H}_1$ or of a signal s defined by

$$s\,\Lambda = \bar{H}_1 = h\,\Lambda + \frac{1}{c}\,\mu\,\lambda_\mu$$

when this equation is differentiated and the previous results used, there comes

$$\Lambda\left(\frac{ds}{dt} - u\,h + 2\,u\,s + \frac{1}{c}\,a\,\xi\,\mu\,s\right) + (s - h)\,h\,\lambda_h\sin\psi = 0$$

This is again homogeneous in Λ and λ_h and, to be compatible with (5.4), where $\lambda_\theta = 0$, requires

$$\frac{ds}{dt} = u\,(h - 2\,s) + (\omega + v)\,(s - h)\tan\psi - \frac{1}{c}\,a\,\xi\,\mu\,s \quad (5.9)$$

Integration of this equation will provide a cutoff signal whenever s crosses from positive to negative values, a relighting signal if it becomes positive again.

All this is true whether the transfer time is specified or not.

Let us discuss the last case first. The end values of the signal cannot be positive for then $\xi = 1$ and $a\,\xi\,\mu\,\bar{H}_1 > 0$, which is incompatible with (3.21) and (3.23) or (3.24).

On the other hand $s\,(a) < 0$ means that we have to wait in orbit untill we get a lighting signal. During this time no propellants are used, hence, except a difference in elapsed time which is by definition unimportant, we can adopt $s\,(a) = 0$ for the beginning of the trajectory. Similarly we can end the trajectory, without loss of significance, when $s\,(b) = 0$.

Since θ is an ignorable coordinate we can also adopt $\theta\,(a) = 0$ without loss of generality. With $\mu\,(a) = 1$, the only initial values that remain to be chosen are $u\,(a)$, $v\,(a)$, $h\,(a)$, $\psi\,(a)$ and $\omega\,(a)$. They are already related by the constraints (3.10—11) but also by a third constraint stemming from (3.23). Indeed, using eqs. (4.6)

$$H_0 = -u\,h\,\lambda_h + \Lambda\,[(v^2 - u^2 - h^3)\sin\psi - 2\,u\,v\cos\psi]$$

Hence $H_0 = 0$ is a homogeneous relation between λ_h and Λ; its compatibility with (5.4) requires

$$u\,(\omega + v) - 2\,u\,v\cos^2\psi + (v^2 - u^2 - h^3)\sin\psi\cos\psi = 0 \quad (5.10)$$

and this has to be satisfied by the initial values. We see that there are finally two degrees of freedom left in the choice of initial values. At the cutoff signal $s\,(b) = 0$, marking the end of the trajectory, the end values of the variables should satisfy the constraints (3.14—15), this would in principle be allowed by the two initial degrees of freedom.

Although (3.24) indicates that the end values must also satisfy (5.10), this is automatic because (3.9) is only a consequence of eqs. (3.8). Once $H\,(a) = 0$ has been enforced by $s\,(a) = 0$ and $H_0\,(a) = 0$, H will automatically remain zero and enforce $H_0\,(b) = 0$ when $s\,(b) = 0$.

Let us pass to the case where the transit time is specified.

This has no meaning unless we fix the state of the system from which we start to count the time. Hence the initial state variables $h\,(a)$, $u\,(a)$, $v\,(a)$ must all be specified and not only be limited by the constraints (3.10—11). This removes the transversality condition (3.23).

This time $\psi\,(a)$, $\omega\,(a)$ and $s\,(a)$ are three degrees of freedom. The constraint (3.24) remains, hence (5.10) and (3.14—15) must be satisfied when the integration is stopped at the specified end time. Those three final constraints correspond in principle to the three initial degrees of freedom.

On the basis of the preceding results it is easily proven that, if neither polar transit angle nor duration are prescribed, there are no intermediate thrust extremal arcs. The possibility of intermediate thrust hinges on

$$\bar{H}_1 = 0 \text{ or } s = 0 \quad (5.11)$$

being satisfied during a finite time interval, so that the application of the maximum principle leaves ξ undetermined. The level of thrust is actually determined by exploring the consequences of (5.11); first on (5.9) that reduces to

$$u = (\omega + v)\tan\psi \quad (5.12)$$

This, in turn, reduces (5.6) to

$$\frac{d\omega}{dt} = -3\,h^3\sin\psi\,\cos\psi \quad (5.13)$$

Being satisfied during a finite time interval (5.12) can be differentiated and all the derivatives substituted from the hamiltonian equations of motion, (5.3) and (5.13). The result is

$$\omega^2 = h^3\,(1 - 3\,\sin^2\psi) \quad (5.14)$$

If, in addition to $\lambda_\theta = 0$, we add the condition of free transfer duration $H = 0$, this reduces by virtue of (5.11) to $H_0 = 0$. Hence (5.10) is satisfied during the same time interval. Substitution of (5.12) into (5.10) yields

