STATIC-GEOMETRIC ANALOGIES AND FINITE ELEMENT MODELS

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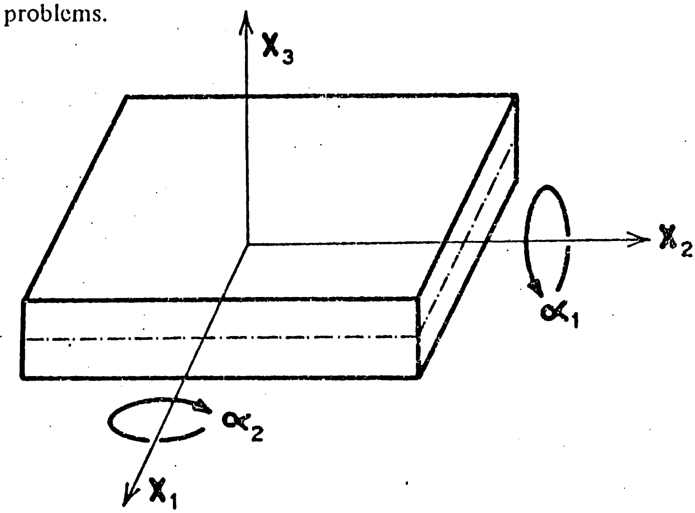
KIRCHHOFF PLATE BENDING THEORY

The variational principles underlying the Kirchhoff plate bending theory are elaborations of the general principles [1] that take into account the simplifying assumptions reducing the problem from three to two dimensions. They will be presented in a form outlining the static-geometric analogies with plane stress problems.

Figure 1.



(x_1, x_2) are cartesian coordinates in the middle plane. Basic assumptions are:

$$\tau_{33} = 0 \qquad u_1 = x_3\alpha_1(x_1,x_2) \qquad u_2 = x_3\alpha_2(x_1,x_2) \qquad u_3 = w(x_1,x_2) \tag{1}$$

Bending and twisting moments are defined as

$$M_{11} = \int_{-h}^{h} \tau_{11}x_3dx_3 \qquad M_{12} = \int_{-h}^{h} \tau_{12}x_3dx_3 \qquad M_{22} = \int_{-h}^{h} \tau_{22}x_3dx_3 \tag{2}$$

and the shear loads as

$$Q_1 = \int_{-h}^{h} \tau_{13} dx_3 \qquad Q_2 = \int_{-h}^{h} \tau_{12} dx_2 \tag{3}$$

For the variation of strain energy per unit plate area one obtains then

$$\delta W = M_{11} D_1 \delta\alpha_1 + M_{12} D_1 \delta\alpha_2 + M_{12} D_2 \delta\alpha_1 + M_{22} D_2 \delta\alpha_2$$
$$+ Q_1 (\delta\alpha_1 + D_1 \delta w) + Q_2 (\delta\alpha_2 + D_2 \delta w) \tag{4}$$

so that W is a function of the strains

$$D_1\alpha_1, D_1\alpha_2 + D_2\alpha_1, D_2\alpha_2, \alpha_1 + D_1 w, \alpha_2 + D_2 w$$

and the bending and twisitng moments and shear loads are the corresponding partial derivatives. For isotropic materials we have, explicitly

$$W = \frac{1}{2} B \{(D_1\alpha_1)^2 + (D_2\alpha_2)^2 + 2\nu D_1\alpha_1 D_2\alpha_2 + \frac{1-\nu}{2}(D_1\alpha_2 + D_2\alpha_1)^2\}$$
$$+ Gh\{(\alpha_1 + D_1 w)^2 + (\alpha_2 + D_2 w)^2\} \tag{5}$$

where G is the shear modulus and

$$B = \frac{2}{3} \frac{Eh^3}{1-\nu^2}$$

is the so-called bending rigidity of the plate.

The plate bending theory corresponding to the foregoing assumptions is that of Hencky [27]; it includes the effect of shearing deformations. The additional Kirchhoff-Love assumptions

$$\alpha_1 + D_1 w = 0 \qquad \alpha_2 + D_2 w = 0 \tag{6}$$

whereby shearing strains are ignored, eliminates the calculation of shear loads as partial derivatives of the strain energy. Nevertheless, one can retain them in expression (4) as lagrangian multipliers liberating the Kirchhoff-Love constraints.

Conservation of energy requires that the total increase in strain energy be equal to the virtual work performed by the external forces acting on the plate, transverse pressure $\bar{q}(x_1, x_2)$, edge bending moment $\bar{M}_{nn}$, edge shear load $\bar{K}_n$ and concentrated transverse forces $\bar{Z}_i$ at angular points of the edge contour.

$$\int_A \delta W dA = \int_A \bar{q} \delta W dA + \int_C (\bar{M}_{nn} \delta\alpha_n + \bar{K}_n \delta w) ds + \Sigma \bar{Z}_i \delta w_i \tag{7}$$

The angle α_n is the rotation about the axis $\vec{t}$ tangent to the edge contour. The outward normal $\vec{n}$

to the edge contour forms with $\vec{t}$ a system of local axes whose orientation corresponds to a rotation of the axes Ox_1 and Ox_2 about Ox_3 by an angle θ. Thus the direction cosines of the outward normal are

$$n_1 = \cos\theta \qquad n_2 = \sin\theta \tag{8}$$

and

$$\alpha_n = n_1\alpha_1 + n_2\alpha_2 \tag{9}$$

while the rotation of the edge about the normal is

$$\alpha_t = n_1\alpha_2 - n_2\alpha_1 \tag{10}$$

Similarly, the derivatives along the outward normal and along the tangent to the contour will be given by

$$D_n = n_1 D_1 + n_2 D_2 \tag{11}$$

$$D_s = n_1 D_2 - n_2 D_1 \tag{12}$$

Turning back to the equation of conservation of energy (7), we substitute (4) and integrate by parts obtaining

$$\int_A \{\delta\alpha_1(Q_1 - D_1M_{11} - D_2M_{12}) + \delta\alpha_2(Q_2 - D_1M_{12} - D_2M_{22}) - \delta w(D_1Q_1 + D_2Q_2$$

$$+ \overline{q})\}dA + \int_c \{\delta\alpha_1(n_1M_{11} + n_2M_{12}) + \delta\alpha_2(n_1M_{12} + n_2M_{22}) + \delta w(Q_n - \overline{K}_n)\}ds$$

$$- \int_c \delta\alpha_n\overline{M}_{nn}ds - \Sigma\,\overline{Z}_i\,\delta w_i = 0 \tag{13}$$

