BENDING AND STRETCHING OF PLATES SPECIAL MODELS FOR UPPER AND LOWER BOUNDS*

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The purpose of this report is to deal with certain difficulties arising in the construction of equilibrium models and displacement models of finite elements, yielding upper bounds or lower bounds to static influence coefficients.

The constraints imposed on the number of parameters of the displacement field or the stress field, in relation to the number of generalized coordinates, in order to achieve either continuity of displacements or continuity of stress transmission within the structure, are sometimes hard to satisfy. Two cases are presented, a displacement model for bending and an equilibrium model for stretching, where the difficulties are removed by a particular grouping of elements in a larger building block.

To make the report self-contained, the dual theories of displacement and equilibrium models are oriefly reviewed.

DISPLACEMENT MODELS (Reference 1, 2 and 4)

The displacement field within the element is approximated by a linear superposition of a finite number of displacement modes, including the rigid body modes. The unknown intensities α_i of the assumed modes are the parameters of the field and form the coordinates of a column matrix α. From the parameters of the field we pass to a set of generalized displacements q_j according to the fillowing rules:

a. Along each boundary, where the element is to be joined to a neighboring element, a complete set of boundary displacement modes, compatible with the parametric field, is chosen. The generalized displacements pertaining to this boundary are defined to be the intensities of these boundary modes.

b. The same boundary modes are valid for the neighboring element. In this manner, equating the corresponding generalized displacements secures complete continuity of the displacement field across the elements.

The justification of these rules is that the resulting displacement field for the whole structure is piece-wise differentiable and lower bounds are obtained for static influence coefficients.

Since the boundary modes are deduced from the parametric displacement field, a linear relationship is always available between the parameters α_i and the generalized displacements q_j ; in matrix form

$$q = S\,\alpha \tag{1}$$

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On the other hand the strain energy of the element is always available as a quadratic form in the parameters:

$$U = \frac{1}{2}\, \alpha^T A \alpha \qquad (A = A^T) \tag{2}$$

Three cases must then be distinguished:

Case (a)

The simplest case is when the number m of generalized displacements is equal to the number n of parameters and Equation 1 appears as a linear transformation with a non-singular matrix S. There is a reciprocal transformation

$$\alpha = T q \qquad \text{with} \qquad T = S^{-1} \tag{3}$$

The strain energy can be written as a quadratic form in the generalized displacements

$$U = \frac{1}{2} q^T K q' \qquad \text{with} \qquad K = T^T A T \tag{4}$$

Let $\mathbf{g}$ denote the column matrix of static generalized loads associated to the generalized displacements. Then, according to the theorem of CLAPEYRON

$$U = \frac{1}{2} q^T K q = \frac{1}{2} q^T \mathbf{g}$$

and, since this is true for any $\mathbf{q}$ matrix,

$$\mathbf{g} = K q \tag{5}$$

and $\mathbf{K}$ is the "stiffness matrix" of the element.

Furthermore, considering Equation 3, the displacement field $\vec{u}$ is expressible as a linear superposition of "q - modes" $\vec{W}_j$

$$\vec{u} = \sum q_j \vec{W}_j (x, y, z) \tag{6}$$

The physical significance of generalized loads is obtained by substitution of this expression into the virtual work equation

$$q^T \mathbf{g} = \sum q_j g_j = \int \vec{X} \cdot \vec{u}\, dv + \int \vec{p} \cdot \vec{u}\, dS \tag{7}$$

where $\vec{X}$ denotes the actually applied body forces and $\vec{p}$ the actually applied surface tractions. Identification of the coefficients of the q_j on both sides gives

$$g_j = \int \vec{X} \cdot \vec{W}_j\, dv + \int \vec{p} \cdot \vec{W}_j\, dS \tag{8}$$

so that the $\mathbf{q}$ - modes appear also to be the weighting functions of the applied loads.

Case (b)

When the number of parameters exceeds the number of generalized displacements required for continuity purposes (n > m), one possible procedure is as follows:

Take a complementary set of (n - m) generalized displacements, represented by the column-matrix $\mathbf{r}$, so that

$$\begin{bmatrix} q \\ r \end{bmatrix} = S\,\alpha$$

is a non-singular transformation, whose reciprocal can be written

$$\alpha = T_q\, q + T_r\, r \tag{9}$$

The displacement field is then expressible in terms of $\mathbf{q}$ - modes and $\mathbf{r}$ - modes

$$\vec{u} = \sum q_j\, \vec{W}_j + \sum r_k\, \vec{V}_k \tag{10}$$

Substitution of Equation 9 into 2 shows the stiffness relations to be

$$g_q = K_{qq}\, q + K_{qr}\, r \tag{11}$$

$$g_r = K_{rq}\, q + K_{rr}\, r \tag{12}$$

with

$$K_{qq} = T_q'\, A T_q$$

$$K_{qr} \quad T_q' A\, T_r \quad K_{rq}'$$

$$K_{rr} = T_r'\, A T_r$$

where g_q is the set of generalized loads related to $\mathbf{q}$ and g_r to $\mathbf{r}$. If one operates with the complete stiffness matrix

$$\begin{bmatrix} K_{qq} & K_{qr} \\ K_{rq} & K_{rr} \end{bmatrix}$$

one introduces unnecessary constraints of continuity between the displacement fields of the elements, which is equivalent to lowering the bounds obtained for the influence coefficients. A final step in the procedure is therefore the elimination of the complementary set of generalized displacements. Because the $\mathbf{q}$ - modes necessarily contain the rigid body degrees of freedom, the matrix K_{rr} is non-singular and r can be taken from Equation 12 and substituted into Equation 11, yielding

$$g = g_q - K_{rr}^{-1}\quad g_r = K\, q \tag{13}$$

where

$$K = K_{qq} - K_{qr}\quad K_{rr}^{-1}\quad K_{rq} \tag{14}$$

This expression of $\mathbf{K}$ is the operational stiffness matrix of the element. It must be noted that the set of generalized loads associated to $\mathbf{q}$ is thereby modified. With the notation

$$K_{rr}^{-1} = (f_{jk})$$

the weighting functions of the new loads are displayed in the result

$$g_j = \int \vec{X} \cdot (\vec{W}_j - \sum f_{jk}\, \vec{V}_k)\, dv + \int \vec{p} \cdot (\vec{W}_j - \sum f_{jk}\, \vec{V}_k)\, dS \tag{15}$$

The procedure described has the advantage of keeping track of the physical significance of the generalized loads and also of yielding information on complementary generalized displacements of physical interest. To this purpose the equations

$$r = K_{rr}^{-1} \quad (g_r - K_{rq} \; q)$$

can be kept in slow memory and computed after solving for the q - modes. The spar element model, whose correct behavior in bending has been a source of difficulties, is an example where this procedure gives excellent results (Reference 6 and 8).

Case (c)

The worst case occurs when the number of generalized displacements, necessary for continuity purposes, exceeds the number of parameters $(m > n)$. In most of these cases an increase in the number of displacements modes (parameters), besides complicating the model, also increases the number of boundary modes, so that the inequality persists. The boundary modes never form an independent set and it proves impossible to set up a stiffness matrix. However, as will be shown later, it is sometimes possible by direct analysis to obtain continuity of displacements and independent boundary modes for a group of such elements. The success depends on the geometry of the boundaries internal to the group. A case in point is represented by the quadrilateral plate in bending (Reference 7).

