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# BASIS OF A WELL CONDITIONED FORCE PROGRAM FOR EQUILIBRIUM MODELS VIA THE SOUTHWELL SLAB ANALOGIES



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## TABLE OF CONTENTS.

| | Page. |
|---|---|
| ABSTRACT | I |
| I. FIRST ANALOGY | 2 |
| 2. SECOND ANALOGY | 3 |
| 3. EQUILIBRIUM MODEL THEORY | |
| 3.I. The Stiffness Matrix | 6 |
| 3.2. The External Interface Loads | I2 |
| 3.3. The Solution of the Boundary Value Problem | I3 |
| 4. THE HELP OF THE ANALOGIES IN SOLVING THE MAJOR PROBLEM | |
| 4.I. Interior Values of the Stress Function(s) as Intensities of Minimal States of Selfstraining | I6 |
| 4.2. Development of a Particular State of Stress due to an External load | I8 |
| REFERENCES | 2I |

## MAIN LIST OF SYMBOLS.

| | |
|---|---|
| $\phi(x,y)$ | Airy stress function for equilibrium state of plane stress. |
| $U(x,y)$ , $V(x,y)$ | Stress functions for equilibrium state of bending. |
| $t_{xx}$ , $t_{xy}$ , $t_{yy}$ | normal force and shear flows in plane stress. |
| $M_x$ , $M_{xy}$ , $M_y$ | bending and twisting moments in bending. |
| $Q_x$ , $Q_y$ | transverse shears in bending. |
| $K_n$ | equivalent Kirchhoff transverse shear, defined by eq. (8). |
| $b_m$ | column matrix of stress parameters of element $m$ . |
| $h_m$ | ditto but in equilibrium with body loads. |
| $F_m$ | with various superscripts : flexibility matrices of element $m$ . |
| $g_m$ | generalized loads along connecting boundaries of element $m$ . |
| $q_m$ | associated generalized displacements. |
| $p_m$ | other generalized loads of element $m$ . |
| $r_m$ | associated generalized displacements. |
| $C_m$ | loads connection (incidence) matrix defined in eqs. (26). |
| $(B_m$ , $G_m$ , $H_m)$ | submatrices of $C_m$ defined in eq. (27). |
| $K_m$ | stiffness matrix of element $m$ . |
| $L_m$ | localizing (incidence) matrix for displacements defined by eq. (34). |
| $P_m$ | incidence matrix for the $p_m$ loads. |
| $P$ | defined by eq. (38). |

Abstract.-

For the purpose of obtaining upper bounds to displacements in a structural analysis into finite elements, the structure must be subdivided into equilibrium models (see for instance references 6 and 7).
It has already been noted that, while stiffness matrices can be obtained for such elements, the use of a stiffness program can be wasteful because the number of nodal displacements can be considerably larger than with displacement models.
On the contrary, the number of self-stressing states becomes much smaller and a solution by a Force program would be efficient, provided the coupling between self-stressing states be kept to a minimum.

It seems that the analogies noted by Southwell (reference 5) between displacements in the extension problem of a slab and stress functions for the flexure problem of the slab on the one hand, and between transverse displacements in flexure and the Airy stress function for extension on the other hand, provide an ideal set up for a good force program in such two-dimensional cases.
The nodal values of the stress function(s) are really force-type unknowns which define minimally coupled states of self-stressing.
Considerable thought was also given to the problem of introducing body-force type external loading into the program, as well as interface type external loads.

## I. First Analogy.

Let $t_{xx}$ , $t_{yy}$ and $t_{xy}$ denote the normal force flows and shear flow (products of stresses by the local thickness) in a state of plane stress within a finite element.
If they derive from an Airy stress function $\phi(x,y)$

$$t_{xx} = \frac{\partial^2\phi}{\partial y^2} \qquad t_{yy} = \frac{\partial^2\phi}{\partial x^2} \qquad t_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad \text{(I)}$$

they satisfy automatically the equilibrium equations

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} = 0 \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0 \qquad (2)$$

Hence, provided we can also secure continuity in the stress transmission at interfaces, we have the ingredients of an equilibrium model.
The first analogy will consist in showing that, if the Airy function is identical to the transverse deflection $w(x,y)$ of a conforming plate bending element, the continuity of stress transmission at the interfaces is fulfilled.
Indeed, along a straight interface boundary,

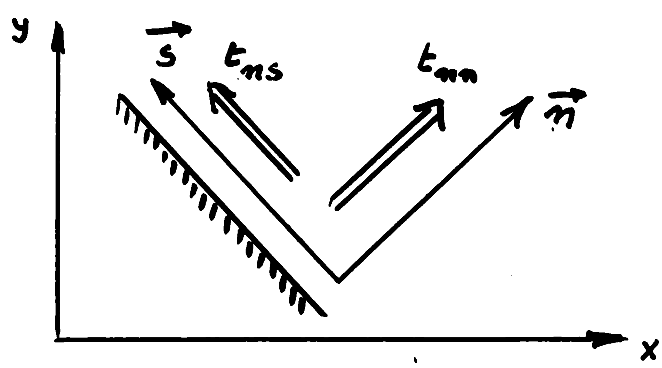



Fig. I.