$$\sin\psi\,\cos\psi\,(\omega^2 - h^3) = 0 \quad (5.15)$$

The only solution to (5.12), (5.14) and (5.15) is

$$\sin\psi = 0 \qquad u = 0 \qquad \omega^2 = h^3$$

Then $du/dt = 0$, and from the corresponding hamiltonian equation

$$v^2 = h^3$$

Also from $u = 0$ follows

$$\frac{dh}{dt} = 0 \qquad h = \text{constant.}$$

Finally $dv/dt = 0$ and from the corresponding hamiltonian equation of motion

$$\xi = 0.$$

The only case where (6.1) can remain true for some finite time under free polar transit angle and free duration is a coasting phase along a circular orbit. This agrees with the conclusions obtained earlier in a slightly different way [3].

6. Canonical Transformation to ψ As a State Variable. Intermediate Thrust Arcs Under Specified Polar Angle and/or Transfer Time

Differential equations for the angular velocity of thrust orientation and rate of change of a cutoff signal can be obtained directly in the general case by the canonical transformation

$$\lambda_{u_r}\, du_r + \lambda_{u_\theta}\, du_\theta = \lambda\, du_\lambda + \lambda_\psi\, d\,\psi \qquad (6.1)$$

generated by the following change of multipliers:

$$\lambda_{u_r} = \lambda \sin\psi \qquad \lambda_{u_\theta} = \lambda \cos\psi \qquad (6.2)$$

Substitution of (6.2) into (6.1) and identification produces the corresponding transformation of variables

$$\begin{aligned} u_\lambda &= u_r \sin\psi + u_\theta \cos\psi \\ \lambda_\psi &= \lambda\,(u_\theta \sin\psi - u_r \cos\psi) \end{aligned} \qquad (6.3)$$

It is a requirement of optimality that the multipliers remain continous along the trajectory. Hence the angle ψ, defined by the transformation (6.2) will also be continuous. Eq. (6.1) shows that it becomes a state variable and, by virtue of the maximum principle, it is identical with the optimal angle of thrust orientation. Inverting eqs. (6.3)

$$\begin{aligned} u_r &= u_\lambda \sin\psi - \lambda_\psi \frac{\cos\psi}{\lambda} \\ u_\theta &= u_\lambda \cos\psi + \lambda_\psi \frac{\sin\psi}{\lambda} \end{aligned} \qquad (6.4)$$

Eqs. (6.4) and (6.2) enable us to express the hamiltonian in the new variables:

$$\begin{aligned} H_0 = \lambda_r \left(u_\lambda \sin\psi - \lambda_\psi \frac{\cos\psi}{\lambda}\right) + \frac{\lambda_\theta}{r}\, \cdot \\ \cdot \left(u_\lambda \cos\psi + \lambda_\psi \frac{\sin\psi}{\lambda}\right) - \\ - \lambda \frac{\sin\psi}{r^2} + u_\lambda \lambda_\psi \frac{\cos\psi}{r} + \lambda_\psi{}^2 \frac{\sin\psi}{\lambda r} \end{aligned} \qquad (6.5)$$

$$H_1 = \lambda + \frac{1}{c}\mu\,\lambda_\mu \qquad (6.6)$$

Among the hamiltonian equations of motion we have now

$$\frac{d\psi}{dt} = \frac{\partial H}{\partial \lambda_\psi} \qquad \text{and} \qquad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial u_\lambda}$$

while it is easily established that

$$\frac{d}{dt}(\mu\,\lambda_\mu) = -\,a\,\xi\,\mu\,H_1 \qquad (6.7)$$

so that the rate of change of cutoff signal is

$$\frac{dH_1}{dt} = -\frac{\partial H}{\partial u_\lambda} - \frac{1}{c}\,a\,\xi\,\mu\,H_1 \qquad (6.8)$$

This formulation is rather convenient to examine, in the general case, the consequences of the chattering condition

$$H_1 = 0 \qquad t_1 \leqslant t \leqslant t_2 \qquad (6.9)$$

that leads to intermediate thrust arcs. First consequences of (6.9) are: from (6.7)

$$\mu\,\lambda_\mu = \text{constant}$$

then, from (6.6)

$$\lambda = \text{constant}$$

This is a property discovered by Lawden [2] and called by him "the constancy of the primer".

The constant cannot vanish; otherwise we would have from (6.2) that both λ_{u_r} and λ_{u_θ} remain zero during the chattering interval.

This was proved impossible at the end of section 3.

