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Optimal Steering and Cutoff-Relight Programs for Orbital Transfers

By

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With 3 Figures

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Abstract

Optimal Steering and Cutoff-Relight Programs for Orbital Transfers. The use of orbital parameters as state variables has advantages both from the viewpoint of accuracy of numerical integration with low acceleration thrust and simplicity of correlation between cutoff values and restarting values. The transformation to such orbital parameters as z, the inverse angular momentum, and $A = z\,e\cos\theta_0$, $B = z\,e\sin\theta_0$, where e is the eccentricity and θ_0 the argument of pericenter, is effected by a canonical transformation. The new adjoint variables turn out to be themselves orbital constants if the transfer is open with respect to time. A general geometrical interpretation of the optimal steering angle is given. The reduced characteristic velocity is retained as a final choice for the independant variable and the choice of initial values discussed. In particular it is shown that the initial values of steering angle and true anomaly present advantages. The boundaries of the domains in which those initial values should be taken in order to obtain a positive gradient of the switching function can be given in analytical form.

Résumé

Programmes optimaux de guidage et d'extinction-réallumage pour transferts orbitaux. La variation de paramètres orbitaux est avantageuse du point de vue de la précision d'intégration numérique en cas de faibles accélérations propulsives et du point de vue de la simplicité des relations entre valeurs d'extinction et de réallumage. Le passage à des paramètres d'orbite tels que z, moment angulaire inverse, $A = z\,e\cos\theta_0$, $B = z\,e\sin\theta_0$, où e est l'excentricité et θ_0 l'argument du péricentre, est effectué par une transformation canonique. Les nouvelles variables adjointes sont elles-mêmes des constantes orbitales, si la durée du transfert est indifférente. Une interprétation géométrique générale de l'orientation optimale de la poussée est exposée. La vitesse caractéristique réduite est adoptée comme choix final de variable independante et un choix des valeurs d'initiation de l'intégration est proposé. Il préconise l'emploi de l'orientation de la poussée et de l'anomalie vraie au départ comme inconnues initiales et présente l'avantage de pouvoir définir analytiquement les frontières des domaines dans lesquels ces inconnues assurent un gradient positif du signal d'allumage.

Zusammenfassung

Optimale Schubvektor-Steuerprogramme und Programme für das Aus- und Einschalten der Triebwerke bei Umlaufbahnübergängen. Die Verwendung von Umlaufbahnparametern als Zustandsvariable bietet sowohl vom Standpunkt der Genauigkeit der numerischen Integration mit geringer Antriebsbeschleunigung als auch von der Einfachheit der Zuordnung zwischen Brennschluß-Werten und Wiederstart-Werten Vorteile. Die Transformation in solche Umlaufbahn-Parameter, wie z, das reziproke Drehmoment, und $A = z\,e\cos\theta_0$, $B = z\,e\sin\theta_0$, wobei e die Exzentrizität und θ_0 das Argument des nächsten Punktes zum Zentrum ist, wird durch eine kanonische Transformation erreicht. Die neu zugeordneten Variablen erweisen sich selbst als Umlaufbahnkonstante, wenn der Übergang in Bezug auf die Zeit unbestimmt ist. Eine allgemeine geometrische Interpretation des optimalen Schubwinkels wird gegeben. Die reduzierte charakteristische Geschwindigkeit wird als letzte, unabhängige Variable ausgewählt und die Wahl der Anfangsbedingungen diskutiert. Besonders wird aufgezeigt, daß als Anfangsbedingungen der Schubwinkel und die wahre Anomalie Vorteile bringen. Die Grenzen der Bereiche, in der diese Anfangsbedingungen angewendet werden sollten, um einen positiven Gradienten der Schaltfunktion zu erreichen, können in analytischer Form dargestellt werden.

1. Variation of Orbital Parameters

In [1] various formulations of the optimal thrust-coast-thrust transfer problems were investigated with the fundamental purpose of establishing analytical properties. For the practical computation of trajectories a formulation in terms of orbital parameters is generally more accurate, especially if the ratio of maximum thrust to reference vehicle weight is small. It presents also obvious advantages for the transfer of variables between cutoff and relighting conditions during the orbital coasting phases. The orbital parameters selected are those of section 7 of [1]

$$z = \frac{1}{r\,u_\theta} \tag{1.1}$$

the inverse angular momentum

$$A = z\,e\cos\theta_0 \quad B = z\,e\sin\theta_0 \tag{1.2}$$

were e is the eccentricity and θ_0 the argument of pericenter.