EQUILIBRIUM MODELS (Reference 6)

The stress field within the element is approximated by a linear superposition of stress-modes. Each stress-mode satisfies internal equilibrium conditions. For the sake of simplicity it will be assumed that these conditions are homogeneous, although the theory can be extended to cover the case of equilibrium in the presence of body forces (Reference 6). The external loading of the structure must then be conceived to take place through surface tractions applied along the boundaries of the elements. The unknown intensities β_i of the stress-modes are the parameters of the field and form the coordinates of a column matrix β.

From the parameters of the field we pass to a set of generalized loads g_j according to the following rules:

(a) Along each boundary, where the element is to be joined to a neighboring element, a complete set of boundary surface traction modes, compatible with the parametric field, is chosen. The generalized loads pertaining to this boundary are defined to be the intensities of these surface traction-modes.

(b) The same boundary modes are valid for the neighboring element. In this manner we can obtain, either by reciprocity of generalized loads, or by equilibrium with an external loading mode of the same nature, complete continuity in the transmission of stresses.

The justification of these rules is that the resulting stress-field of the whole structure is an equilibrium field, thereby providing upper bounds to static influence coefficients. Since the boundary surface traction-modes are deduced from the parametric field, a linear relationship is always available between the parameters β_i of the field and the generalized loads g; in matrix form

$$g = C\beta \tag{16}$$

where C is the load connection matrix. As will become evident later, it is a rectangular matrix, the number of rows (generalized loads) exceeding the number of columns (parameters) by at least the number of rigid body degrees of freedom.

On the other hand the complementary energy of the element can always be expressed as a quadratic form in the parameters:

$$\psi = \frac{1}{2} \beta^T F \beta \tag{17}$$

The symmetric matrix F, the flexibility matrix of the element, is necessarily non-singular.

The physical interpretation of the generalized displacements is obtained by virtual work considerations. Let

$$\vec{p} = \sum g_j \vec{p}_j \tag{18}$$

express the decomposition of surface tractions in modes for a given boundary b. Then by definition of virtual work along this boundary,

$$\int_b \vec{p} \cdot \vec{u} \, dS = \sum g_j \int_b \vec{P}_j \cdot \vec{u} \, dS = \sum g_j q_j \tag{19}$$

Hence

$$q_j = \int_b \vec{P}_j \cdot \vec{u} \, dS \tag{20}$$

and the generalized displacements are weighted means of the displacements along the boundary under consideration, the surface traction modes playing the role of weighting functions. It should be noted that, unless the internal deformation field turns out to be integrable, no other information is available about displacements than the above weighted averages.

A stiffness matrix for the equilibrium model is now easily built up by an appeal to the complementary energy principle. To this purpose, all the generalized displacements are considered to be specified quantities and the complementary potential energy expressed in terms of the parameters:

$$- g^T q = - \beta^T C^T q$$

The principle

$$\frac{1}{2} \beta^T F \beta - \beta^T C^T q \qquad \text{minimum}$$

where the parameters can be varied independently, yields the compatibility conditions

$$F \beta = C^T q \tag{21}$$

From Equations 16 and 21 follows then

$$g = C \beta = (C F^{-1} C^T) q \tag{22}$$

In other words, the stiffness matrix of the element is

$$K = C F^{-1} C^T \tag{23}$$

In contrast with displacement models, a discrepancy between the number of parameters and the number of generalized loads does not effect the usefulness of the model through the possible appearance of spurious kinematical deformation modes. Observe first that the homogeneous system

$$C \beta = 0 \longrightarrow \beta = 0 \tag{24}$$

has only the trivial solution $\beta = o$. Indeed, referring back to the non-homogeneous system Equation 16, it is tantamount to stating that in the absence of external loads ($\mathbf{g} = \mathbf{o}$) the internal stresses must vanish. As a result, and because $\mathbf{F}$ is non-singular, the solutions of the homogeneous problem

$$\mathbf{K}\,\mathbf{q} = \mathbf{o} \tag{25}$$

are all contained in the homogeneous problem

$$\mathbf{C}^T\,\mathbf{q} = \mathbf{o} \tag{26}$$

Problem Equation 25 aims at discovering the displacement systems that leave the element unstressed. Rigid body modes

$$\mathbf{q} = \mathbf{u}_{(i)} \qquad i = 1, 2 \ldots r \tag{27}$$

are certainly solutions

$$\mathbf{C}^T\,\mathbf{u}_{(i)} = \mathbf{o} \tag{28}$$

C - m x n

and are generally to be found by elementary considerations. Hence the number m of generalized loads certainly exceeds the number n of parameters by the number r of rigid body freedoms:

$$m \geq n + r \tag{29}$$

However, because of the property Equation 24, $\mathbf{C}^T$ is a (n x m) matrix of rank n, so that the number of solutions of Equations 26 and 25 is exactly m - n. Therefore, if Equation 29 is not an equality, other displacement systems appear that leave the element unstressed, they are the kinematical deformation modes:

$$\mathbf{q} = \mathbf{z}_{(j)} \qquad \mathbf{C}^T\,\mathbf{z}_{(j)} = \mathbf{o} \qquad j = r+1, \ldots, m-(n+r) \tag{30}$$

From the structure Equation 23 of the stiffness matrix it is also obvious that

$$\mathbf{u}^T_{(i)}\,\mathbf{K} = \mathbf{o} \qquad \mathbf{z}'_{(j)}\,\mathbf{K} = \mathbf{o}$$

and, consequently, that

$$\mathbf{u}^T_{(i)}\,\mathbf{g} = \mathbf{o} \tag{31}$$

$$\mathbf{z}^T_{(j)}\,\mathbf{g} = \mathbf{o} \tag{32}$$

Equation 31 expresses the external equilibrium conditions of the element under the applied loads; Equation 32 the additional constraints on the applied loads due to the kinematical deformability of the element. These constraints constitute the undesirable feature of some equilibrium models.

Another feature of equilibrium models is the appearance of kinematical freedoms in a group of assembled elements. Each element undergoes a rigid body motion but the group is able to distort. A case in point is the quadrilateral plate formed of four triangular panels, each of which is under a plane state of constant stress (Reference 3). A way out of this difficulty is a suitable geometric pattern for the assemblage, whereby the constraints on the external loads can be satisfied simply by avoiding loads on the internal boundaries of the group.

If the elements are sufficiently sophisticated to prevent relative translations and rotations when assembled, the previous difficulty does not arise. However such elements have a tendancy to produce their own kinematical modes. Two ways are open to remove the difficulties. The first consists in sacrificing perfect stress transmission across boundaries by a reduction in the number of generalized loads. The deficiency in stress transmission should be statically equivalent to zero along a boundary so that its effect can be expected to be small, by appeal to the de Saint-Venant's principle. The potential energy of the removed loads should be converted into complementary energy, if the model is still required to give upper bounds to the influence coefficients. This can be done when the internal deformation field is integrable (Reference 6).

The second way out is a grouping of the elements in a larger building block, where, by suitable geometry and avoidance of external loading on the internal boundaries, the constraints on the loading of the external boundaries of the group are removed. This is very similar to the procedure for avoiding group kinematics and will be illustrated by the triangular panel under a plane state of linearly varying stresses.

A DISPLACEMENT MODEL FOR PLATE BENDING

This model illustrates the idea of grouping a small number of simple elements, with a suitable geometry, to provide the correct balance between number of parameters and number of generalized coordinates.

