STRESS FUNCTION APPROACH

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Summary

The stress function approach allows in principle to construct systematically finite elements satisfying stress equilibrium in finite regions and surface traction reciprocity at their interfaces. Some difficulties are experienced with the requirements of C_1 continuity of the stress functions, when rigorous rotational equilibrium is desired. They disappear when only first order stress functions are used, whereby translational equilibrium is maintained, but rotational equilibrium must be separately enforced by a Lagrange multiplier technique. The paper discusses in detail this new generation of finite elements with discretized rotational equilibrium. In particular the extent to which rotational equilibrium can be enforced without inducing mechanisms. Because diffusivity involves only C_o continuity of the first order stress functions, the new elements lend themselves very flexibly to isoparametric coordinate transformations.

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1. FIRST ORDER STRESS FUNCTIONS

Following the Euler-Cauchy assumption the short-range internal forces in a continuum are representable by surface integrals of surface tractions. Thus a simply connected volume of the continuum will be in translational equilibrium if

$$\iint_S \vec{t}\, dS + \iiint_V \rho\, \vec{g}\, dV = 0 \qquad (1)$$

where $\vec{t}$ is the surface traction vector acting on the boundary of the volume, ρ the specific mass of the continuum and $\vec{g}$ an accelaration of gravitational or inertial origin. The application of this equilibrium property to an infinitesimal volume leads to the well known Euler-Cauchy formula, expressed here in Cartesian coordinates,

$$t_j = n_i\, \sigma_{ij} \qquad (2)$$

where n_i are the components of the unit outward normal $\vec{n}$ to a facet of the continuum and σ_{ij} the stress tensor.
Let us now consider the equilibrium property for a finite volume under the assumption $\vec{g} = 0$. Drawing a simple closed contour c on the boundary surface S that separates it into two parts S_1 and S_2 and reversing the orientation of the normal on S_2, the equilibrium property takes the form

$$\iint_{S_1} t_j\, dS = \iint_{S_2} t_j\, dS \qquad (j = 1, 2, 3) \quad .$$

This equality will hold for any surface S that is limited by the same contour c and on which the orientation of the normal $\vec{n}$ follows the same "screw" convention with respect to a prescribed sense of circulation on c. Consequently the surface integrals will only depend on the circulation of some tensor field on c

$$\iint_S t_j\, dS = \oint_c A_{mj}\, dx_m \qquad (3.a)$$

In the left hand side we substitute (2), and transform the right hand side by Stokes theorem to obtain

$$\iint_S n_i \sigma_{ij} \, dS = \iint_S n_i e_{ipq} D_p A_{qj} \, dS \qquad (j=1,2,3)$$

where e_{ipq} is the alternating tensor and D_p stands for $\partial/\partial x_p$. Because this relation will hold for an arbitrarily small and arbitrarily oriented surface element dS, it follows that

$$\sigma_{ij} = e_{ipq} D_p A_{qj} \qquad (3.b)$$

It is indeed immediatly verified that this expression of the stress tensor identically satisfies the local translational equilibrium equations without inertial loading

$$D_i \sigma_{ij} = 0 \qquad (j = 1, 2, 3) \qquad (4)$$

since

$$e_{ipq} D_i D_p \equiv 0$$

on account of the skew symmetry of the alternating tensor. We call A_{qj} a tensor of first order stress functions, because the stresses are combinations of their first derivatives.

2. DIFFUSIVITY AND C_0 CONTINUITY OF THE FIRST ORDER STRESS FUNCTIONS

From (2) and (3.b) follows for the surface tractions

$$t_j = \partial_q A_{qj} \qquad (5)$$

where

$$\partial_q = e_{ipq} n_i D_p \qquad (6)$$

are surface derivative operators; they are the components of a vector operator $\vec{n} \times \text{grad}$ that projects the gradient in the surface normal to $\vec{n}$. It is clear therefore that, provided the first order stress function tensor A_{qj} is continuous across an interface separating regions where it receives different approximations, the surface tractions will also be continuous. Taking into account the opposite character of outward normals to the regions meeting at the interface, it may be concluded that C_o continuity of the first order stress function tensor at the interface guarantees the property of diffusivity of the surface tractions, that is the action-reaction reciprocity rule of the short-range internal forces.

3. SECOND ORDER STRESS FUNCTIONS

A similar approach can be followed as regards rotational equilibrium. The resultant moment with respect to the origin of the surface tractions on a finite volume has the components

$$\iint_S e_{umj}\, x_m\, t_j\, dS \qquad (u = 1, 2, 3)$$

It must also vanish in the absence of inertial loads and, by similar reasoning we conclude that, for all surfaces S limited by the same contour c,

$$\iint_S e_{umj}\, x_m\, t_j\, dS = \oint_c B_{qu}\, dx_q \qquad (u=1,2,3) \qquad (7.a)$$

Again, substituting (2) to the left, using Stokes theorem on the right, and comparing the integrands

$$e_{umj}\, x_m\, \sigma_{ij} = e_{ipq}\, D_p\, B_{qu} \qquad \forall(u,i) \qquad (7.b)$$

If the operator D_i is applied to this equation, the right-hand side vanishes because of the skew symmetry of the alternating tensor. On the left-hand side we note that $D_i\, x_m = \delta_{im}$ and obtain

$$e_{uij}\ \sigma_{ij} + e_{umj}\ x_m\ D_i\ \sigma_{ij} = 0 \qquad (u=1,2,3)$$

Thus, if the stresses are already in translational equilibrium, we obtain the well known local rotational equilibrium conditions

$$e_{uij}\ \sigma_{ij} = 0 \qquad (u = 1,\ 2,\ 3) \qquad (8)$$

equivalent to the symmetry property of the stress tensor

$$\sigma_{ji} = \sigma_{ij} \qquad (9)$$

Since precisely the translational equilibrium conditions (4) are satisfied by the use of first order stress functions, equations (8) are transformed into

$$e_{uij}\ e_{ipq}\ D_p\ A_{qj} = 0$$

or, since

$$e_{uij}\ e_{ipq} = \delta_{jp}\ \delta_{uq} - \delta_{jq}\ \delta_{up} \qquad (10)$$

into

$$D_j\ A_{uj} - D_u\ A_{pp} = 0 \qquad (u = 1,\ 2,\ 3) \qquad (11)$$

The problem is now to obtain a general solution of (11).
To this effect we go back to (7.b) where (3.b) is now substituted

$$e_{ipq}\ \left[\ e_{umj}\ x_m\ D_p\ A_{qj} - D_p\ B_{qu}\ \right] = 0$$

Transforming

$$x_m\ D_p\ A_{qj} = D_p\ (x_m\ A_{qj}) - \delta_{mp}\ A_{qj}$$

and using the formula of type (10), this can be placed in the form

$$e_{ipq} D_p \left[e_{umj} x_m A_{qj} - B_{qu} \right] - \delta_{iu} A_{qq} + A_{ui} = 0$$