From eqs. (7.10) and (7.11) of [1] the parameters A and B are expressible in terms of the polar variables as

$$A = u_r \sin\theta + \left(u_\theta - \frac{1}{r\,u_\theta}\right)\cos\theta$$
$$B = -\,u_r \cos\theta + \left(u_\theta - \frac{1}{r\,u_\theta}\right)\sin\theta \qquad (1.3)$$

With θ as new independant variable we set up the canonical transformation

$$\lambda_r\,dr + \lambda_\theta\,d\theta + \lambda_{u_r}\,du_r + \lambda_{u\theta}\,du_\theta + \lambda_\mu\,d\mu - H\,dt =$$
$$= \lambda_t\,dt + \lambda_z\,dz + \lambda_A\,dA + \lambda_B\,dB + \lambda_\mu\,d\mu - L\,d\theta$$

Replacing the differentials of (1.1) and (1.3) in the right-hand side and identifying with the left-hand side, the old multipliers are found in terms of the new ones

$$\lambda_r = \frac{1}{r^2\,u_\theta}\,(-\,\lambda_z + \lambda_A\cos\theta + \lambda_B\sin\theta) \qquad (1.4)$$

$$\lambda_\theta = A\,\lambda_B - B\,\lambda_A - L \qquad (1.5)$$

$$\lambda_{u_r} = \lambda_A\sin\theta - \lambda_B\cos\theta \qquad (1.6)$$

$$\lambda_{u\theta} = -\frac{z}{u_\theta}\,\lambda_z + \left(1 + \frac{z}{u_\theta}\right)(\lambda_A\cos\theta + \lambda_B\sin\theta) \qquad (1.7)$$

$$H = -\,\lambda_t \qquad (1.8)$$

Substitution of these values in the old hamiltonian

$$H = H_0 + a\,\xi\,\mu\,H_1 \qquad (1.9)$$

where H_0 and H_1 are given by eqs. (3.2) and (3.3) of [1], gives

$$H_0 = -\,z\,(z + A\cos\theta + B\sin\theta)^2\,L \qquad (1.10)$$

$$H_1 = \frac{1}{c}\,\mu\,\lambda_\mu + \sin\psi\,(\lambda_A\sin\theta - \lambda_B\cos\theta) + $$
$$+ \cos\psi\left(\lambda_A\cos\theta + \lambda_B\sin\theta + \frac{z\,(-\,\lambda_z + \lambda_A\cos\theta + \lambda_B\sin\theta)}{z + A\cos\theta + B\sin\theta}\right) \qquad (1.11)$$

Eliminating H and H_0 between (1.8), (1.9) and (1.10) and solving for the new hamiltonian L:

$$L = \frac{\lambda_t + a\,\xi\,\mu\,H_1}{z\,(z + A\cos\theta + B\sin\theta)^2} \qquad (1.12)$$

where H_1 is given in terms of the new variables by (1.11).

In this new formulation the time equation

$$\frac{dt}{d\theta} = \frac{\partial L}{\partial \lambda_t} = \frac{1}{z\,(z + A\cos\theta + B\sin\theta)^2} \qquad (1.13)$$

is separable. It can be left out of the search for an optimal trajectory except if the transfer duration is specified. The conjugate equation

$$d\,\lambda_t/d\,\theta = \partial\,L/\partial\,t = 0 \qquad (1.14)$$

shows λ_t to be the isoperimetrical time constant as is otherwise obvious from (1.8).

The case $\lambda_t = 0$, which occurs if the flight duration is left open, is particularly simple since then the new hamiltonian is proportional to the control ξ and the multipliers λ_z, λ_A and λ_B become orbital constants just like their conjugate state variables.