As a modulus, λ is thus a positive constant; by writing

$$\lambda = 1 \qquad (6.10)$$

we simply fix the scale of the multipliers during the chattering period. Differentiation of (6.10) produces

$$\frac{\partial H}{\partial u_\lambda} = \lambda_r \sin\psi + (\lambda_\theta + \lambda_\psi)\frac{\cos\psi}{r} = 0 \qquad (6.11)$$

If this is used to eliminate λ_r in the hamiltonian equations, we obtain considerable simplifications:

$$\begin{aligned} \frac{du_\lambda}{dt} &= \frac{\partial H}{\partial \lambda} = -\frac{\lambda_\psi(\lambda_\theta + \lambda_\psi)}{r \sin\psi} - \frac{\sin\psi}{r^2} + a\,\xi\,\mu \\ \frac{d\psi}{dt} &= \frac{\partial H}{\partial \lambda_\psi} = \frac{\lambda_\theta + \lambda_\psi}{r\sin\psi} + u_\lambda \frac{\cos\psi}{r} + \lambda_\psi \frac{\sin\psi}{r} \\ \frac{d\lambda_\psi}{dt} &= -\frac{\partial H}{\partial \psi} = \frac{u_\lambda(\lambda_\theta + \lambda_\psi)}{r\sin\psi} + \frac{\cos\psi}{r^2} \end{aligned}$$

We differentiate (6.11) again in the form

$$\lambda_r + (\lambda_\theta + \lambda_\psi)\frac{\cot\psi}{r} = 0$$

and replace all differential coefficients to obtain after reduction

$$\sin^2\psi\,(1 - 3\sin^2\psi) = r\,(\lambda_\theta + \lambda_\psi)^2 \qquad (6.12)$$

Finally, because of (6.3)

$$H = H_0$$

and we know that H is the isoperimetrical time constant.

Replacing H_0 by (6.5), simplifying by (6.10) and (6.11), and rearranging:

$$\sin\psi\,(\sin\psi + r^2 H) = r\,\lambda_\psi\,(\lambda_\theta + \lambda_\psi) \qquad (6.13)$$

Equations (6.12) and (6.13) yield r and λ_ψ as algebraic functions of $\sin\psi$. They are fully equivalent to LAWDEN's equations (89) and (90) of [2].

The correspondence with LAWDEN's notation appears to be

$\lambda_\psi = w$ the orthogonal projection of the velocity vector on the normal to the thrust vector,

$\lambda_\theta = -A \qquad H = -C$

so that LAWDEN's constants A and C are recognized to be the isoperimetrical constants of the problem.

Further differentiation of (6.12) produces

$$\tan\psi\,(3 - 5\sin^2\psi)\,u_\lambda + 4\,\lambda_\theta + (3 + 5\sin^2\psi)\,\lambda_\psi = 0 \qquad (6.14)$$

so that u_λ is also expressible as an algebraic function of $\sin\psi$. The next differentiation gives the value of $a\,\xi\,\mu$, that is the level of thrust required. The differentiation procedure can be stopped at this stage for, if continued, it would merely furnish the successive differential coefficients of ξ. To complete the integration of the problem we can construct $dt/d\psi$, $d\theta/d\psi$ and $d\,(1/\mu)\,/d\psi$ as functions of ψ, so that t, θ and $1/\mu$ are solved by quadratures.

LAWDEN has examined in detail the case $H = 0$, which is now seen to correspond to a specification on the polar transit angle only. The other special case $\lambda_\theta = 0$ where the only specification is on transit time is also simple. The intermediate thrust arc is also a kind of spiral with the property

$$3\sin^3\psi = -r^2 H$$

7. An Orbitally Linear Formulation

A formulation that is linear, except for the propulsive terms, is obtained by taking θ as the independant variable and by the single transformation

$$z = \frac{1}{r\,u_\theta} \qquad (7.1)$$

The new variable z is the reciprocal of the angular momentum. Denoting by K the new hamiltonian we perform the canonical transformation

$$\lambda_r\,dr + \lambda_\theta\,d\theta + \lambda_{u_r}\,du_r + \lambda_{u\theta}\,du_\theta + \lambda_\mu\,d\mu - H\,dt = \Lambda_z\,dz + \Lambda_t\,dt + \Lambda_{u_r}\,du_r + \Lambda_{u\theta}\,du_\theta + \Lambda_\mu\,d\mu - K d\theta$$

Differentiation of (7.1), substitution and identification produces

$$\begin{aligned} \lambda_r &= -z^2\,u_\theta\,\Lambda_z \\ \lambda_\theta &= -K \\ \lambda_{u_r} &= \Lambda_{u_r} \\ \lambda_{u_\theta} &= \Lambda_{u_\theta} - \frac{z}{u_\theta}\Lambda_z \\ \lambda_\mu &= \Lambda_\mu \\ H &= -\Lambda_t \end{aligned} \qquad (7.2)$$