the normal load flow $t_{nn}$ and shear flow $t_{ns}$ are given by

$$t_{nn} = \frac{\partial^2\phi}{\partial s^2} \qquad t_{ns} = -\frac{\partial^2\phi}{\partial n\partial s} \qquad (3)$$

Now in a conforming plate bending element the transverse deflection $w$ and the normal slope $\partial w/\partial n$ are continuous across an interface. Hence the same is true of their derivatives along the interface $\partial w/\partial s$ , $\partial^2 w/\partial s^2$ and $\partial^2 w/\partial s \partial n$ . Thus if the Airy function is everywhere identical to the transverse deflection, it follows from (3) that the flows $t_{nn}$ and $t_{ns}$ are continuously transmitted across the interface.

This analogy provides a direct conversion from a conforming displacement model for plate bending into an equilibrium model for plate stretching.

However one should observe that there is no room for the introduction of external loading except at the boundaries of the assembled structure : there are no body loads and no external interface loading modes. Those should be introduced by superimposing to the equilibrium field generated by the Airy function a particular field in equilibrium with the desired body loading modes and interface loading modes.

## 2. The second analogy.

Let the bending and twisting moments of a finite plate element be generated by two stress functions $U(x,y)$ and $V(x,y)$ as follows :

$$M_x = \frac{\partial V}{\partial y} \qquad M_y = \frac{\partial U}{\partial x} \qquad M_{xy} = -\frac{1}{2}\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) \tag{4}$$

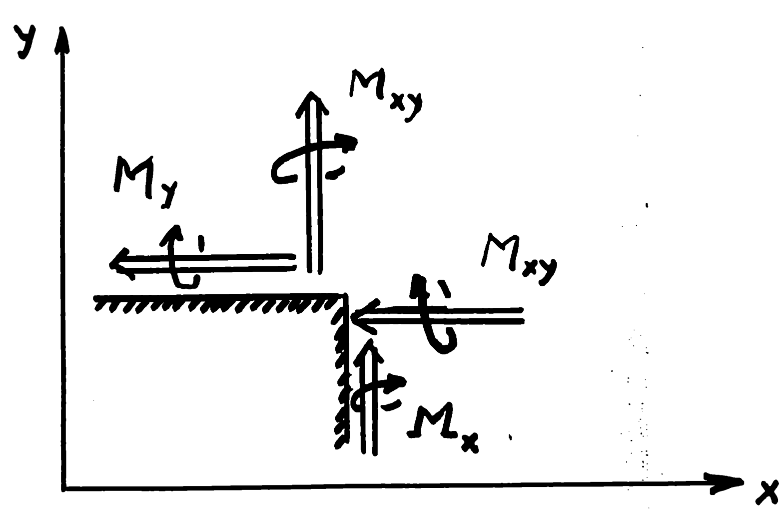



Fig. 2.

z axis upwards

Then, if the transverse shears $Q_x$ and $Q_y$ , defined in the same positive sense as $\tau_{xz}$ and $\tau_{yz}$ , are generated by the equations

$$Q_x = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) \qquad Q_y = -\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) \tag{5}$$

the equilibrium equations for moments

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y \tag{6}$$

are automatically satisfied. Furthermore, the shear loads (5) verify the equilibrium equation

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{7}$$

indicating that there is no transverse load applied to the element.
To obtain an equilibrium model for plate bending, in the framework of the Kirchhoff theory, we must still secure continuity of transmission at an interface of the bending moment $M_n$ and the equivalent Kirchhoff transverse shear

$$K_n = Q_n + \frac{\partial}{\partial s} M_{sn} \tag{8}$$

where $Q_n$ is the resultant of the $\tau_{nz}$ shear stresses and $M_{ns}$ is the twisting moment at the interface