Thus, setting

$$\phi_{qu} = B_{qu} - e_{umj} x_m A_{qj} \tag{12}$$

we find

$$A_{ui} = e_{ipq} D_p \phi_{qu} + \delta_{iu} A_{qq}$$

In particular, setting u = i,

$$A_{uu} = e_{upq} D_p \phi_{qu} + 3 A_{qq}$$

whereby

$$A_{qq} = - \frac{1}{2} e_{upq} D_p \phi_{qu} \tag{13}$$

that, after back substitution, finally expresses the first order stress function tensor in terms of a second order stress function tensor ϕ_{qu}

$$A_{uj} = e_{jpq} D_p \phi_{qu} - \frac{1}{2} \delta_{ju} e_{mpq} D_p \phi_{qm}$$

It is easily verified that this satisfies equations (11). After a change of notation for the subscripts

$$A_{qj} = e_{jrs} D_r \phi_{sq} - \frac{1}{2} \delta_{qj} e_{mrt} D_r \phi_{tm} \tag{14}$$

and, substituting into (3.b), the stresses are given by

$$\sigma_{ij} = e_{ipq} e_{jrs} D_p D_r \phi_{sq} - \frac{1}{2} e_{ipj} e_{mrt} D_p D_r \phi_{tm}$$

The second term is clearly skew symmetrical in the subscripts i and j and will disappear if we write

$$\sigma_{ij} = \frac{1}{2} (\sigma_{ij} + \sigma_{ji})$$

an operation that is licit since we known that rotational equilibrium is now satisfied. Then, by also exchanging the dummy subscripts p and q, and also q and s in the first term, there comes

$$\sigma_{ij} = e_{ipq} e_{jrs} D_p D_r (\frac{\phi_{qs} + \phi_{sq}}{2})$$

so that the stresses depend only on the symmetrical part of the second order stress function tensor. This establishes that the result of B. FINZI [1]

$$\sigma_{ij} = e_{ipq} e_{jrs} D_p D_r \phi_{qs} \qquad \phi_{sq} = \phi_{qs} \tag{15}$$

for stresses satisfying both translational and rotational equilibrium is general. Because of the assumed symmetry of the tensor

$$e_{mrs} \phi_{sm} = 0$$

and (14) is seen to reduce to

$$A_{qj} = e_{jrs} D_r \phi_{sq} \tag{16}$$

In particular we have now that

$$A_{qq} = D_r (e_{qrs} \phi_{sq}) = 0 \tag{17}$$

4. THE INDETERMINACY OF THE FINZI TENSOR

For a given stress field in equilibrium without inertial loads the FINZI tensor is not unique. In other words the homogeneous problem

$$\sigma_{ij} = e_{ipq}\, e_{jrs}\, D_p\, D_r\, \phi_{qs} \equiv 0$$

has non trivial solutions. They can be found by observing that since

$$\oint A_{mj}\, dx_m = 0 \qquad\qquad \oint B_{qi}\, dx_q = 0 ,$$

must be satisfied for any closed curve drawn within the simply connected continuum, the integrands must be perfect differentials of single valued functions

$$A_{mj} = D_m\, \Omega_j \tag{18}$$

$$B_{qi} = D_q\, V_i \tag{19}$$

If this is used to transform (12)

$$\phi_{qi} = D_q\, V_i - e_{imj}\, x_m\, D_q\, \Omega_j = D_q\, (V_i - e_{imj}\, x_m\, \Omega_j) + e_{iqj}\, \Omega_j$$

Exchanging the subscripts q and i and using $e_{qij} = - e_{iqj}$

$$\phi_{iq} = D_i\, (V_q - e_{qmj}\, x_m\, \Omega_j) - e_{iqj}\, \Omega_j$$

As ϕ_{qi} may be taken to be symmetrical, the arithmetic mean of ϕ_{qi} and ϕ_{iq} gives

$$\phi_{qi} = \frac{1}{2}\, (D_q\, U_i + D_i\, U_q) \tag{20}$$

where

$$U_i = V_i - e_{imj}\, x_m\, \Omega_j \tag{21}$$

while, by substraction,

$$0 = D_q U_i - D_i U_q + 2 e_{iqj} \Omega_j$$

This result, multiplied by e_{mqi}, gives

$$2 \Omega_m = e_{mqi} D_q U_i \qquad (22)$$

Hence, if the second order stress functions derive from a vector field U_i in the same manner as strains from a displacement field, no stresses are generated and the first order stress functions are the gradients of the associated field of rotations. As a matter of fact if the U_i are considered as displacements the conditions $\sigma_{ij} \equiv 0$ are equivalent to the de Saint-Venant compatibility conditions for strains, and the equations

$$D_m \Omega_j = A_{mj} = e_{jrs} D_r \phi_{sm}$$

are equivalent to the BELTRAMI equations for the integration of the rotation field from the knowledge of the strain field.
The indeterminacy of the FINZI tensor may be used to reduce it to its non-diagonal components. Since

$$\hat{\phi}_{qs} = \phi_{qs} - \frac{1}{2} (D_q U_s + D_s U_q)$$

generates the same stresses, a U_i field can always be constructed from the 3 equations

$$\phi_{11} = D_1 U_1 \qquad \phi_{22} = D_2 U_2 \qquad \phi_{33} = D_3 U_3$$

so that $\hat{\phi}_{qs}$ has its diagonal terms identically zero. The non-diagonal terms constitute the stress functions of MORERA. Conversely it is easily verified that a U_i field can always be constructed from the 3 equations

$$\phi_{12} = \frac{1}{2} (D_1 U_2 + D_2 U_1) \qquad \phi_{23} = \frac{1}{2} (D_2 U_3 + D_3 U_2) \qquad \phi_{31} = \frac{1}{2}(D_3 U_1 + D_1 U_3)$$

so that the non diagonal terms of $\hat{\phi}_{qs}$ vanish identically. The diagonal terms then constitute a system of stress functions of MAXWELL.

5. DIFFUSIVITY AND C_1 CONTINUITY OF THE SECOND ORDER STRESS FUNCTIONS

From (2), (6) and (15) the surface tractions generated by the second order stress functions are

$$t_j = \partial_q \left[e_{jrs} D_r \phi_{qs} \right]$$

Any cartesian derivative is expressible in terms of surface derivatives and of the derivative Δ along the normal itself

$$D_r = n_r \Delta - e_{ruv} n_u \partial_v \qquad (23)$$

This formula is justified by the fact that, multiplied by $n_p e_{pqr}$ it reduces to the definition

$$e_{pqr} n_p D_r = - e_{pqr} e_{ruv} n_p n_u \partial_v = - n_q n_p \partial_p + n_p n_p \partial_q = \partial_q$$

of the surface derivative ($n_p \partial_p \equiv 0$ and $n_p n_p = 1$), while, multiplied by n_r, it reduces to the definition of the normal derivative

$$n_r D_r = \Delta \qquad (24)$$

The expression of surface tractions in terms of normal and surface derivatives of the FINZI tensor is thus

$$t_j = \partial_q \left[e_{jrs} n_r \Delta\phi_{qs} \right] + \partial_q \left[n_j \partial_s \phi_{qs} - n_s \partial_j \phi_{qs} \right] \qquad (25)$$

It shows that diffusivity at an interface requires continuity not only of the FINZI tensor itself but also of its normal derivative.

This has an important bearing on the development of isoparametric stress elements because, if an isoparametric coordinate transformation will preserve C_o continuity at interfaces, it will in general not preserve continuity of the normal derivatives and will consequently destroy diffusivity.