The hamiltonian itself is no more an isoperimetrical constant as

$$d\,L/d\,\theta = \partial\,L/\partial\,\theta$$

is not zero. However a first integral is immediatly available from (1.5) in terms of the old isoperimetrical constant λ_θ related to the total polar angle of the trajectory. In particular, if this angle is left open

$$L = A\,\lambda_B - B\,\lambda_A \qquad (\lambda_\theta = 0) \qquad (1.15)$$

The optimal steering control is still determined by

$$\lambda\sin\psi = \lambda_{u_r} \qquad \lambda\cos\psi = \lambda_{u\theta} \qquad \lambda = \sqrt{\lambda_{u_r}^2 + \lambda_{u\theta}^2} \qquad (1.16)$$

with λ_{u_r} and $\lambda_{u\theta}$ calculated from (1.6) and (1.7) as auxiliary variables together with

$$u_\theta = z + A\cos\theta + B\sin\theta \qquad (1.17)$$

The cutoff and relighting signal is still given by the maximum value of H_1

$$\overline{H}_1 = \lambda + \frac{1}{c}\,\mu\,\lambda_\mu \qquad (1.18)$$

$$\xi = 0 \quad \text{if} \quad \overline{H}_1 < 0$$
$$\xi = 1 \quad \text{if} \quad \overline{H}_1 > 0$$

2. A Geometrical Construction for the Optimal Steering Angle

When the sine and cosine of the optimal steering angle are expressed in terms of the new variables

$$N\sin\psi = (\lambda_A\sin\theta - \lambda_B\cos\theta)\,(z + A\cos\theta + B\sin\theta)$$
$$N\cos\psi = (\lambda_A\cos\theta + \lambda_B\sin\theta)\cdot$$
$$\cdot\,(2\,z + A\cos\theta + B\sin\theta) - z\,\lambda_z \qquad (2.1)$$

where the modulus N is of course the square root of the sum of the squares of the right-hand sides. Using the known interpretation (1.2) of A and B and defining a new modulus m and angle θ_1 by

$$\lambda_A = m\cos\theta_1 \qquad \lambda_B = m\sin\theta_1 \qquad (2.2)$$

we can write

$$\frac{N}{m\,z}\sin\psi = \sin(\theta - \theta_1)\,[1 + e\cos(\theta - \theta_0)]$$
$$\frac{N}{m\,z}\cos\psi = \cos(\theta - \theta_1)\cdot$$
$$\cdot\,[2 + e\cos(\theta - \theta_0)] - \lambda_z/m \qquad (2.3)$$

or, with $f = \theta - \theta_0$ the true anomaly, and $\theta_2 = \theta_1 - \theta_0$

$$\frac{N}{m\,z}\sin\psi = \sin(f - \theta_2) - $$
$$- \frac{e}{2}\sin\theta_2 + \frac{e}{2}\sin(2\,f - \theta_2) \qquad (2.4)$$

$$\frac{N}{m\,z}\cos\psi = 2\cos(f - \theta_2) + $$
$$+ \frac{e}{2}\cos\theta_2 + \frac{e}{2}\cos(2\,f - \theta_2) - \lambda_z/m$$

These expressions lead to the geometrical interpretation of Fig. 1, which generalizes that due to MAREC [2] for the case of infinitesimally close transfers.

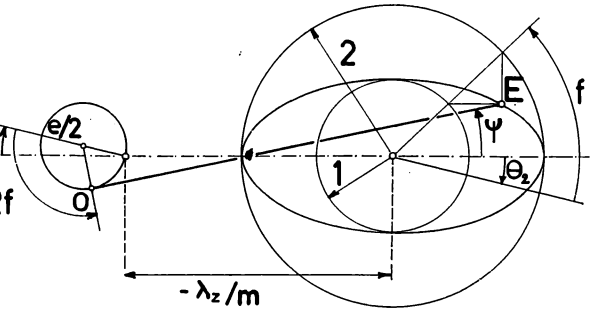

Fig. 1. Geometrical interpretation of the optimal steering angle Ψ

Under orbital conditions with $\lambda_t = 0$ all the parameters of the figure remain constant except the true anomaly. The origin 0 of the vector describes the small circle of radius $e/2$ at twice the angular rate of the radius positioning the extremity E on the ellipse (with fixed semi-axes of lengths 1 and 2).

It is apparent that the amplitude of oscillation of ψ is governed by the ratio λ_z/m and that λ_z must be sufficiently negative (as on the drawing) to ensure a forward thrust at a small inclination on the local horizon.

