Reprinted from
ADVANCED PROBLEMS AND METHODS FOR SPACE FLIGHT OPTIMIZATION
Edited by
FRAEIJS de VEUBEKE
PERGAMON PRESS — OXFORD & NEW YORK — 1969

THE SECOND VARIATION TEST WITH ALGEBRAIC AND DIFFERENTIAL CONSTRAINTS

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ABSTRACT

The second variation test for the algebraic constrained minimum problem can be placed in eigenvalue form by the introduction of a norm in the space of coordinate perturbations. It appears then as a Jacobi type of accessory minimum problem. A similar objective can be reached in the control problem, where the constraints are of differential type, by norming the weakly varied trajectories in the neighborhood of the optimal. If the norm is expressed in terms of control perturbations only, the eigenvalue type of second variation test and the optimum linear feedback control problem are very closely related. The case of broken extremals is considered and solved by the use of total perturbations, involving a shift in the independant variable. The rate of shift is introduced as an additional control. The eigenvalue test remains applicable to extremals containing singular arcs and constitutes a different approach to the ones attempted previously. An example of such a case is treated in detail.

1. ALGEBRAIC CONSTRAINTS

1.1. First order conditions for a constrained minimum

Let the coordinates $(q_1, \ldots, q_n)$ of a vector q satisfy m algebraic constraints

$$g_j(q) = 0 \qquad (j = 1, \ldots, m < n) \tag{1.1}$$

and consider the problem of finding a point $q = \hat{q}$ where a function $f(q)$ presents a relative minimum. It will be assumed that the functions f and g_j are twice differentiable at $\hat{q}$. Let δq denote the column matrix of the small increments in the coordinates. The constraints (1.1) will be satisfied to first order in the neighborhood of $\hat{q}$ if

$$g_j(\hat{q}) = 0 \qquad (j = 1, \ldots, m) \tag{1.2}$$

and

$$g_j^q \delta q = 0 \qquad (j = 1, \ldots, m) \tag{1.3}$$

where g_j^q denotes the row matrix $\left(\frac{\partial g_j}{\partial q_1}, \ldots, \frac{\partial g_j}{\partial q_n}\right)$ evaluated at $\hat{q}$. Only the case where the m row matrices in (1.3) are linearly independent will be considered; otherwise the problem is "abnormal". The assumption amounts to stating that all the constraints are "effective" at $\hat{q}$; then, the set (1.3) of linear homogeneous equations has $(n-m)$ independent and nontrivial solutions for δq, defined except for a scale factor. They are the *admissible directions* for the variation of coordinates. To the first order

$$f(\hat{q}+\delta q)-f(\hat{q}) = f^q\,\delta q \tag{1.4}$$

and the first order requirement for a minimum will be that

$$f^q\,\delta q = \delta f \geqslant 0 \tag{1.5}$$

This requirement will not be satisfied if the row matrix f^q is linearly independent from the row matrices g_j^q, for then the $m+1$ linear equations (1.3) and (1.5) would have a matrix of maximal rank $m+1 \leqslant n$ and a nontrivial solution δq could be found for any arbitrary value of δf. Hence f^q must be expressible as a linear combination

$$f^q = -\sum_1^m \lambda_j g_j^q \tag{1.6}$$

of the m independant row matrices of (1.3). Postmultiplying by δq and taking (1.3) into account, there follows

$$f^q\,\delta q = \delta f = 0 \tag{1.7}$$

The first order condition (1.6) is usually expressed in the equivalent form

$$\frac{\partial F}{\partial q_i} = 0 \quad (i = 1, \ldots, n) \quad \text{or} \quad F^q = 0 \tag{1.8}$$

after introduction of the "augmented function"

$$F(q) = f+\sum_1^m \lambda_j g_j \tag{1.9}$$

The $m+n$ conditions (1.2) and (1.8) constitute a set of $m+n$ algebraic equations to be satisfied by the n coordinates and the m Lagrangian multipliers λ_j. It will be appreciated that the solution for the multipliers is unique, otherwise, by difference of two solutions we would obtain a statement of abnormality (linear dependence of the g_j^q rows). The stationarity condition (1.7) is thus necessary for a minimum. To aim at a sufficiency condition, the left-hand side of (1.4) must now be calculated up to second order:

$$f(\hat{q}+\delta q)-f(\hat{q}) = f^q\,\delta q+\frac{1}{2}\,\delta q' f_q^q\,\delta q \tag{1.10}$$

with f_q^q denoting the $(n\times n)$ matrix of partial second derivatives at $\hat{q}$. It would be incorrect to suppress the first term by invoking (1.7). Second order terms, stemming from the requirement to satisfy also the constraints to second order:

$$g_j^q \delta q + \frac{1}{2} \delta q' g_{j_q^q} \delta q = 0 \qquad (1.11)$$

would be lost. In fact, combining (1.6) and (1.11), there comes a second order evaluation for this term

$$f^q \delta q = -\sum_1^m \lambda_j g_j^q \delta q = \frac{1}{2} \delta q' \left(\sum_1^m \lambda_j g_{j_q^q} \right) \delta q$$

Substitution of this result into (1.10) gives, in view of the definition (1.9), the second order expression

$$f(\hat{q}+\delta q) - f(\hat{q}) = \frac{1}{2} \delta q' F_q^q \delta q \qquad (1.12)$$

1.2. The second variation test as a Jacobi accessory minimum problem

The second variation just obtained

$$\phi = \frac{1}{2} \delta q' F_q^q \delta q \qquad (1.13)$$

is a quadratic form to be evaluated for the admissible variations defined by the constraints (1.3).

A convenient compact formulation of these constraints is

$$G\delta q = 0 \qquad (1.14)$$

introducing the $(m \times n)$ matrix

$$G = \begin{pmatrix} g_1^q \\ \cdots \\ g_m^q \end{pmatrix}$$

If, for some admissible nonvanishing variation, ϕ takes a negative value, $\hat{q}$ is certainly not a point of minimum for f. A sufficient condition for a relative minimum is that ϕ be strictly positive for all admissible nonzero variations. If ϕ is only nonnegative, the second variation test is not conclusive and higher order estimates become necessary. It was already observed that admissible variations can be arbitrarily scaled. In fact, if the admissible variation δq gives to the quadratic form the value ϕ, the admissible variation $\alpha \delta q$ (α real and nonzero) gives to the quadratic form the value $\alpha^2 \phi$. Hence scaling does not alter the property of δq to yield a positive, a negative or the zero value to the second variation. Consequently the second variation test is not altered by the addition of an imposed norm on the scaling, such as the Euclidian one

$$\frac{1}{2} \delta q' \delta q = d\varepsilon^2 \qquad (1.15)$$

In fact the test is simplified by restricting the scanning to the admissible directions of variation. Another advantage of the introduction of a norm is to reduce the test to the solution of a Jacobi-type of accessory minimum problem. Instead of scanning all the admissible directions we try to find the most critical one: the one giving to ϕ its minimum value. If the minimum value of ϕ is negative, the test has failed; positive, the test is successful; zero, the test is not conclusive.

The accessory problem

$$\frac{1}{2}\,\delta q' F^q_q\,\delta q \quad \text{minimum}$$

under the constraints (1.14) and (1.15) can be solved by the Lagrangian technique of section 1.1. Set up the augmented function

$$\frac{1}{2}\,\delta q' F^q_q\,\delta q + b' G\,\delta q + \zeta\left(d\varepsilon^2 - \frac{1}{2}\,\delta q'\,\delta q\right)$$

where $$b' = (\beta_1, \ldots, \beta_m)$$

is a row matrix of Lagrangian multipliers and obtain the necessary first order conditions by equating to zero the partial derivatives with respect to each δq_i. The resulting equations can be written in matrix form as

$$F^q_q\,\delta q + G' b = \zeta\,\delta q \tag{1.16}$$

The problem (1.16) and (1.14) can now be placed in the form

$$\begin{pmatrix} F^q_q - \zeta E & G' \\ G & 0 \end{pmatrix} \begin{pmatrix} \delta q \\ b \end{pmatrix} = 0 \tag{1.17}$$

where E denotes the $(n \times n)$ identity matrix. It is an eigenvalue problem for the Lagrangian multiplier ζ, associated to the norming constraint. It is also self-adjoint and its $n-m$ eigenvectors for δq, denoted by h_r, constitute an orthonormal basis from which an arbitrary admissible variation can be expanded:(1)

$$\delta q = \sum_{1}^{n-m} \alpha_r h_r \tag{1.18}$$

with $$h_r' h_s = 0 \quad \text{and} \quad h_r' F^q_q h_s = 0 \quad \text{for} \quad r \neq s \tag{1.19}$$