The theory developed by G. SANDER and the author (Reference 7) is presented here with considerable simplifications due to the adoption of oblique coordinates. In rectangular coordinates (ξ, η), the strain energy per unit area according to the Kirchhoff theory is

$$W = D\left(\frac{1}{2}(w_{\xi\xi} + w_{\eta\eta})^2 - (1-\nu)(w_{\xi\xi} w_{\eta\eta} - w_{\xi\eta}^2)\right) \tag{33}$$

Now let the η axis turn to form an angle α with the other. This will be our (x, y) oblique coordinate system. We have

$$\frac{\partial}{\partial\xi} = \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial\eta} = \frac{1}{\sin\alpha}\left(\frac{\partial}{\partial y} - \cos\alpha\frac{\partial}{\partial x}\right)$$

and consequently

$$\begin{aligned} w_{\xi\xi} &= w_{xx} \\ w_{\xi\eta} &= \frac{1}{\sin\alpha}(w_{xy} - \cos\alpha\, w_{xx}) \\ w_{\eta\eta} &= \frac{1}{\sin^2\alpha}(w_{yy} - 2\cos\alpha\, w_{xy} + \cos^2\alpha\, w_{xx}) \end{aligned} \tag{34}$$

Noting that the correspondence between surface elements is

$$d\xi\, d\eta = \sin\alpha\, dx\, dy$$

the strain energy of the plate in oblique coordinates turns out to be

$$U = \frac{1}{2\sin^3\alpha}\iint D\left\{(w_{xx} + w_{yy} - 2\cos\alpha\, w_{xy})^2 - 2\sin^2\alpha(1-\nu)(w_{xx}w_{yy} - w_{xy}^2)\right\}dx\,dy \tag{35}$$

We adopt for the vertical deflection a complete cubic parametric field:

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x + 2\alpha_5 xy + \alpha_6 y^2$$

$$+ 4(\alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3) \qquad (36)$$

Suppose the plate element to be triangular, two of the sides being defined to be x = o and y = o respectively.

Along y = o we have

$$w = \alpha_1 + \alpha_2 x + \alpha_4 x^2 + 4\alpha_7 x^3$$

$$w_y = \alpha_3 + 2\alpha_5 x + 4\alpha_8 x^2 \qquad (37)$$

Hence deflections and slopes along this side depend on 7 parameters or 7 boundary displacement modes. We can in fact express the parameters in terms of a triplet (deflection + 2 slopes) at one vertex, a second triplet at the second vertex and a transverse slope at mid-distance. The total number of generalized displacements required for the triangle is then equal to 12 (3 triplets at the vertices and 3 mid-distance slopes), while the total number of field parameters is only 10. One easily convinces oneself that there is no escape from this difficulty by modifying the geometry of the element. If we complicate the field by quartic terms, we need only two out of five possible and there is no guidance as to which one should be chosen. Indeed there is every reason to suspect that, unless some preferential direction is desirable in the approximation, we would ruin the isotropic behavior of the element by some arbitrary choice.

It will now be shown that if the element is grouped at the outset with three other ones, as indicated on Figure 1, perfect compatibility in deflections and slopes is achievable for the quadrilateral panel as a whole. Let Equation 36 be the field of triangle 1. Then, according to Equation 37, the same values must be retained for the parameters ($\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7$ and α_8) of the field of triangle 2, in order to preserve continuity of deflections and slopes along the internal boundary between the two. But the three remaining parameters can be arbitrarily changed to some other values (α_6' α_9' and α_{10}'). Similarly, by noting that along x = o

$$w = \alpha_1 + \alpha_3 y + \alpha_6 y^2 + 4\alpha_{10} y^3$$

$$w_x = \alpha_2 + 2\alpha_5 y + 4\alpha_9 y^2 \qquad (38)$$

Compatibility between fields 1 and 3 is achieved by keeping ($\alpha_1, \alpha_2, \alpha_3, \ , \alpha_5, \alpha_6, \alpha_9$ and α_{10}) but changing the remaining three to new arbitrary values (α_4', α_7' and α_8'). The last field, field of triangle 4, is now compatible with both fields 2 and 3, if described by the parameters ($\alpha_1, \alpha_2, \alpha_3, \alpha_4', \alpha_5, \alpha_6', \alpha_7', \alpha_8', \alpha_9'$ and α_{10}'). In other words it must incorporate both changes and this is possible because the changes affect different sets of parameters.

The total number of parameters is now increased by six units and equal to 16. The total number of generalized coordinates needed along the four sides of the quadrilateral is also 16 (four triplets at the vertices and four mid-distance transverse slopes). The major burden of the analysis is to invert the $\mathbf{q} = S\,\boldsymbol{\alpha}$ relationship. It is desirable to do so analytically, both for accuracy in the numerical work and to establish the weighting functions capable of translating any external load into generalized loads. This burden is considerably reduced by the simple geometry provided by the oblique coordinates. One method consists in using the triplet

$$\alpha_1 = w_0 \qquad \alpha_2 = w_{x,0} \qquad \alpha_3 = w_{y,0}$$

(the deflection and slopes at the origin) as auxiliary unknowns. In each triangle the seven remaining parameters can then be solved in terms of these unknowns, the triplets at the two other vertices and a transverse slope at mid-distance along the external boundary. This program entails no more than the solution of systems of order two.

Identification of the expressions found for the same parameters furnishes a system to solve for the remaining unknowns. The final result is broken down for presentation as follows: a column matrix of vertex displacements is denoted by $\mathbf{w}$, column matrices for vertex slopes are denoted by $\mathbf{w}_x$ and $\mathbf{w}_y$, a column matrix for x-slopes halfway is denoted by $\boldsymbol{\phi}$.

For the corresponding row matrices:

$$\mathbf{w}' = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

$$\mathbf{w}_x' = \begin{bmatrix} w_{x,1} & w_{x,2} & w_{x,3} & w_{x,4} \end{bmatrix}$$

$$\mathbf{w}_y' = \begin{bmatrix} w_{y,1} & w_{y,2} & w_{y,3} & w_{y,4} \end{bmatrix}$$

$$\boldsymbol{\phi}' = \begin{bmatrix} \phi_{12} & \phi_{23} & \phi_{34} & \phi_{41} \end{bmatrix}$$

The first three parameters are needed only for establishing weighting functions, not for energy computations:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{A}\,\mathbf{w} + \mathbf{A}_x\,\mathbf{w}_x + \mathbf{A}_y\,\mathbf{w}_y + \mathbf{A}_\phi\,\boldsymbol{\phi}$$

with

$$\mathbf{A} = \frac{1}{x_1 - x_3} \begin{bmatrix} -x_3 & 0 & x_1 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_x = \frac{1}{6(x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} x_1 x_3 (y_2 - y_4) & 0 & -x_1 x_3 (y_2 - y_4) & 0 \\ -3x_1 (y_2 - y_4) & -3y_4 (x_1 - x_3) & 3x_3 (y_2 - y_4) & 3y_2 (x_1 - x_3) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_y = \frac{1}{x_1 - x_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 \end{bmatrix}$$

$$A_\phi = \frac{2}{3(x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} -x_1\ x_3\ y_4 & x_1\ x_3\ y_4 & -x_1\ x_3 y_4 & x_1\ x_3\ y_4 \\ 3x_1\ y_4 & -3x_3\ y_4 & 3\,x_3\ y_2 & -3x_1\ y_2 \\ 3x_1\ x_3 & -3\,x_1\ x_3 & 3x_1\ x_3 & -3\,x_1\ x_3 \end{bmatrix}$$

The curvature matrix

$$\gamma = \begin{bmatrix} w_{xx} \\ w_{xy} \\ w_{yy} \end{bmatrix} = 2\,\gamma_0 + 8\,x\,\gamma_x + 8\,y\,\gamma_y \tag{39}$$

leads to a presentation of the other parameters as:

$$\gamma_0 = \begin{bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \Gamma_0^o\, w + \Gamma_0^x\, w_x + \Gamma_0^y\, w_y + \Gamma_0^\phi\, \phi \tag{40}$$