6. THE TWO-DIMENSIONAL STRESS FUNCTIONS

H. SHAEFER's treatment of the two-dimensional case [4] is attractive by its elegance and simplicity. The basic idea is that, although stress functions of type (20) are normally improductive, they are able to generate stresses if the vector field U_i contains singularities. Consider then a plate of middle surface $x_3 = 0$ that is not loaded on its faces $x_3 = \pm \varepsilon$. The plate is considered to be embedded in the indefinite continuum and, while the vector field U_i is regular outside of the plate, it presents singularities inside of it. In fact, if we are content with the knowledge of internal stresses integrated throughout the thickness, we may go to the limit $\varepsilon \to 0$ and consider the middle plate as a sheet of discontinuity of the field U_i and its rotational Ω_m.

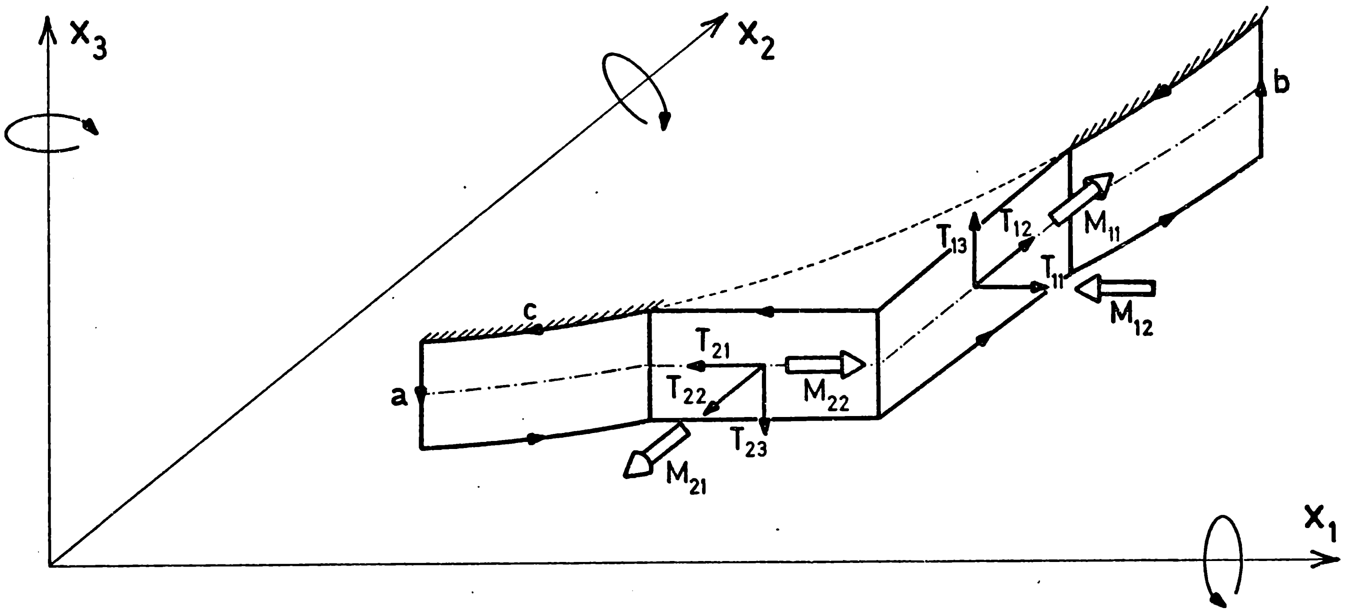

FIG. 1

Thus we introduce the following functions of the inplane variables (x_1 , x_2)

$$\lim_{\varepsilon\to 0} \left[\Omega_i(x_1,x_2,-\varepsilon) - \Omega_i(x_1,x_2,+\varepsilon) \right] = \Omega_i(x_1,x_2) \quad (26)$$

$$\lim_{\varepsilon\to 0} \left[U_i(x_1,x_2,-\varepsilon) - U_i(x_1,x_2,+\varepsilon) \right] = U_i(x_1,x_2) \quad (27)$$

and consider the closed contour c of figure 1, defined by a curve (a,b) drawn in the middle plane. For any integrand we replace

$$\lim_{\varepsilon\to 0} \oint_c f(x_1,x_2,x_3)\, ds = \int_a^b \lim_{\varepsilon\to 0} \left[f(x_1,x_2,-\varepsilon) - f(x_1,x_2,+\varepsilon) \right] ds \quad (28)$$

and apply the considerations on the computation of resultant stresses as outlined in sections 1 and 3. The total force components due to short range internal forces acting on the cross-section of the plate defined by the contour c are given by

$$\int_a^b T_{1j}dx_2 - T_{2j}dx_1 = \lim_{\varepsilon\to 0} \oint_c A_{mj}dx_m = \int_a^b D_1\Omega_j dx_1 + D_2\Omega_j dx_2 \quad (29)$$

where use has been made of (18) and (26). Since this holds for an arbitrary curve (a,b) we must have

$$T_{1j} = D_2\Omega_j \qquad T_{2j} = - D_1\Omega_j \quad (30)$$

Similarly the total moment of these forces with respect to the cartesian axes will be given by

$$\int_a^b (M_{22}-x_2T_{23})dx_1 + (-M_{12}+x_2T_{13})dx_2 = \lim_{\varepsilon\to 0} \oint_c B_{m1}dx_m$$

$$= \int_a^b D_1(U_1+x_2\Omega_3)dx_1 + D_2(U_1+x_2\Omega_3)dx_2$$

$$\int_a^b (-M_{21}+x_1T_{23})dx_1 + (M_{11}-x_1T_{13})dx_2 = \lim_{\varepsilon\to 0} \oint B_{m2}dx_m$$

$$= \int_a^b D_1(U_2-x_1\Omega_3)dx_1 + D_2(U_2-x_1\Omega_3)dx_2$$

$$\int_a^b (x_2T_{21}-x_1T_{22})dx_1 + (x_1T_{12}-x_2T_{11})dx_2 = \lim_{\varepsilon\to 0} \oint_c B_{m3}dx_m$$

$$= \int_a^b D_1(U_3-x_2\Omega_1+x_1\Omega_2)dx_1+D_2(U_3-x_2\Omega_1+x_1\Omega_2)dx_2$$

after the use of (19), (21) and (26) and (27).
The first equation furnishes the pair of results

$$M_{22}-x_2T_{23} = D_1U_1 + x_2D_1\Omega_3 \qquad -M_{12}+x_2T_{13} = D_2U_1+\Omega_3+x_2D_2\Omega_3$$

that can be simplified by noting that the terms still containing the coordinates x_i cancel by virtue of (30), leaving

$$M_{22} = D_1\,U_1 \qquad \text{and} \qquad M_{12} = -\,D_2\,U_1 - \Omega_3$$

Similar simplifications occur in the two other pairs. The complete set of results may be split into two groups :

<u>The plate stretching group</u>

$$T_{11} = D_2\,\Omega_1 \qquad T_{21} = -\,D_1\,\Omega_1$$
$$T_{12} = D_2\,\Omega_2 \qquad T_{22} = -\,D_1\,\Omega_2 \qquad (31)$$

satisfying identically the translational membrane equilibrium equations

$$D_1\,T_{11} + D_2\,T_{21} = 0 \qquad D_1\,T_{12} + D_2\,T_{22} = 0 \qquad (32)$$

so that Ω_1 and Ω_2 are first order stress functions.