A simple proof of this is obtained by reducing the problem to the classical one of the simultaneous reduction of two quadratic forms, one of which is positive definite, to their diagonal form. This is achieved by the nonsingular transformation in the variations

$$\begin{pmatrix} \delta p \\ dr \end{pmatrix} = \begin{pmatrix} G \\ V \end{pmatrix} \delta q \tag{1.20}$$

where the $n-m$ rows of the submatrix V are taken to complete the already linearly independent rows of G. The constraints (1.14) reduce to

$$\delta p = 0 \quad \text{or} \quad \delta p_i = 0 \qquad (i = 1, \ldots, m) \tag{1.21}$$

The inversion of (1.20) can be written in the form

$$\delta q = A\delta p + B\delta r$$

and the admissible variations are

$$\delta q = B\delta r$$

The second variation test reduces to that of positive definiteness of the quadratic form

$$\phi = \frac{1}{2}\,\delta r' B' F_q^q B\,\delta r \tag{1.22}$$

The Euclidian norm that was imposed on δq is transformed into the norming constraint on the admissible variations:

$$d\varepsilon^2 = \frac{1}{2}\,\delta r' B' B\,\delta r$$

The eigenvalues ζ_r of problem (1.17) are those of the $n-m$ dimensional problem

$$(B' F_q^q B - \zeta B' B)\,\delta r = 0$$

which has the classical form announced ($B'B$ is positive definite). The eigenvectors u_r of this problem are related to the h_r of problem (1.17) by

$$h_r = Bu_r$$

whence the orthogonality properties (1.19) are evident. Substitution of the expansion (1.18) into (1.13), use of the orthogonality properties (1.19) and of norming constraints

$$h_r' F_q^q h_r = \zeta_r h_r' h_r = 2(d\varepsilon)^2 \zeta_r \qquad (r = 1, \ldots, n-m) \tag{1.23}$$

on the eigenvectors gives the following structure to the second variation

$$\phi = (d\varepsilon)^2 \sum_1^{n-m} \alpha_r^2 \zeta_r \tag{1.24}$$

From (1.18) and (1.23) follows also that the coordinates α_r of a normed admissible vector are related by

$$\sum_1^{n-m} \alpha_r^2 = 1 \tag{1.25}$$

To examine the result in the spirit of the Jacobi accessory problem, the critical direction (minimum of ϕ under (1.25)) is obtained for

$$\alpha_1 = 1 \qquad \alpha_i = 0 \qquad (i = 2, \ldots, n-m)$$

assuming of course that ζ_1 is the smallest eigenvalue. The failure, success or inconclusiveness of the second variation test is equivalent to the property of the smallest eigenvalue to be negative, positive or zero.

In practice it is most convenient to obtain the eigenvalues from problem (1.17) as roots of the algebraic equation

$$\begin{vmatrix} F^q_q - \zeta E & G' \\ G & 0 \end{vmatrix} = 0 \qquad (1.26)$$

The algorithms of Routh and Hurwitz can also be applied to test the positive character of all the eigenvalues. As a final observation, the Euclidian norm is not mandatory and was used for simplicity. Any positive definite quadratic form will serve; it may modify the critical direction but not the outcome of the test.

An example of application of this eigenvalue type of analysis to an optimization problem will be found in the article by P. Beckers[2]. It plays there an important role in eliminating many solutions provided by the simple stationarity conditions.

1.3. Inequality constraints

The algebraic minimum problem with inequality constraints is important in view of its applications to the maximum principle with bounded controls. In addition to the algebraic constraints (1.1), let the minimum problem for $f(q)$ be further constrained by a set of inequalities

$$c_k(q) \geqslant 0 \qquad k = 1, \ldots, M \qquad (1.27)$$

At any point $q = \hat{q}$ where relative minimality is to be tested, only those inequality constraints for which $c_k(\hat{q}) = 0$ need be considered. To first order then, assuming the functions (1.27) to be differentiable at $\hat{q}$, we have a set of inequalities in the increments

$$c_k(\hat{q}+\delta q) = c^q_k \delta q \geqslant 0 \qquad k = 1, \ldots, K \leqslant M \qquad (1.28)$$

It is again convenient to make a nonsingular transformation of incremental variables

$$\begin{pmatrix} \delta p \\ \delta r \\ \delta s \end{pmatrix} = \begin{pmatrix} G \\ V \\ W \end{pmatrix} \delta q \qquad (1.29)$$

The submatrix V is chosen to generate, together with G, a set of maximal rank $m+\varrho \leqslant n$ in the linear forms (1.3) and (1.28). W, required if $m+\varrho < n$, is taken to obtain a nonsingular transformation. The algebraic constraints (1.3) are eliminated by substitution of (1.21) into the other forms. Hence, the inequalities (1.28) can now be written in the form

$$a^r_k \, \delta r \geqslant 0 \qquad k = 1, \ldots, K \geqslant \varrho \qquad (1.30)$$

ϱ being the rank of the set of linear forms, and the first order minimality condition can be written as

$$f^r \, \delta r + f^s \, \delta s \geqslant 0 \qquad (1.31)$$

Proposition 1. A necessary first order condition for relative minimality is that the row matrix f^s be identically zero. This is obvious since $\delta r = 0$ and δs arbitrary, represents an admissible set of variations and, unless $f^s = 0$, δs can always be so chosen that $f^s\,\delta s < 0$.

The condition $f^s = 0$ is equivalent to stating that

$$\delta f = f^p\,\delta p + f^r\,\delta r = (f^p G + f^r V)\,\delta q$$

is a linear combination of the linear forms (1.3) and (1.28), since it is a linear combination of a set of maximal rank in these forms. This justifies the following.

Proposition 2. First order necessary conditions for a relative minimum are that at $q = \hat{q}$

$$F^q = 0 \quad \text{or} \quad \frac{\partial F}{\partial q_i} = 0 \qquad i = 1, \ldots, n \tag{1.32}$$

where
$$F = f + \sum_1^m \lambda_j g_j + \sum_1^K \mu_k c_k$$

This proposition, extending the technique of Lagrange to inequality conditions, is more constructive as a first step towards the search for coordinates where a relative minimum can occur. The μ_k are the Kuhn–Tucker multipliers. It should be realized that, for a given point $\hat{q}$ where (1.32) is satisfied, the set of multipliers need not be unique.

However, if the augmentation of the function is restricted to take into account only those constraints which yield for (1.3) and (1.28) a set of maximal rank, then obviously the corresponding set of multipliers is unique. Letting

$$k_1, \ldots, k_\varrho$$

denote the set of indices k corresponding to a set of maximal rank, we obtain by virtue of (1.32) and (1.3)

$$\delta f = -\sum_k \mu_k c_k^q\,\delta q \qquad k = k_1, \ldots, k_\varrho$$

Then, by virtue of (1.28), the first order condition for a minimum is seen to require that the Kuhn–Tucker multipliers be nonpositive

$$\mu_k \leqslant 0 \qquad k = k_1, \ldots, k_\varrho \tag{1.33}$$

A particular and frequent case is that where the complete set of linear forms (1.28) constitutes with (1.3) a set of maximal rank, so that there is a single augmented function and a unique set of multipliers.

If, furthermore, the Kuhn–Tucker multipliers turn out to be strictly negative, the first order minimality criterion is strongly verified ($\delta f > 0$) for

$$c_k^q\,\delta q > 0 \qquad k = 1, \ldots, K$$

and weakly verified ($\delta f = 0$) for

$$c_k^q\,\delta q = 0 \qquad k = 1, \ldots, K \tag{1.34}$$

Then, either $m+K = n$, and (1.3) plus (1.34) have only the trivial solution $\delta q = 0$; or $m+K < n$ and there are nontrivial solutions. In the former case the minimality of f at $q = \hat{q}$ is proved. In the latter case the proof of minimality still requires a second variation test

$$\frac{1}{2}\,\delta q' F_q^q\,\delta q \geqslant 0$$

under the first order algebraic constraints (1.3) and (1.34). The technique of section 1.2 is applicable to this purpose.

When some of the Kuhn–Tucker multipliers turn out to be zero, we have the following weaker statements:

Proposition 3. If the first order conditions (1.32) are satisfied with a unique set of multipliers and if the K–T multipliers are nonpositive, the first order minimality condition is satisfied.

Let ψ denote the second variation

$$\psi = \frac{1}{2}\,\delta q' F_q^q\,\delta q$$

with the constraints on the variations δq restricted to (1.3) and those of (1.34) whose K–T multiplier is strictly negative. Then the strong second order condition

$$\psi > 0 \tag{1.35}$$

is sufficient for relative minimality, but the corresponding weak condition

$$\psi \geqslant 0 \tag{1.36}$$

is only necessary when at most one of the K–T multipliers is zero.

Proof: If no K–T multiplier vanishes, the proposition is equivalent to the result obtained before. If one K–T multiplier, which can be taken to be μ_1, is zero, the first order minimality condition is only weakly satisfied for $c_1^q\,\delta q > 0$.