The (3 x 4) matrices $\Gamma_o^o, \Gamma_o^x, \Gamma_o^y, \Gamma_o^\phi$ are given in Appendix B. They are submatrices of a (3 x 16)

$$\Gamma_o = (\Gamma_o^o\ \Gamma_o^x\ \Gamma_o^y\ \Gamma_o^\phi)$$

such that

$$\gamma_0 = \Gamma_0\ q \tag{41}$$

where the order of generalized displacements is taken to be

$$q^T = (w^T\ \ w_x^T\ \ w_y^T\ \ \phi^T)$$

Similarly

$$\gamma_x = \begin{bmatrix} 3\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix} = \Gamma_x\ q \tag{42}$$

and

$$\gamma_y = \begin{bmatrix} \alpha_8 \\ \alpha_9 \\ 3\alpha_{10} \end{bmatrix} = \Gamma_y\ q \tag{43}$$

where the submatrices of Γ_x and Γ_y

$$\Gamma_x = (\Gamma_x^o\ \Gamma_x^x\ \Gamma_x^y\ \Gamma_x^\phi) \qquad \Gamma_y = (\Gamma_y^o\ \Gamma_y^x\ \Gamma_y^y\ \Gamma_y^\phi)$$

are given in Appendix B.

The strain energy Equation 35 can be written in matrix form, using the curvature matrix, as

$$U = \frac{1}{2} \iint D(\gamma^T H \gamma)\, dx\, dy \tag{44}$$

with

$$H = \frac{1}{\sin^3 \alpha} \begin{bmatrix} 1 & -2\cos\alpha & (\nu \sin^2\alpha + \cos^2\alpha) \\ -2\cos\alpha & 4\cos^2\alpha + 2(1-\nu) & -2\cos\alpha \\ (\nu \sin^2\alpha + \cos^2\alpha) & -2\cos\alpha & 1 \end{bmatrix} \tag{45}$$

To obtain the contribution of triangle 1 to the stiffness matrix, we substitute Equations 39, 41, 42, and 43, obtaining

$$\begin{aligned} K_1 = & (\Gamma_0^T H \Gamma_0) A_1^0 + (\Gamma_0^T H \Gamma_x + \Gamma_x^T H \Gamma_0) A_1^x + (\Gamma_0^T H \Gamma_y + \Gamma_y^T H \Gamma_0) A_1^y \\ & + (\Gamma_x^T H \Gamma_x) A_1^{xx} + (\Gamma_x^T H \Gamma_y + \Gamma_y^T H \Gamma_x) A_1^{yy} + (\Gamma_y^T H \Gamma_y) A_1^{yy} \end{aligned} \tag{46}$$

with geometrical constants evaluated over the area of triangle 1:

$$A_1^0 = 4 \iint_1 D\, dx\, dy$$

$$A_1^x = 16 \iint_1 Dx\, dx\, dy \qquad A_1^y = 16 \iint_1 Dy\, dx\, dy$$

$$A_1^{xx} = 64 \iint_1 Dx^2\, dx\, dy \qquad A_1^{xy} = 64 \iint_1 Dxy\, dx\, dy \qquad A_1^{yy} = 64 \iint_1 Dy^2\, dx\, dy$$

For constant bending rigidity D, we have simply

$$A_1^0 = 2 Dx_1 y_2 \qquad A_1^x = \frac{8}{3} Dx_1^2 y_2 \qquad A_1^y = \frac{8}{3} Dx_1 y_2^2$$

$$A_1^{xx} = \frac{16}{3} Dx_1^3 y_2 \qquad A_1^{xy} = \frac{8}{3} Dx_1^2 y_2^2 \qquad A_1^{yy} = \frac{16}{3} Dx_1 y_2^3$$

The matrix operations leading to K_1 are best performed numerically. To include the contributions of the three remaining triangles to the stiffness matrix, new matrices Γ should be displayed, containing the expressions of α_4', α_6', α_7', α_8', α_9' and α_{10}' in terms of the generalized displacements.

To save space, a simple rule can be given to perform the necessary modifications on the submatrices of Appendix B. To pass from triangle 1 to triangle 2 it is enough to exchange the subscripts 2 and 4 everywhere; this leaves invariant the expressions of the parameters which are still applicable (a control on the analytical inversion) but changes α_6 to α_6', α_9 to α_9' and α_{10} to α_{10}'. It also exchanges the order of the generalized displacements $w_{x,2}$, $w_{x,4}$, $w_{y,2}$, $w_{y,4}$, ϕ_{12}, ϕ_{23}, ϕ_{34} and ϕ_{41}, so that one is cautioned to restore this order by the corresponding exchange of columns in the submatrices. Similarly the exchange of subscripts 1 and 3 in the field of triangle 1, produces the expressions valid for triangle 3. Finally the double exchange produces the field for triangle 4. It should be observed that in the calculation of the geometrical constants, each exchange of subscripts must be accompanied by a change in sign. The complete (16 x 16) stiffness matrix

$$K = K_1 + K_2 + K_3 + K_4$$

is now built up but expressed in "local directions", those of the diagonals of the quadrilateral. Should they be the same for all quadrilaterals, the stiffness matrix can be used directly. If they are not, the slope directions should be aligned on two reference directions, common to all the structural elements.

AN EQUILIBRIUM MODEL FOR PLATE STRETCHING WITH A LINEARLY VARYING STRESS FIELD

Remarkable raising of lower bounds of influence coefficients were obtained in displacement models for plate stretching, when passing from the linear displacement field of Turner (Reference 1) to a quadratic one, keeping the basic triangular building block (Reference 6). It is then natural to ask whether a similar lowering of upper bounds could be obtained by passing from the constant stress equilibrium model (Reference 6), to an equilibrium model with linearly varying stresses. Besides the obvious bettering of the approximation, one might expect that such a model could avoid the kinematical deformation freedoms encountered when assembling the constant stress models (Reference 6).

It turns out that the assumption of linearly varying stresses leads to kinematical deformation modes within the element itself and that the way of overcoming this difficulty by a suitable assemblage of linear fields is exactly the same as that used for the plate bending displacement model. This is perhaps not surprising in view of the known analogy between transverse displacements in plate bending and Airy function in plate stretching. An equilibrium field, without body forces, is obtained for triangle 1, by the cubic Airy function

$$A(x, y) = \frac{1}{2} a x^2 + b\, xy + \frac{1}{2} c y^2 + \frac{1}{6} m x^3 + \frac{1}{2} p x^2 y + \frac{1}{2} q x y^2 + \frac{1}{6} r y^3$$

This equilibrium field is

$$\begin{aligned} c_x &= c + q x + r y \\ -\tau_{xy} &= b + p x + q y \\ \sigma_y &= a + m x + p y \end{aligned} \qquad (47)$$

As shown in Appendix A, all this is valid in oblique coordinates, provided stresses are defined properly. Along each side of triangle 1, we have 4 surface traction boundary modes, as shown in Figure 2. According to the equilibrium model theory we need 12 generalized loads to ensure perfect transmission of these surface traction modes to the neighboring triangles. Substracting the three rigid body modes, or overall equilibrium conditions, we should dispose of 9 parameters in the stress field and we have only 7. The triangular element has consequently 2 kinematical deformation modes. Without bothering to exhibit these, we shall make a direct analysis of the equilibrium conditions involved in the grouping of four of these elements in a quadrilateral, and find them suppressed.