In the last equation the hamiltonian H can be replaced by its expression (3.1, 2, 3) and the old multipliers replaced by the new ones. Solving for K, we find

$$K = K_0 + a\,\xi\,\mu\,K_1 \qquad (7.3)$$

$$K_0 = \frac{\Lambda_t}{z\,u_\theta^2} + (u_\theta - z)\,\Lambda_{u_r} - u_r\,\Lambda_{u\theta} \qquad (7.4)$$

$$K_1 = \frac{1}{z\,u_\theta^2}\,(\Lambda_{u_r}\sin\psi + \Lambda_{u\theta}\cos\psi) - \frac{\Lambda_z}{u_\theta^3}\cos\psi + \frac{\mu}{c\,z\,u_\theta^2}\Lambda_\mu \qquad (7.5)$$

In this formulation

$$\frac{dK}{d\theta} = \frac{\partial K}{\partial\theta} = 0 \qquad \frac{d\Lambda_t}{d\theta} = -\frac{\partial K}{\partial t} = 0$$

K becomes the isoperimetrical constant for polar angle specification, Λ_t the constant for time specification.

This is also clearly indicated in (7.2). Just as θ was previously an ignorable coordinate, so becomes t:

$$\frac{dt}{d\theta} = \frac{\partial K}{\partial \Lambda_t} = \frac{1}{z\,u_\theta^2} \qquad (7.6)$$

can be integrated separately when we require informations on the timing of the trajectory.

The other hamiltonian equations of motion are

$$\begin{aligned} \frac{dz}{d\theta} &= -a\,\xi\,\mu\frac{\cos\psi}{u_\theta^3} \\ \frac{du_r}{d\theta} &= u_\theta - z + a\,\xi\,\mu\frac{\sin\psi}{z\,u_\theta^2} \\ \frac{du_\theta}{d\theta} &= -u_r + a\,\xi\,\mu\frac{\cos\psi}{z\,u_\theta^2} \\ \frac{d\mu}{d\theta} &= a\,\xi\,\mu^2\frac{1}{c\,z\,u_\theta^2} \end{aligned} \qquad (7.7)$$

and

$$\begin{aligned} \frac{d\Lambda_z}{d\theta} &= \frac{\Lambda_t}{z^2\,u_\theta^2} + \Lambda_{u_r} - a\,\xi\,\mu\frac{\partial K_1}{\partial z} \\ \frac{d\Lambda_{u_r}}{d\theta} &= \Lambda_{u\theta} \\ \frac{d\Lambda_{u\theta}}{d\theta} &= \frac{2\,\Lambda_t}{z\,u_\theta^3} - \Lambda_{u_r} - a\,\xi\,\mu\frac{\partial K_1}{\partial u_\theta} \\ \frac{d\Lambda_\mu}{d\theta} &= -a\,\xi\,K_1 - \frac{a\,\xi\,\mu}{c\,z\,u_\theta^2}\Lambda_\mu \end{aligned} \qquad (7.8)$$

This formulation is specially convenient for the integration of the orbital motion $\xi = 0$. The following integrals are readily obtained

$$z = Z \qquad (7.9)$$

$$u_r = A\sin\theta - B\cos\theta \qquad (7.10)$$

$$u_\theta = Z + A\cos\theta + B\sin\theta \qquad (7.11)$$

$$\mu = M \qquad (7.12)$$

Z is the reciprocal of the constant angular momentum, A and B are orbital constants related to eccentricity and argument of pericenter by the identification

$$\frac{1}{r} = z\,u_\theta = Z\,(Z + A\cos\theta + B\sin\theta) = Z^2\,[1 + e\cos(\theta - \theta_0)]$$

whence

$$A = Z\,e\cos\theta_0 \qquad B = Z\,e\sin\theta_0 \qquad (7.13)$$

The energy integral is

$$\frac{1}{2}(u_r^2 + u_\theta^2) - z\,u_\theta = \frac{1}{2}(A^2 + B^2 - Z^2) \quad (7.14)$$

The corresponding integrals for the adjoint system are obtained as follows. We set

$$\theta - \theta_0 = f$$

the true anomaly, and

$$\begin{aligned} \Lambda_{u_r} &= P \sin f + Q \cos f \\ \Lambda_{u\theta} &= P \cos f - Q \sin f \end{aligned} \quad (7.15)$$

Then from eqs. (7.8) with $\xi = 0$, P and Q must satisfy the differential equations

$$\begin{aligned} \frac{dP}{d\theta} &= \frac{2\,\Lambda_t}{Z^4} \frac{\cos f}{(1 + e\cos f)^3} \\ \frac{dQ}{d\theta} &= -\frac{2\,\Lambda_t}{Z^4} \frac{\sin f}{(1 + e\cos f)^3} \end{aligned} \quad (7.16)$$

The required quadratures are best obtained by transforming to the eccentric anomaly E, defined in the elliptic case $0 < e < 1$ by