It becomes necessary to extend the admissible variations in the second variation test to those satisfying

$$c_1^q\,\delta q \geqslant 0 \tag{1.37a}$$

$$c_k^q\,\delta q = 0 \qquad k = 2, \ldots, K \tag{1.37b}$$

Let ϕ denote the values that the second variation can assume under these conditions. By disregarding (1.37a) we reduce the test to the ψ test of Proposition 3. But, since the set of admissible variations has been thereby further extended, ϕ is positive definite with ψ and (1.35) is sufficient. Further, if a critical variation $\delta q = \delta a$, satisfying (1.37b), is found to render ψ negative, the same will be true of $\delta q = -\delta a$. But then either the first, or the second, will also satisfy (1.37a). Hence the possibility that $\psi < 0$ implies the possibility of $\phi < 0$ and (1.36) is necessary.

If more than one K–T multiplier vanishes, the extension of the proof of sufficiency of (1.35) is obvious. The following counterexample shows that it is useless to attempt a proof of the necessity of (1.36):

$$f = q_1^2 - q_2^2$$

$$c_1 = a_{11}q_1 + a_{12}q_2 \geqslant 0$$

$$c_2 = a_{21}q_1 + a_{22}q_2 \geqslant 0$$

The first order conditions (1.32) are satisfied at point $\hat{q}_1 = \hat{q}_2 = 0$ with $\mu_1 = \mu_2 = 0$. (In fact f is stationary there on its own.) The second variation test ψ is given by

$$\psi = \delta q_1^2 - \delta q_2^2$$

with free δq_1 and δq_2, it obviously can assume negative values. The ϕ test is

$$\phi = \delta q_1^2 - \delta q_2^2$$

$$a_{11}\delta q_1 + a_{12}\delta q_2 \geqslant 0$$

$$a_{21}\delta q_1 + a_{22}\delta q_2 \geqslant 0$$

but ϕ can only assume positive values if, for instance, $a_{11} > 0$, $a_{11} + a_{12} < < 0$, $a_{21} > 0$ and $a_{22} - a_{21} > 0$. A necessary condition is that the second variation be nonnegative under conditions (1.34) since they are more restrictive than (1.37a and b). These tests are helpful in that they can be conclusive, but if they are not we must examine the more general second variation test on

$$\chi = \frac{1}{2}\, \delta q' F_q^q\, \delta q \tag{1.38}$$

with a set of inequalities

$$c_k^q\, \delta q \geqslant 0 \qquad k = 1, \ldots, t \leqslant n - m \tag{1.38a}$$

corresponding to the zero K–T multipliers of one of the solutions of (1.32), and a complementary set of equalities

$$g_j^q\, \delta q = 0 \quad j = 1, \ldots, m \quad c_k^q\, \delta q = 0 \quad k = t+1, \ldots, K \tag{1.38b}$$

As the intersection of half-spaces and hyperplanes the admissible variations δq form a convex set. Since furthermore $\alpha\delta q$ is admissible, if and only if δq is admissible, and $\alpha \geqslant 0$, it is a convex cone C with vertex at $\delta q = 0$. On the other hand $\chi \geqslant 0$ is a convex cone Γ_1 with its anticone Γ_2 (since an arbitrary change in scale and sign does not alter the sign of χ). Convexity results from the following:

Let $\delta q = \delta a$ belong to Γ_1 so that $\chi_a \geqslant 0$; let also $\chi_b \geqslant 0$ so that $\delta q = \delta b$ belongs to either Γ_1 or Γ_2. Then for $\delta q = \alpha\delta a + \beta\delta b$ we find

$$\chi = \alpha^2\chi_a + \beta^2\chi_b + \alpha\beta\, \delta a' F_q^q\, \delta b$$

If $\delta a' F_q^q \delta b > 0$ we consider δb to belong to Γ_1 and then $\alpha\delta a + \beta\delta b$ belongs to Γ_1 ($\chi \geqslant 0$) for arbitrary $\alpha \geqslant 0$ and $\beta \geqslant 0$ (convexity), while $-\alpha\delta a - \beta\delta b$ belongs to Γ_2. If $\delta a' F_q^q \delta b \leqslant 0$ we must consider $-\delta b$ to belong to Γ_1.

From a geometrical standpoint the second variation test consists in verifying that the convex cone C is contained in either of the convex cones Γ_1 or Γ_2. Obviously it is sufficient to this purpose to verify the sign of χ for variations lying along the ridges of C. The development of an algorithm to obtain the ridge variations will not be developed here.

2. DIFFERENTIAL CONSTRAINTS

2.1. *Tightly controlled systems*

Consider now trajectory or control problems wherein the coordinates of a system are subject to differential constraints

$$\dot{q}_i = f_i(q_1, \ldots, q_n, t;\ v_1, \ldots, v_r) \qquad (i = 1, \ldots, n) \tag{2.1}$$

to boundary value constraints

$$U_j[q(\alpha), t(\alpha), q(\beta), t(\beta)] = 0 \qquad j = 1, \ldots, p < 2n+2 \tag{2.2}$$

and the cost function to be minimized is

$$I[q(\alpha), t(\alpha), q(\beta), t(\beta)] \tag{2.3}$$

An optimal trajectory of this problem will be denoted in parametric form as

$$\begin{aligned} q_i &= \hat{q}_i(\theta) \qquad i = 1, \ldots, n \\ t &= \hat{t}(\theta) \end{aligned} \qquad \alpha \leqslant \theta \leqslant \beta \tag{2.4}$$

For a given state and time, equations (2.1) constitute a mapping of the set $\{v_1, \ldots, v_r\}$ of admissible control vectors into the hodograph space. The range of this mapping, or the set $\{\dot{q}_1, \ldots, \dot{q}_n\}$ of admissible velocity vectors may be nonconvex. In this case, however, chattering of the controls extends the set of attainable velocity vectors to the convex hull of the range.[(3)] This merely requires the functions f_i to be differentiable with respect to state and time. It will be assumed that, by the addition of appropriate, sectionally linear, artificial controls, chattering has been "regularized"[(4)] or, equivalently, the variational problem "relaxed".[(5)] The extended set of admissible controls will map into the convex hull and the optimal controls $(\hat{v}_1, \ldots, \hat{v}_r)$ will be sectionally continuous functions of the independant variable t.

Under the usual assumption that the f_i functions are differentiable with respect to the control variables, the relations between weak control variations and the correspondingly small changes in the velocity vector are given, in matrix form, by

$$\delta \dot{q} = F \, \delta v \tag{2.5}$$

where δv and $\delta \dot{q}$ are column matrices and F is the Jacobian matrix $F = (\partial f_i / \partial v_j)$. If the optimal trajectory contains a segment along which the equations

$$F \, \delta v = 0 \tag{2.6}$$

admit a continuous non-trivial solution, the set of optimal controls is not well determined in the sense that, neglecting second order changes, the same trajectory can be generated by appropriate first order modifications in the control history. This situation cannot prevail if, along the optimal, F is of maximal rank and this rank equal to the number r of controls (which implies $n \geqslant r$). In this case the system will be said to be "tightly controlled". This concept means, intuitively, that the system remains sensitive to any combination of small changes in the control variables. It is essential to set up a metric in the space of weakly varied trajectories in terms of control variations only. Such a metric is for instance defined by the norm

$$\| dN \|^2 = \int_\alpha^\beta \delta v' M \, \delta v \, dt \tag{2.7}$$

where M is a matrix of constant or variable elements, everywhere positive definite. If the system were not tightly controlled along the optimal, one of the axioms of distance would be violated: any weak control variation, consistent with (2.6), would produce a nonzero distance between the optimal trajectory and itself.

2.2. Change of control variables

Up to now we have disregarded possible constraints arising on the control variations, due to boundedness of certain of the controls. In a relaxed variational problem the domain of admissible controls in E^r has no isolated point, it has only interior points and boundary points. More than often optimal trajectories contain segments along which the control vector lies on the boundary; its variation is then restricted by a set of equations of the form

$$y_k^v \, \delta v = 0 \qquad k = 1, \ldots, p < r \tag{2.8a}$$

$$y_k^v \, \delta v \geqslant 0 \qquad k = p+1, \ldots, K \tag{2.8b}$$

In a wide variety of problems a change in control variables

$$v_j = V_j(q, t; w) \tag{2.9}$$

can be devised mapping a domain D of the new controls $(w_1, \ldots, w_m)$ (often the whole E^m space) onto the set of all admissible controls $\{v_1, \ldots, v_r\}$ and such that the boundary points used along the optimal trajectory become interior points of D. One has then the advantage that the weak variations δw on the new controls are unconstrained. In fact the relation between weak variations is

$$\delta v = V\,\delta w \qquad V = (\partial v_j / \partial w_l) \tag{2.10}$$

with the $(r \times m)$ Jacobian matrix V evaluated along the optimal. Since the transformation is only required when constraints on δv prevail, we are led to assume that the set of linear equations

$$V'y = 0$$

has non-trivial, linearly independent solutions,

$$y = y_k \qquad k = 1, \ldots, L \quad (\text{hence } L \leqslant r)$$

so that the equations (2.10) can be solved for δw if, and only if,

$$y_k'\,\delta v = 0 \qquad k = 1, \ldots, L \tag{2.11}$$

Then, provided the set of row matrices y_k' be equivalent to a set of independent row matrices y_k^v of (2.8a and b), arbitrary δw variations generate only admissible δv variations. In fact they generate a restricted class of admissible variations, since the more stringent equality sign prevails for (2.8b). Thus, while the transformation to new control variables does not restrict the overall controllability of the system, it does restrict the class of weakly varied trajectories about the optimal. To some extent this may increase the existence possibilities of weak relative minima for the cost function. However a weak relative minimum is of little practical significance and nothing should be changed regarding the existence of strong relative minima.