Also, from the third moment equation,

$$\Omega_1 = D_2 U_3 \qquad \Omega_2 = - D_1 U_3 \tag{33}$$

from which follows $T_{12} = T_{21}$, or equilibrium of rotation. U_3 is a second order stress function. It is in fact the Airy stress fonction as easily recognized by the equations

$$T_{11} = D_2^2 U_3 \qquad T_{21} = T_{12} = - D_1 D_2 U_3 \qquad T_{22} = D_1^2 U_3 \tag{34}$$

obtained after the elimination of the first order functions.

<u>The plate bending group</u>

$$\begin{aligned} M_{11} &= D_2 U_2 & - M_{21} &= D_1 U_2 - \Omega_3 \\ - M_{12} &= D_2 U_1 + \Omega_3 & M_{22} &= D_1 U_1 \\ T_{13} &= D_2 \Omega_3 & T_{23} &= - D_1 \Omega_3 \end{aligned} \tag{35}$$

satisfying identically the plate bending equilibrium equations

$$\begin{gathered} D_1 M_{11} + D_2 M_{21} = T_{13} \qquad D_1 M_{12} + D_2 M_{22} = T_{23} \\ D_1 T_{13} + D_2 T_{23} = 0 \end{gathered} \tag{36}$$

As in the limiting process the property

$$\Omega_3 = \frac{1}{2} (D_1 U_2 - D_2 U_1) \tag{37}$$

is preserved, it is observed that

$$\begin{aligned} M_{11} &= D_2 U_2 \\ M_{22} &= D_1 U_1 \end{aligned} \qquad M_{12} = M_{21} = - \frac{1}{2} (D_1 U_2 + D_2 U_1) \tag{38}$$

so that the pair (U_1,U_2) is to be recognized as the SOUTHWELL stress function vector [2,5,9].

It is already clear from the expression of surface tractions on facets perpendicular to the axes that reduction to two-dimensional problems, has not alleviated the C_1 continuity requirements on the stress functions when complete equilibrium is postulated.

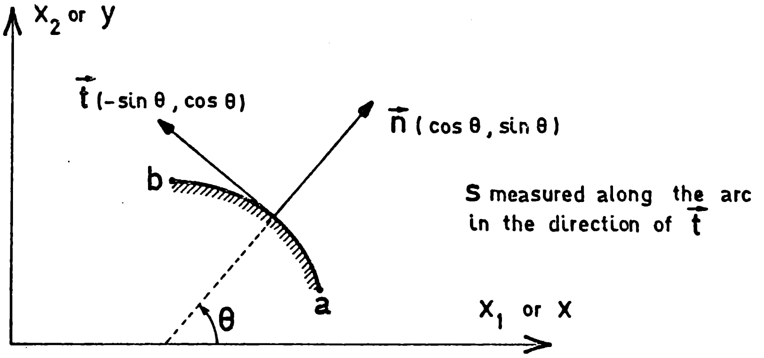

FIG. 2

It becomes even clearer in local coordinates n and s (see figure 2). In the case of the Airy function, hence forward denoted by ϕ instead of U_3, the normal and tangential components of the surface tractions (integrated throughout the thickness) are respectively

$$T_n = \frac{\partial^2\phi}{\partial s^2} + \dot{\theta}\,\frac{\partial\phi}{\partial n} \qquad T_t = -\frac{\partial^2\phi}{\partial n\partial s} \quad (\dot{\theta} = \frac{d\theta}{ds}) \tag{39}$$

If the curve ab is an interface, the continuity of the normal derivative is essential for the reciprocity of tangential loads and even necessary for that of normal loads if there is curvature present.

In the case of the Southwell vector, local resolution into its normal component U_n and tangential compoenen U_t gives

$$M_n = \frac{\partial}{\partial s}\,U_t + \dot{\theta}\,U_n \tag{40}$$

for the bending moment about the $\vec{t}$ axis

$$M_{nt} = -\frac{1}{2}\left(\frac{\partial}{\partial s}\,U_n + \frac{\partial}{\partial n}\,U_t\right) + \frac{1}{2}\,\dot{\theta}\,U_t \tag{41}$$

for the twisting moment and

$$T_n = \frac{\partial \Omega}{\partial s} = \frac{1}{2}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial n} U_t - \frac{\partial}{\partial s} U_n + \dot{\theta}\, U_t\right) \qquad (42)$$

for the transverse shear load. The continuity of the normal derivative of U_t is necessary to implement that of M_{nt} and T_n. This complicates the development of equilibrium elements for plate bending based on REISSNER's theory.

7. KIRCHHOFF PLATE BENDING

It is rather remarkable that the KIRCHHOFF assumption according to which fibers originally normal to the middle plane remain normal to the deformed middle surface, that is purely kinematical, should reduce diffusivity at an interface to C_o continuity of the SOUTHWELL vector. The twisting moment distribution along the arc ab becomes equivalent,by virtual work considerations, to corner loads (positively oriented as the transverse axis) equal to the local value of M_{nt} in a and $- M_{nt}$ in b, together with an equivalent additional transverse shear $\frac{\partial}{\partial s} M_{nt}$.

The total or KIRCHHOFF shear load becomes thus

$$K_n = T_n + \frac{\partial M_{nt}}{\partial s} = - \frac{\partial^2}{\partial s^2} U_n + \frac{\partial}{\partial s}(\dot{\theta}\, U_t) \qquad (43)$$

and, as was already the case for the bending moment M_n, requires only for its diffusivity that U_n and U_t be continuous. Moreover the corner loads at an internal boundary node automatically add up to zero. To show this we note that at a corner of a region (say region 1 of figure 3)

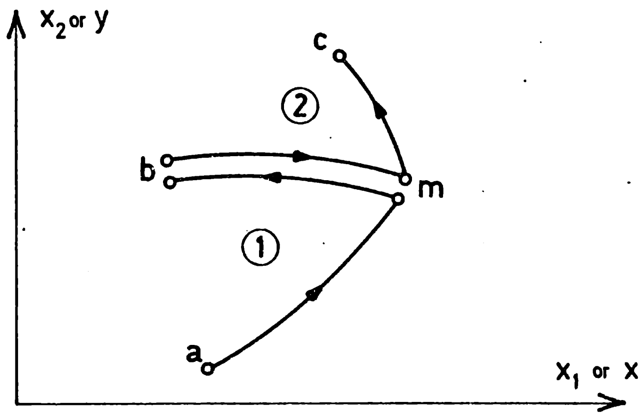

FIG. 3

the total corner load in m, resulting from the replacement of the twisting moment distributions along am and mb, is

$$\lim_{\substack{a \to m \\ b \to m}} \{ (M_{nt})_b - (M_{nt})_a \} = \Delta M_{nt}$$

or the jump in twisting moment as we turn around the corner in the positive sense of description of the boundary.
Writing the twisting moment in the form

$$M_{nt} = - \Omega - \frac{\partial}{\partial s} U_n + \dot{\theta} U_t \qquad (44)$$

and recognizing that for any given region where the stress vector is uniquely defined, Ω is also single-valued, the jump in twisting moment is equivalent to the jump

$$\Delta \left(- \frac{\partial}{\partial s} U_n + \dot{\theta} U_t\right) \qquad (45)$$

Let region 2 become adjacent to region 1 with bm as interface, the quantity $- \frac{\partial}{\partial s} U_n + \dot{\theta} U_t$ is single-valued at the interface if the stress vector is continuous (but the twisting moments will not be reciprocal because in general the boundary values of Ω for region 2 will differ from those of region 1). Hence, in adding up the corner loads in m, the contributions due to the interface in (45) will cancel and the formula remains valid for the combined corner load of the two regions, the jump being mesured at the corner of the combined regions. Obviously, if m becomes an internal node, there is complete cancellation of the contributions of all regions meeting in this point.