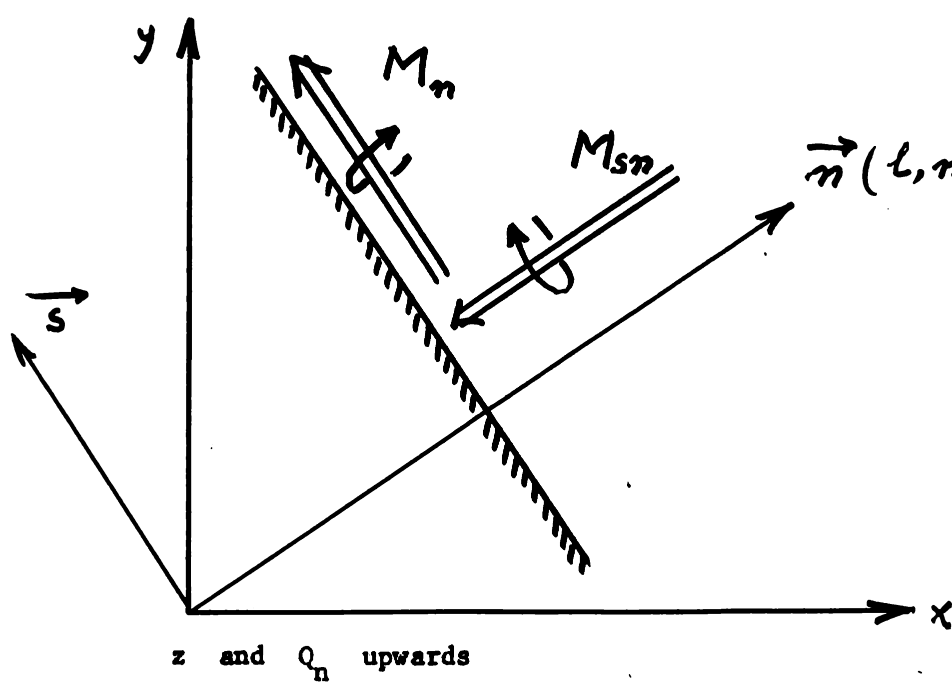



Fig. 3.

If $(l,m)$ denote the direction cosines of the outward normal to the interface, the change of orientation from the $(x,y)$ axes to the $(n,s)$ axes introduces the stress functions

$$N = \ell\, U + m\, V \qquad S = -\, m\, U + \ell\, V \tag{9}$$

from which we find that

$$M_n = \frac{\partial S}{\partial s} \tag{IO}$$

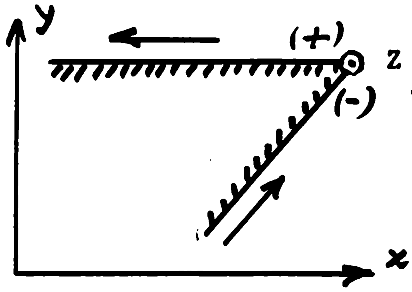
$$M_{sn} = -\frac{1}{2}\left(\frac{\partial S}{\partial n} + \frac{\partial N}{\partial s}\right) \tag{II}$$

$$Q_n = \frac{1}{2}\ \frac{\partial}{\partial s}\left(\frac{\partial S}{\partial n} - \frac{\partial N}{\partial s}\right) \tag{I2}$$

and so, from (8), that

$$K_n = \frac{\partial^2 N}{\partial s^2} \tag{I3}$$

The second analogy consists in taking for the stress functions $U(x,y)$ and $V(x,y)$ the displacement components $u(x,y)$ and $v(x,y)$ of a conforming displacement model for plate stretching. Then the property of conformity ensures that $U$ and $V$ are continuous across an interface.
The same is true of the combinations (9) and of their derivatives in the $\vec{s}$ direction. It follows then from (IO) and (I3) that the normal bending moment $M_n$ and the equivalent Kirchhoff shear load $K_n$ are continuously transmitted across the interface.
Exactly as in the case of the first analogy there appears to be no provision for external loads (except at the external boundaries of the assembled elements). However, to complete the proof of this, it must still be shown that at a common vertex the corner loads add up to zero. The proof follows easily from the analogy itself. The corner load on a single element is produced by the jump in the value of $M_{sn}$ as we turn around the corner (fig. 4).




Z (corner load positive upwards)

$= M_{sn(+)} - M_{sn(-)}$

Fig. 4.

Now, from (II) it appears that, except for a factor $-1/2$, the analog of the twisting moment is the shearing strain $\gamma_{sn}$.
The jump in shearing strain, as we turn around the corner, is equal to the reduction in the wedge angle of the strained element. Since it is obvious that, if the vertex is an interior point of the structure, the sum of all the wedge angles of the elements meeting there must remain equal to $2\pi$, the sum of all the wedge angle reductions must be zero.
So then, by the analogy, is the sum of corner loads at an interior point.
The conclusion does not hold at a vertex on the boundary of the structure, the concentrated external load that must balance there the sum of the corner loads is part of the boundary value elements of the problem.
Again, a particular stress field in equilibrium with external loading modes must be superimposed if other external loads than boundary loads are contemplated.

## 3. Equilibrium model theory.

A general theory of equilibrium models, adapted to the use of direct stiffness programs, was given in references 6 and 7. Our purpose will be to develop a new theory adapted to a force type program, because the topology of connections between equilibrium models suggests better computational efficiency with redundant forces as the basic unknowns. Also, as will appear later, the Southwell analogies provide a direct approach to the best choice of redundancies. A better picture will emerge if both theories are developed simultaneously and the opportunity will be taken to slightly modify and clarify previously used notations.

### 3.I. The stiffness matrix of an equilibrium model.

Let the subscript m denote a particular finite element of the structure. The equilibrium stress field within the element is taken to be a linear superposition of stress modes which fall into two groups. In the first group the stress modes satisfy homogeneous equilibrium equations at interior points (no body forces) and at boundary surfaces b.s. which are not potential interfaces i.f. (there are no surface tractions there).