Special care should be exercised in introducing a norm of type (2.7) when changing control variables. The system remains tightly controlled in w if $m < r$ and if V has maximal rank $\varrho = m$, for then the equations

$$V\,dw = 0 \tag{2.12}$$

possess only the trivial solution $\delta w = 0$, while $L = r - m$. But if either $\varrho < m < r$ or $\varrho < r \leqslant m$, (2.12) has $m - \varrho$ independent non-trivial solutions, while $L = r - \varrho$.

A suitable metric when the system is no more tightly controlled in w is of course directly provided by

$$\|dN\|^2 = \int_\alpha^\beta \delta w' N\,\delta w\,dt \tag{2.13}$$

where

$$N = V'MV \tag{2.14}$$

With respect to δw it is in fact a semi-norm, but it guarantees that $\|dN\| \neq 0$ implies a truly varied trajectory.

A simple illustration for a change of control variables is the following. Let (v_1, v_2) be controls which appear linearly in the f_i functions of a problem so that the system is certainly tightly controlled. Let them be constrained by

$$v_1^2+v_2^2 \leqslant F^2$$

They could be the thrust components of a single rocket engine in a plane motion problem, F being then the maximum thrust.

Along a segment of fully thrusted trajectory, the variational constraint is

$$\hat{v}_1\,\delta v_1+\hat{v}_2\,\delta v_2 \leqslant 0$$

The transformation of control variables

$$\begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} = F\,\frac{1+\cos w_2}{2}\begin{pmatrix} \cos w_1 \\ \\ \sin w_1 \end{pmatrix}$$

does not restrict the controllability of the system and the complete Euclidian space of the w controls is mapped onto the admissible v controls. But the matrix:

$$V = -\frac{F}{2}\begin{pmatrix} (1+\cos w_2)\sin w_1 & \cos w_1 \sin w_2 \\ \\ -(1+\cos w_2)\cos w_1 & \sin w_1 \sin w_2 \end{pmatrix}$$

degenerates in the following cases:

(1) For $\cos w_2 = 1$ (full thrust) it has rank 1 and restricts the variational constraint to

$$\hat{v}_1\,\delta v_1+\hat{v}_2\,\delta v_2 = 0$$

while, simultaneously, $V\,\delta w = 0$ admits the solution $\delta w_1 = 0$, δw_2 arbitrary. The system is no more tightly controlled. First order changes in the thrust orientation are possible but not a change in the thrust level.

(2) For $\cos w_2 = -1$ (zero thrust) it has rank zero and introduces artificial constraints $\delta v_1 = 0$, $\delta v_2 = 0$. The system becomes completely insensitive to first order changes in the w controls. (This situation stems of course from the fact that we cannot modify the zero thrust level; that we cannot change the thrust orientation becomes then immaterial.)

2.3. *Total variations. First order conditions*

The small changes incurred by a state variable or a control, when passing from the optimal to a neighboring trajectory, are usually stated as differences measured for the same value of the independent variable (time).

These *isochronic variations* δq and δw can be generalized to differences involving a time shift s. Such *total variations* will be denoted by

$$x = \delta q + \dot{q}s \quad \text{for state variables} \tag{2.15}$$

$$u = \delta w + \dot{w}s \quad \text{for controls.} \tag{2.16}$$

The notation, which is purposely simple to alleviate the formulation of subsequent analytical developments, should not lead us to forget that x, u and s represent small quantities of the same order as δq and δw. Total variations, already useful in handling problems with variable end values of the time, yield essential simplifications in the presence of broken extremals. Indeed, if an optimal trajectory contains a certain number of control discontinuities, an important part of the controllability of the system by small perturbations is involved in advancing or retarding the epochs of such discontinuities. If $s(\tau)$ denotes the time shift of a discontinuity in the control vector, occurring at $t = \tau$ along the optimal, the isochronic variation δw is not defined in the interval between τ and $\tau + s(\tau)$. In this interval the difference in the control vectors is finite (Fig. 1). One way to reduce

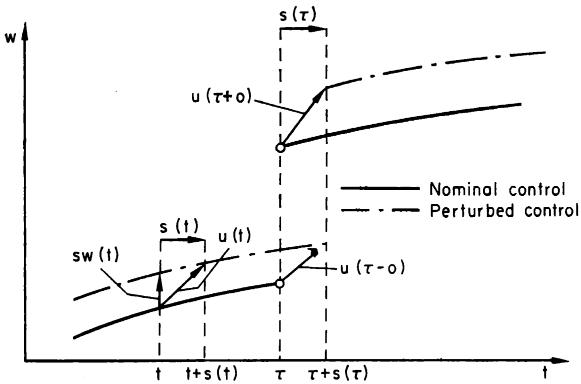

FIG. 1. Isochromic control perturbation $\delta w(t)$ and total control perturbation $u(t)$ involving a time-shift $s(t)$. At the epoch $t = \tau$ of a control discontinuity along the nominal trajectory the time-shift $s(\tau)$ is that of the discontinuity. Then $u(t)$ is a sectionally continuous first order perturbation.

this case to small perturbations is to allow first order discontinuities in the state variables at $t = \tau$ (see section 2.7). But, by far the simplest way is to consider only the total variations based on the actual time shift. The total variations in state variables are continuous, the total variations in the controls are well defined as sectionally continuous functions of time. In principle the time shifts need only be known at the ends and at the epochs of control discontinuities along the optimal, However one of the purposes of extending the optimum linear feedback control theory,[6] based on the second variation, to the case of broken extremals would be to obtain a time shifting rule all along. Some results of first order optimality must first be recalled.

$$H(\lambda, q, t; w) = \lambda' f(q, t; w) \tag{2.17}$$

is the variational Hamiltonian built up from the functions f_i of eqs. (2.1) (except possibly for a transformation of control variables) and from a set of Lagrangian multipliers represented by the row matrix λ'. An equivalent form of equations (2.1) is then

$$\dot{q} = H_\lambda \tag{2.18}$$

where the subscript notation indicates the formation of a column matrix of the corresponding partial derivatives. Similarly the superscript notation will indicate the formation of a row matrix. When the Hamiltonian or derivatives of it are merely considered as functions of time, by substitution of the arguments λ, q and w as time functions along the optimal, the notations

$$\hat{H}, \hat{H}_\lambda \quad \text{etc.}$$

will be used. As an immediate example the equations (2.15) should be written as

$$x = \delta q + s\hat{H}_\lambda \tag{2.19}$$

The adjoint differential system is, in general,

$$\dot{\lambda} = -H_q$$

and, considered along the optimal, gives

$$\frac{d}{dt}\hat{\lambda} = -\hat{H}_q \tag{2.20}$$

We shall not consider abnormal problems, where the set of optimal multipliers $\hat{\lambda}$ is not uniquely defined (except for scale).

First order optimality conditions are: the maximum principle

$$\hat{w} = \arg\sup H(\hat{\lambda}, \hat{q}, t, w) \tag{2.21}$$

and the transversality conditions

$$\begin{aligned} \hat{\lambda}(\alpha) &= \hat{J}_{q(\alpha)} & \hat{\lambda}(\beta) &= -\hat{J}_{q(\beta)} \\ \hat{H}(\alpha) &= -\hat{J}_{t(\alpha)} & \hat{H}(\beta) &= \hat{J}_{t(\beta)} \end{aligned} \tag{2.22}$$

where J is the augmented functional

$$J(q(\alpha), t(\alpha), q(\beta), t(\beta)) = I + \sum_1^p \mu_j U_j \tag{2.23}$$

At a discontinuity of the control vector, the Weierstrass–Erdmann corner conditions are equivalent to continuity of the multipliers and of the Hamiltonian. If control variations are unrestricted, (2.21) implies

$$\hat{H}_w = 0 \tag{2.24}$$

in which case (2.18) and (2.20) have the consequence that

$$\frac{d}{dt}\hat{H} = \hat{H}_t \tag{2.25}$$

In the sequel we shall assume that the control variables are such that they have unrestricted variations along the optimal and use equations (2.24) and (2.25). The transversality conditions are equivalent to

$$(\hat{\lambda}'x - \hat{H}s)_\alpha^\beta = -\delta J \tag{2.26}$$

where δJ denotes the first order variation of the augmented functional

$$\delta J = \hat{J}^{q(\alpha)}x(\alpha) + \hat{J}^{t(\alpha)}s(\alpha) + \hat{J}^{q(\beta)}x(\beta) + \hat{J}^{t(\beta)}s(\beta) \tag{2.27}$$

The first order conditions result in δJ being zero to first order, so that δI is also zero to first order if the variational boundary constraints $\delta U_j = 0$ are satisfied to first order.