Exact transmission of τ_{xy} and σ_y along the boundary y = o between triangles 1 and 2 implies that the parameters a, b, m and p remain valid for field 2. Parameters c, q and r can however take different values c', q' and r'. Similarly, perfect transmission of τ_{xy} and σ_x along x = o between triangles 1 and 3, implies conservation of b, c, r and q but leaves a, m and p free to take new values d', m' and p'. Finally, in the field of triangle 4, perfect stress transmission with its neighbors is achieved if and only if we adopt the values a', b, c', p', q', r' and m' corresponding to the double change. The total number of generalized loads needed for the external boundaries of the quadrilateral is now equal to 16. Reduced by the three overall equilibrium conditions it becomes exactly equal to the total of 13 parameters, now at our disposal.

If the generalized loads are defined as resultants of the surface traction modes described on Figure 2, the loads connection matrix **C** is extremely simple. Naturally, inversion of the flexibility matrix **F**, obtained by integration of the complementary energy of Appendix A, should be performed numerically to construct **K** according to formula 28.

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APPENDIX A

NATURAL STRESSES AND STRAINS IN OBLIQUE COORDINATES

As shown on Figure 3, the covariant components (u, v) of a displacement vector $\overrightarrow{PP'}$ are defined as the orthogonal projections of the vector on the coordinate axes. The contravariant components ($\bar{u}$, $\bar{v}$), defined by the parallelogram law, are related to (u, v) by the following transformation:

$$u = \bar{u} + \bar{v}\cos\alpha \qquad \bar{u}\sin^2\alpha = u - v\cos\alpha$$

$$v = \bar{v} + \bar{u}\cos\alpha \qquad \bar{v}\sin\alpha = v - u\cos\alpha \tag{48}$$

Using contravariant (parallelogram law) components (F_x, F_y) for any force vector and covariant components for displacements, the virtual work expression keeps the simple form

$$u F_x + v F_y$$

Thus, if the surface tractions on the elementary parallelogram are defined according to Figure 4, the work done by these forces during a small variation of the displacement field is given by:

$$\frac{\partial}{\partial x} (\sigma_x \, \delta u + \tau_{xy} \, \delta v) \, dx \, dy + \frac{\partial}{\partial y} (\tau_{yx} \, \delta u + \sigma_y \, \delta v) \, dx \, dy$$

Similarly, if there is a body force field of contravariant components (X, Y) per unit area, it produces an amount of work

$$(\sin \alpha \, dx \, dy) \, (X \, \delta u + Y \, \delta \nu)$$

The total work is stored as an increase in strain energy

$$(\sin \alpha \, dx \, dy) \, \delta W$$

where W is the strain energy per unit area.

The work equation is therefore

$$\sin \alpha \, \delta W = \frac{\partial}{\partial x} (\sigma_x \, \delta u + \tau_{xy} \, \delta v) + \frac{\partial}{\partial y} (\tau_{yx} \, \delta u + \sigma_y \, \delta v) + \sin \alpha \, (X \, \delta u + Y \, \delta v) \qquad (49)$$

We first apply this equation to the particular case

$$\delta u = \delta u_o \text{ (a constant)} \qquad \delta v = o \quad ,$$

which represents a simple translation of the element in a direction perpendicular to the y axis. Since there is no additional deformation, we must have $\delta W = o$; equation 49 reduces to the equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X \sin \alpha = o \qquad (50)$$

Similarly, a translation perpendicular to the x axis, yields

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y \sin \alpha = o \qquad (51)$$

We can use Equations 50 and 51 to simplify Equation 49

$$\sin \alpha \, \delta W = \sigma_x \frac{\partial}{\partial x} (\delta u) + \tau_{xy} \frac{\partial}{\partial x} (\delta v) + \tau_{yx} \frac{\partial}{\partial y} (\delta u) + \sigma_y \frac{\partial}{\partial y} (\delta v)$$

and consider now a small rotation $\delta \omega$ of the element about the origin:

$$\delta u = -y \sin \alpha \, \delta \omega \qquad \delta v = x \sin \alpha \, \delta \omega$$

Again we must have $\delta W = o$ and the simplified work equation reduces to the rotational equilibrium condition:

$$\tau_{xy} = \tau_{yx} \tag{52}$$

Furthermore, commuting the δ operator with partial derivatives, the work equation is placed in the form

$$\sin\alpha \, \delta \, W = \sigma_x \, \delta \, \epsilon_x + \tau_{xy} \, \delta \, \gamma_{xy} + \sigma_y \, \delta \, \epsilon_y \tag{53}$$

where

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \tag{54}$$

Equation 53 shows that the strain energy per unit area is a function of the strains defined by Equation 54, and that the stress-strain relations are

$$\sigma_x = \sin\alpha \frac{\partial W}{\partial \epsilon_x}$$

$$\tau_{xy} = \sin\alpha \frac{\partial W}{\partial \gamma_{xy}} \tag{55}$$

$$\sigma_y = \sin\alpha \frac{\partial W}{\partial \epsilon_y}$$

The definitions are such that there is an almost perfect formal identity of equilibrium equations, definition of strains and stress-strain relations, compared to the formulation in rectangular coordinates. The formal identity would indeed be perfect if the constant factor $\sin\alpha$ were absorbed in the definition of the stresses.

There remains to obtain explicit expressions for the strain energy and the stress-strain relations. This we proceed to do for an isotropic plate in a state of plane stress. Let (σ_ξ, $\tau_{\xi\eta}$, σ_η) and ($\epsilon_{\xi\eta}$, $\gamma_{\xi\eta}$, ϵ_η) denote the surface tractions and strains in rectangular coordinates (ξ, η) and let the axes x and ξ coincide; the η axis lying in the same half-plane as the y axis. Consider a segment OP, of original length dx, lying on the x axis. The orthogonal projection of a displacement vector on this direction is, by definition, u. Hence

$$\epsilon_\xi = \frac{\partial u}{\partial x} = \epsilon_x \tag{56}$$

For a segment OQ, of original length $d\eta$, lying on the η axis, the orthogonal displacement projection is

$$\bar{v} \sin\alpha = \frac{1}{\sin\alpha}(v - u\cos\alpha)$$

Hence

$$\epsilon_\eta = \frac{1}{\sin\alpha} \frac{\partial}{\partial \eta}(v - u\cos\alpha)$$

and, since the contravariant components of the segment are $dx = -d\eta \cot\alpha$ and $dy = \frac{1}{\sin\alpha} d\eta$,

$$\epsilon_\xi = -\frac{1}{\sin\alpha}\frac{\partial}{\partial x}(v - u\cos\alpha)\cot\alpha + \frac{1}{\sin\alpha}\frac{\partial}{\partial y}(v - u\cos\alpha)\frac{1}{\sin\alpha}$$

or, after reduction,

$$\epsilon_\xi = \frac{1}{\sin^2\alpha}(\cos^2\alpha\,\epsilon_x - \cos\alpha\,\gamma_{xy} + \epsilon_y) \tag{57}$$

The displacement component normal to OP is $\bar{v}\sin\alpha$, hence the rotation of OP towards OQ is measured by

$$\frac{\partial}{\partial x}(\bar{v}\sin\alpha) = \frac{1}{\sin\alpha}\frac{\partial}{\partial x}(v - u\cos\alpha)$$

The displacement component normal to OQ is u, hence the rotation of OQ towards OP is measured by

$$\frac{\partial u}{\partial\eta} = -\cot\alpha\frac{\partial u}{\partial x} + \frac{1}{\sin\alpha}\frac{\partial u}{\partial y}$$

Adding the two contributions we find

$$\gamma_{\xi\eta} = \frac{1}{\sin\alpha}(-2\cos\alpha\,\epsilon_x + \gamma_{xy}) \tag{58}$$