In the case of our finite plate elements, in flexure or in extension, each assumed stress mode of the first group will thus generate no surface tractions except on the cylindrical boundary with generators parallel to the $z$ axis.
In the second group the stress modes satisfy non homogeneous equilibrium equations. Thus each of them determines an external loading mode consisting either in a given pattern of body forces, or in a given pattern of surface tractions on b.s. or both.
The reason for the distinction is obvious : each loading mode of the second group is also an independant external loading mode of the assembled structure, while the surface tractions applied to the interfaces will combine when the elements are assembled to produce another type of external loading of the structure (unless they are required to add up to zero).
With each stress mode suitably normed, the coefficients of the linear superposition or parameters of the field are grouped in a conventional sequence into column matrices. For the first group, whose parameters are denoted by $\beta_i$ , the transposed matrix will be denoted

$$b'_m = (\beta_1 , \beta_2 \dots )$$

For the second group, whose parameters are denoted $\eta_j$

$$h'_m = (\eta_1 , \eta_2 \dots )$$

The stress energy of the element can now be calculated and becomes a quadratic homogeneous form in the parameters

$$\Psi_m = \frac{1}{2} b'_m F^{bb}_m b_m + b'_m F^{bh}_m h_m + \frac{1}{2} h'_m F^{hh}_m h_m \tag{I4}$$

The matrices $F^{bb}_m$ , $F^{bh}_m$ and $F^{hh}_m$ are the flexibility matrices of the element.
One can also write

$$\Psi_m = \frac{1}{2} \begin{vmatrix} b_m \\ \\ h_m \end{vmatrix}' F_m \begin{vmatrix} b_m \\ \\ h_m \end{vmatrix} \tag{I5}$$

with the complete, non-singular, flexibility matrix

$$F_m = \begin{vmatrix} F_m^{bb} & F_m^{bh} \\ F_m^{hb} & F_m^{bb} \end{vmatrix} = F_m' \qquad (I6)$$

$$(F_m^{bb})' = F_m^{bb} \qquad (F_m^{bh})' = F_m^{hb} \qquad (F_m^{hh})' = F_m^{hh}$$

Now, for each interface boundary, we specify a complete set of surface traction modes. A linear superposition of these modes must be able to reproduce any surface traction pattern generated by the parameters of the stress field. The choice of the modes is largely governed by simplicity in the subsequent interpretation of the stress output and physical significance of the corresponding generalized loads. A generalized load $\gamma_j$ , defined at an interface, is in fact the coefficient in the expansion

$$\vec{p} = \sum_j \gamma_j \vec{P}_j \qquad (I7)$$

of the surface traction $\vec{p}$ at this interface in terms of the suitably normed modes $\vec{P}_j$ .

The corresponding generalized displacements $\kappa_j$ are interpreted by the virtual work equation on this interface

$$\int_{i.f.} \vec{p}.\vec{u} \, dArea = \sum_j \gamma_j \int_{i.f.} \vec{P}_j.\vec{u} \, dArea = \sum_j \gamma_j \kappa_j$$

Hence

$$\kappa_j = \int_{i.f.} \vec{P}_j.\vec{u} \, dArea \qquad (I8)$$

is in general a weighted average of the displacement field on the interface. In exceptional cases there are generalized loads belonging to more than one interface. A case in point is provided by the corner loads of the Kirchhoff plate bending theory. Such a corner load is necessarily a generalized load and the associated generalized displacement is obviously the local bending deflection. All the generalized loads defined at the interfaces are grouped in a conventional sequence in a column matrix denoted by $g_m$ , the corresponding generalized displacements in a column matrix $q_m$ . The virtual work of loads on the interfaces of the element is therefore

$$q_m' \, g_m = g_m' \, q_m \tag{I9}$$

Since the interface modes are generated by the stress modes a linear relationship is always available between the parameters and the generalized loads

$$g_m = B_m \, b_m + G_m \, h_m \tag{20}$$

Some important characteristics of the "load connection matrices" $B_m$ and $G_m$ will become apparent later.

Generalized loads due to body forces and surface tractions on the b.s. boundaries are necessarily linear combination of the $\eta_j$ parameters of the stress modes of the second group. If we denote them by $\pi_j$ and their column matrix by $p_m$ , we shall have

$$p_m = H_m \, h_m \tag{2I}$$

with a non-singular matrix $H_m$ . In most cases the second group of stress modes can be so devised that the $\eta_j$ parameters themselves are suitable definitions for generalized loads, the $H_m$ matrix then reduces to an identity matrix. With body forces

$$\vec{X} = \Sigma \, \pi_j \, \vec{X}_j \tag{22}$$

and surface tractions

$$\vec{p} = \Sigma \, \pi_j \, \vec{Q}_j \tag{23}$$



the virtual work equation

$$\int_{Vol} \vec{X}.\vec{u} \, dVol + \int_{b.s.} \vec{p}.\vec{u} \, dArea = \Sigma \, \pi_j \, \rho_j$$

yields the interpretation of the generalized displacements

$$\rho_j = \int_{Vol} \vec{X}_j . \vec{u} \; dVol + \int_{b.s.} \vec{Q}_j . \vec{u} \; dArea \qquad (24)$$

conjugate to the $\pi_j$ loads. The virtual work is then, in matrix form

$$r'_m \, p_m = p'_m \, r_m \qquad (25)$$

with $r'_m = (\rho_1 , \rho_2 \ldots)$ .