3. Orbital Transfer of Variables

The orbital transfer of variables was established and discussed in [1] in the so-called orbitally linear formulation. Its final form is actually much simpler in the present one. To convert the multipliers of the orbitally linear formulation in the new ones we set up the canonical transformation

$$\Lambda_z\,dz + \Lambda_t\,dt + \Lambda_{u_r}\,du_r + \Lambda_{u\theta}\,du_\theta + \Lambda_\mu\,d\mu - K\,d\theta = \\ = \lambda_z\,dz + \lambda_t\,dt + \lambda_A\,dA + \lambda_B\,dB + \lambda_\mu\,d\mu - L\,d\theta$$

where the differentials of

$$\begin{aligned} A &= u_r \sin\theta + (u_\theta - z)\cos\theta \\ B &= -\,u_r \cos\theta + (u_\theta - z)\sin\theta \end{aligned} \tag{3.1}$$

have to be substituted. There follows by identification

$$\Lambda_z = \lambda_z - (\lambda_A \cos\theta + \lambda_B \sin\theta) \tag{3.2}$$

$$\Lambda_t = \lambda_t \tag{3.3}$$

$$\Lambda_{u_r} = \lambda_A \sin\theta - \lambda_B \cos\theta = \lambda_{u_r} \tag{3.4}$$

$$\Lambda_{u\theta} = \lambda_A \cos\theta + \lambda_B \sin\theta = \lambda_z - \Lambda_z \tag{3.5}$$

$$\Lambda_\mu = \lambda_\mu \tag{3.6}$$

$$K = L + B\,\lambda_A - A\,\lambda_B \tag{3.7}$$

Since we are in the case $\lambda_t = 0$, so that λ_μ, λ_A, λ_B and λ_z are constant along the coasting orbit, just as μ, A, B and z, the transfer from cutoff to relighting values is governed simply by the relation between θ_c (cutoff) and θ_r (relight). Accordingly in the next equations the variables without a cutoff or relight subscript are orbital constants actually determined at cutoff. From eqs. (1.2) follow

$$\begin{aligned} z\,e\sin(\theta_c - \theta_0) &= A\sin\theta_c - B\cos\theta_c \\ z\,e\cos(\theta_c - \theta_0) &= A\cos\theta_c + B\sin\theta_c \end{aligned} \tag{3.8}$$

and the true anomaly at cutoff

$$f_c = \theta_c - \theta_0 \qquad -\pi \leqslant f_c \leqslant +\pi \tag{3.9}$$

is uniquely determined by eqs. (3.8) as, consequently, the argument θ_0 of pericenter (even after many turns of the polar angle θ).

In the case of a symmetrical relight (so-called because it occurs at a point symmetrical to the cutoff point with respect to the major axis of the coasting orbit)

$$\theta_r = 2\pi - \theta_c + 2\,\theta_0 \tag{3.10}$$

In the case of an asymmetrical relight we must first estimate the quantity σ, defined by eqs. (8.17) of [1]. In view of eqs. (7.8) of [1] and, using (1.2) and (3.5)

$$e\,\frac{R_0}{P_0} = (\Lambda_z + \Lambda_{u\theta})\,\frac{e\cos f_c}{\Lambda_{u\theta}} = \\ = \frac{\lambda_z\,(A\cos\theta_c + B\sin\theta_c)}{z\,(\lambda_A\cos\theta_c + \lambda_B\sin\theta_c)} \tag{3.11}$$

This can be further simplified if account is taken of the fact that the theory applies to the case $\lambda_\theta = 0$ so that (1.15) holds true. However, at cutoff, the hamiltonian L vanishes together with the signal $\overline{H}_1\,(\lambda_t = 0)$ hence

$$A\,\lambda_B - B\,\lambda_A = 0$$

and (3.11) reduces to

$$e\,\frac{R_0}{P_0} = \frac{A\,\lambda_z}{z\,\lambda_A} = \frac{B\,\lambda_z}{z\,\lambda_B}$$

and we can compute σ from

$$\sigma = \frac{z}{4\,u_{\theta,c}^2}\left(z + \lambda_z\,\frac{A}{\lambda_A}\right)^2 \tag{3.12}$$

with

$$u_{\theta,c} = z + A\cos\theta_c + B\sin\theta_c \tag{3.13}$$

We recall the tests required to distinguish between the occurrence of a symmetrical or an asymmetrical relight.