To summarize, the strain components of the cartesian system are related to the oblique strain components by the matrix transformation

$$\epsilon = M e \tag{59}$$

where

$$\epsilon^T = (\epsilon_\xi \quad \gamma_{\xi\eta} \quad \epsilon_\eta)$$

$$e^T = (\epsilon_x \quad \gamma_{xy} \quad \epsilon_y)$$

$$M = \frac{1}{\sin^2\alpha}\begin{bmatrix} \sin^2\alpha & 0 & 0 \\ -2\sin\alpha\cos\alpha & \sin\alpha & 0 \\ \cos^2\alpha & -2\cos\alpha & 1 \end{bmatrix}$$

If t(x, y) denotes the thickness of the plate, the strain energy per unit area is known to be

$$W = \frac{Et}{2(1-\nu^2)}\epsilon^T\begin{bmatrix} 1 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 1 \end{bmatrix}\epsilon$$

Substitution oı Equation 59 expresses it in terms of the oblique strain components. Then, application of Equatior 55 yields

$$\begin{bmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{bmatrix} = \frac{Et}{(1-\nu^2)\sin^3\alpha} \begin{bmatrix} 1 & -\cos\alpha & \cos^2\alpha+\nu\sin^2\alpha \\ -\cos\alpha & \frac{1-\nu}{2}\sin^2\alpha+\cos^2\alpha & -\cos\alpha \\ \cos^2\alpha+\nu\sin^2\alpha & -\cos\alpha & 1 \end{bmatrix} \times \begin{bmatrix} \epsilon_x \\ \gamma_{xy} \\ \epsilon_y \end{bmatrix} \tag{60}$$

A dual procedure, based on the complementaryenergy, furnishes easily the inversion of this relation. The complementary energy per unit area ϕ, will be defined by the following Legendre transformation:

$$(\sin\alpha)\phi = \sigma_x\,\epsilon_x + \tau_{xy}\,\gamma_{xy} + \sigma_y\,\epsilon_y - W\sin\alpha$$

Whence, in view of Equation 53

$$(\sin\alpha)\delta\phi = \epsilon_x\,\delta\sigma_x + \gamma_{xy}\,\delta\tau_{xy} + \epsilon_y\,\delta\sigma_y$$

and the inverse stress-strain relations

$$\begin{aligned} \epsilon_x &= \sin\alpha\,\frac{\partial\phi}{\partial\sigma_x} \\ \gamma_{xy} &= \sin\alpha\,\frac{\partial\phi}{\partial\tau_{xy}} \\ \epsilon_y &= \sin\alpha\,\frac{\partial\phi}{\partial\sigma_y} \end{aligned} \tag{61}$$

Now, by elementary equilibrium considerations, illustrated on Figure 5, we have

$$\boldsymbol{\sigma} = \mathbf{N}\,\mathbf{s} \tag{62}$$

where

$$\sigma^T \left[\sigma_\xi \quad \tau_{\xi\eta} \quad \sigma_\eta\right]$$

$$s^T = \left[\sigma_x \quad \tau_{xy} \quad \sigma_y\right]$$

$$\mathbf{N} = \frac{1}{\sin\alpha} \begin{bmatrix} 1 & 2\cos\alpha & \cos^2\alpha \\ 0 & \sin\alpha & \sin\alpha\ \cos\alpha \\ 0 & 0 & \sin^2\alpha \end{bmatrix}$$

The complementary strain energy per unit area of plate is known to be

$$\phi = \frac{1}{2Et}\,\sigma^T \begin{bmatrix} 1 & 0 & -\nu \\ 0 & 2(1+\nu) & 0 \\ -\nu & 0 & 1 \end{bmatrix} \sigma$$

Substitution of Equation 62 expresses this energy in terms of the oblique stress system. Then, application of Equation 61 yields:

$$\begin{bmatrix} \epsilon_x \\ \gamma_{xy} \\ \epsilon_y \end{bmatrix} = \frac{1}{Et\ \sin\alpha} \begin{bmatrix} 1 & 2\cos\alpha & \cos^2\alpha - \nu\sin^2\alpha \\ 2\cos\alpha & 4\cos^2\alpha + 2(1+\nu)\sin^2\alpha & 2\cos\alpha \\ \cos^2\alpha - \nu\ \sin^2\alpha & 2\cos & 1 \end{bmatrix} \times \begin{bmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{bmatrix} \qquad (63)$$

The reciprocal character of Equations 60 and 63 can now be checked by matrix multiplication.

APPENDIX B

$$\Gamma_o^o = \frac{3}{x_1 y_2^2 (x_1 - x_3)} \begin{bmatrix} -y_2^2 & 0 & y_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ x_1 x_3 & x_1(x_1 - x_3) & -x_1^2 & 0 \end{bmatrix}$$

$$\Gamma_o^x = \frac{1}{2x_1 y_2^2 (x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} x_3 y_2^2 (y_2 - y_4) & 2y_4 y_2^2 (x_1 - x_3) & -x_3 y_2^2 (y_2 - y_4) & -2y_2^3 (x_1 - x_3) \\ 0 & x_1 y_2^2 (x_1 - x_3) & 0 & -x_1 y_2^2 (x_1 - x_3) \\ -x_1^2 x_3 (y_2 - y_4) & 0 & x_1^2 x_3 (y_2 - y_4) & 0 \end{bmatrix}$$

$$\Gamma_o^y = \frac{1}{y_2 (x_1 - x_3)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_2 & 0 & -y_2 & 0 \\ 2x_3 & -(x_1 - x_3) & -2x_1 & 0 \end{bmatrix}$$

$$\Gamma_o^\phi = \frac{2}{x_1 y_2^2 (x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} -y_4 y_2^2 (2x_1 - x_3) & x_3 y_2^2 y_4 & -x_3 y_2^3 & y_2^3 (2x_1 - x_3) \\ -x_1^2 y_2^2 & x_1 x_3 y_2^2 & -x_1 x_3 y_2^2 & x_1^2 y_2^2 \\ x_1^2 x_3 (y_4 - 2y_2) & x_1^2 x_3 (2y_2 - y_4) & -x_1^2 x_3 y_2 & x_1^2 x_3 y_2 \end{bmatrix}$$

$$\Gamma_x^o = \frac{3}{4x_1^2 y_2^2 (x_1 - x_3)} \begin{bmatrix} y_2^2 & 0 & -y_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ -x_1^2 & 0 & x_1^2 & 0 \end{bmatrix}$$

$$\Gamma_x^x = \frac{1}{8x_1^2 y_2^2 (x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} y_2^2 (y_2 - y_4)(3x_1 - 4x_3) & -3y_2^2 y_4 (x_1 - x_3) & x_3 y_2^2 (y_2 - y_4) & 3y_2^3 (x_1 - x_3) \\ 0 & -2x_1 y_2^2 (x_1 - x_3) & 0 & 2x_1 y_2^2 (x_1 - x_3) \\ x_1^3 (y_2 - y_4) & -x_1^2 y_4 (x_1 - x_3) & -x_1^2 x_3 (y_2 - y_4) & x_1^2 y_2 (x_1 - x_3) \end{bmatrix}$$

$$\Gamma_x^y = \frac{1}{4x_1 y_2 (x_1 - x_3)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -y_2 & 0 & y_2 & 0 \\ -2x_1 & 0 & 2x_1 & 0 \end{bmatrix}$$