A deeper understanding of the reduction to C_o continuity is provided by the following virtual work consideration. Closer inspection of the plate bending group of equations (35) shows that U_1, U_2 and Ω_3 form a first order stress functions group with diffusivity under C_o continuity if we discard the rotational equilibrium condition

$$M_{12} = M_{21} \tag{46}$$

and consequently the necessity for Ω_3 to be the rotational

$$\Omega_3 = \frac{1}{2} (D_1 U_2 - D_2 U_1)$$

of the stress function vector (U_1, U_2). This equilibrium condition is the result of the virtual work equation

$$\int_{-\varepsilon}^{\varepsilon} \omega_3 (\sigma_{12} - \sigma_{21}) \, dx_3 = 0$$

where ω_3 is the material rotation of the continuum about the third axis :

$$\omega_3 = \frac{1}{2} (D_1 u_2 - D_2 u_1)$$

For small plate thickness we retain the first terms of a Taylor expansion of the displacements, which are antisymmetrical with respect to the middle plane

$$u_1 = x_3 \alpha_1 (x_1 , x_2) \qquad u_2 = x_3 \alpha_2 (x_1 , x_2)$$

and obtain the virtual work condition in the form

$$\frac{1}{2} (D_1 \alpha_2 - D_2 \alpha_1)(M_{12} - M_{21}) = 0$$

If the transverse fiber twist

$$\frac{1}{2} (D_1 \alpha_2 - D_2 \alpha_1) = D_3 \omega_3$$

is unconstrained, the virtual work condition will require the satisfaction of (46) and consequently introduce C_1 continuity. However under the KIRCHHOFF assumption there is no transverse shear

$$\alpha_1 + D_1 u_3 = 0 \qquad \alpha_2 + D_2 u_3 = 0$$

and the virtual work condition is satisfied without the need of (46) because the transverse twist becomes identically zero.

The first plate bending equilibrium model based on KIRCHHOFF's theory, although originally developped differently [7,8], corresponds in fact to full quadratic polynomial approximations of the stress function vector in triangular regions (linearly varying bending moments and uniform Kirchhoff shear loads). As was first pointed out by J. ROBINSON [13], it may be subjected to an isoparametric transformation, using the same interpolation functions (associated for instance to the nodal values at vertices and mid-edges) for in-plane displacements, leading to parabolically curved boundaries without loss of diffusivity. This statement remains true for complete polynomial approximations of higher degree.

8. DISCRETIZATION OF ROTATIONAL EQUILIBRIUM

The property of symmetry of the stress tensor, resulting from local rotational equilibrium, is so firmly taken for granted that little thought was spent up to now on the possibility of relaxing this constraint by discretization, in the same manner as translational equilibrium is relaxed in ordinary displacement elements. Such a relaxation, obtained by a discretization of the material rotation, allows the use of first order stress functions to enforce exact translational equilibrium. This method not only simplifies the construction of stress elements by a reduction to C_o continuity but opens the field to three- or two-dimensional stress elements with curved boundaries. It is furthermore applicable to the geometrical non linearity introduced by the consideration of finite elastic displacements [12]. A complete discussion of the two-dimensional case will be given, which is an improved version of an unpublished paper presented in Calgary.
Reverting to the more usual (x,y) notation, the translational equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_y}{\partial y} = 0 \tag{47}$$

are solved by using the first order stress function vector (A,B)

$$\sigma_x = \frac{\partial A}{\partial y} \qquad \tau_{yx} = - \frac{\partial A}{\partial x} \qquad \tau_{xy} = \frac{\partial B}{\partial y} \qquad \sigma_y = - \frac{\partial B}{\partial x} \tag{48}$$

From this we find for the surface tractions

$$t_x ds = \sigma_x dy - \tau_{yx} dx = dA \qquad t_y ds = \tau_{xy} dy - \sigma_y dx = dB \tag{49}$$

The rotational equilibrium condition

$$\tau_{xy} - \tau_{yx} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = 0 \tag{50}$$

is incorporated as a constraint in the complementary energy principle by means of a Lagrangian multiplier $\omega(x,y)$ so that, for given boundary displacements $(\bar{u},\bar{v})$

$$\iint \{ \Phi + \omega(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}) \} dxdy - \oint \bar{u}\, dA + \bar{v}\, dB \tag{51}$$

should be stationary with respect to arbitrary variations on A,B and ω . The stress energy Φ will be considered as a positive definite function of the arguments $\{ \sigma_x , \frac{1}{2}(\tau_{xy} + \tau_{yx}) , \sigma_y \}$ so that the constitutive equations (stress-strain relations) will be

$$\varepsilon_x = \frac{\partial \Phi}{\partial \sigma_x} \qquad \varepsilon_{xy} = \frac{\partial \Phi}{\partial \tau_{xy}} = \frac{\partial \Phi}{\partial \tau_{yx}} = \varepsilon_{yx} \qquad \varepsilon_y = \frac{\partial \Phi}{\partial \sigma_y} \tag{52}$$

It should be observed that the symmetry of the strain tensor is a consequence of the fact that the stress energy is a symmetric function of the stresses, without requiring that (50) be satisfied. Equation (50) is obviously the Euler equation resulting from variations on ω . The Euler equations due to variations on A and B (the arguments of Φ are to be replaced by (48) in the variational principle) are respectively

$$\frac{\partial \omega}{\partial x} = -\frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial x} \qquad \frac{\partial \omega}{\partial y} = -\frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} \tag{53}$$

and are recognized to be the Beltrami equations for the integration of ω, that must be identified with the material rotation. After the transformation

$$-\oint \bar{u}\, dA + \bar{v}\, dB = \oint A\, d\bar{u} + B\, d\bar{v}$$

justified by single valuedness of (A,B) and the data $(\bar{u},\bar{v})$, the boundary terms are found to be

$$\oint \delta A \{ d\bar{u} + (\sin\theta\varepsilon_x - \cos\theta\varepsilon_{xy} + \omega\cos\theta)\, ds \} = 0$$

$$\oint \delta B \{ d\bar{v} + (\sin\theta\varepsilon_{xy} - \cos\theta\varepsilon_y + \omega\sin\theta)\, ds \} = 0$$

By linear combinations, corresponding to the resolution of the stress function vector into its local components A_n and A_t, there follows

$$\cos\theta \frac{d\bar{u}}{ds} + \sin\theta \frac{d\bar{v}}{ds} + \omega = \sin\theta\cos\theta(\varepsilon_y - \varepsilon_x) + (\cos^2\theta - \sin^2\theta)\varepsilon_{xy} = \varepsilon_{nt}$$

$$-\sin\theta \frac{d\bar{u}}{ds} + \cos\theta \frac{d\bar{v}}{ds} = \cos^2\theta\varepsilon_y + \sin^2\theta\varepsilon_x - 2\sin\theta\cos\theta\varepsilon_{xy} = \varepsilon_t$$

Or, finally, also resolving the boundary displacement vector into its local components,

$$\frac{d\bar{u}_n}{ds} - \dot{\theta}\, \bar{u}_t = \varepsilon_{nt} - \omega \qquad \frac{d\bar{u}_t}{ds} + \dot{\theta}\, \bar{u}_n = \varepsilon_t \tag{54}$$

After substitution of (52) and (48) into (53) and (54) we obtain, together with (50), a system of three partial differential equations for the unknowns (A,B) and ω and two boundary conditions. As, by elimination of ω between equations (53), the compatibility of strains will be satisfied in a simply connected domain, a displacement field will exist with boundary values corresponding to the data.