2.4. *Second variation*

To introduce easily a continuous time shift the parametric form of the differential equations is useful. Along the optimal

$$\frac{d\hat{q}}{d\theta} = \frac{d\hat{t}}{d\theta} f(\hat{q}, \hat{t};\, \hat{w})$$

and, for a neighboring trajectory,

$$\frac{d\hat{q}}{d\theta} + \frac{dx}{d\theta} = \left(\frac{d\hat{t}}{d\theta} + \frac{ds}{d\theta}\right) f(\hat{q}+x, \hat{t}+s;\, \hat{w}+u)$$

Subtracting, and identifying θ with the independant variable, we find

$$\dot{x} = f(\hat{q}+x, t+s;\, \hat{w}+u) - f(\hat{q}, t;\, \hat{w}) + \dot{s}f(\hat{q}+x, t+s;\, \hat{w}+u) \tag{2.28}$$

All total variations and the time shift s are now considered as functions of the time. Retaining only first order terms, we obtain the linear perturbation equations with time-varying coefficients

$$\begin{aligned} \dot{x} &= \hat{f}^q x + \hat{f}^t s + \hat{f}^w u + \dot{s}\hat{f} \\ &= \hat{H}_\lambda^q x + \hat{H}_\lambda^t s + \hat{H}_\lambda^w u + \dot{s}H_\lambda \end{aligned} \tag{2.29}$$

To calculate the effect of perturbations on the functional to the second order we combine equations (2.28) in the single one

$$\hat{\lambda}'\dot{x} = H(\lambda, \hat{q}+x, t+s, \hat{w}+u) - H(\hat{\lambda}, \hat{q}, t, \hat{w}) + \dot{s}H(\hat{\lambda}, \hat{q}+x, t+s, \hat{w}+u)$$

and expand the right-hand side up to second order terms:

$$\hat{\lambda}'\dot{x} = \hat{H}^q x + \hat{H}^t s + \hat{H}^w u + \dot{s}\hat{H} \qquad \text{(first order terms)}$$
$$+ \Delta^2 H + \dot{s}(\hat{H}^q x + \hat{H}^t s + \hat{H}^w u) \qquad \text{(second order terms)}$$

where

$$\Delta^2 H = \tfrac{1}{2}x'\hat{H}^q_q x + \tfrac{1}{2}s\hat{H}^t_t s + \tfrac{1}{2}u'\hat{H}^w_w u + x'\hat{H}^w_q u + s\hat{H}^q_t x + s\hat{H}^w_t u \tag{2.30}$$

Then, inserting $\hat{H}^w = 0$ and integrating along a segment $t = \tau_m$ to τ_{m+1} along which the optimal controls are continuous

$$\int_{\tau_m}^{\tau_{m+1}} (\hat{\lambda}'\dot{x} - \hat{H}\dot{s})dt = \int_{\tau_m}^{\tau_{m+1}} (\hat{H}^q x + \hat{H}^t s)\, dt + D^2 \tag{2.31}$$

where
$$D^2 = \int_{\tau_m}^{\tau_{m+1}} \{\Delta^2 H + \dot{s}(\hat{H}^q x + \hat{H}^t s)\}\, dt \tag{2.32}$$

is a second order quantity. The left-hand side is manipulated by partial integration and use of (2.25) and (2.20)

$$\int_{\tau_m}^{\tau_{m+1}} (\hat{\lambda}'\dot{x} - \hat{H}\dot{s})dt = \left(\hat{\lambda}'x - \hat{H}s\right)_{\tau_m}^{\tau_{m+1}} - \int_{\tau_m}^{\tau_{m+1}} \left(x'\frac{d}{dt}\hat{\lambda} - s\frac{d}{dt}\hat{H}\right) dt$$
$$= \left(\hat{\lambda}'x - \hat{H}s\right)_{\tau_m}^{\tau_{m+1}} + \int_{\tau_m}^{\tau_{m+1}} (\hat{H}^q x + \hat{H}^t s)\, dt$$

when this result is substituted into (2.31), there comes

$$\left(\hat{\lambda}'x - \hat{H}s\right)_{\tau_m}^{\tau_{m+1}} = D^2$$

This formula may be extended to the whole trajectory by simple addition, remembering that, even at the level of control discontinuities, $\hat{\lambda}$ and $\hat{H}$ are continuous and x and s are required to be continuous. Hence, combining with (2.26) we obtain the second order estimate

$$-\delta J = \left(\hat{\lambda}'x - \hat{H}s\right)_{\alpha}^{\beta} = \int_{\alpha}^{\beta} \{\Delta^2 H + \dot{s}(\hat{H}^q x + \hat{H}^t s)\}\, dt \tag{2.33}$$

What we are really aiming at is the second order estimate of the increase in the functional I:

$$\Delta^2 I = \delta I + \delta^2 I \tag{2.34}$$

where, in the Taylor expansion of

$$I[\hat{q}(\alpha) + x(\alpha),\ \hat{t}(\alpha) + s(\alpha),\ \hat{q}(\beta) + x(\beta),\ \hat{t}(\beta) + s(\beta)]$$

δI and $\delta^2 I$ are respectively the groups of first and second degree terms in the perturbations. $\delta^2 I$ is clearly of second order, and so must be δI since it vanishes to first order. Indeed

$$\delta I = \delta J - \sum_1^p \mu_j \delta U_j$$

and because the boundary conditions must also be enforced to second order

$$\delta U_j = -\delta^2 U_j$$

so that finally

$$\delta I = \delta J + \sum_1^p \mu_j \delta^2 U_j$$

Substitution of this into (2.34) yields

$$\Delta^2 I = \delta J + \delta^2 J \tag{2.35}$$

where the second order estimate of δJ is given by (2.33).

2.5. *The Jacobi accessory minimum problem*

A necessary condition for a relative minimum is that

$$\Delta^2 I \geqslant 0$$

for all perturbed trajectories satisfying the linear perturbation equations (2.29) and the perturbed boundary conditions

$$\delta U_j = 0 \qquad j = 1, \ldots, p \tag{2.36}$$

The idea of the accessory problem is to determine the most critically perturbed trajectory, the one that minimizes $\Delta^2 I$. As in the algebraic case this problem is meaningless because any admissible perturbed trajectory can be scaled arbitrarily (including negative scale factors) without ceasing to be admissible.

This change of scale affects the value of the functional but not its sign (it is homogeneous quadratic in the perturbations). No most critically perturbed trajectory can exist since a more critical can always be found by a change of scale. In fact the accessory minimum problem is generally deviated from its original purpose towards a search for conjugate points. Its original purpose can be restored by introducing a concept of distance between the perturbed trajectory and the optimal one. There is then generally a most critically perturbed trajectory lying at a prescribed distance from the optimal; if the minimum it provides for $\Delta^2 I$ is positive or zero, the necessary second variation condition is fulfilled; if this minimum is negative the extremal is not a relative minimum.