In this equation Q_n stands for the shear load resultant

$$Q_n = n_1Q_1 + n_2Q_2 = \int_{-h}^{h} \tau_{n3}dx_3 \tag{14}$$

Since α_1 and α_2 were liberated from the constraints (6), the variations $\delta\alpha_1$, $\delta\alpha_2$ and δw are independent in the surface integral, and the vanishing of their coefficients provides the classical equilibrium equations

$$D_1M_{11} + D_2M_{12} = Q_1 \qquad D_1M_{12} + D_2M_{22} = Q_2 \tag{15}$$

$$D_1 Q_1 + D_2 Q_2 = -\bar{q} \tag{16}$$

The remaining equation can be placed in the form

$$\int_c \{\delta\alpha_n (M_{nn} - \bar{M}_{nn}) + \delta\alpha_t M_{nt} + \delta w (Q_n - \bar{K}_n)\} ds - \Sigma \bar{Z}_i \delta w_i = 0 \tag{17}$$

by using (9) and (10) and noting the tensor transformation rules to the local axes

$$\begin{aligned} n_1 M_{11} + n_2 M_{12} &= n_1 M_{nn} - n_2 M_{nt} \\ n_1 M_{12} + n_2 M_{22} &= n_2 M_{nn} + n_1 M_{nt} \end{aligned} \tag{18}$$

The contour integral must be manipulated to take into account the constraints

$$a_n = -D_n w \qquad \alpha_t = -D_s w \tag{19}$$

deriving from (6), (9), (11) and (12). A further integration by parts is then indicated, and (17) takes the final form

$$\int_c \{D_n \delta w (\bar{M}_{nn} - M_{nn}) + \delta w (Q_n + D_s M_{nt} - \bar{K}_n)\} ds - [M_{nt} \delta w]_c - \Sigma \bar{Z}_i \delta w_i = 0$$

Since $D_n \delta w$ and δw are independent contour variations, the boundary conditions for applied loads are

$$M_{nn} = \bar{M}_{nn} \tag{20}$$

$$K_n = Q_n + D_s M_{nt} = \bar{K}_n \tag{21}$$

$$(M_{nt})_+ - (M_{nt})_- = \bar{Z}_i \quad \text{at angular points.} \tag{22}$$

The last condition stems from the fact that δw must be single valued at an angular point, the subscripts - and + indicating respectively the limiting values of the twisting moment as we approach or leave the angular point. The same analysis shows that the displacements that can be imposed along the contour are w and $\alpha_t = -D_n w$.

THE FIRST STATIC-GEOMETRIC ANALOGY

Several authors [3,4] noted some of the analogies between the formulation of Kirchhoff plate bending theory in terms of the transverse displacement w and the plane stress (or plane strain) problem as formulated in terms of an Airy function. This analogy is particularly transparent when comparing the variational approach of the previous section with the complementary energy principle for plane stress [5]. Since in the complementary energy formulation the equilibrium equations are to be satisfied a priori, the use of the Airy stress function is natural. Assuming the body forces to derive from a potential ψ

$$X_1 = -D_1\psi \qquad X_2 = -D_2\psi \tag{23}$$

the volume equilibrium equations for the normal forces and shear flow are given by

$$\begin{aligned} D_1(\tau_{11} - \psi) + D_2\tau_{21} &= 0 \\ D_1\tau_{12} + D_2(\tau_{22} - \psi) &= 0 \end{aligned} \tag{24}$$

and are satisfied first by introducing two stress functions α_1 and α_2:

$$\begin{aligned} \tau_{11} - \psi &= -D_2\alpha_2 \qquad & \tau_{21} &= D_1\alpha_2 \\ \tau_{12} &= D_2\alpha_1 \qquad & \tau_{22} - \psi &= -D_1\alpha_1 \end{aligned} \tag{25}$$

Those equations show that $\psi - \tau_{11}$, $\psi - \tau_{22}$ and $\tau_{21} + \tau_{12}$ play roles analogous to the strains of the Kirchhoff theory. In order to satisfy the rotational equilibrium condition $\tau_{21} = \tau_{12}$, one should introduce the Airy stress function w

$$\alpha_1 = -D_1 w \qquad \alpha_2 = -D_2 w \tag{26}$$

and obtain the classical relations

$$\begin{aligned} \tau_{11} &= \psi + D_2^2 w \qquad & \tau_{12} = \tau_{21} &= -D_1 D_2 w \\ \tau_{22} &= \psi + D_1^2 w \end{aligned} \tag{27}$$

However, noting the analogy between (26) and the Kirchhoff-Love assumptions (6), we shall keep (26) as constraints liberated by the use of lagrangian multipliers q_1 and q_2 analogous to the shear loads Q_1 and Q_2 of the Kirchhoff theory.

The variation of the complementary energy per unit area will then be

$$\begin{aligned} \delta\phi &= \epsilon_{11}\delta\tau_{11} + \epsilon_{21}\delta\tau_{21} + \epsilon_{12}\delta\tau_{12} + \epsilon_{22}\delta\tau_{22} \\ &= -\epsilon_{11}D_2\delta\alpha_2 + \epsilon_{21}D_1\delta\alpha_2 + \epsilon_{12}D_2\delta\alpha_1 - \epsilon_{22}D_1\alpha_1 \\ &\quad + q_1(\delta\alpha_1 + D_1\delta w) + q_2(\delta\alpha_2 + D_2\delta w) \end{aligned} \tag{28}$$

In this expression the ϵ_{ij} are the partial derivatives of ϕ, considered as a quadratic function in the variables τ_{11}, τ_{22} and $\tau_{12} + \tau_{21}$, so that $\epsilon_{12} = \epsilon_{21}$. By conservation of energy, the variation of the total stress energy must be equal to the complementary virtual work of the displacements $(\bar{u}_1, \bar{u}_2)$ imposed along the boundary on the surface traction variations

$$\int_A \delta\phi dA = \int_c \{\bar{u}_1 (n_1 \delta\tau_{11} + n_2 \delta\tau_{21}) + \bar{u}_2 (n_1 \delta\tau_{12} + n_2 \delta\tau_{22})\} ds$$

$$= \int_c \{\bar{u}_1 (-n_1 D_2 \delta\alpha_2 + n_2 D_1 \delta\alpha_2) + \bar{u}_2 (n_1 D_2 \delta\alpha_1 - n_2 D_1 \delta\alpha_1)\} ds$$

$$= \int_c (-\bar{u}_1 D_s \delta\alpha_2 + \bar{u}_2 D_s \delta\alpha_1) ds \tag{29}$$

Substituting (28) and integrating by parts

$$\int_A \{\delta\alpha_2 (D_2 \epsilon_{11} - D_1 \epsilon_{21} + q_2) + \delta\alpha_1 (-D_2 \epsilon_{12} + D_1 \epsilon_{22} + q_1) - \delta w (D_1 q_1 + D_2 q_2)\} dA$$
$$+ \int_c \{\delta\alpha_2 (n_1 \epsilon_{21} - n_2 \epsilon_{11}) + \epsilon\alpha_1 (n_2 \epsilon_{12} - n_1 \epsilon_{22}) + \bar{u}_1 D_s \delta\alpha_2 - \bar{u}_2 D_s \delta\alpha_1 + q_n \delta w\} ds = 0 \tag{30}$$

Again the vanishing of coefficients of the independent variations $\delta\alpha_1$, $\delta\alpha_2$, δw in the surface integral produces the compatibility requirements

$$- D_1 \epsilon_{22} + D_2 \epsilon_{12} = q_1 \qquad D_1 \epsilon_{21} - D_2 \epsilon_{11} = q_2 \tag{31}$$

$$D_1 q_1 + D_2 q_2 = 0 \tag{32}$$

By comparison with (15) and (16) they show that $(-\epsilon_{22}, \epsilon_{12}, -\epsilon_{11})$ are the respective analogues of (M_{11}, M_{12}, M_{22}).