An existence test (8.18) of [1] for the asymmetrical case. In our new variables this test to be verified by the cutoff values is

$$[(z + A\cos\theta_c + B\sin\theta_c)\,(\sigma + \sqrt{\sigma^2 + 2\,\sigma}) - z]^2 \leqslant \\ \leqslant A^2 + B^2 \tag{3.14}$$

It really depends on the eccentricity of the coasting orbit and is equivalent to

$$\frac{1 - e}{1 + e\cos f_c} \leqslant \sigma + \sqrt{\sigma^2 + 2\,\sigma} \leqslant \frac{1 + e}{1 + e\cos f_c} \tag{3.15}$$

If it fails, the relighting condition is of the symmetrical type. If it succeeds the second test on

$$\varepsilon = (\sigma - 0.25)\,(A\sin\theta_c - B\cos\theta_c) \tag{3.16}$$

is required. If sgn $\varepsilon > 0$ the relighting condition is of symmetrical type; if sgn $\varepsilon < 0$ of asymmetrical type.

In the last case the polar angle of relight is the smallest angle $\theta_r > \theta_c$ satisfying

$$z + A\cos\theta_r + B\sin\theta_r = \\ = (\sigma + \sqrt{\sigma^2 + 2\sigma})(z + A\cos\theta_c + B\sin\theta_c) \tag{3.17}$$

4. The Characteristic Velocity as Independant Variable

The differential system of canonical equations, using the polar angle as independant variables, can immediatly be written down from the hamiltonian L.

Moreover, the rules of orbital transfer of variables allow to limit their integration to the powered phases of flight, at least when time and polar transit angle are open. It is then immediatly apparent that further simplifications are available by turning to the reduced characteristic velocity

$$\Phi = c\ln\mu \tag{4.1}$$

as the independant variable. In fact the canonical transformation

$$\lambda_\mu\, d\mu - L\, d\theta = \Lambda_\theta\, d\theta - M\, d\Phi$$

yields the following relations

$$\Lambda_\theta = -L \tag{4.2}$$

$$M = -\frac{1}{c}\mu\lambda_\mu \tag{4.3}$$

Thus, eliminating L, H_1 and λ_μ between the above equations and (1.11) and (1.12), the new hamiltonian is obtained:

$$M = \frac{\exp(-\Phi/c)}{a\xi}[\lambda_t + z(z + A\cos\theta + B\sin\theta)^2\Lambda_\theta] + \\ + \sin\psi(\lambda_A\sin\theta - \lambda_B\cos\theta) + \\ + \cos\psi\Big[\lambda_A\cos\theta + \lambda_B\sin\theta + \\ + \frac{z(-\lambda_z + \lambda_A\cos\theta + \lambda_B\sin\theta)}{z + A\cos\theta + B\sin\theta}\Big] \tag{4.4}$$

In powered flight ($\xi = 1$), the canonical equations for the state variables are:

$$\frac{dt}{d\Phi} = \frac{\partial M}{\partial\lambda_t} = \frac{1}{a}\exp(-\Phi/c)$$
$$\frac{d\theta}{d\Phi} = \frac{\partial M}{\partial\lambda_\theta} = \frac{z u_\theta^2}{a}\exp(-\Phi/c)$$
$$\frac{dz}{d\Phi} = \frac{\partial M}{\partial\lambda_z} = -\frac{z}{u_\theta}\cos\psi \tag{4.5}$$
$$\frac{dA}{d\Phi} = \frac{\partial M}{\partial\lambda_A} = \sin\psi\sin\theta + \left(1 + \frac{z}{u_\theta}\right)\cos\psi\cos\theta$$
$$\frac{dB}{d\Phi} = \frac{\partial M}{\partial\lambda_B} = -\sin\psi\cos\theta + \left(1 + \frac{z}{u_\theta}\right)\cos\psi\sin\theta$$

where

$$u_\theta = z + A\cos\theta + B\sin\theta \tag{4.6}$$

is used as an auxiliary combination of state variables.