The distance can naturally be defined directly in terms of a positive definite integral over some quadratic form in the perturbations of state variables. This would however modify the adjoint equations of the perturbed problem. A simpler correlation between the accessory minimum problem and the optimum linear feedback control problem is achieved by stating the distance, or the norm of the perturbed trajectory in terms of the control perturbations alone:

$$d\varepsilon^2 = \int_\alpha^\beta (\tfrac{1}{2}u'Nu+\tfrac{1}{2}\sigma^2)dt \tag{2.37}$$

To account for broken extremals (control discontinuities occurring along the optimal) the total control perturbation u, which is sectionally continuous, is used instead of δw. As already observed, N is taken positive definite for a tightly controlled system but is a suitable non-negative matrix if the transformation to new control variables, required to free the control perturbations, destroys tight control. Furthermore, a new perturbation control σ has been introduced to govern the time shift; it is defined by the additional perturbation equation

$$\dot{s} = \sigma \tag{2.38}$$

To reduce the accessory minimum problem to the same Mayer standard form as the original one, we further introduce the new state variables h and n, governed by the differential equations

$$\dot{h} = \Delta^2 H+\sigma(\hat{H}^q x+\hat{H}^t s) \tag{2.39}$$

$$\dot{n} = \tfrac{1}{2}u'Nu+\tfrac{1}{2}\sigma^2 \tag{2.40}$$

So that the functional to be minimized becomes

$$\Delta^2 I = \delta^2 J+h(\alpha)-h(\beta)$$

and the augmented functional

$$F = \Delta^2 I+\sum_1^p \nu_j\,\delta U_j-\zeta[n(\beta)-n(\alpha)] \tag{2.41}$$

As in the algebraic case, the constant multiplier ζ associated with the norming condition (2.37) will play the role of an eigenvalue parameter. Denote by p the column matrix of adjoint multipliers associated with the perturbation equations of state (2.29); by γ the adjoint to h, by ϱ the adjoint to n and by η the adjoint to s. The variational Hamiltonian will be

$$\begin{aligned} K = {} & p'(\hat{H}^q_\lambda x+\hat{H}^t_\lambda s+\hat{H}^w_\lambda u+\hat{H}_\lambda\sigma) \\ & +\gamma(\Delta^2 H+\sigma\hat{H}^q x+\sigma\hat{H}^t s)+\tfrac{1}{2}\varrho(u'Nu+\sigma^2)+\eta\sigma \end{aligned} \tag{2.42}$$

Two of the equations of the adjoint system are elementary:

$$\dot{\gamma} = -\frac{\partial K}{\partial h} = 0 \quad \text{and} \quad \dot{\varrho} = -\frac{\partial K}{\partial n} = 0$$

Hence γ and ϱ are constants that can be found from the transversality conditions:

$$\gamma = \frac{\partial F}{\partial h(\alpha)} = -\frac{\partial F}{\partial h(\beta)} = 1$$

which serves to scale the adjoint variables and

$$\varrho = \frac{\partial F}{\partial n(\alpha)} = -\frac{\partial F}{\partial n(\beta)} = \zeta$$

identifying ϱ with the eigenvalue parameter. The remaining equations of the adjoint system are then

$$\dot{p} = -K_x = -\hat{H}^\lambda_q p - \hat{H}^q_q x - \hat{H}^w_q u - \hat{H}^t_q s - \hat{H}_q \sigma \tag{2.43}$$

$$\dot{\eta} = -K_s = -\hat{H}^\lambda_t p - \hat{H}^q_t x - \hat{H}^w_t u - \hat{H}^t_t s - \hat{H}_t \sigma \tag{2.44}$$

The multipliers p are obviously the perturbations of the λ and equations (2.43) the perturbed adjoint differential equations (2.20).

It will be observed that the adjoint equations do not depend on the eigenvalue parameter; they differ from the classical perturbations of the adjoint equations of the original problem[6] only through the presence of the terms involving time shift and rate of time shift. The adjoint variable η is seen to become constant for autonomous systems. The remaining transversality conditions, supplementing the variational boundary conditions (2.36), are easily obtained from the partial derivatives of the augmented functional and will not be written in detail; they are also homogeneous of the first degree. In fact, because equations (2.39) and (2.40) are ignorable and all the others are homogeneous of degree one, the critical perturbations will only exist for a spectrum of values of the parameter ζ.

2.6 Critically perturbed trajectories

The notion of critically perturbed trajectories implies of course that, in order to minimize the second variation, the control perturbations are optimal. Necessary conditions therefore are

$$K_u = 0 \quad \text{and} \quad K_\sigma = 0$$

or, explicitly, the homogeneous equations

$$\hat{H}^\lambda_w p + (\hat{H}^w_w + \zeta N) u + \hat{H}^q_w x + \hat{H}^t_w s = 0 \tag{2.45}$$

$$\hat{H}^\lambda p + \hat{H}^q x + \hat{H}_t s + \sigma\zeta + \eta = 0 \tag{2.46}$$

The last equation contains the first order perturbation of the Hamiltonian which is related to the adjoint to the time perturbation, just like $-H$ is the adjoint variable to the time. In principle, unless ζ is zero, the optimal rate of time shift can be calculated from (2.46). The optimal control perturbations can be obtained from (2.45), if the matrix $(\hat{H}^w_w + \zeta N)$ is nonsingular.

The optimal control perturbations are generally discontinuous at precisely the same corner points as the broken extremal tested. This is due to the discontinuities occuring in the coefficients of the various time-varying matrices. Conversely, at points of continuity of the matrices the optimal control perturbations remain continuous and are uniquely determined by the eigenvalue. Hence it is not conceivable to introduce strong variations in the accessory problem and the maximum principle does not apply. Note however the strong Legendre–Clebsch condition that $\hat{H}^w_w$ should be negative definite; in which case the optimal controls will indeed maximize the Hamiltonian K for $\zeta = 0$. This case corresponds to the optimum linear feedback controls of guidance theory.(6) But in the critical perturbation theory, the optimal controls need not maximize the Hamiltonian when the eigenvalue parameter is different from zero.

For a similar reason the Hamiltonian K is not required to be continuous at the corner points; the position of those in time on the perturbed trajectories is not a free variation. Instead, it is the continuity of the adjoint variable to the time shift, η, that plays the essential role in determining this shift at the corner points (see section 2.7).

The main significance of the eigenvalue parameter is obtained through the following tedious but straightforward calculation. Compute the time derivative

$$\frac{d}{dt}(p'x+\eta s)$$

by substituting equations (2.29, 38, 43 and 44). Group the terms proportional to σ and those postmultiplying u' and substitute η from (2.46) and $\hat{H}^\lambda_w p$ from (2.45). After cancellations and consideration of (2.39 and 40), the result can be written in the form

$$\frac{d}{dt}(p'x+\eta s) = -2\dot{h}-2\dot{n}\zeta$$

Integration along the trajectory and application of the norming condition (2.37) gives

$$(p'x+\eta s)^\beta_\alpha = 2h(\alpha)-2h(\beta)-2\zeta(d\varepsilon)^2$$

The left-hand side can be evaluated from the transversality conditions and the Euler theorem on homogeneous forms to produce

$$-2\delta^2 J-\sum \nu_j \delta U_j = 2h(\alpha)-2h(\beta)-2\zeta(d\varepsilon)^2$$

Finally, considering equations (2.36)

$$\Delta^2 I = \delta^2 J+h(\alpha)-h(\beta) = \zeta(d\varepsilon)^2 \tag{2.47}$$

From this can be concluded that the positive definite, nonnegative or indefinite character of the second variation depends on the positive, zero or negative character of the smallest eigenvalue. The method has the advantage of remaining applicable to trajectories containing a singular arc ($\hat{H}^w_w$ singular) and constitutes a different approach to other tests devised to this purpose.(7) An example will be treated in section 2.8.

2.7. Optimal time shift

Some insight into the continuous time shift process, implicit in the perturbational equations, is provided by the elimination of η between equations (2.44 and 2.46). All the time derivatives appearing in the differentiation of (2.46) can be substituted except those of $\hat{w}$ and σ. Considerable simplification occurs by noting that (2.24) can be differentiated:

$$\frac{d}{dt}H_w = H_w^\lambda\dot{\lambda}+H_w^q\dot{q}+H_w^t+H_w^w\dot{w}$$

equated to zero along the optimal yields

$$-\hat{H}_w^\lambda\hat{H}_q+\hat{H}_w^q\hat{H}_\lambda+\hat{H}_w^t+H_w^w\frac{d}{dt}(\hat{w}) = 0$$

This and equation (2.45) finally puts the result of elimination of η in the form:

$$\zeta\left(\dot{\sigma}+u'N\frac{d}{dt}\hat{w}\right) = 0$$

In the study of critical perturbations of nonzero eigenvalue, this implies the rather simple rule:

$$\ddot{s} = -u'N\frac{d}{dt}\hat{w} \tag{2.48}$$

associating the acceleration of time shift to the control perturbations. In the theory of optimal perturbation guidance, or in the search for conjugate points, we have to set $\zeta = 0$. In view of the preceding result we have then to consider (2.46) as a first integral and (2.44) can be ignored as identically satisfied. Simultaneously (2.46) does not yield any more the information about the instantaneous rate of time shift.

Returning then to the Hamiltonian K of equation (2.42) we can observe that it becomes linear in σ and that (2.46) can be considered as the chattering condition on this control. In such cases the control value is usually found by repeated differentiation of this condition. But as we have observed that the first differential is identically zero, we have a case where the actual value of the control is indifferent.

Fortunately the theory still predicts implicitly the optimal time shift at the corner points. This is due to the requirement of continuity of the adjoint η at corner points.