Note further that elimination of the multipliers between (31) and (32) furnishes the classical local compatibility condition for strains

$$D_1^2 \epsilon_{22} + D_2^2 \epsilon_{11} = 2 D_1 D_2 \epsilon_{12} \tag{33}$$

Furthermore, should we satisfy (32) by setting

$$q_1 = -D_2 \omega \qquad q_2 = D_1 \omega \tag{34}$$

equations (31) become the Beltrami equations for the integration of the material rotation ω, (33) being their integrability condition. After disappearance of the surface integral in (30), the remainder can first be manipulated as in the previous section by using

$$n_1 \epsilon_{21} - n_2 \epsilon_{11} = n_1 \epsilon_{nt} - n_2 \epsilon_{tt}$$
$$n_1 \epsilon_{22} - n_2 \epsilon_{12} = n_2 \epsilon_{nt} + n_1 \epsilon_{tt} \tag{35}$$

There comes

$$\int_c (\epsilon_{nt}\delta\alpha_t - \epsilon_{tt}\delta\alpha_n + \bar{u}_1 D_s \delta\alpha_2 - \bar{u}_2 D_s \delta\alpha_1 + q_n \delta w)ds = 0 \tag{36}$$

This is integrated by parts, noting that the single-valuedness of $\bar{u}_1$, $\bar{u}_2$ and $\delta\alpha_1$ and $\delta\alpha_2$ causes the integrated terms to vanish, producing

$$\int_c \{\delta\alpha_t(\epsilon_{nt} - n_1 D_s \bar{u}_1 - n_2 D_s \bar{u}_2)' + \delta\alpha_n(-\epsilon_{tt} - n_2 D_s \bar{u}_1 + n_1 D_s \bar{u}_2) + q_n \delta w\} ds = 0$$

Finally, we note that, in accordance with (26),

$$a_n = -D_n w \qquad \alpha_t = -D_s w \tag{37}$$

so that a further integration by parts is indicated to obtain coefficients of the independent variations δw and $D_n \delta w$. In carrying out this integration the single-valuedness of δw indicates that

$$\epsilon_{nt} - n_1 D_1 \bar{u}_1 - n_2 D_s \bar{u}_2 \quad \text{single-valued along c.} \tag{38}$$

Setting the coefficients of independent variations equal to zero, we obtain

$$D_s(c_{nt} - n_1 D_s \bar{u}_1 - n_2 D_s \bar{u}_2) + q_n = 0 \tag{39}$$

$$\epsilon_{tt} = n_1 D_s \bar{u}_2 - n_2 D_s \bar{u}_1 \qquad \text{along c} \tag{40}$$

The interpretation of those boundary conditions is perhaps clearer if the displacement vector is decomposed in local coordinates

$$\bar{u}_1 = n_1 \bar{u}_n - n_2 \bar{u}_t$$

$$\bar{u}_2 = n_2 \bar{u}_n + n_1 \bar{u}_t$$

Then, using (8), the notation $\dot{\theta} = d\theta/ds$, and noting that

$$q_n = n_1 q_1 + n_2 q_2 = (n_2 D_1 - n_1 D_2)\omega \tag{41}$$

we transform (38), (39) and (40) respectively in

$$\epsilon_{nt} - D_s \bar{u}_n + \dot{\theta}\bar{u}_t \qquad \text{single-valued along c} \tag{38'}$$

$$D_s(\epsilon_{nt} - \omega - D_s \bar{u}_n + \dot{\theta}\bar{u}_t) = 0 \tag{39'}$$

$$\epsilon_{tt} = D_s \bar{u}_t + \dot{\theta} \bar{u}_n \tag{40'}$$

showing that, as it should, the single-valuedness of ω along c. Also, in view of the Beltrami equations (31) and (34) it can be shown that

$$D_s(\epsilon_{nt} - \omega) = 2D_s \epsilon_{nt} - D_n \epsilon_{tt} + \dot{\theta}(\epsilon_{nn} - \epsilon_{tt}) \tag{42}$$

MULTI-VALUEDNESS OF THE AIRY FUNCTION AND ITS DERIVATIVES

Consider the following integrals extended to any closed contour γ, which is the boundary of a simply connected domain D

$$\begin{aligned} \int_\gamma \tau_{11} dx_2 - \tau_{21} dx_1 &= \int_\gamma (\psi dx_2 - d\alpha_2) \\ \int_\gamma \tau_{12} dx_2 - \tau_{22} dx_1 &= \int_\gamma (-\psi dx_1 + d\alpha_1) \end{aligned} \tag{43}$$

where the right-hand sides are obtained from relations (25). The left-hand sides represent respectively the total force components F_1 and F_2 due to the surface tractions along the boudary. By global equilibrium considerations we must have

$$\begin{aligned} F_1 &= -\int_D X_1 dx_1 dx_2 = \int_D D_1 \psi dx_1 dx_2 = \int_\gamma \psi dx_2 \\ F_2 &= -\int_D X_2 dx_1 dx_2 = \int_D D_2 \psi dx_1 dx_2 = -\int_\gamma \psi dx_1 \end{aligned}$$

the successive forms deriving from (23) and application of Stokes theorem.

The conclusions

$$\int_\gamma d\alpha_1 = 0 \qquad \int_\gamma d\alpha_2 = 0 \tag{44}$$

can also be obtained directly by observing that the stresses, generated by α_1 and α_2 only, are in equilibrium without body forces. In a similar manner, considering the moment with respect to the origin of the surface traction forces

$$\int_\gamma x_1(\tau_{12} dx_2 - \tau_{22} dx_1) - x_2(\tau_{11} dx_2 - \tau_{21} dx_1)$$

$$= -\int_{\gamma} \psi(x_1 dx_1 + x_2 dx_2) + \int_{\gamma} x_1 d\alpha_1 + x_2 d\alpha_2 \tag{45}$$

and observing that global equilibrium requires that it should also be equal to

$$-\int_{D} (x_1 X_2 - x_2 X_1) dx_1 dx_2 = -\int_{D} (x_2 D_1 \psi - x_1 D_2 \psi) dx_1 dx_2 = -\int_{\gamma} \psi(x_1 dx_1 + x_2 dx_2)$$

we obtain

$$-\int_{\gamma} x_1 d\alpha_1 + x_2 d\alpha_2 = 0 \tag{46}$$

In view of (44) and (26) this is equivalent to

$$-\int_{\gamma} \alpha_1 dx_1 + \alpha_2 dx_2 = \int_{\gamma} dw = 0 \tag{47}$$

Consequently, in any simple connected domain the Airy function and its first partial derivatives are single-valued. Consider now a multiply-connected domain with a single internal cavity of contour ζ as shown in Fig. 2.