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{f}{2} \quad (7.17)$$

With P_0 and Q_0 denoting the values of P and Q at pericenter $E = f = 0$, we find

$$\begin{aligned} P = P_0 + \frac{2\,\Lambda_t}{Z^4 (1 - e^2)^{5/2}} [(1 + e^2) \sin E - \\ - \frac{3}{2} E - \frac{e}{4} \sin 2\,E] \\ Q = Q_0 - \frac{2\,\Lambda_t}{Z^4 (1 - e^2)^2} [1 - \cos E - \\ - \frac{e}{4} (1 - \cos 2\,E)] \end{aligned} \quad (7.18)$$

Hence finally

$$\begin{aligned} & (1 - e\cos E)\,\Lambda_{u_r} = \\ & = P_0 (1 - e^2)^{1/2} \sin E + Q_0 (\cos E - e) + \\ & + \frac{\Lambda_t}{Z^4 (1 - e^2)^2} \Big[2\,(1 + e)\,(1 - \cos E) + \\ & + \frac{e^2}{2} (1 - \cos 2\,E) - 3\,E \sin E\Big] \end{aligned} \quad (7.19)$$

$$\begin{aligned} & (1 - e\cos E)\,\Lambda_{u\theta} = P_0 (\cos E - e) - \\ & - Q_0 (1 - e^2)^{1/2} \sin E + \frac{\Lambda_t}{Z^4 (1 - e^2)^{5/2}} \Big[3\,e\,E - \\ & - 3\,E \cos E + \left(2 - 2\,e - 2\,e^2 - \frac{9}{4} e^3\right) \sin E + \\ & + \frac{5}{2} e^2 \sin 2\,E - \frac{e^3}{4} \sin 3\,E\Big] \end{aligned} \quad (7.20)$$

P_0 and Q_0 are also the respective values of $\Lambda_{u\theta}$ and Λ_{u_r} at pericenter. To integrate Λ_z it is easier to form the combination

$$\begin{aligned} \frac{d}{d\theta} (\Lambda_z + \Lambda_{u\theta}) &= \Lambda_t \frac{u_\theta + 2\,z}{z^2 u_\theta^3} \\ &= \Lambda_t \frac{3 + e\cos f}{Z^4 (1 + e\cos f)^3} \end{aligned} \quad (7.21)$$

We find

$$\begin{aligned} \Lambda_z = R_0 - \Lambda_{u\theta} + \frac{\Lambda_t}{Z^4 (1 - e^2)^{5/2}} \Big[(3\,E + \\ + e\,(e^2 - 5) \sin E + \frac{e^2}{2} \sin 2\,E\Big] \end{aligned} \quad (7.22)$$

and at pericenter the value is $R_0 - P_0$.

The orbital behaviour of the multipliers is now completely known; Λ_μ and Λ_t are constants, Λ_{u_r}, $\Lambda_{u\theta}$ and Λ_z are given in terms of integration constants P_0, Q_0 and R_0.

Since the hamiltonian K is constant, we must find that for $\xi = 0$, K_0 is constant. Indeed, substituting the integrals into (7.4) it is found that

$$K_0 = K = e\,Z\,Q_0 + \frac{\Lambda_t}{Z^3 (1 + e)^2} \quad (7.23)$$

The KEPLER equation resulting from the integration of (7.6) takes the form

$$t = t_0 + \frac{E - e\sin E}{Z^2 (1 - e^2)^{3/2}} \quad (7.24)$$

where t_0 is the epoch of pericenter.

These results allow to write down immediately the orbital integrals in other coordinate systems by means of the canonical transformation equations. Simplifications occur if:

The polar transit angle is not specified,

$$K = 0 \quad (7.25)$$

Then (7.23) furnishes Q_0 in terms of Λ_t.

The transfer time is not specified,

$$\Lambda_t = 0 \quad (7.26)$$

P and Q reduce to constant values P_0 and Q_0. Hence very simply

$$\begin{aligned} \Lambda_{u_r} &= P_0 \sin f + Q_0 \cos f \\ \Lambda_{u\theta} &= P_0 \cos f - Q_0 \sin f \\ \Lambda_z &= R_0 - \Lambda_{u\theta} \\ K &= e\,Z\,Q_0 \end{aligned} \quad (7.27)$$

Neither polar angle nor transfer time are specified,

$$\begin{aligned} \Lambda_t &= 0 \\ K &= 0 \\ \Lambda_{u_r} &= P_0 \sin f \\ \Lambda_{u\theta} &= P_0 \cos f \\ \Lambda_z &= R_0 - \Lambda_{u\theta} \end{aligned} \quad (7.28)$$

8. Orbital Transfer of Variables

The analytical integration of an orbital coasting phase makes it possible to correlate the state variables and adjoint variables at engine relight with the same variables at the preceding engine cutoff. Such a procedure, which accelerates the numerical integration of an optimal trajectory, will be called an orbital transfer of variables. It will be discussed in the simplest case, that of free polar transit angle and free transfer time.