At an interface between two regions, we must assume that the data for the connectors $(\bar{u},\bar{v})$ are coherent, that is $\bar{u}_n$ and $\bar{u}_t$ must be single-valued. The left-hand sides of (54) are then also single-valued and C_o continuity of the stress function vector will conduce to single-valuedness of the quantities $(\varepsilon_{nt} - \omega \, , \, \varepsilon_t)$.
As the variation of ω does not appear in the boundary terms, the continuity of $\omega(x,y)$ is not required at interfaces. Finite elements will now be considered in which the ω field is discretized together with the stress function vector.

9. THE ZERO ENERGY STRESS STATE

As we assume the energy density Φ to be everywhere a positive definite function of its arguments, the stress energy $\iint \Phi \, dx \, dy$ will vanish if and only if

$$\sigma_x \equiv 0 \qquad \sigma_y \equiv 0 \qquad \tau_{xy} + \tau_{yx} \equiv 0$$

In terms of first order stress functions this implies

$$\frac{\partial A}{\partial y} = 0 \qquad \text{or} \qquad A = A(x)$$

$$\frac{\partial B}{\partial x} = 0 \qquad \text{or} \qquad B = B(y)$$

and finally

$$\frac{\partial B}{\partial y} = \frac{\partial A}{\partial x} = \gamma \qquad \text{that is} \qquad \tau_{xy} = - \tau_{yx} = \gamma$$

a constant since it must be both a function of y alone and x alone. There is thus one zero energy stress field, of arbitrary uniform intensity, representing in its purest form the violation of rotational equilibrium.

10. GLOBAL ROTATIONAL EQUILIBRIUM ELEMENTS

It will be shown that if A and B are discretized as complete polynomials of degree $n+1 \geqslant 2$ (n is the degree of the generated stress field) and if the Lagrangian multiplier (the material rotation ω) is reduced to a constant $\omega(x,y) = \theta$ (so that the element will only satisfy global rotational equilibrium), the stiffness matrix of simple triangular element is well-behaved. For definiteness some of the characteristics of the element will be explicited in the lowest degree case $n = 1$.
For this case

$$A = \alpha_o + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + 2\alpha_4 xy + \alpha_5 y^2$$

$$B = \beta_o + \beta_1 x + \beta_2 y + \beta_3 x^2 + 2\beta_4 xy + \beta_5 y^2$$

In forming a stress parameters vector s, we keep only the active parameters, discarding the "idle" parameters α_o and β_o (that produce no stresses). We may chose for instance

$$s^T = (\alpha_1, \alpha_2 \ldots \alpha_5 \mid \beta_1 \ldots \beta_5)$$

and note that the vector s_o associated to the zero energy case is then

$$s_o^T = \frac{1}{2\Delta} (1 \quad 0 \quad 0 \quad 0 \quad 0 \mid 0 \quad 1 \quad 0 \quad 0 \quad 0) \tag{55}$$

The convenience of having it normed so that

$$\alpha_1 = \beta_2 = \gamma = \frac{1}{2\Delta} \qquad \Delta = \iint_E dx\, dy$$

where Δ is the area of the triangular element, will appear later.
For linear elastic constitutive equations we will have

$$\iint_E \Phi \, dx\, dy = \frac{1}{2} s^T F s \tag{56}$$

where, on account of the existence of a zero energy state, the flexibility matrix will not be positive definite but merely non negative. In fact (Appendix A) $s_o^T F s_o = 0$ induces

$$F s_o = 0 \tag{57}$$

and s_o will be the only (normed) non trivial solution of the homogeneous equation $F s = 0$.
A constant multiplier enforces global rotational equilibrium of the element as can be seen from

$$\iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right) dx\, dy = \oint A dy - B dx = \oint x dB - y dA = \oint (x t_y - y t_x) ds = 0 \tag{58}$$

In the discretization

$$\iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right) dxdy = w^T s \tag{59}$$

a linear form in the active stress parameters; explicitly for $n = 1$

$$w^T = (\Delta \;\; 0 \;\; 2\iint_E x\, dxdy \;\; 2\iint_E y\, dxdy \;\; 0 \mid 0 \;\; \Delta \;\; 0 \;\; 2\iint_E x\, dxdy \;\; 2\iint_E y dxdy) \tag{60}$$

From (60) and (55) follows the essential property

$$w^T s_o = 1 \tag{61}$$

that obviously holds for arbitrary polynomial degree n.
To obtain the complete discretized form of the variational principle (51) we must still express

$$\oint \bar{u}\, dA + \bar{v}\, dB = g^T q$$

in terms of a boundary loads vector g with attendant virtual work definitions of conjugate displacements, collected in the (weak) boundary displacements vector q.

The definition of boundary loads follows the pattern of equilibrium models. Reciprocity of boundary loads on each interface the element can have with a neighbour must entail complete diffusivity. As an example for the case n = 1