From (43) we obtain that

$$\int_{\zeta} d\alpha_2 = \int_{\zeta} \psi dx_2 - F_1 \qquad \int_{\zeta} d\alpha_1 = \int_{\zeta} \psi dx_1 + F_2 \tag{48}$$

where F_1 and F_2 are the components of the total force exerted by the surface tractions applied along the internal boundary. Furthermore, there are also contributions from the body force potential that do not in general cancel F_1 and F_2. Hence, the first partial derivatives of the Airy function have in general nonzero cyclic constants. The jumps $\Delta\alpha_1$ and $\Delta\alpha_2$ incurred across any barrier devised to render the domains simply connected remain equal to the values of (48) because the additional integrals along $(PA)_-$ and $(AP)_+$ cancel in view of the single-valuedness of the body force potential and the reciprocity of the surface tractions

$$t_1 = n_1 \tau_{11} + n_2 \tau_{21} \qquad \text{and} \qquad t_2 = n_1 \tau_{12} + n_2 \tau_{22}$$

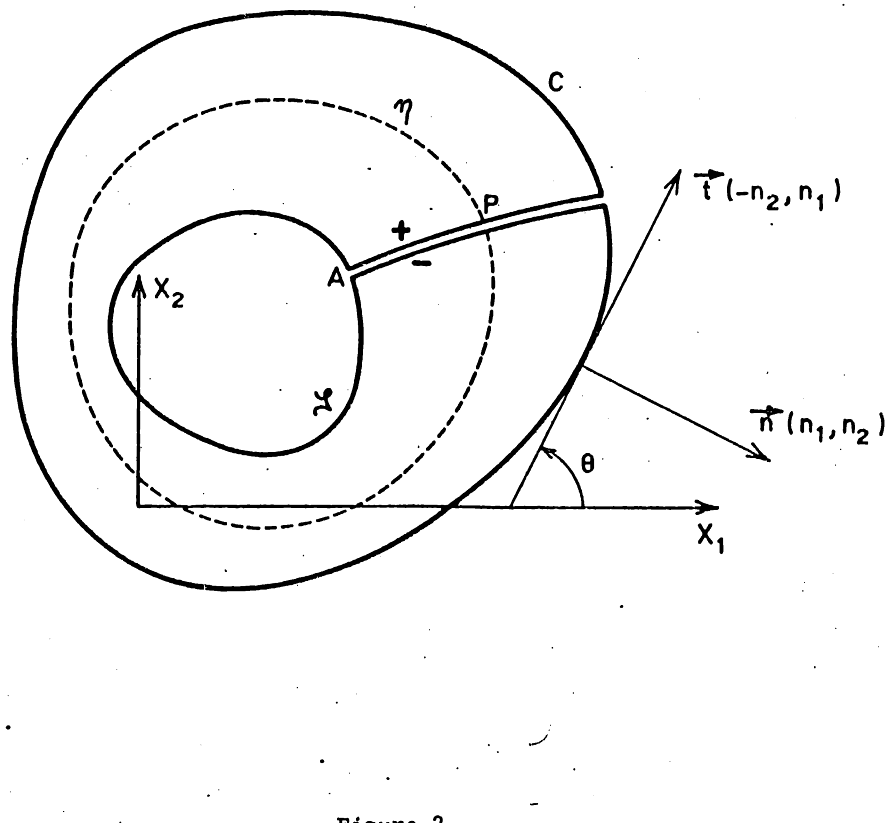


Figure 2.

Similarly we find

$$\int_{\zeta} dw = -\int_{\zeta} \alpha_1 dx_1 + \alpha_2 dx_2 = -X_1^A \int_{\zeta} d\alpha_1 - x_2^A \int_{\zeta} d\alpha_2 + \int_{\zeta} x_1 d\alpha_1 + x_2 d\alpha_2 \tag{49}$$

and the last term is, from (45),

$$\int_{\zeta} x_1 d\alpha_1 + x_2 d\alpha_2 = M_0 + \int_{\zeta} \psi(x_1 dx_1 + x_2 dx_2) \tag{50}$$

where M_0 is the moment with respect to the origin of the surface tractions along the internal boundary. This result furnishes the jump in value of the Airy function itself when the inner contour is described from A_- to A_+.

Consider now any point P in the domain and a closed contour q, passing through P and enclosing the inner cavity. We join it to the contour ζ by a two faced barrier $(AP)_-$ and $(AP)_+$. The domain enclosed in $A_-\ A_+\ P_+\ P_-\ A_-$ is simply connected so that the total jumps in w, α_1 and α_2 are zero. In other words, the jumps in $P(x_1, x_2)$ by describing the contour η are the same as those calculated by describing the contour $P_-\ A_-\ A_+\ P_+$. Because, again, the contributions from the two faces of the barrier cancel, we obtain the same result as (49) and (50), except that the coordinates of A must be replaced by those of P. Hence in P

$$\Delta\alpha_1 = \int_{\zeta} d\alpha_1 \qquad \Delta\alpha_2 = \int_{\zeta} d\alpha_2 \qquad \text{as given by (48)} \tag{51}$$

$$\Delta w = -x_1 \Delta\alpha_1 - x_2 \Delta\alpha_2 + M_0 + \int_{\zeta} \psi(x_1 dx_1 + x_2 dx_2) \tag{52}$$

In the analogy with plate flexure, where w is the deflection and α_1 and α_2 the slopes of this deflection, results (51) and (52) show that both faces of a cut from inner to outer boundaries can undergo a relative kinematical displacement. The analogue of the multivaluedness of the Airy function and its first derivatives are a "dislocation."

The extension of equations (51) and (52) to multiply-connected domains with more than one internal cavity presents no difficulties.

There is, of course, a straight forward derivation of the results (51) and (52) that, however, does not yield the physical interpretation of the constants. It consists in observing that the surface tractions

$$t_1 = n_1(\psi - D_2\alpha_2) + n_2 D_1 \alpha_2 = n_1 \psi - D_s \alpha_2$$

$$t_2 = n_1 D_2 \alpha_1 + n_2(\psi - D_1\alpha_1) = n_2 \psi + D_s \alpha_1$$

of reciprocal across a barrier, and in case of single valuedness of the body force potential, entail immediately the constancy of the jumps in α_1 and α_2. Thus the jump in w is necessarily of the form (52)

$$\Delta w = -x_1 \Delta\alpha_1 - x_2 \Delta\alpha_2 + \Delta w_o \tag{53}$$

as can be verified from (26).