Equations (20) and (2I) can be combined into a single relation

$$\begin{vmatrix} g_m \\ p_m \end{vmatrix} = C_m \begin{vmatrix} b_m \\ h_m \end{vmatrix} \qquad (26)$$

with the complete load connection matrix

$$C_m = \begin{vmatrix} B_m & G_m \\ 0 & H_m \end{vmatrix} \qquad (27)$$

The complementary energy principle will now be used to obtain the best compatibility conditions. To this purpose we assume the generalized displacements specified and determine the stress parameters giving to

$$\Psi_m - (q'_m \, g_m + r'_m \, p_m)$$

its minimum value. With the stress energy expressed by (I4) and the generalized loads by (20) and (2I) in terms of the stress parameters, the minimum conditions are

$$F_m^{bb} \, b_m + F_m^{bh} \, h_m = B'_m \, q_m \qquad (28)$$

$$F_m^{hb} \, b_m + F_m^{hh} \, h_m = G'_m \, q_m + H'_m \, r_m \qquad (29)$$

Or, in equivalent form,

$$F_m \begin{vmatrix} b_m \\ h_m \end{vmatrix} = C'_m \begin{vmatrix} q_m \\ r_m \end{vmatrix} \tag{30}$$

Solving for the stress parameters

$$\begin{vmatrix} b_m \\ h_m \end{vmatrix} = F_m^{-1} \; C'_m \begin{vmatrix} q_m \\ r_m \end{vmatrix} \tag{31}$$

This, substituted into (26), gives the stiffness relations of the equilibrium element

$$\begin{vmatrix} g_m \\ p_m \end{vmatrix} = K_m \begin{vmatrix} q_m \\ r_m \end{vmatrix} \tag{32}$$

with the following stiffness matrix

$$K_m = C_m \; F_m^{-1} \; C'_m = K'_m \tag{33}$$

The load connection matrix is easily obtained, the only troublesome operation in setting up the stiffness matrix is the inversion of the flexibility matrix. For simple models this can be done analytically, for more sophisticated ones it must be done numerically and some loss in accuracy is to be feared. Furthermore, because in equilibrium models most of the interface connections are between pairs of elements only, the number of modal displacements for the structure tends to be considerably higher than for a similar set of displacement models. In counterpart there is the advantage of being able to use the same computer program.

### 3.2. The external interface loads.

The interface boundary loads $g_m$ of individual elements are added up in the process of assembling the elements and should equilibrate the interface boundary loads applied from the outside. This process can be described mathematically by first expressing the geometrical connection between elements :

$$q_m = L_m\, q \tag{34}$$

Equations like (34) state that the generalized displacements defined at the boundaries of the m-th element can be identified with certain nodal displacements of the structure.

Those are listed in a conventional sequence in the column matrix $q$ . If the identification do not involve changes of reference frame, the localizing matrix $L_m$ of the element is only composed of zeros and ones. In the more general case where the local definitions of the $q_m$ coordinates require transformation to a common reference frame at the structural level, the matrix of coordinate transformation is incorporated in $L_m$ .

The total virtual work of the loads $g_m$ must equal the virtual work of the externally applied interface loads $g$ , conjugate to $q$ . Hence

$$\sum_m g'_m\, q_m = g'\, q$$

Substituting equations (34) and noting that

$$(\sum_m g'_m\, L_m) q = g'\, q$$

must hold for any set of nodal displacements

$$g' = \sum_m g'_m\, L_m \qquad \text{or}$$

$$g = \sum_m L'_m\, g_m \tag{35}$$

We now make use of the load connection matrices defined by (36) and (37) in order to express the external interface loads in terms of the stress parameters :

$$g = \sum_m L'_m\, (B_m\, b_m + G_m\, h_m)$$

with $$h_m = H_m^{-1} \, p_m$$

so that

$$g = \sum_m L'_m \, (B_m \, b_m + G_m \, H_m^{-1} \, p_m) \tag{36}$$

The external generalized loads $p_m$ do not add up in the assembling process. We list them in a conventional sequence in a complete column matrix $p$ for the complete structure and write for each element the incidence relation

$$p_m = P_m \, p$$

Hence $P_m$ is simply an identity matrix with additional columns of zeros. Finally (36) can be placed in the form

$$g - P \, p = \sum_m L'_m \, B_m \, b_m \tag{37}$$

where $$P = \sum_m L'_m \, G_m \, H_m^{-1} \, P_m \tag{38}$$

## 3.3. The solution of the boundary value problem.

The major problem is the determination of the general solution of equation (38) for the stress parameters $b_m$, the external loads $g$ and $p$ being considered as given. Of course the $g$ and $p$ loads are not independant but should satisfy overall equilibrium. This is expressed by zero virtual work conditions

$$q'_\alpha \, g + r'_\alpha \, p = o \qquad \alpha = 1, 2 \ldots \tag{39}$$

when $q_\alpha$ and $r_\alpha$ represent a set of generalized displacements corresponding to a rigid body motion. We note that such sets of $q_\alpha$ and $r_\alpha$ can be obtained by introducing a rigid body motion $\vec{u}_\alpha$ into the interpretations of the generalized displacements.