$$\Gamma_x^\phi = \frac{1}{2x_1^2 y_2^2 (x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} y_2^2 y_4 (3x_1 - 2x_3) & -x_3 y_2^2 y_4 & x_3 y_2^3 & -y_2^3 (3x_1 - 2x_3) \\ -x_1 y_2^2 (x_3 - 2x_1) & -x_1 x_3 y_2^2 & x_1 x_3 y_2^2 & x_1 y_2^2 (x_3 - 2x_1) \\ x_1^3 (2y_2 - y_4) & -x_1^2 x_3 (2y_2 - y_4) & x_1^2 x_3 y_2 & -x_1^3 y_2 \end{bmatrix}$$

$$\Gamma_y^o = \frac{3}{4y_2^3(x_1 - x_3)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -y_2 & 0 & y_2 & 0 \\ -2x_3 & -2(x_1 - x_3) & 2x_1 & 0 \end{bmatrix}$$

$$\Gamma_y^x = \frac{1}{8x_1 y_2^3(x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} 0 & -2y_2^3(x_1 - x_3) & 0 & 2y_2^3(x_1 - x_3) \\ x_1^2 y_2(y_2 - y_4) & -x_1 y_2 y_4(x_1 - x_3) & -x_1 x_3 y_2(y_2 - y_4) & x_1 y_2^2(x_1 - x_3) \\ 2x_1^2 x_3(y_2 - y_4) & 0 & -2x_1^2 x_3(y_2 - y_4) & 0 \end{bmatrix}$$

$$\Gamma_y^y = \frac{1}{4x_1 y_2^2(x_1 - x_3)} \begin{bmatrix} -y_2^2 & 0 & y_2^2 & 0 \\ -2x_1 y_2 & 0 & 2x_1 y_2 & 0 \\ -3x_1 x_3 & 3x_1(x_1 - x_3) & 3x_1^2 & 0 \end{bmatrix}$$

$$\Gamma_y^\phi = \frac{1}{2x_1 y_2^3(x_1 - x_3)(y_2 - y_4)} \begin{bmatrix} -y_2^3(x_3 - 2x_1) & -x_3 y_2^3 & x_3 y_2^3 & y_2^3(x_3 - 2x_1) \\ x_1^2 y_2(2y_2 - y_4) & -x_1 x_3 y_2(2y_2 - y_4) & x_1 x_3 y_2^2 & -x_1^2 y_2^2 \\ x_1^2 x_3(3y_2 - 2y_4) & -x_1^2 x_3(3y_2 - 2y_4) & x_1^2 x_3 y_2 & -x_1^2 x_3 y_2 \end{bmatrix}$$

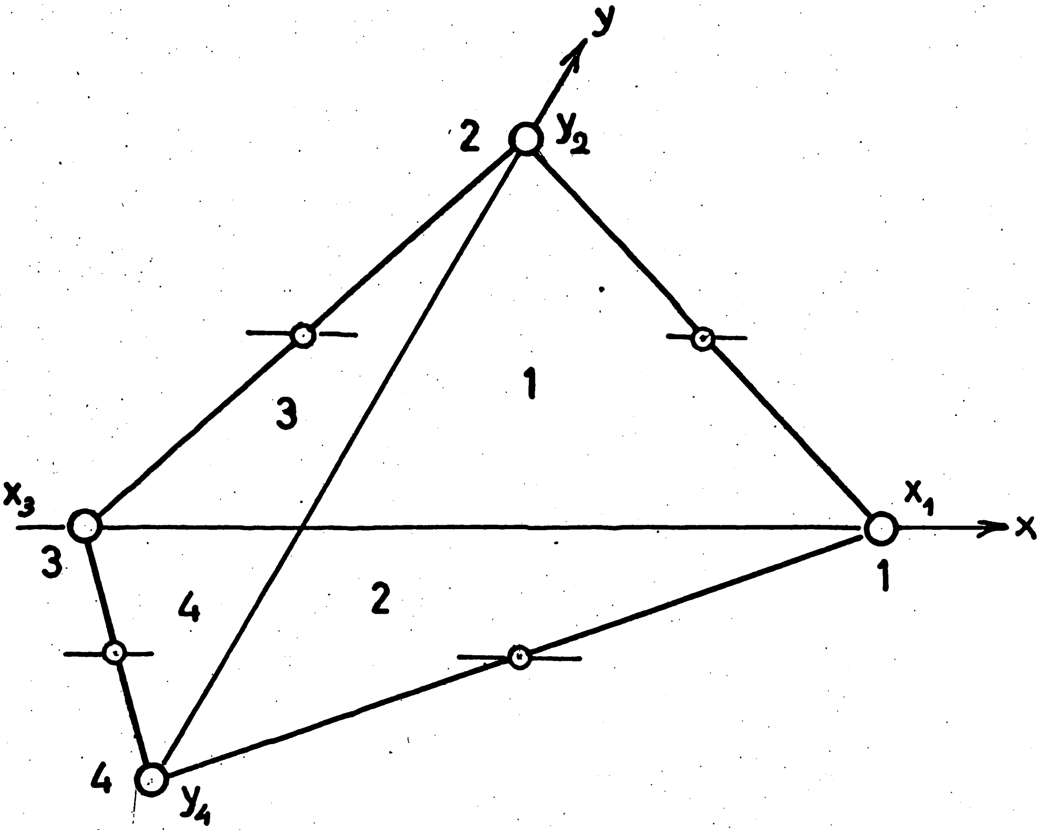

Figure 1. The Triangular Field of a Quadrilateral Plate in Oblique Coordinates

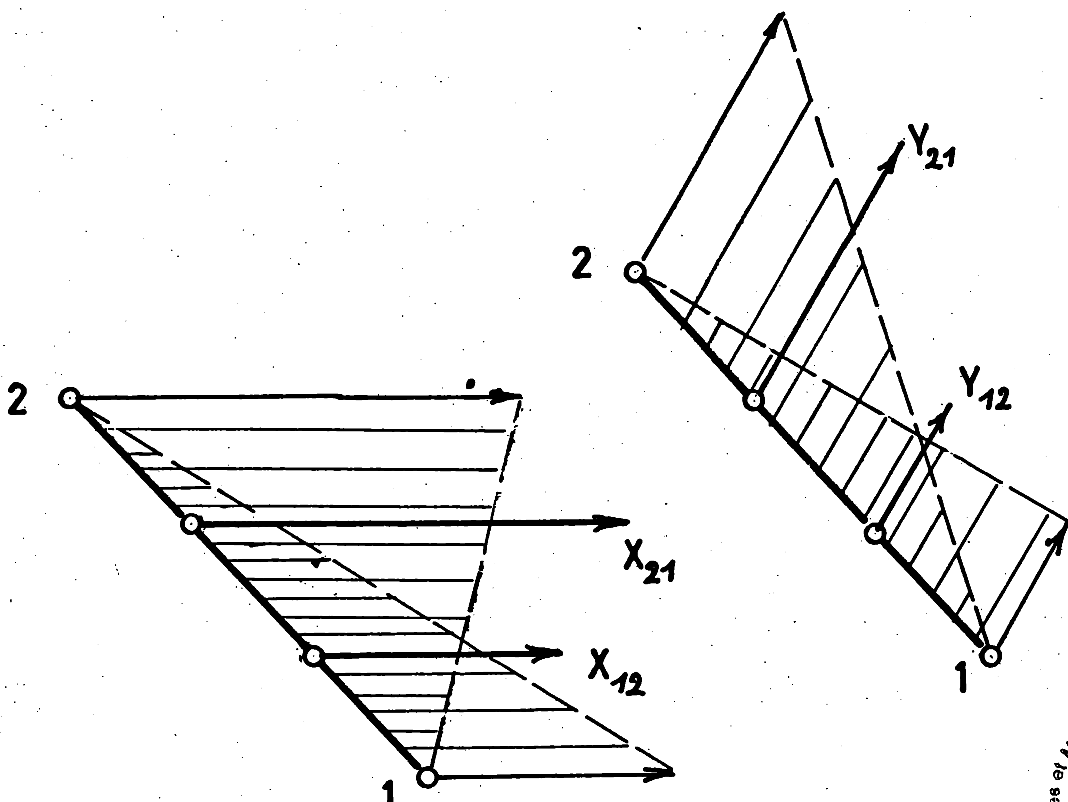

Figure 2. Surface Traction Modes and Resultants Along Boundary 1-2.