DISLOCATIONS IN THE PLANE STRESS PROBLEM

Since the application of the complementary energy principle requires a simply-connected domain, we can imagine the multiply-connected domain to be converted to a simply connected one by physically cutting the domain by the barriers. There are no specified displacements at the interfaces; thus invoking the reciprocity of the surface tractions, we can write on the strength of the previous results that

$$(\delta\alpha_t)_+ = (\delta\alpha_t)_- \qquad (\delta\alpha_n)_+ = (\delta\alpha_n)_- \qquad (\delta w)_+ = (\delta w)_-$$

By the lagrangian variation technique that was used, those variations are independent. Hence, grouping terms at both interfaces in (36) and setting the coefficients of the variations equal to zero, we get as natural transition conditions:

$$(\epsilon_{nt})_+ = (\epsilon_{nt})_- \qquad (\epsilon_{tt})_+ = (\epsilon_{tt})_- \tag{54}$$

and, in view of (41),

$$D_s \omega_+ = D_s \omega_- \tag{55}$$

Those conditions are satisfied by the following jump conditions

$$\Delta u_1 = \Delta v_1 - x_2 \Delta\omega$$
$$\Delta u_2 = \Delta v_2 + x_1 \Delta\omega \tag{56}$$

Hence, in the multiply connected case, satisfaction of the local compatibility equation (33) is not sufficient to produce single-valued displacements.

To each barrier one should add the compatibility conditions in the large

$$\Delta v_1 = 0 \qquad \Delta v_2 = 0 \qquad \Delta\omega = 0 \tag{57}$$

that suppress the dislocation or relative kinematical displacement possibilities discovered by Weingarten [6] and Volterra [7].

THE SECOND ANALOGY

The stress approach is now applied to plate bending and compared to the displacement approach in membrane stretching, thus reversing the roles played by the variational principles in the first analogy. The plate equilibrium equation (16) is satisfied by setting

$$Q_1 = -D_1 \mu + D_2 \omega \qquad Q_2 = -D_2 \mu - D_1 \omega \tag{58}$$

provided the potential μ obeys the Poisson equation

$$(D_1^2 + D_2^2)\mu = \bar{q} \tag{59}$$

The other two equilibrium equations (15) become then

$$D_1(M_{11}+\mu) + D_2(M_{21} - \omega) + 0$$

$$D_1(M_{12} + \omega) + D_2(M_{22}+\mu) = 0$$

and can be satisfied by introducing a stress function vector (u_1, u_2) such that

$$M_{11} + \mu = D_2u_2 \qquad M_{22} + \mu = D_1u_1$$

$$M_{21} - \omega = -D_1u_2 \qquad M_{12} + \omega = -D_2u_1 \tag{60}$$

Finally the reciprocity of twisting moments requires that

$$\omega = \frac{1}{2}(D_1u_2 - D_2u_1) \tag{61}$$

in which case

$$M_{12} = M_{21} = -\frac{1}{2}(D_1u_2 + D_2u_2) \tag{62}$$

The notations u_1, u_2 and ω for the stress functions of plate bending theory were used on purpose to stress their analogy with the displacements and the material rotation of the membrane stretching problem. The analogues of the strains ϵ_{11}, ϵ_{22}, ϵ_{12} in this last problem are clearly the quantities $M_{22}+\mu$, $M_{11}+\mu$ and $-M_{12}$.

Introduce now the elements of the curvature tensor of plate bending

$$k_{11} = D_1\alpha_1 = -D_1^2w$$

$$k_{12} = \frac{1}{2}(D_1\alpha_2 + D_2\alpha_1) = -D_1D_2w$$

$$k_{22} = D_2\alpha_2 = -D_2^2w \tag{63}$$

which, according to (27), are the respective analogues of $\psi - \tau_{22}$, τ_{12} and $\psi - \tau_{11}$ in the membrane problem.

While the strain energy of plate bending per unit area was a function of those curvatures, the complementary strain energy, or stress-energy, is a function of the dual variables M_{11}, M_{12} and M_{22}. Its variation is

$$\delta S = k_{11}\delta M_{11} + 2k_{12}\delta M_{12} + k_{22}\delta M_{22}$$

$$= k_{11}D_2\delta u_2 - k_{12}(D_2\delta u_1 + D_1\delta u_2) + k_{22}D_1\delta u_1 \tag{64}$$

Along a boundary curve we can calculate

$$M_{ss} = n_1^2M_{11} + 2n_1n_2M_{12} + n_2^2M_{22} = -\mu + D_su_t + \dot{\theta}u_n \tag{65}$$

$$M_{nt} = n_1 n_2 (M_{22} - M_{11}) + (n_1^2 - n_2^2) M_{12} = -\frac{1}{2}(D_s u_n + D_n u_t) + \frac{1}{2}\dot{\theta} u_t \tag{66}$$

$$Q_n = n_1 Q_1 + N_2 Q_2 = -D_n \mu + D_s \omega$$

$$K_n = Q_n + D_s M_{nt} = -D_n \mu - D_s (D_s u_n - \dot{\theta} u_t) \tag{67}$$

where (u_n, u_t) are the components of the stress function vector in local axes. Assuming all displacements to be prescribed along the contour c, the statement of energy conservation is found to be

$$\int_A \delta\phi dA = \int_c (\bar{\alpha}_n \delta M_{nn} + \bar{w} \delta K_n) ds - [\bar{w} \delta M_{nt}]_o \tag{68}$$

After substitution of (64), (65), (66) and (67) and integration by parts it can be placed in the following form

$$\int_A \{\delta u_1 (-D_1 k_{22} + D_2 k_{12}) + \delta u_2 (D_1 k_{12} - D_2 k_{11})\} dA$$

$$+ \int_c \{\delta u_n (k_{tt} + D_s^2 \bar{w} - \dot{\theta} \bar{\alpha}_n) + \delta u_t (-k_{nt} + D_s \bar{\alpha}_n + \dot{\theta} D_s \bar{w})\} ds$$

$$+ [D_1 \bar{w} \delta u_2 - D_2 \bar{w} \delta u_1 - \bar{w} \delta \omega]_o = 0 \tag{69}$$

Use was made of the tensor transformation rules

$$n_1 k_{22} - n_2 k_{12} = n_1 k_{tt} + n_2 k_{nt}$$

$$- n_1 k_{12} + n_2 k_{11} = n_2 k_{tt} - n_1 k_{nt}$$

From the coefficients of the independent variations in the surface integral we obtain compatibility equations for the curvatures

$$- D_1 k_{22} + D_2 k_{12} = 0 \qquad D_1 k_{12} - D_2 k_{11} = 0 \tag{70}$$

that represent the integrability conditions for a transverse displacement; they are the analogues of the equilibrium equations (24).