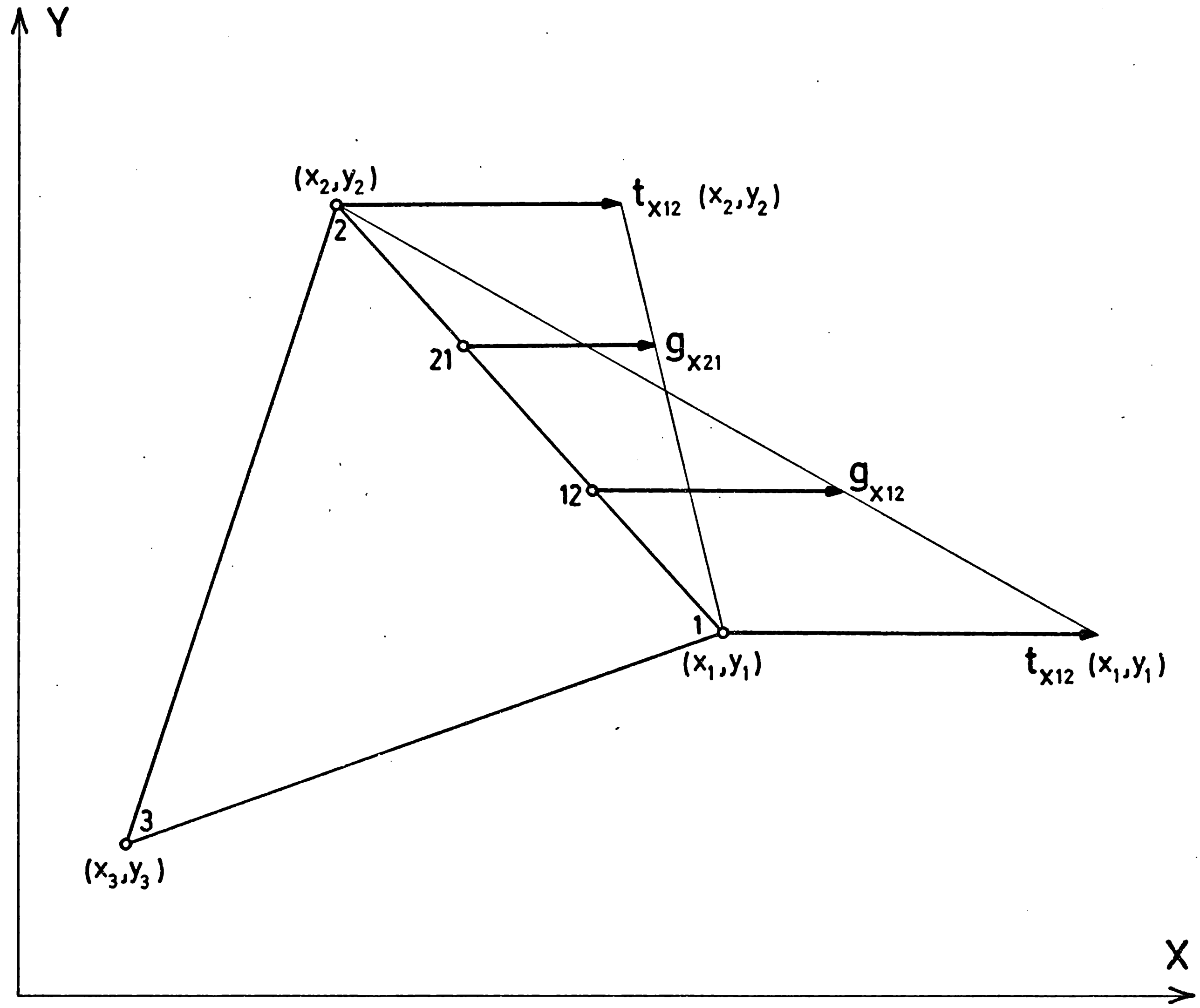

FIG. 4

we can define the loads g_{x12} and g_{x21} of the boundary 12 (figure 4) as the resultants of the two linear interpolation functions of the surface traction t_x. By virtual work the conjugate displacements are

$$u_{12} = \frac{2}{c_{12}} \int_1^2 \left(1 - \frac{s}{c_{12}}\right) u(s)\,ds \qquad u_{21} = \frac{2}{c_{12}} \int_1^2 \frac{s}{c_{12}}\, u(s)\,ds$$

$$c_{12} = \int_1^2 ds$$

where s is measured along the boundary from 1 towards 2. Similar definitions are valid for g_{y12} and g_{y21}, to define uniquely the linearly varying t_y traction, and for the conjugate vertical displacements v_{12} and v_{21}.
From the very definitions adopted for the boundary loads one obtains their value in terms of the active stress parameters

$$g = S\ s \tag{62}$$

whence finally

$$\oint \bar{u}\ dA + \bar{v}\ dB = q^T\ S\ s \tag{63}$$

where, in applying the variational principle, it must be considered that the vector q is prescribed. Collecting the results (63), (59) and (56) the stationarity of

$$\frac{1}{2}\ s^T\ F\ s + \theta\ w^T\ s - q^T\ S\ s$$

with respect to the choice of s and θ , implies

$$\begin{aligned} F\ s + \theta\ w &= S^T\ q \\ w^T\ s &= 0 \end{aligned} \tag{64}$$

Although F is singular, the banded symmetrical matrix of the complete linear system

$$\begin{pmatrix} F & w \\ \hline w^T & 0 \end{pmatrix} \begin{pmatrix} s \\ \hline \theta \end{pmatrix} = \begin{pmatrix} S^T q \\ \hline 0 \end{pmatrix} \tag{65}$$

is non singular.

Indeed the homogeneous system

$$F\ s + \theta\ w = 0 \qquad w^T\ s = 0$$

possesses only the trivial solution as can be seen by premultiplying the first equation by s_o^T, whereby, on account of (57) and (61), we must have $\theta = 0$. Then $F\ s = 0$ has a non trivial solution $s = s_o$, but it fails to satisfy the second equation.
Thus the reciprocal matrix exists and is of the form

$$\begin{pmatrix} F & w \\ w^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} F^{\#} & v \\ v^T & \mu \end{pmatrix} \qquad F^{\#} = (F^{\#})^T$$

As a right inverse its elements must satisfy

$$F\ F^{\#} + w\ v^T = I \qquad F\ v + \mu\ w = 0 \qquad w^T\ F^{\#} = 0 \qquad w^T\ v = 1$$

and we find that $\mu = 0$ and $v = s_o$
so that

$$F\ F^{\#} + w\ s_o^T = I \qquad F^{\#}\ w = 0 \tag{66}$$

and the solution is

$$s = F^{\#}\ S^T\ q \qquad \theta = s_o^T\ S^T q \tag{67}$$

In practice, of course, the pseudo-inverse $F^{\#}$ of F is obtained by numerical inversion of the banded matrix. The second of equations (67) gives the average rotation experienced by the element as a result of the given boundary displacements. When the first is substituted into (62) we obtain the stiffness matrix of the element

$$g = K\ q \qquad K = S\ F^{\#}\ S^T \tag{68}$$

and it remains to verify that it is well-behaved in the sense that the only solutions of

$$K\ q = 0 \tag{69}$$

are the rigid body displacement modes of the element.

As will be shown, the translational rigid body modes stem from the solutions of $S^T q = 0$; the rotational mode from $F^{\#} S^T q = 0$.

11. SELF-STRESSING STATES AND THE SOLUTIONS OF $S^T q = 0$

The stresses generated by A and B satisfy identically the local translational equilibrium equations. As a consequence the boundary loads g should also be in global translational equilibrium. By virtual work this can be formulated as

$$g^T q_x = 0 \qquad g^T q_y = 0 \qquad \text{for any s vector}$$

where q_x and q_y are rigid body translation modes. In view of (62) this shows that

$$S^T q_x = 0 \qquad\qquad S^T q_y = 0 \qquad\qquad (70)$$

are solutions of $S^T q = 0$. A direct proof also follows from (63) by inputting $\bar{u} = 1$ and $\bar{v} = 0$ and observing that the left hand side vanishes for any single valued A and similarly for $\bar{u} = 0$, $\bar{v} = 1$ and any single valued B.
There are no other solutions (linearly independent) of $S^T q = 0$ than those translation modes as can be seen by counting the number of solutions n(x) of the adjoint problem

$$S\ s = 0 \qquad\qquad (71)$$

As S is a n(g) x n(s) matrix we know from algebra that the number of linearly independent solutions of $S^T q = 0$ is

$$m = n(x) + n(g) - n(s)$$

and must be shown to be equal to 2.
Problem (71) is the problem of self-stressing states of the element; it enquires about the active stress parameters that would produce a state of stress whose surface tractions vanish at the boundary.