Suppose the problem solved and let

$$b^*_m = \Pi_m \, (g - P \, p) \tag{40}$$

be a particular solution, and add the general solution of the homogeneous problem

$$\sum_m L'_m B_m b_m = o \tag{4I}$$

in the form

$$b_m = X_m x \tag{42}$$

where x is a column of independant unknowns.
Then

$$b_m = \Pi_m (g - P p) + X_m x \tag{43}$$

represents, together with

$$h_m = H_m^{-1} P_m p \tag{44}$$

the most general state of stress in the assembled structure, satisfying equilibrium with the externally applied loads.
The corresponding stress energy is

$$\Psi = \frac{1}{2} (g' F_{gg} g + p' F_{pp} p + x' F_{xx} x) + g' F_{gp} p + g' F_{gx} x + p' F_{px} x$$

with flexibility matrices

$$F_{gg} = \sum_m \Pi'_m F_m^{bb} \Pi_m$$

$$F_{pp} = \sum_m (H_m^{-1} P_m)' F_m^{hh} (H_m^{-1} P_m)$$

$$- \frac{1}{2} P' (\sum_m \Pi'_m F_m^{bh} H_m^{-1} P_m) - \frac{1}{2} (\sum_m (H_m^{-1} P_m)' F_m^{hb} \Pi_m) P$$

$$F_{xx} = \sum_m X'_m F_m^{bb} X_m$$

$$F_{gp} = \sum_m \Pi'_m F_m^{bh} H_m^{-1} P_m - (\sum_m \Pi'_m F_m^{bb} \Pi_m) P$$

$$F_{gx} = \sum_m \Pi'_m F^{bb}_m X_m$$

$$F_{px} = \sum_m (H^{-1}_m P_m)' F^{hb}_m X_m$$

$$- \frac{1}{2} P' (\sum_m \Pi'_m F^{bb}_m X_m) - \frac{1}{2} (\sum_m X'_m F^{bb}_m \Pi_m) P$$

The principle of minimum of the complementary energy is then written in the form

$$\delta \left\{ \Psi - q' g - r' p + \sum_\alpha \lambda_\alpha (g' q + p' r) \right\} = o$$

Where the displacements $q$ and $r$ are assumed to be specified and the equilibrium constraints (39) added with lagrangean multipliers to allow independant variations on all elements of $g$ and $p$.
The independant variations on the unknowns (redundant forces) $x$ furnish

$$F_{xx}\, x + F'_{gx}\, g + F'_{px}\, p = o \qquad (45)$$

Hence the unknowns are determined in terms of the external loads by

$$x = - F^{-1}_{xx} (F'_{gx}\, g + F'_{px}\, p) \qquad (46)$$

and the complete state of stress can be determined.
The independant variations on $g$ and $p$ furnish the generalized displacements

$$q = F_{gg}\, g + F_{gp}\, p + F_{gx}\, x + \sum_\alpha \lambda_\alpha q_\alpha \qquad (47)$$

$$r = F'_{gp}\, g + F_{pp}\, p + F_{px}\, x + \sum_\alpha \lambda_\alpha r_\alpha \qquad (48)$$

Once the unknowns $x$ are substituted the displacements are determined except for the undetermined rigid body modes. The solution is in fact completed for the case of an unsupported structure. Introduction of the support conditions and determination of the reactions is a straight forward final step.

In practice it may be of advantage to group the external loads g and p in a single column matrix and reduce to three the number of flexibility matrices involved.

## 4. The help of the analogies in solving the major problem.

We have seen in sections I and 2 that stress functions with continuity properties analogous to those of displacement type models can secure satisfaction of equilibrium conditions within each element and continuity of stress transmission at interfaces. However this procedure does not accept body loads and does not generate interface external loads but only loads at the boundary of the structure. This suggests immediatly that it provides at least a direct answer to the problem of finding the general solution of equation (37) in the homogeneous case $g = o$ and $p = o$ .