The signal function $\overline{H}_1$, as given by eq. (3.5), is first transformed by (7.2) in

$$\overline{H}_1 = \left[\Lambda_{u_r}^2 + \left(\Lambda_{u\theta} - \frac{z}{u_\theta}\Lambda_z\right)^2\right]^{1/2} + \frac{1}{c}\mu\,\Lambda_\mu$$

The signal becomes zero at cutoff and again zero at relight. But, during the intermediate coasting phase, both μ and Λ stay constant, consequently

$$\Lambda_{u_r}^2 + \left(\Lambda_{u_\theta} - \frac{z}{u_\theta}\Lambda_z\right)^2 \tag{8.1}$$

takes the same value at cutoff and relight.

If we replace in (8.1) the multipliers by the values (7.28) they take along the coasting phase and $u_\theta = z\,(1 + e\cos f)$, we find, after dropping a constant term P_0^2, that

$$F\,(f) = \frac{(P_0\cos f - R_0)\;(3\,P_0\cos f + 2\,e\,P_0\cos^2 f - R_0)}{(1 + e\cos f)^2} \tag{8.2}$$

must take the same value at cutoff and relight. If f_c denotes the true anomaly at cutoff and f_r the true anomaly at relight, the equation

$$F\,(f_r) = F\,(f_c) \tag{8.3}$$

is algebraic of the third degree in the unknown $\cos f_r$. It can in principle be reduced to a second degree equation by removing the root $\cos f_r = \cos f_c$. However, before we discuss the general case, we shall first investigate two limiting cases.

(a) The Case $e = 0$

The coasting phase would be along a circular orbit and (8.2) reduced to

$$F\,(f) = (P_0\cos f - R_0)\;(3\,P_0\cos f - R_0) \tag{8.4}$$

Eq. (8.3) yields in this case two roots

$$\cos f_r' = \cos f_c \qquad \cos f_r'' = -\cos f_c + \frac{4\,R_0}{3\,P_0} \tag{8.5}$$

Moreover, since the true anomaly increases along the coasting phase, we have the additional necessary conditions

$$\frac{dF}{df} < 0 \quad \text{for } f = f_c \qquad > 0 \quad \text{for } f = f_r \tag{8.6}$$

for the signal function $\overline{H}_1$ to exhibit the correct changes in sign.

In the present case

$$\frac{dF}{df} = -\,6\,P_0^2\sin f\left(\cos f - \frac{2\,R_0}{3\,P_0}\right) \tag{8.7}$$

and we shall prove that this derivative actually vanishes at cutoff in the limiting case where the eccentricity also vanishes. Indeed, at any time during powered or unpowered flight, the true anomaly of the osculating orbit can be determined from

$$\tan f = \frac{z\,e\sin f}{z\,e\cos f} = \frac{u_r}{u_\theta - z}$$

with

$$0 < f < \pi \text{ if } u_r > 0$$

$$-\pi < f < 0 \text{ if } u_r < 0$$

This expression becomes indeterminate as $u_r \to 0$ and $u \to z$ with $e \to 0$. But the true value is then

$$\tan f = \frac{du_r/d\theta}{d\,(u - z)/d\theta} = \frac{1}{2}\tan\psi$$

according to eqs. (7.7). This can be transformed by (3.4) and (7.2) into

$$\tan f = \frac{1}{2}\,\lambda_{u_r}/\lambda_{u\theta} = \frac{\Lambda_{u_r}}{2\,(\Lambda_{u\theta} - \Lambda_z)}$$

Thus, if cutoff occurs for $e = 0$, we see from (7.28), that

$$\tan f_c = \frac{P_0\sin f_c}{2\,(2\,P_0\cos f_c - R_0)} \tag{8.8}$$

This relationship between true anomaly at cutoff and the parameters P_0 and R_0 can actually be satisfied in two ways:

Provided $|\,2\,R_0/3\,P_0\,| < 1$ one possible solution is

$$\cos f_c = \frac{2\,R_0}{3\,P_0} \tag{8.9}$$

it makes (8.7) vanish at cutoff and both roots (8.5) are seen to become identical.

Since then

$$F\,(f) - F\,(f_c) = 3\,P_0^2\,(\cos f - \cos f_c)^2 \geqslant 0 \tag{8.10}$$

there is no finite time interval during which the $\overline{H}_1$ signal remains negative. The second possible solution is

$$\sin f_c = 0 \tag{8.11}$$

which also causes (8.7) to vanish at cutoff.