To simplify the adjoint canonical equations and the computation of the optimal controls we also use the following auxiliary quantities, previously encountered

$$\lambda_{u_r} = \lambda_A\sin\theta - \lambda_B\cos\theta \tag{4.7}$$

$$\lambda_{u\theta} = \lambda_z - \left(1 + \frac{z}{u_\theta}\right)\Lambda_z \tag{4.8}$$

$$\Lambda_z = \lambda_z - (\lambda_A\cos\theta + \lambda_B\sin\theta) \tag{4.9}$$

$$\lambda = \sqrt{\lambda_{u_r}^2 + \lambda_{u\theta}^2} \tag{4.10}$$

$$\lambda\sin\psi = \lambda_{u_r} \qquad \lambda\cos\psi = \lambda_{u\theta} \tag{4.11}$$

Inserted into the hamiltonian (4.4) they give

$$M = \frac{\exp(-\Phi/c)}{a\xi}(\lambda_t + z u_\theta^2\Lambda_\theta) + \lambda \tag{4.12}$$

and this is used to eliminate Λ_θ in the adjoint canonical equations. Again, for powered flight ($\xi = 1$) and open time ($\lambda_t = 0$), those are found to be

$$\frac{d\lambda_z}{d\Phi} = -\frac{\partial M}{\partial z} = \left(\frac{2}{u_\theta} + \frac{1}{z}\right)s + \\ + \Lambda_z\frac{1}{u_\theta}\cdot\left(1 - \frac{z}{u_\theta}\right)\cos\psi$$
$$\frac{d\lambda_A}{d\Phi} = -\frac{\partial M}{\partial A} = \frac{\cos\theta}{u_\theta}\left(2s - \frac{z}{u_\theta}\Lambda_z\cos\psi\right) \tag{4.13}$$
$$\frac{d\lambda_B}{d\Phi} = -\frac{\partial M}{\partial B} = \frac{\sin\theta}{u_\theta}\left(2s - \frac{z}{u_\theta}\Lambda_z\cos\psi\right)$$
$$\frac{dM}{d\Phi} = \frac{\partial M}{\partial\Phi} = \frac{1}{c}s$$

where

$$s = \lambda - M \tag{4.14}$$

It is obvious from (4.3) that s is an equivalent form of the switching function (1.18), so that the powered flight phases are characterized by $s > 0$. Moreover, from (1.5) and (4.2) follows the first integral

$$A\lambda_B - B\lambda_A + \Lambda_\theta = \lambda_\theta \tag{4.15}$$

Substituting Λ_θ from this into (4.12) and specializing to powered flight, open time and open polar transit angle, we obtain

$$z u_\theta^2(A\lambda_B - B\lambda_A) = a s\exp(\Phi/c) \tag{4.16}$$

This can be used, either as a control on the numerical integration, or to calculate s without integrating the last of eqs. (4.13) and obtaining s from (4.14). It also follows from (4.16) that at cutoff or relight ($s = 0$)

$$A\lambda_B - B\lambda_A = 0 \tag{4.17}$$

A condition that persists during the intermediate coasting phases, since the left-hand side is composed of orbital constants. It was shown in section 8 of [1] that the eccentricity of those intermediate orbital phases cannot vanish. Hence (4.17) is never satisfied by the simultaneous disappearance of A and B but always by the condition

$$m\sin(\theta_1 - \theta_0) = 0$$

Then, either m is non zero and the angle θ_2 of Fig. 1 stays equal to zero or to π during an intermediate coasting phase, or m itself vanishes ($\lambda_A = 0$ and $\lambda_B = 0$); in which case it is apparent from (4.7) that $\lambda_{u_r} = 0$ and the steering angle ψ remains equal to zero or to π.

5. Initial Values

The boundary and transversality conditions for an open transfer ($\lambda_t = 0$, $\lambda_\theta = 0$) can be taken as follows.

Supposing, as in [1], the departure orbit to be specified by its reduced apocenter $\alpha(a)$ and pericenter $\beta(a)$, the initial value of z

$$z(a) = \sqrt{\frac{\alpha(a) + \beta(a)}{2\,\alpha(a)\,\beta(a)}} \tag{5.1}$$

is known, together with the initial eccentricity

$$e(a) = \frac{\alpha(a) - \beta(a)}{\alpha(a) + \beta(a)} \tag{5.2}$$

The reference direction in the orbital plane from which the polar angle is measured is taken to be that of the pericenter