### 4.I. Interior values of the stress function(s) as intensities of minimal states of selfstraining.

Let $f_m$ denote the column matrix of local values of the stress function(s) for the m-th element, corresponding to the local values of displacements in the analogous displacement model. There is a linear relationship between the stress parameters $\beta_i^{'}$ of the equilibrium model and the stress function(s) values :

$$b_m = A_m f_m \qquad (49)$$

In the displacement model, linked by the analogy, the continuity of displacements is expressed by the use of localizing matrices $M_m$ (different from the previous $L_m$ ). So that continuity of the stress function is expressed by

$$f_m = M_m f \qquad (50)$$

where f is the set of local values of the stress function(s) at nodal points of the structure. We shall have to distinguish in f , the set of values x , defined at interior points of the structure, and the complementary set y , defined at points lying along a boundary of the structure.
Relation (50) will then be replaced by

$$f_m = M_m^x x + M_m^y y \qquad (51)$$

Naturally $M_m^y$ is identically zero for any element that has no boundary in common with the structure (an interior element). Combining (49) and (5I)

$$b_m = (A_m M_m^x)\ x + (A_m M_m^y)\ y \qquad (52)$$

Our first observation is that we can, as the identity in notation suggested, identify the set x with the internal load redundancies and consequently, by reference to equation (42), adopt

$$X_m = A_m M_m^x \qquad (53)$$

Indeed (see figures 4 and 5), if $y = o$, the stress function(s) is (are) identically zero along the boundary of the structure which is then unstressed.
If at an internal point, a local stress function value $x_i$ is not zero, but all other local values are taken to be zero, internal generalized loads are generated on those interfaces between the elements which meet at that point.

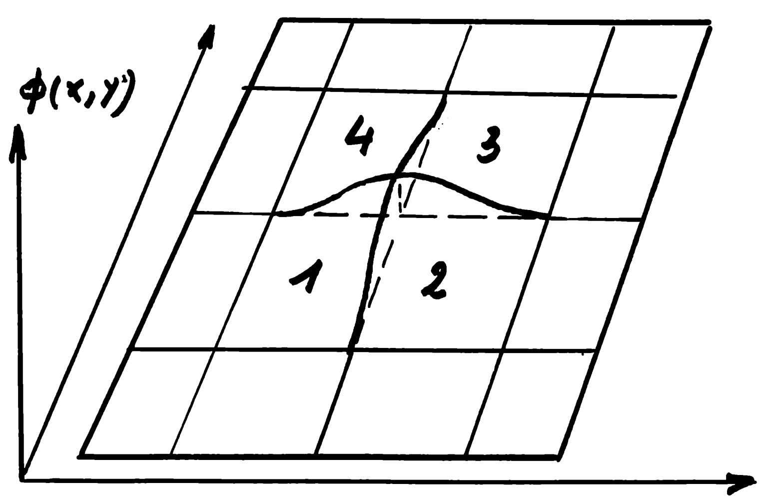



Fig. 4. Internal generalized loads generated along the interfaces between elements 1, 2, 3 and 4.

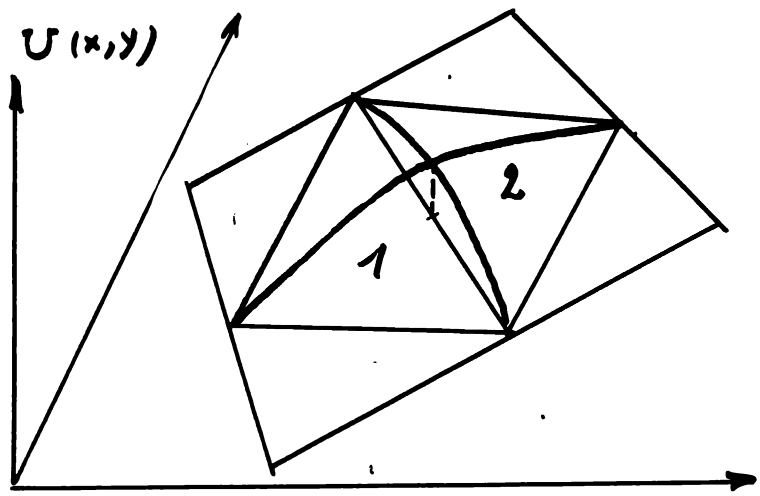



Fig. 5. Internal generalized loads are generated along the interfaces between elements 1 and 2 .

It was shown that because of the continuity of the stress function(s) those internal loads were reciprocal across each interface; no external generalized loads are produced.
Consequently, each local value of a stress function at an interior point represents a state of self-straining. Furthermore this state is of a minimal type; it induces self-equilibrating stresses in the smallest number of elements. This property is extremely valuable since it decreases the coupling of redundancies to a minimum and produces the best conditioned equations to solve.
In the case of figure 5, which is one of a local value defined at mid distance of an interface, only two finite elements are stressed. Each generalized load along the common interface must already be statically equivalent to zero (pinch type load).

## 4.2. Development of a particular state of stress due to an external load.

This problem, which is that of finding a $\Pi_m$ matrix for each element, is also simplified by the analogies, i.e. by the introduction of local stress function values.
As depicted on figure 6, we can select a chain of elements to transmit an external load $\gamma$ up to the boundary of the structure. Preferably we choose a segment of the boundary which is supported. Only the elements of the chain will be stressed.