From

$$\frac{dF}{d\cos f} = 6\,P_0^2\left(\cos f - \frac{2}{3}\,\frac{R_0}{P_0}\right) \tag{8.12}$$

it can then be concluded that the signal $\overline{H}_1$ will become negative after cutoff provided

$$\frac{2\,R_0}{3\,P_0} < 1 \qquad \text{if } \cos f_c = +1$$
$$\frac{2\,R_0}{3\,P_0} < -1 \quad \text{if } \cos f_c = -1 \tag{8.13}$$

Now, from (3.5), (3.7) and (3.8), remembering that $\lambda_\theta = 0$, we find

$$\frac{d\overline{H}_1}{dt} = \frac{d\lambda}{dt} - \frac{1}{c}\,a\,\xi\,\mu\,\overline{H}_1$$

and evaluation of $d\lambda/dt$ under the assumptions $e = 0$ ($u_r = 0$, $u_\theta^2 = 1/r$), $\sin f = 0$ ($\sin\psi = 0$, $\lambda_{u_r} = 0$) produces

$$\frac{d\lambda}{dt} = 0$$

Hence at cutoff $\overline{H}_1 = 0$

$$\frac{d\overline{H}_1}{dt} = 0 \quad \text{for} \quad \xi = 0 \quad \text{or} \quad \xi = 1$$

which agrees with $dF/df = 0$. Moreover

$$\frac{d^2\bar{H}_1}{dt^2} = \frac{d^2\lambda}{dt^2} \quad \text{for} \quad \xi = 0 \quad \text{or} \quad \xi = 1$$

and, after evaluation,

$$\frac{d^2\bar{H}_1}{dt^2} = \frac{3\,P_0^2}{\lambda\,u_\theta^6} \cos f_c \left(\frac{2\,R_0}{3\,P_0} - \cos f_c\right) \qquad (8.14)$$

This second derivative is continuous at the presumed cutoff point and, according to (8.13), is negative.

But then the signal $\bar{H}_1$ was also negative before cutoff, which contradiction finally rules the case $e = 0$ completely out.

(b) The Case $P_0 = 0$

Expression (8.2) is reduced to

$$F(f) = \frac{R_0^2}{(1 + e\cos f)^2}$$

For $e < 1$, the only root of (8.3) is

$$\cos f_r = \cos f_c$$

one of the solutions of the general case.

The signal $\bar{H}_1$ will be negative between f_c and f_r provided $\cos f > \cos f_c$. The true anomaly at cutoff will therefore be negative $-\pi < f_c < 0$ and f will reach $f_r = -f_c$ after passing through the pericenter $f = 0$. This conclusion will be verified by the discussion of the general case.

It should be noted here that P_0 and R_0 cannot vanish simultaneously; this would induce λ_{u_r} and $\lambda_{u\theta}$ to vanish for a finite period of time (the coasting phase) and this situation was recognized to be impossible at the end of section 3.

To discuss the general case, we substitute

$$\Lambda_{u_r} = \frac{P_0}{z\,e}\,u_r$$

$$\Lambda_{u\theta} - \frac{z}{u\theta}\Lambda_z = \frac{P_0}{z\,e}\left[u_\theta - \frac{z^2}{u\theta}\left(1 + \frac{e\,R_0}{P_0}\right)\right]$$

into (8.1) to produce

$$\left(\frac{P_0}{z\,e}\right)^2\left[u_r^2 + u_\theta^2 - 2\,z^2\left(1 + \frac{e\,R_0}{P_0}\right) + \frac{z^4}{u_\theta^2}\left(1 + \frac{e\,R_0}{P_0}\right)^2\right]$$

Having dealt with the special cases $P_0 = 0$ and $e = 0$ we cancel the constant factor in front, drop the constant term of the second bracket and conclude that

$$u_r^2 + u_\theta^2 + \frac{z^4}{u_\theta^2}\left(1 + \frac{e\,R_0}{P_0}\right)^2$$

takes the same value at cutoff and relight.

However, from the energy integral,

$$u_r^2 + u_\theta^2 - 2\,z\,u_\theta$$

remains constant during the coasting phase. Hence by substraction and cancellation of the constant factor z, there comes

$$2\,u_\theta + \frac{z^3}{u_\theta^2}\left(1 + \frac{e\,R_0}{P_0}\right)^2$$

Equating this expression for the cutoff value $u_\theta = x$ and the relight value $u_\theta = y$ we solve the third degree equation in y. It has two significant roots

$$y' = x \qquad (8.15)$$

$$y'' = x\,(\sigma + \sqrt{\sigma^2 + 2\,\sigma}) \qquad (8.16)$$

where

$$\sigma = \frac{1}{4}\left(\frac{z}{x}\right)^3\left(1 + \frac{e\,R_0}{P_0}\right)^2 > 0 \qquad (8.17)$$

The third root, corresponding to a minus sign before the radical in (8.16), must be discarded as negative.

The cosine of the true anomaly f'', corresponding to y'', is given by

$$y'' - z = z\,e\cos f''$$

It exists only if

$$(y'' - z)^2 \leqslant z^2 e^2 = (u_r^2 + (u_\theta - z)^2)_c \qquad (8.18)$$

the right-hand side being evaluated in the cutoff condition. If this existence test fails, the only solution is (8.15). If it succeeds, the choice between (8.15) and (8.16) can be decided by a second test, illustrated on Fig. 2.