$$\theta_0(a) = 0 \tag{5.3}$$

which yields the following initial values

$$A(a) = z(a)\,e(a) \qquad B(a) = 0 \tag{5.4}$$

The vehicle mass at departure is taken as unit $[\mu(a) = 1]$ so that

$$\Phi(a) = 0 \tag{5.5}$$

There is no loss of generality in taking

$$t(a) = 0 \tag{5.6}$$

This accounts for all the initial values of state variables except for the initial polar angle, identical, by virtue of (5.3) to the true anomaly at departure

$$\theta(a) = f(a) \qquad \text{unknown} \tag{5.7}$$

As discussed in [1] on the basis of the transversality condition $H_0(a) = 0$, the departure takes place when the switching function becomes positive, initiating the first powered arc of trajectory. Thus (4.17) holds true at departure with (5.4). In the general case when $e(a)$ and consequently $A(a)$ are non zero, satisfaction of (4.17) requires

$$\lambda_B(a) = 0 \tag{5.8}$$

At this stage we look at the transversality condition $\lambda_\mu(b) = -1$, which in view of (4.3) requires that the multipliers be so scaled that the end value of M be strictly positive. Now $M = \lambda$ at a cutoff or relight, which is a non negative value, and M remains constant during the orbital phases. During the powered phases ($s > 0$) it is increasing as shown by the last of eqs. (4.13). Hence M is an increasing function, constant during the orbital phases.

It is easily seen that $\lambda = 0$, and thus $M = 0$, at departure would result in all the multipliers being zero. Consequently we can scale the multipliers so that

$$M(a) = \lambda(a) = 1 \tag{5.9}$$

Then eqs. (4.7) to (4.11) reduce at departure to

$$\sin\psi(a) = \lambda_A(a)\sin f(a) \tag{5.10}$$

$$\frac{\Lambda_z(a)}{1 + e(a)\cos f(a)} = \lambda_A(a)\cos f(a) - \cos\psi(a)$$

$$= \sin\psi(a)\cot f(a) - \cos\psi(a) \tag{5.11}$$

$$\lambda_z(a) = \Lambda_z(a) + \lambda_A(a)\cos f(a) \tag{5.12}$$

A choice of $f(a)$ and $\psi(a)$ at departure successively determines $\lambda_A(a)$, $\Lambda_z(a)$ and $\lambda_z(a)$ so that all initial values of the adjoint variables are then determined. There is one exceptional case; $f(a) = 0$, which requires to adopt either $\psi(a) = 0$ or $\psi(a) = \pi$; but then the indeterminate value of $\lambda_A(a)$ become the second arbitrary choice.

The two arbitrary choices are in principle related to the satisfaction of two terminal conditions. After an undetermined number of intermediate coasting phases, a terminal point is reached when a final cutoff signal is received from the switching function (a consequnece of the last transversality condition $H_0(b) = 0$). At this point the two final values of z and e specify the terminal orbit. If they are not the values required, iteration procedures will hopefully modify $f(a)$ and $\psi(a)$ untill they are obtained.

When the system of initial conditions just developed is applied to the limiting case of zero initial eccentricity, it seems surprizing at first sight that (5.8) would still be generally valid, because (4.17) is now verified by $A(a) = B(a) = 0$. It must however be remembered that for $\lambda = 1$ and $e = 0$

$$\sin\psi = \lambda_A\sin\theta - \lambda_B\cos\theta$$
$$\cos\psi = \lambda_z - 2\,\Lambda_z$$
$$\Lambda_z = \lambda_z - (\lambda_A\cos\theta + \lambda_B\sin\theta)$$

So that

$$\lambda_A = \sin\theta\sin\psi + \cos\theta\,(\Lambda_z + \cos\psi)$$
$$\lambda_B = -\cos\theta\sin\psi + \sin\theta\,(\Lambda_z + \cos\psi)$$

and the initial values of λ_A and λ_B clearly depend on the initial value chosen for θ. This choice makes the initial value of λ_B zero, provided

$$\Lambda_z(a) = \sin\psi(a)\cot\theta(a) - \cos\psi(a)$$

and when this is substituted

$$\lambda_A(a) = \frac{\sin\psi(a)}{\sin\theta(a)}$$

This agrees with the limiting case $e(a) = 0$ for (5.11) and (5.10) and (5.12). There is however a difference in interpretation because, as will be shown, (5.3) is no more true so that $\theta(a)$ is not the same thing as the initial true anomaly. As a matter of fact the true anomaly is undetermined on the circular departure orbit but, as shown in [1], its instantaneous value becomes determinate as soon as thrust is applied:

$$\tan f(a) = \frac{1}{2}\tan\psi(a)$$

$$0 < f(a) < \pi \quad \text{if} \quad \sin\psi(a) < 0 \tag{5.13}$$
$$-\pi < f(a) < 0 \quad \text{if} \quad \sin\psi(a) > 0$$