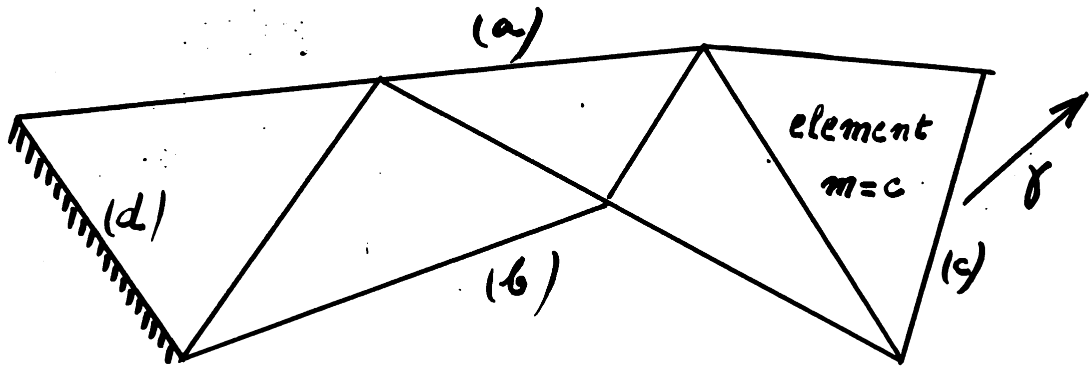



Fig. 6.

Let us distinguish the partial boundaries of the chain denoted by (a) , (b) , (c) and (d) . In the elements composing the chain, the stress parameters are calculated by

$$b_m^o = A_m f_m = A_m M_m^z z \quad \text{(m belongs to the chain)}$$

Where $M_m^z$ is a localizing matrix, a part of $M_m$ , which expresses continuity of the stress function(s) on the partial structure represented by the chain. In fact $z$ is the set of nodal local values on this chain. The values of the elements $\zeta_i$ of the matrix $z$ are taken as follows

I) $\zeta_i = o$ along the boundary (a) including the end points.
   This ensures that this boundary is unstressed.

2) Along the boundary (b), end points included, the $\zeta_i$ values form a rigid body displacement mode of the stress function(s). Then, according to the analogies, the boundary (b) is also unstressed.

3) At all nodal points interior to the chain we can take $\zeta_i = o$ .

4) Independant arbitrary values are assigned at nodal points along (c) and (d) which are not end points.

According to either one of the analogies, there are 3 independant degrees of freedom in a rigid body mode. Hence the complete matrix $z$ can be written in the form

$$z = \theta_1 z_1 + \theta_2 z_2 + \theta_3 z_3 + \Sigma \zeta_i e_i$$

where $z_1$ , $z_2$ and $z_3$ are rigid body type modes and the $e_i$ unit vectors for the independant values assigned along (c) and (d).

Hence we have

$$b_m^{\circ} = D_m\ t \tag{54}$$

where $t' = (\theta_1\ \theta_2\ \theta_3\ \ldots\ \zeta_i\ \ldots)$

and

$$D_m = A_m\ M_m^z\ (z_1\ z_2\ z_3\ \ldots\ e_i\ \ldots) \tag{55}$$

The $\zeta_i$ pertaining to (d) generate only pinch loads along (d), associated to stress parameters in the last element of the chain only. Even then no loads are generated along the other interfaces of this last element. Their determination from compatibility conditions can be left to the final adjustment of boundary conditions for the complete structure.
Hence we particularize further our particular solution by setting those $\zeta_i$ values equal to zero and in $D_m$ retain only the $\zeta_i$ pertaining to (c).
Let $m = c$ be the subscript of the first element of the chain, adjacent to (c), and denote by $G_c^c$ that part of the loads connection matrix generating the generalized loads $g_{(c)}$ along (c). The $g_{(c)}$ loads are, in our case, the external interface loads to transmit statically along the chain. Then

$$g_{(c)} = G_c^c\ b_c^{\circ} = G_c^c\ D_c\ t \tag{56}$$

The matrix $G_c^c\ D_c$ is non singular and

$$t = (G_c^c\ D_c)^{-1}\ g_{(c)}$$

This inversion operation is not costly since the total number of generalized loads along a single interface is not large. We finally obtain the required particular solution in all the elements of the chain in terms of the external loads $g_{(c)}$ as

$$b_m^{\circ} = D_m\ (G_c^c\ D_c)^{-1}\ g_{(c)} \tag{57}$$

Again this procedure is economical because the coupling between the particular solution and the hyperstatic unknowns is reduced to a small number of elements. We can deal in a similar fashion with external loads of type $p_m$. For loads of this type in the element $m = c$ we must first determine the associated reaction loads of $g$ type in this element

$$g_c = G_c h_c = G_c H_c^{-1} p_c$$

We have, to this purpose, used equation (20) under the assumption that in this element $b_c = o$. Hence in the element c the stress parameters reduce to

$$h_c = H_c^{-1} p_c$$

The reaction loads $g_c$ are then considered as external interface loads applied to the adjacent elements and transmitted by the previous procedure.

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