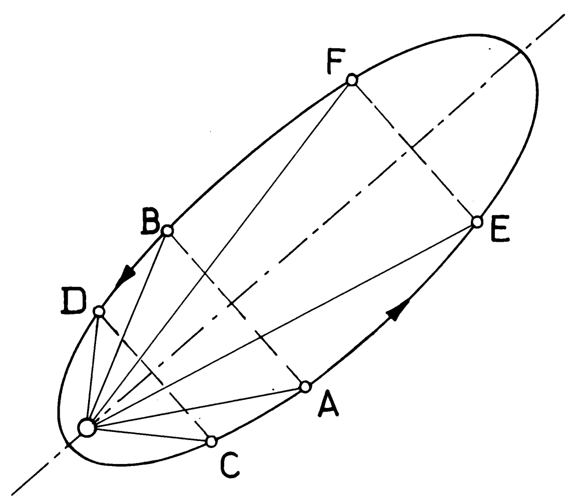

Fig. 2. Symmetrical and asymmetrical relighting conditions

(1) $(\sigma - 0.25)\,u_{r,\,c} > 0$

This occurs if, for example, the radial velocity at cutoff is positive and σ larger than 0.25, which makes $y'' > x$.

The cutoff condition is located in A, on Fig. 2, where the radial velocity is positive. The relighting conditions based on (8.16) are located in C and D.

The vehicle passes first through B which is the relighting condition associated with (8.15).

Similarly, if the radial velocity at cutoff is negative and σ smaller than 0.25, the cutoff condition is in B. Because $y'' < x$ the relighting condition (8.16) is located in E and F. The vehicle passes first through A, associated with (8.15).

(2) $(\sigma - 0.25)\,u_{r,\,c} < 0$

For a positive radial velocity (cutoff point A) the vehicle passes first through E, which is ($\sigma < 0.25$ and $y'' < x$) a relighting condition given by (8.16).

For a negative radial velocity (cutoff point B) the vehicle passes first through D, which is ($\sigma < 0.25$ and $y'' > x$) a relighting condition given by (8.16).

As $e \to 0$ and consequently $z \to u_\theta$, σ tends to 0.25 and y'' tends to y'. Due to this double root the period during which the signal can be negative vanishes. This agrees with the discussion of the case $e = 0$. As $P_0 \to 0$, σ becomes very large and y'' also, causing the true anomaly associated with (8.16) to become imaginary. The only solution is (8.15). This also agrees with the conclusions of the case $P_0 = 0$.

In conclusion it is seen that we can rely on the two tests (8.18) and sgn $(\sigma - 0.25)\, u_{r,\,c}$ to decide on the relight conditions:

(1) The Symmetrical Relight Condition

It is located symmetrically with respect to the major axis of the coasting orbit.

The variables z, u_θ, μ, $\Lambda_{u\theta}$, Λ_z and Λ_μ retain their cutoff values. The variables u_r, ψ and Λ_{u_r} change sign.

(2) The Asymmetrical Relight Condition

$$u_{\theta,\,r} = u_\theta \left(\sigma + \sqrt{\sigma^2 + 2\,\sigma}\right)$$

$$\sigma = \frac{z}{4\,u_\theta^3} \cdot \left(z + \operatorname{sgn}\,(u_r\,\Lambda_{u_r})\,\frac{(\Lambda_z + \Lambda_{u\theta})\sqrt{u_r^2 + (u_\theta - z)^2}}{\sqrt{\Lambda_{ur}^2 + \Lambda_{u\theta}^2}}\right)^2 \qquad (8.19)$$

$$u_{r,\,r} = \operatorname{sgn}\,(u_r)\sqrt{u_r^2 + u_\theta^2 - u_{\theta,\,r}^2 + 2\,z\,(u_{\theta,\,r} - u_\theta)} \qquad (8.20)$$

All right-hand side values being cutoff values except where indicated by the subscript r.

z, μ and Λ_μ retain their values. The other multipliers can be calculated from the following invariant expressions

$$\frac{\Lambda_{u\theta}}{u_\theta - z} \qquad \frac{\Lambda_{u_r}}{u_r} \qquad \Lambda_z + \Lambda_{u\theta} \qquad (8.21)$$

References

1. C. R. Faulders, Minimum-Time Steering Programs for Orbital Transfer with Low-Thrust Rockets. Astronaut. Acta **7**, 35—49 (1961).
2. D. F. Lawden, Optimal Intermediate-Thrust Arcs in a Gravitational Field. Astronaut. Acta **8**, 106—123 (1962).
3. B. Fraeijs de Veubeke and J. Geerts, Optimization of Multiple-Impulse Orbital Transfers by the Maximum Principle. Proceedings of the XVth IAF Congress, Warsaw, 1964, Vol. I, pp. 589 — 616.

Printed in Austria
Druck: Paul Gerin, Wien II.