With $\zeta = 0$ in (2.46)

$$\eta = -\delta H$$

as already observed, and continuity requires at a corner point $t = \tau$ that

$$\delta H|_{\tau-0} = \delta H|_{\tau+0} \tag{2.49}$$

despite the discontinuities in the coefficient of the matrices of derivatives of H. Considering that the perturbed trajectories are here neighboring optimals of the original problem, and that continuity of the variational Hamiltonian is a necessary condition, already satisfied along the original optimal, continuity of the first order perturbation of the Hamiltonian is a natural request and could have been justified directly.

The expansion of (2.49) taking into account the continuity of x, s and p, gives

$$\begin{aligned}(\hat{H}^\lambda(\tau+0)-\hat{H}^\lambda(\tau-0))p(\tau)+(\hat{H}^q(\tau+0)-\hat{H}^q(\tau-0))x(\tau)\\ = -(\hat{H}^t(\tau+0)-\hat{H}^t(\tau-0))s(\tau)\end{aligned} \tag{2.50}$$

In particular, if the system is autonomous, the right-hand side vanishes. To make the calculation of $s(\tau)$ explicit from (2.50) we can introduce the isochronic perturbations

$$\begin{aligned}\delta q(\tau-0) &= x(\tau)-\hat{H}_\lambda(\tau-0)s(\tau)\\ \delta\lambda(\tau-0) &= p(\tau)+\hat{H}_q(\tau-0)s(\tau)\end{aligned} \tag{2.51}$$

whereby (2.50) furnishes

$$Ds(\tau) = (\hat{H}^\lambda(\tau+0)-\hat{H}^\lambda(\tau-0))\delta\lambda(\tau-0)+(\hat{H}^q(\tau+0)-\hat{H}^q(\tau-0))\delta q(\tau-0) \tag{2.52}$$

where

$$D = \hat{H}^t(\tau-0)-\hat{H}^t(\tau+0)+\hat{H}^\lambda(\tau+0)\hat{H}_q(\tau-0)-\hat{H}^q(\tau+0)\hat{H}_\lambda(\tau-0)$$

The integration of the isochronic perturbations is performed with the simplified perturbation equations ($\sigma = 0$)

$$\begin{aligned}\delta\dot{q} &= \hat{H}^q_\lambda\delta q+\hat{H}^w_\lambda\delta w\\ \delta\dot{\lambda} &= -\hat{H}^\lambda_q\delta\lambda-\hat{H}^q_q\delta q-\hat{H}^w_q\delta w\\ \hat{H}^\lambda_w\delta\lambda+\hat{H}^q_w\delta q+\hat{H}^w_w\delta w &= 0\end{aligned}$$

the last one being solvable for δw if the strong Legendre–Clebsch condition is satisfied.

The formula (2.52) is adapted to forward integration of the perturbations. If repeated backward integration is used to prepare a tube of neighboring optimals in order to establish the matrix synthetizing the optimal control perturbations as functions of the state,[6] we can replace (2.51) by

$$\begin{aligned}\delta q(\tau+0) &= x(\tau)-\hat{H}_\lambda(\tau+0)s(\tau)\\ \delta\lambda(\tau+0) &= p(\tau)+\hat{H}_q(\tau+0)s(\tau)\end{aligned} \tag{2.53}$$

whereby (2.52) takes exactly the same form except that we replace

$$\begin{aligned}\delta\lambda(\tau-0) &\to \delta\lambda(\tau+0)\\ \delta q(\tau-0) &\to \delta q(\tau+0)\end{aligned}$$

and D remains in fact the same.

The jump in the isochronic perturbations as we pass a corner point is found by noting that x and p are continuous. Then, from (2.51) and (2.53)

$$\begin{aligned} \delta q(\tau+0)-\delta q(\tau-0) &= \big(\hat{H}_\lambda(\tau-0)-\hat{H}_\lambda(\tau+0)\big)s(\tau) \\ \delta\lambda(\tau+0)-\delta\lambda(\tau-0) &= \big(\hat{H}_q(\tau+0)-\hat{H}_q(\tau-0)\big)s(\tau) \end{aligned} \tag{2.54}$$

This can be used in forward integration in conjunction with (2.52) and in backward integration with its indicated modification. The interpretation of (2.54) is of course that the finite difference in the controls during the time interval between τ and $\tau+s(\tau)$, which represents an impulse, can be replaced by a Dirac measure at $t = \tau$ on the velocity vector of the system.

The interpretation of the guidance command can be put forward as follows. Replace the nominal optimal command $\hat{w}$ by the sum of a continuous vector valued function $\hat{w}_1$ and a vector valued step function $\hat{w}_2$. The perturbed optimal command consists in replacing $\hat{w}_1$ by $\hat{w}_1+\delta w$ and displace from $s(\tau)$ in the time scale the steps of $\hat{w}_2$.

As regards the conjugate point condition, the situation can be sketched as follows. We start a forward integration with zero initial perturbations on state and time $\big(\delta q(\alpha) = 0,\ s(\alpha) = 0\big)$ but a nonzero perturbation vector on the multipliers $\delta\lambda(\alpha)$.

If the problem is not abnormal (if the set of optimal multipliers $\lambda(t)$ is unique), the system will leave the optimal trajectory to follow a neighboring one. A conjugate point (to the origin) exsits at $t = t(\theta)$ if, for a suitable $\delta\lambda(\alpha)$, the perturbation on the state vanishes again: $\delta q(\theta) = 0$. Stated otherwise: a conjugate point occurs there where the $(n\times n)$ matrix of a fundamental system

$$\delta_m q(t) \qquad m = 1, \ldots, n$$

of state perturbations, generated by n linearly independent perturbations $\delta_m\lambda(\alpha)$, degenerates. The implications of this on optimal perturbation guidance is obvious. One can also prove, by an extension of an argument by Bliss,[8, 9] that the interdiction to trespass a conjugate point is necessary for the non-negative character of the second variation. To this purpose one takes from $t(\alpha)$ to $t(\theta)$ the perturbed trajectory leading to $\delta q(\theta) = 0$ completed from $t(\theta)$ to $t(\beta)$ by the nominal trajectory itself. This perturbation can be normed by scaling to $(d\varepsilon)^2$. One then shows that

(1) the second variation is zero;
(2) this perturbation does not give to the second variation its minimum value because a necessary condition to this effect is not satisfied.

(1) is obvious because the contribution of the first portion of trajectory to the second variation is $\zeta(d\varepsilon)^2$ and ζ is zero.

(2) is also true because $\delta\lambda(\theta-0)$ cannot have all its components zero (otherwise, backward integration with also $\delta q(\theta) = 0$ shows all perturbations to vanish identically). But $\delta\lambda\ (\theta+0) = 0$ and the necessary condition of continuity of the perturbed multipliers is violated.

2.8. Application to an extremal with a singular arc

The following example, that can be treated analytically, illustrates the application of the eigenvalue type of second variation analysis to an optimal trajectory involving a singular extremal. The system is described by the differential equations:

$$\dot{q}_1 = v \qquad \dot{q}_2 = q_1^2$$

with the boundary conditions:

$$q_1(\alpha) = A > 0 \qquad q_1(\beta) = B > 0$$
$$t(\alpha) = 0 \qquad t(\beta) = T > A+B$$

The functional to be minimized is:

$$I = q_2(\beta) - q_2(\alpha) \text{ minimum.}$$

The single control v is originally a flip-flop $v = \pm 1$. Typically the solution, which can of course be analyzed by elementary methods,(4) involves chattering of the control in the time interval (A, $C = T-B$). The relaxed variational problem, which regularizes chattering and makes the absolute minimum accessible, consists in extending the possible control values to the segment $v \in (-1, 1)$. The variational constraints on the control along the segments where $v = \pm 1$ are avoided by the transformation $v = \cos w$. This requires a semi-norm for the metric of weakly varied trajectories:

$$2(d\varepsilon)^2 = \int_0^T (\sin^2 wu^2 + \sigma^2)dt$$

The optimal trajectory in (q_1, t) space is shown on Fig. 2. Its characteristics, needed to calculate the critical perturbations, are summarized below.

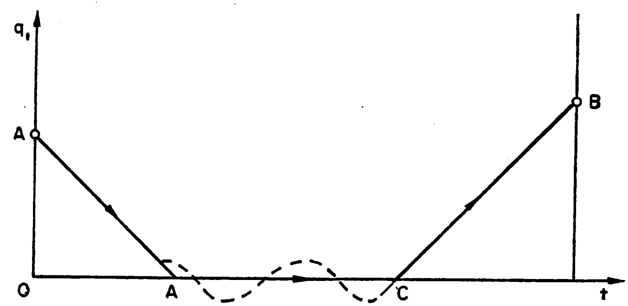

FIG. 2. The three segments of the optimal trajectory in (q_1, t) space. Segment AC is a singular arc. To each eigenvalue of the necessary minimum problem corresponds a perturbed trajectory of the type shown in dotted line.