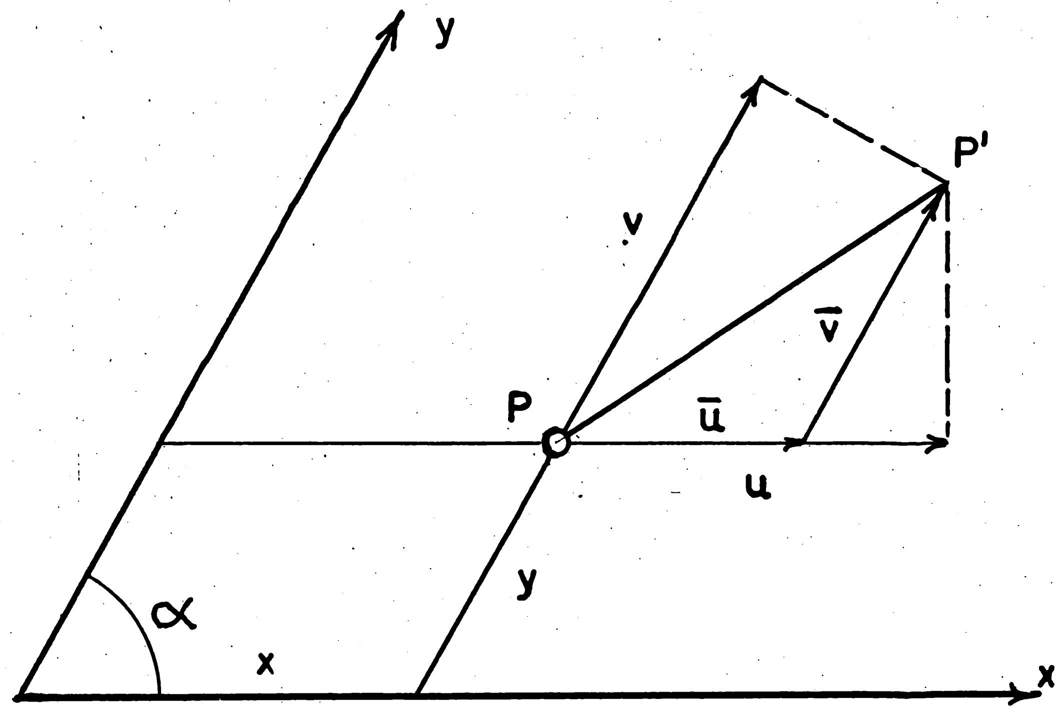

Figure 3. Covariant and Contravariant Displacement Components in Oblique Coordinates

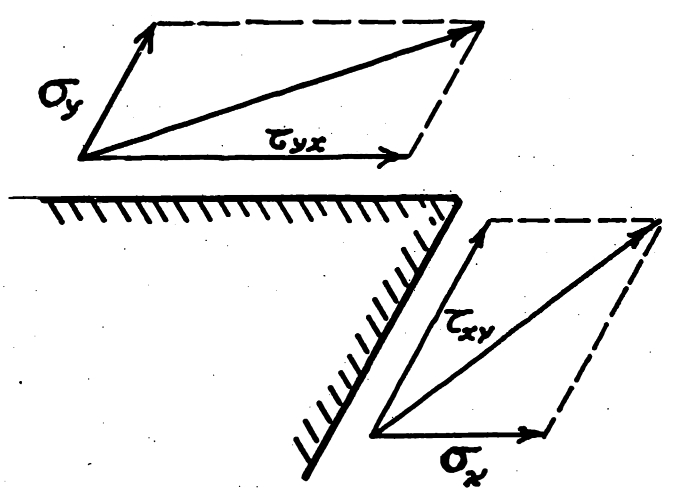

Figure 4. Definition of Stresses in Oblique Coordinates

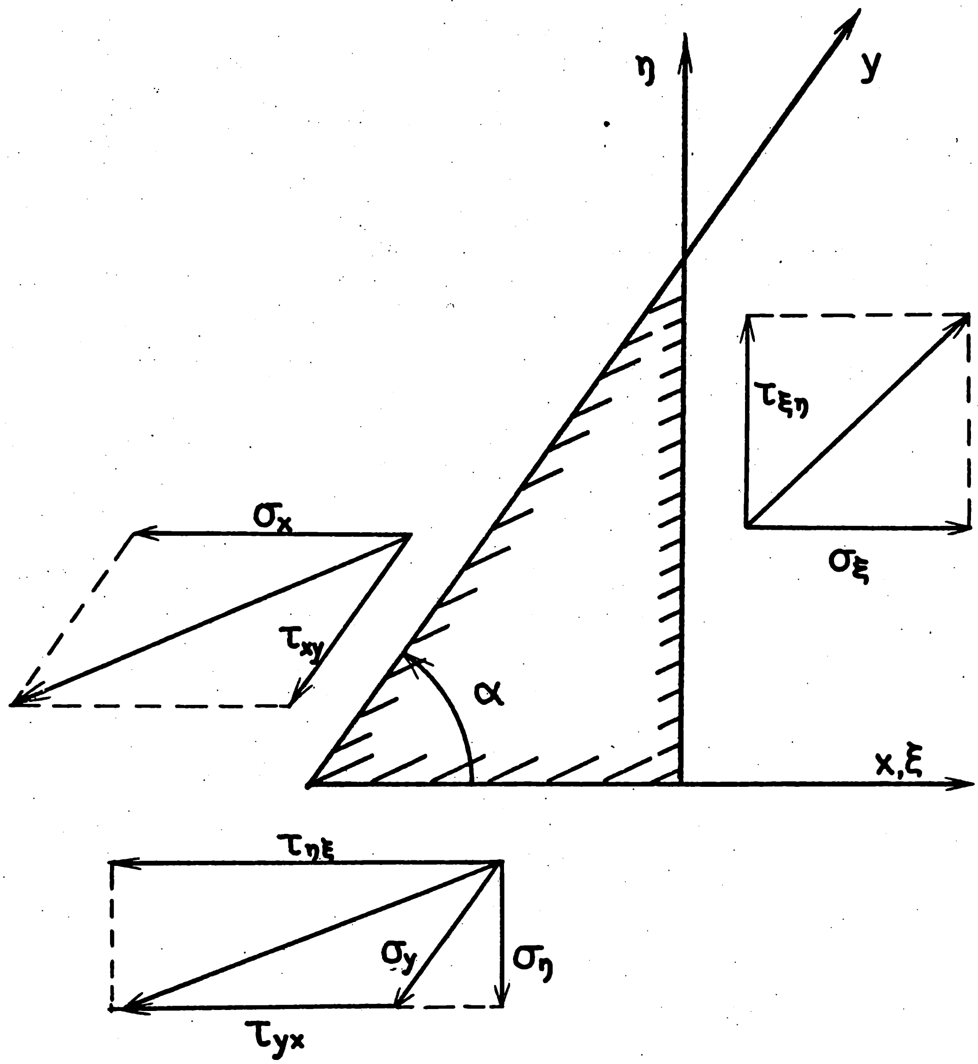

Figure 5. Relationship Between Oblique and Cartesian Stresses