Along each segment of boundary with continuously turning tangent we must satisfy the boundary conditions

$$k_{tt} = -D_s^2\bar{w} + \dot{\theta}\alpha_n$$

$$k_{nt} = D_s\bar{\alpha}_n - \dot{\theta}D_s\bar{w} \qquad (71)$$

At angular points the deflection and both slopes of the deflecfion are known from the boundary data, and it was found advantageous to express the last term of (69) in the fixed axes. This term vanishes by virtue of the single valuedness of the prescribed deflection and its slopes and the assumed single-valuedness of the stress functions u_1, u_2 and ω.

MULTI-VALUEDNESS OF THE STRESS FUNCTIONS u_1, u_2 and ω

The potential μ generates a particular state of stress in equilibrium with the transverse pressure $\bar{q}$. In particular the force system consisting in the transverse pressure distribution and the surface traction forces generated by μ along the boundaries is statically equivalent to zero. The stress functions (u_1, u_2) generate stresses in equilibrium without transverse pressure, and the corresponding system of surface tractions along the boundary is statically equivalent to zero. Considering a simply connected domain of boundary γ, this implies that the total transverse force is zero:

$$\int_\gamma -Q_2\,dx_1 + Q_1\,dx_2 = 0$$

Transformed by equations (58) with $\mu = 0$, this gives

$$\int_\gamma D_1\omega dx_1 + D_2\omega dx_2 = \int_\gamma d\omega = 0 \qquad (72)$$

Similarly for the moments with respect to the Ox_1 and Ox_2

$$\int_\gamma M_{22}\,dx_1 - M_{12}\,dx_2 + x_2\,d\omega = 0$$

$$\int_\gamma - M_{21}\,dx_1 + M_{11}\,dx_2 - x_1\,d\omega = 0$$

Transformed by equations (60) with $\mu = 0$, they give

$$\int_\gamma d(u_1 + x_2\omega) = 0 \qquad \int_\gamma d(u_2 - x_1\omega) = 0$$

and, taking (72) into account, reduce to

$$\int_{\gamma} du_1 = 0 \qquad \int_{\gamma} du_2 = 0 \tag{73}$$

This establishes the single-valuedness of u_1, u_2 and ω for a simply connected domain.

Across a barrier in a multiply-connected domain, the reciprocity of M_{nn} and K_n entails, by virtue of (65) and (67), and provided μ and $D_n\mu$ be single-valued, the continuity of

$$D_s u_t + \dot{\theta} u_n \quad \text{and} \quad D_s(D_s u_n - \dot{\theta} u_t) \quad \text{continuous at interface} \tag{74}$$

As a matter of fact this corresponds exactly to the same jump conditions for the stress functions as the displacement dislocation expressed by equations (56) and constitutes another facet of the second analogy. Indeed, adopting (56), a simple calculation shows that

$$D_s \Delta u_t + \dot{\theta} \Delta u_n = 0 \quad \text{and} \quad D_s \Delta u_n - \dot{\theta} \Delta u_t = -\Delta\omega \tag{75}$$

DISLOCATIONS IN THE PLATE BENDING PROBLEM

From the results just obtained concerning the jumps in stress functions across a barrier in a multiply-connected domain, we conclude that the variations of the stress functions are continuous. The transition conditions provided by those variations are the continuity of k_{tt} and k_{tn}; or, equivalently, of

$$D_s^2 w + \dot{\theta} D_n w \quad \text{and} \quad D_s D_n w - \dot{\theta} D_s w \quad \text{continuous at interfaces} \tag{76}$$

The corresponding jump condition for w turns out to be exactly the same as that given in (53) for the Airy stress function; it is a facet of the first analogy.

EQUILIBRIUM MODELS OF FINITE ELEMENTS BASED ON DISCRETIZATION OF STRESS FUNCTIONS

To construct equilibrium models of finite elements for plane stress problems one can use an Airy stress function containing a finite number of unknown parameters. For plate bending problems the same procedure is applicable to the vector stress function.

An interesting property associated with stress functions is diffusivity at interfaces. In the case of the Airy function, the fact that

$$\tau_{nn} = D_s^2 w + \dot{\theta} D_n w \qquad \tau_{nt} = -D_s D_n w + \dot{\theta} D_s w$$

shows that continuity of w and $D_n w$, equivalent to that of w and both its slopes, is sufficient to guarantee diffusivity; that is, continuous transmission of surface tractions. By the first analogy it follows that any displacement model for plate bending that is conforming will correspond to a diffusive equilibrium model for plate stretching. However, those continuity requirements, if sufficient, are not quite necessary. As was shown in the section pertaining to multi-valuedness of the Airy functions and its derivatives, the Airy function may undergo a jump of rigid body type ed across the interface without impairing diffusivity. In multiply-connected domains this does actually occur across the cuts devised to restore simple connectivity. Similarly for the vector stress function, the fact that

$$M_{sn} = D_s u_t + \dot{\theta} u_n \qquad K_n = -D_s(D_s u_n - \dot{\theta} u_t)$$

shows that simple continuity of the vector function is sufficient to guarantee diffusivity. Hence, by the second analogy, any conforming displacement model for plate stretching will correspond to a diffusive equilibrium model for plate bending. Again, continuity of the vector stress function may be replaced by jump conditions of the type (56) without altering diffusivity.

Suppression of Kinematical Freedoms in Equilibrium Finite Elements

It was observed [1] that kinematical freedoms or "mechanisms" can appear either within an equilibrium model itself or as a consequence of its connection to neighbouring elements. An important feature of equilibrium models derived from displacement analogues is the absence of such undesirable characteristics. To this effect we must start from a displacement model, whose generalized boundary displacements, devised to guarantee conformity, are independent. It is known [1] that in such a case the displacement field parameters, represented by the column matrix a, are expressible in terms of the generalized boundary displacements q and a set of bubble coordinates b:

$$a = Qq + Bb \tag{77}$$

with a non-singular transformation matrix (QB). The choice of field parameters can always be such that we distinguish between the set r, generating rigid body motions, and a complementary set s, generating a true deformation field:

$$a' = (r's')$$

In the analogy, a becomes a set of parameters for either the Airy function or the vector stress function (u_1, u_2). The subset r is that which describes "rigid body motions of the stress functions", e.g.

$$v = -\gamma_1 x_1 - \gamma_2 x_2 + w_0 \qquad r' = (\gamma_1 \gamma_2 w_0) \tag{53'}$$

for the Airy function, or

$$u_1 = v_1 - x_2\omega \qquad u_2 = v_2 + x_1\omega \tag{56'}$$

$$r' = (v_1 v_2 \omega)$$

for the vector stress function. Inasmuch as the set r does not generate deformations in the displacement models, it does not generate stresses in the analogue equilibrium models. With this subdivision of a, (77) is split into

$$r = Q_r q + B_r b \tag{78}$$

$$s = Q_s q + B_s b \tag{79}$$

The set s can be identified with the set of stress parameters appearing with the same notation in the general theory of equilibrium models [1]. The set q can be identified with the parameters describing the behavior of the stress functions at the boundaries of the equilibrium element; they are obviously such that interface identification of the elements of q produce, either continuity of the Airy function and its two first derivatives across an interface, or continuity of the vector stress functions. This was recognized to be sufficient for diffusivity. The set b becomes a set of parameters describing an internal behavior of the stress functions that generates no surface tractions at the boundaries. It was shown [1] that, disregarding presently the presence of body loading modes, the generalized boundary forces, devised to insure diffusivity, are connected to the set s by a load connection matrix C:

$$g = Cs \tag{80}$$

We also recognize that the same generalized boundary loads are expressible in terms of the set q of boundary parameters of the stress functions alone, since the b set generates on boundary surface tractions, thus

$$g = \hat{C}q \tag{81}$$

We call $\hat{C}$ the "extended" load connection matrix, and proceed to establish its relationship with C. Substitute (79) into (80).