Segment 1	Segment 2	Segment 3
$t \in (0, A)$	$t \in (A, C)$	$t \in (C, T)$
$\cos \hat{w} = -1$	$\cos \hat{w} = 0$	$\cos \hat{w} = 1$
$\hat{q}_1 = A-t$	$\hat{q}_1 = 0$	$\hat{q}_1 = t-C$
$\hat{\lambda}_1 = -(A-t)^2$	$\hat{\lambda}_1 = 0$	$\hat{\lambda}_1 = (t-C)^2$
$\hat{\lambda}_2 = -1$	$\hat{\lambda}_2 = -1$	$\hat{\lambda}_2 = -1$

The perturbed equations and optimal controls will now be solved for each segment.

Segment 1

$$\dot{x}_1 = -\sigma \qquad \dot{x}_2 = 2(A-t)x_1+(A-t)^2\sigma \qquad \dot{s} = \sigma$$
$$\dot{p}_1 = 2x_1+2(A-t)(\sigma-p_2) \qquad \dot{p}_2 = 0 \qquad \dot{\eta} = 0$$
$$(A-t)^2u = 0 \quad -p_1+(A-t)^2p_2-2(A-t)x_1+\zeta\sigma+\eta = 0$$

The adjoint multiplier η will be constant along the whole perturbed trajectory (the system is in fact autonomous). Differentiating the last equation and substituting the differential coefficients from the other equations (or applying directly equation (2.48)), we find $\zeta\dot{\sigma} = 0$. Hence, if the eigenvalue is not zero, σ is a constant which will be denoted by σ_1. The perturbed equations are now easily integrated. Taking into account the initial values $x_1(0) = 0$ and $s(0) = 0$ needed to satisfy boundary conditions and taking p_1 from the last equation, we obtain the following end values

$$x_1(A) = -\sigma_1 A \quad s(A) = \sigma_1 A \quad p_1(A) = \eta+\zeta\sigma_1$$

The integration of x_2 and p_2 are not required. It is also seen that u must vanish; but this does not affect the other equations. In fact, even if u were nonzero, the system, that is not tightly controlled on this segment, would follow the same trajectory. The only real perturbation here is the shortening or lengthening of the segment by the time shift $s(A)$.

Segment 2

This one is a singular extremal ($\hat{H}^w_w = 0$) but tightly controlled.

$$\dot{x}_1 = -u \qquad \dot{s} = \sigma \qquad \dot{p}_1 = 2x_1$$
$$-p_1+\zeta u = 0 \qquad \zeta\sigma+\eta = 0$$

A differential equation is obtained for the optimal perturbed control by two differentiations:

$$\zeta\dot{u} = \dot{p}_1 = 2x_1 \qquad \zeta\ddot{u} = 2\dot{x}_1 = -2u$$

We change the eigenvalue parameter by the transformation:

$$\omega^2 = 2/\zeta$$

and solve in succession

$$u = a \sin \omega(t-A) + b \cos \omega(t-A)$$
$$\omega x_1 = a \cos \omega(t-A) - b \sin \omega(t-A)$$
$$\omega^2 p_1 = 2a \sin \omega(t-A) + 2b \cos \omega(t-A)$$

Since η is constant and ζ assumed non-zero, σ is a constant, denoted σ_2 and

$$2\sigma_2 + \omega^2 \eta = 0 \tag{a}$$

Integrating s, with continuity at $t = A$, we find

$$s = \sigma_2(t-A) + \sigma_1 A$$

The continuity requirements at $t = A$ of p_1 and x_1 furnish:

$$2b = 2\sigma_1 + \omega^2 \eta \tag{b}$$
$$a = -\omega \sigma_1 A \tag{c}$$

The end values of interest are:

$$s(C) = \sigma_2(C-A) + \sigma_1 A$$
$$x_1(C) = \frac{1}{\omega}[a \cos \omega(C-A) - b \sin \omega(C-A)]$$
$$p_1(C) = \frac{2}{\omega^2}[a \sin \omega(C-A) + b \cos \omega(C-A)]$$

A critical perturbation of this segment is oscillatory.

Segment 3

Segment 3 is simply lengthened or shortened as segment 1.

$$\dot{x}_1 = \sigma \qquad \dot{s} = \sigma \qquad \dot{p}_2 = 0$$
$$\dot{p}_1 = 2x_1 + 2(t-C)\sigma - 2(t-C)p_2$$
$$(t-C)^2 u = 0 \quad p_1 + (t-C)^2 p_2 - 2(t-C)x_1 + \zeta\sigma + \eta = 0$$

Again σ is found to be a constant, denoted σ_3. Integration with continuity at $t = C$ produces:

$$x_1 = (t-C)\sigma_3 + \frac{1}{\omega}(a \cos \omega(C-A) - b \sin \omega(C-A))$$
$$\sigma = (t-C)\sigma_3 + \sigma_2(C-A) + \sigma_1 A$$

and p_1 is taken from the optimal control equation. Continuity of p_1 at $t = C$ requires:

$$\zeta\sigma_3 + \eta + \frac{2}{\omega^2}(a \sin \omega(C-A) + b \cos \omega(C-A)) = 0 \tag{d}$$

Finally the boundary conditions $x_1(T) = 0$ and $\sigma(T) = 0$ yield

$$\omega B\sigma_3 + a\cos\omega(C-A) - b\sin\omega(C-A) = 0 \qquad \text{(e)}$$

$$B\sigma_3 + \sigma_2(C-A) + \sigma_1 A = 0 \qquad \text{(f)}$$

Equations (a)$-$(f) are homogeneous in the six quantities $a, b, \eta, \sigma_1, \sigma_2$ and σ_3. The three first ones are easily eliminated to produce a homogeneous set of conditions in the three rates of time shift:

$$\sigma_1(\cos\phi - \omega A\sin\phi) - \sigma_2(1+\cos\phi) + \sigma_3 = 0$$

$$-\sigma_1(\sin\phi + \omega A\cos\phi) + \sigma_2\sin\phi + \sigma_3\omega B = 0$$

$$\sigma_1 A\omega + \sigma_2\phi + \sigma_3 B\omega = 0 \qquad (\phi = \omega(C-A))$$

Setting the determinant equal to zero, we obtain a transcendental equation to be satisfied by the eigenvalues and which depends on the two parameters:

$$p = \frac{A}{C-A} \quad \text{and} \quad q = \frac{B}{C-A}$$

$$\frac{\sin\phi}{\phi}(1+p+q-pq\phi^2) + \cos\phi(p+q+2pq) + 2pq = 0$$

It is easily seen to possess an infinite sequence of eigenvalues:

$$\phi = \pm\phi_1, \pm\phi_2 \ldots \qquad 0 < \phi_1 < \phi_2 \ldots$$

asymptotic to $\pm n\pi$ for n large.

The corresponding eigenvalues ζ_i are all positive

$$\zeta_i = 2\left(\frac{C-A}{\phi_i}\right)^2 \qquad \zeta_1 > \zeta_2 > \zeta_3 \ldots > 0$$

Although there is no most critically perturbed trajectory, the second variation test is satisfied; zero is an infimum but not a minimum of $\Delta^2 I$.

REFERENCES

1. B. Fraeijs de Veubeke. Une condition suffisante de minimum relatif dans le problème des extrémés liés. *Acad. R. de Belgique, Bull. Cl. des Sciences*, 512–17, 1966–4.
2. P. Beckers. Coplanar impulsive transfers and the second variation test. Proc. Colloquium on Optimization in Space Flight Mechanics, Liège, 1967. Pergamon Press.
3. P. Contensou. Etude théorique des trajectories optimales dans un champ de gravitation. Application au cas d'un centre d'attraction unique. *Astronautica Acta*, **8**, 2–3, 1962.
4. B. Fraeijs de Veubeke. Régularisation des réticences et réduction du Principe du Maximum de Pontrjagin au calcul classique des variations. Comité National de Mécanique Théorique et Appliquée, juin 1964.
5. J. Warga. Relaxed variational problems. *J. Math. Anal. Appl.* **4**, 432–55, 1962.

6. J. V. Breakwell, J. L. Speyer and A. E. Bryson. Optimization and Control of nonlinear systems using the second variation. *SIAM Journal on Control*, **1**, 2, 193–223, 1963.
7. H. J. Kelley, R. E. Kopp and H. G. Moyer. Singular extremals. *Topics in Optimisation*, Chap. 3. Academic Press, 1967.
See also: H. M. Robbins. A generalized Legendre–Clebsch condition for the singular cases of optimal control. IBM report 66–825/2043, September 1966.
8. G. Bliss. *Lectures on the Calculus of Variations*. The University of Chicago Press, 1946.
9. J. V. Breakwell and Y. C. Ho. On the conjugate point condition for the control problem. *Int. J. Engng. Sci.* **2**, 565–79, 1965.