Since $\theta(a)$ and $\psi(a)$ are chosen independantly to generate the complete set of initial values associated with $\lambda_B(a) = 0$, $f(a)$ will in general differ from $\theta(a)$. If so desired, the difference can be computed to obtain the initial value of θ_0.

In conclusion it appears that the general formulas (5.8) to (5.12) are applicable to the limiting case $e(a) = 0$, provided in them $f(a)$ be now interpreted as $\theta(a)$ and $f(a)$ taken from (5.13). This makes the initial θ_0 discontinuous with respect to the initial eccentricity, a fact which is not abnormal in view of the discontinuous behavior of θ_0 when a continuously modified elliptical orbit passes through a state of zero eccentricity.

6. Initial Switching Function Gradient

The possible choices of $f(a)$ and $\psi(a)$ are limited by the obvious condition that the initial rate of growth of the switching function must be positive.

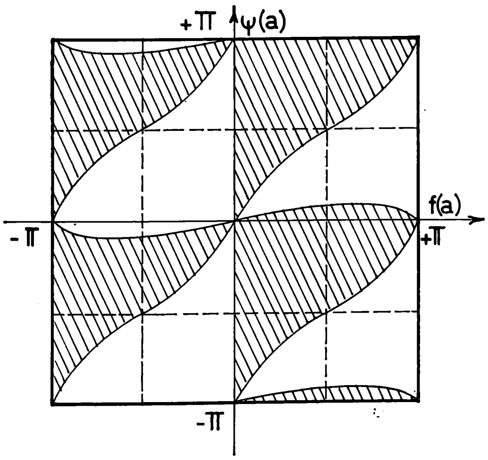

Fig. 2. Hatched domains for positive initial gradient of switching function. Initial eccentricity $e = 0.64$

From (4.16) it can be seen that this condition is equivalent to

$$\frac{d}{d\Phi}(A\,\lambda_B - B\,\lambda_A) > 0 \qquad (x = a) \qquad (6.1)$$

Calculating the left-hand side with the help of the canonical equations and the initial conditions, this gives, after reduction, the condition

$$\cot f\,(\tan\psi)^2 - 2\tan\psi + \frac{e\sin f}{1 + e\cos f} > 0 \qquad (x = a) \qquad (6.2)$$

The roots of the second degree polynomial in $\tan\psi$

$$\tan\psi = \tan f\left(1 \pm \frac{1}{\sqrt{1 + e\cos f}}\right) \qquad (6.3)$$

define the curves of Fig. 2 which are boundaries of the hatched regions where (6.2) is satisfied. Note that

$$\lim_{f \to \pi/2} \tan f\left(1 - \frac{1}{\sqrt{1 + e\cos f}}\right) = \frac{1}{2}e$$

In the limiting case $e(a) = 0$, (6.2) reduces to

$$\tan\psi\,(\cot f \tan\psi - 2) > 0 \qquad (x = a) \qquad (6.4)$$

and, along the boundary curves obtained by setting the bracket equal to zero, θ_0 is either zero or π (Fig. 3).

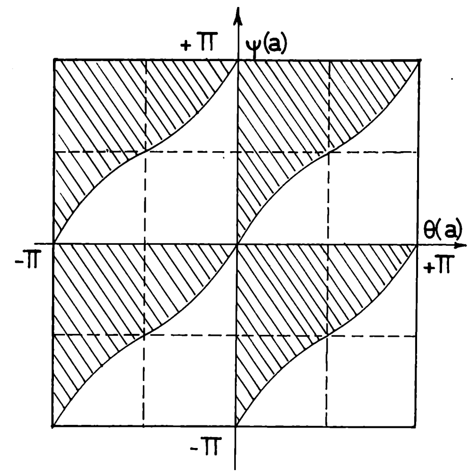

Fig. 3. Hatched domains for positive initial gradient of switching function. Limiting case of zero initial eccentricity

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