$$g = CQ_s q + CB_s b$$

and conclude from the previous considerations that we must have

$$CB_s = 0 \tag{82}$$

$$\hat{C} = CQ_s \tag{83}$$

Similarly, since the system of equations (78), (79) is invertible, there exists a relationship

$$q = M_r r + M_s s \tag{84}$$

which derives in fact directly from the general equation [1] q = M a, after partitioning a into its two subsets. When this is substituted into (81)

$$g = \hat{C}M_r r + \hat{C}M_s s$$

However the set r, that does not generate stresses, cannot generate surface tractions at the boundaries and so

$$\hat{C}M_r = 0 \tag{85}$$

$$C = \hat{C}M_s \tag{86}$$

Equations (83) and (86) are the relationships sought for between the two loan connection matrices. The proof that there are no internal kinematical freedoms hinges on the proposition that the dimensions of the column matrices q and g are identical: n(q) = n(g). This is not only borne out by experience but is susceptible of a general proof. Take first the case of the Airy function. Since one can add to it an arbitrary linear form of type (53'), without altering the stress distribution, there is no loss in generality in considering the Airy function and its two slopes to be zero at one of the vertices V of a finite element

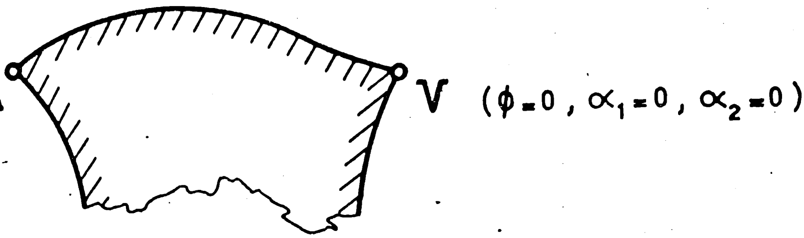


Figure 3.

Suppose now the local values of w and its slopes α_1 and α_2 to be split into two categories: category 1 - contains the values defined between vertices and belonging to only one partial boundary $\partial_\alpha E$; category 2 the values pertaining to the vertices themselves and belonging to two partial boundaries. In category 1 each local value between A and V generates an independent surface traction distribution along A V (that is statically equivalent to zero) and requires the definition of a corresponding generalized boundary load. Hence $n_1(g) = n_1(q)$. In category 2 the surface tractions along A V are generated only by the 3 local values in A (they are no more statically equivalent to zero because complementary surface tractions are also generated along the other boundary issued from A) and require 3 generalized boundary load. Then ($n_2(g) = n_2(q)$ because there are as many partial boundaries as there are vertices. The result $n(g) = n(q)$ follows by addition. A similar argument can be presented for the vector stress function (u_1, u_2). Here there is no loss in generality in assuming $u_1 = 0$ and $u_2 = 0$ in the vertex V and some transverse component to vanish in A , thus preventing the rigid body motions of the stress functions

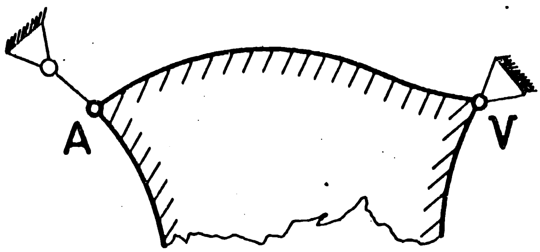


Figure 4.

For category 1 local values the situation is similar to the previous one and $n_1(g) = n_1(q)$. In category 2 however there is only one local value of the stress vector generating a surface traction distribution along AV that requires only one generalized boundary load. Thus $n_2(g) = n(v)$ the number of vertices. However it must be recalled that in this case there are concentrated transverse loads appearing at the vertices that must be added to the generalized boundary loads. Their number $n_3(g) = n(v)$ again. Thus

$$n(g) = n_1(g) + n_2(g) + n_3(g) = n_1(q) + 2n(v)$$

$$= n_1(q) + n_2(q) = n(q)$$

The result $n(g) = n(q)$ implies that in (81) the extended load connextion matrix is square. From a general theorem of algebra it is known that the number of independent solutions of the homogeneous system

$$C'q^* = 0$$

is equal to the number of independent solutions of the adjoint system

$$\hat{C}q = 0$$

and this is in any case equal to 3, representing the rigid body freedoms of the stress functions. Now since from (86)

$$C'q^* = M_s'\hat{C}'q^*$$

and M_s has linearly independent columns it follows that

$$C'q^* = 0$$

has only 3 independent solutions that cannot be other than the 3 rigid body displacement freedoms of the element. This completes the proof that the element is free from spurious mechanisms.[1].

Assembling by the Direct Stiffness Method

The most obvious way to utilize an equilibrium model, established through its static-geometric analogy with a displacement model, is to bring its formulation back to the general thoery. This amounts simply to set up its load connexion matrix C by means of (86). If body force loading modes are required, care must be taken to avoid that the part Gc (see equation (87) of reference 1) of the generalized boundary loads does not increase their number because additional independent surface traction modes become generated. This would introduce again the undesirable feature of internal kinematical freedoms. For prescribed types of body loading modes this poses a problem of careful determination of the part $R(x)c$ in the discretization of the stress field. One example is given in reference 9 for the uniform transverse pressure on a plate bending equilibrium model. As was shown in reference 1, the elastic behavior of any equilibrium model can be represented in terms of discretized stiffness relations, exactly as displacement models. The same assembling process as in the direct stiffness method is thus applicable to them, contrary to the belief that equilibrium models necessarily require a Force Method of solution. No special problems arise with the presence of interface loading modes, provided they conform with the possible shapes of surface traction modes. The same remark applies to the loading modes at the boundaries of the assembled structure.

Assembling by Analogy

The observation was just made that both equilibrium and displacement models can be generated and assembled together, considering nodal displacements as the basic unknowns. The same is true of the Force Method, in which the basic unknowns are intensitites of self-stressing states of the assembled structure; it is applicable to both types of models. However, there is some evidence that the differences in topology of the connexions between both types favors the use of a Direct Stiffness Method for the displacement type and the use of a Force Method for the equilibrium type. Moreover, whenever a static-geometric analogy is at hand, the Force Method applied to a set of equilibrium elements can be presented in a form that is strongly similar to the Direct Stiffness Method as applied to the set of displacement analogues. In other words the static-geometric analogy can be pushed beyond the construction of models to the problem of assembling and solving. The general theory of this procedure is taken up in the next section.

The Direct Force Method

In the general theory of equilibrium models [1], the discretization of the stress field is presented in terms of parameters c, representing body loading modes, and parameters s, representing stress fields in equilibrium without body forces

$$\tau(x) = R(x)c + S(x)s \tag{87}$$

Since the last part is precisely one that can be derived from the use of stress functions, we use (79) to distinguish in s the generalized values q that govern the surface traction distributions on the boundary of the element and the internal generalized values b. Thus

$$\begin{aligned} \tau(x) &= R(x)c + Q(x)q + B(x)b \\ Q(x) &= S(x)Q_s \qquad B(x) = S(x)B_s \end{aligned} \tag{88}$$

The discretization (88) produces a new form of the flexibility matrix of the element

$$\Phi = \frac{1}{2}\int_E \tau' H^{-1} \tau dE = \frac{1}{2}(c'q'b') \begin{pmatrix} F_{cc} & F_{cq} & F_{cb} \\ F_{qc} & F_{qq} & F_{qb} \\ F_{bc} & F_{bq} & F_{bb} \end{pmatrix} \begin{pmatrix} c \\ q \\ b \end{pmatrix} \tag{89}$$

It can also be obtained from the quadratic form in c and s (see equation 93 of reference 1) by substitution of (79). Moreover, because the bubble coordinates b of the stress function are uncoupled to neighbouring elements because they produce no surface tractions, they can be eliminated at the element level by straight forward minimization of the complementary energy. Hence b can be solved from the minimizing condition

$$F_{bc}c + F_{bq}q + F_{bb}b = 0 \tag{90}$$

and substituted to yield a flexibility matrix

$$\Phi = \frac{1}{2}(c'q') \begin{pmatrix} \hat{F}_{cc} & \hat{F}_{cq} \\ \hat{F}_{qc} & \hat{F}_{qq} \end{pmatrix} \begin{pmatrix} c \\ q \end{pmatrix} \tag{91}$$

that only involves the boundary coordinates of the stress function and the body loading parameters. The application of the minimum complementary energy principle at the element level can be achieved by considering the generalized displacements as prescribed. Then

$$\Phi - q^*g - b^*c = \text{minimum}$$

where q^* and b^* are defined by virtual work considerations as in reference 1, section pertaining to Generalized Displacements, and where we have, as already shown

$$g = Gc + \hat{C}q \tag{92}$$

The minimizing conditions with respect to c and q yield

$$\hat{F}_{cc}c + \hat{F}_{oq}q = G'q^* + b^* \tag{93}$$

$$\hat{F}_{qo}c + \hat{F}_{qq}q = \hat{C}'q^* \tag{94}$$

Those generalized flexibility relations, akin to equations (96) and (97) of reference 1, involve generalized strain definitions on the right hand side. To exhibit the analogy in the assembling process with the Direct Stiffness Method in its simpler form we take the case without body loading coordinates and with a simply connected domain. Then, provided there are no interface loads, the stress functions are known to be single-valued so that, without loss of generality, each set q_E of local values can be allocated to a common set w of nodal values:

$$q_E = L_E w \tag{95}$$

Each coordinate in w can be regarded as a hyperstatic unknown [12] since it generates a self-equilibrated stress field. Such hyperstatic unknowns are even the best conceivable because the domain in which their stress field diffuses is strictly limited to those small number of elements whose boundary is influenced by the considered nodal value. In this sense we have a direct answer to the problem of determining the "minimal hyperstatic cells". The coordinates of w that appear along the outer boundary of the assembled structure are to be considered as unknowns only where the boundary displacements are prescribed; we shall also consider this to be the case. Equation (94) is reduced to

$$(\hat{F}_{qq})_E q_E = \hat{C}_E' q_E^* \tag{96}$$

for each element. Hence

$$q_E' (\hat{F}_{qq})_E q_E = q_E' \hat{C}_E' q_E^* = g_E' q_E^* \tag{97}$$

if due account is taken of the reduced form taken by (92). This appears as a Clapeyron statement: on the left twice the strain energy of the element, on the right the product of the externally applied loads g_E and their conjugate displacements. We now use (95) and sum the energies of all elements, obtaining

$$w'\hat{F}w = \sum_E g_E' q_E^* \tag{98}$$

with a master flexibility matrix

$$\hat{F} = \sum_E L_E' (\hat{F}_{qq})_E L_E \tag{99}$$

constructed in a manner that is perfectly analogous to the master stiffness matrix of displacement type elements. The right hand side of (98) can be manipulated to introduce the nodal vector of generalized displacements w^*, through

$$q_E^* = L_E^* w^* \tag{100}$$

Then, noting that by virtual work considerations [1],

$$\sum_E L^{*\prime}_\varepsilon g_\varepsilon = y \qquad (101)$$

where y is the external load vector conjugate to w^*

$$w'\hat{F}w = \sum g_\varepsilon' L_\varepsilon^* w^* = y'w^* \qquad (102)$$

This is again a Clapeyron statement, but this time for the assembled structure. Through (101) the external load vector is expressible as

$$y = (\sum_E L_\varepsilon^{*\prime} \hat{C}_\varepsilon L_\varepsilon) w = \hat{A}w \qquad (103)$$

We now distinguish between the set w_1 of coordinates of w that do not belong to the outer boundary and the complementary set w_2; similarly for y. This leads to a partitioning of A and separate equations

$$y_1 = \hat{A}_{11}w_1 + \hat{A}_{12}w_2$$

$$y_2 = \hat{A}_{21}w_1 + \hat{A}_{22}w_2 \qquad (104)$$

However, since diffusitivity was insured, the interface loads y_1 are always zero, hence $\hat{A}_{11} = 0$ and $\hat{A}_{12} = 0$. We now state the minimum complementary energy principle for the complete structure assuming w_2 to be prescribed

$$\frac{1}{2} w_1'\hat{F}_{11}w_1 + w_1'\hat{F}_{12}w_2 + \frac{1}{2} w_2'\hat{F}_{22}w_2 - (w_1'\hat{A}_{21}' + w_2'\hat{A}_{22}')w_2^* \ \text{min}$$

and this leads to the set of linear equations

$$\hat{F}_{11}w_1 + \hat{F}_{12}w_2 = \hat{A}_{21}'w_2^*$$

$$\hat{F}_{11}w_1 + \hat{F}_{22}w_2 = \hat{A}_{22}'w_2^* \qquad (105)$$

By using also (104) the case of mixed boundary conditions on the outer boundary can be handled. Extensions to multi-connected domains, body loading modes and interface loadings are possible but still remain difficult to automate.

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