From (49) such a situation implies

$$dA = 0 \qquad dB = 0 \qquad \text{along the boundary.}$$

If we add the idle parameters α_o and β_o, so as to form complete polynomials for the stress functions, there is no restriction in setting

$$A = 0 \qquad B = 0 \qquad \text{along the boundary}$$

as characteristic equations describing the self-stressing states. In polynomial approximations, using areal coordinates for simplicity of presentation, each stress function corresponding to a self-stressing state will be of the form

$$L_1 \, L_2 \, L_3 \, P(L_1, L_2, L_3)$$

where P is a linear combination of all monomials in (L_1, L_2, L_3) of degree n+1-3 = n-2. In other terms P will be a complete polynomial in (x,y) of degree n-2 containing $\frac{1}{2}$ n(n-1) terms. Since there are two stress functions the total number of different self-stressing states is

$$n(x) = n(n-1)$$

We have none in the model with linearly varying stresses, then 2 for parabolically varying stresses, 6 for quartic variations, etc... The number of boundary loads is 2(n+1) per side, hence

$$n(g) = 6(n+1)$$

The number of active stress parameters

$$n(s) = (n+2)(n+3) - 2$$

so that in the end we find indeed m = 2, irrespective of the degree n of the stress variations.

The n(n-1) linearly independent solutions $s=x_{(i)}$ of (71) have the important property

$$S\, x_{(i)} = 0 \rightarrow w^T\, x_{(i)} = 0 \tag{72}$$

This follows from writing (59) in the form

$$w^T\, s = \iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right) dxdy = \oint Ady - Bdx$$

and remembering that A = 0 and B = 0 along the boundaries when s is a self-stressing $x_{(i)}$.

12. THE ROTATIONAL RIGID BODY MODE

Equations (67) show $F^{\#}$ to be non negative. For, premultiplying the first by $F^{\#}$ and taking the second into account, we obtain

$$F^{\#}\, F\, F^{\#} = F^{\#} \tag{73}$$

Hence, for an arbitrary vector e,

$e^T\, F^{\#}\, e = (F^{\#}\, e)^T\, F(F^{\#}e) \geqslant 0$ since F is non negative.

This important property, in turn, shows that the solutions of

$$K\, q = S\, F^{\#}\, S^T\, q = 0$$

for which

$$q^T\, K\, q = (S^T q)^T\, F^{\#}(S^T q) = 0$$

are (theorem of Appendix A) those of the linear system

$$F^{\#}\, S^T q = 0 \tag{74}$$

We have already seen that $S^T q = 0$ furnishes only the two translational modes. As it again follows from equations (67), w is, up to arbitrary scale factor, the only solution of $F^{*} e = 0$, so that the other solutions of (69) depend on those of

$$S^T q = w \tag{75}$$

The existence conditions for a particular solution of this singular problem are precisely satisfied by (72), the general solution consisting merely in adding arbitrary translations. The nature of any particular solution is clear from the solution (67) of the variational problem : there are no stresses involved (s=0) but there is a uniform rotation of the element

$$\omega = \theta = s_o^T w = 1$$

This achieves the proof that the stiffness matrix is well behaved.

13. IMPROVEMENT OF ROTATIONAL CONDITIONING FOR $n > 1$

When the degree of polynomial approximation increases above n=1 a lack of balance will develop between the rigorous satisfaction of translational equilibrium and the minimum acceptable state of rotational equilibrium. On the other hand, a too generous improvement of rotational equilibrium may lead to the appearance of mechanisms. This is well known since the earliest attempts at constructing stress elements satisfying rigorously all equilibrium conditions. Taking the case n=1 and implementing exactly equation (50), which is equivalent to the use of an Airy second order stress function represented by a complete cubic, we are left with 7 active parameters only ($\alpha_1 = \beta_2$ $\alpha_3 = \beta_4$ $\alpha_4 = \beta_5$).
Thus, as n(s) = 7, n(g) = 12 and n(x) = 0, there follows

$$m = 5$$

On the other hand F is now positive definite (no non trivial zero energy state).

Three of the solutions of $S^T q = 0$ are the rigid body modes, two other solutions are mechanisms.
In the general case $n > 1$, the degree of the Airy function is $n+2$ so that, discounting its 3 idle parameters (those of the linear part),

$$n(s) = \frac{(n+3)(n+4)}{2} - 3$$

$$n(g) = 6(n+1) \qquad \text{as before}$$

The self-stressing states are now given by Airy functions that are zero together with their normal derivative at the boundary, hence of type

$$\phi = L_1^2 \, L_2^2 \, L_3^2 \, P(L_1, L_2, L_3)$$

P is a complete polynomial of degree $n+2-6 = n-4$

and $n(x) = 0$ for $n \leq 3$

$$n(x) = \tfrac{1}{2}(n-3)(n-2) \qquad \text{for } n > 3$$

In all cases $n > 1$ it is found that

$$m = 6$$

Hence, in addition to the 3 rigid body modes, there are exactly always 3 mechanisms. They may be shown to be of the "floating corner" type.
Although one can live with such mechanisms and inhibit them, either by the composite element technique, or by imposing suitable constraints at the assembled level, it appears more satisfactory to avoid them, inasmuch as, by sticking to first order stress functions we do not thereby inhibit the extension to isoparametricity.

Let us now increase the rotational conditioning of the element by taking as multiplier a complete polynomial of degree r. It is easily seen that for n = 1 the use of a complete linear polynomial (r=1) is equivalent to rigorous rotational equilibrium and would introduce mechanisms. Hence we will impose the restriction $r \leq n-1$ and show that the stiffness matrix remains well behaved. Thus rotational conditioning can be improved in parallel with the improvement of compatibility conditions resulting of a choice of higher degrees n.

The constraints on the active stress parameters will now be

$$\iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)\omega(x,y)\ dxdy = s^T W h = 0 \qquad (76)$$

for an arbitrary vector h, listing the parameters (coefficients) of the polynomial $\omega(x,y)$. The following properties may be proved

1) $W\ h = 0 \ \rightarrow\ h = 0$ (77)

that is to say, the columns of matrix W are linearly independent. If they were not, some non identically zero polynomial $\omega(x,y)$ would exist such that

$$\iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)\ \omega\ dxdy = 0$$

for arbitrary polynomials A and B of degree $n+1 \geq r+2$.
The result should hold in particular for the choice

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = \omega$$

because polynomials A and B of degree r+2 can be so chosen that the terms of degree r+1 on the left-hand side cancel, leaving an arbitrary polynomial of degree r. For such a choice however

$$\iint_E \omega^2\ dxdy = 0 \ \rightarrow\ \omega \equiv 0$$

and the contradiction proves the point.

2) The matrix of the new variational equations to be solved

$$\begin{pmatrix} F & W \\ W^T & 0 \end{pmatrix} \begin{pmatrix} s \\ h \end{pmatrix} = \begin{pmatrix} S^T q \\ 0 \end{pmatrix}$$

is non singular.

The proof consists again in showing that the homogeneous problem

$$F\ s + W\ h = 0 \qquad\qquad W^T\ s = 0$$

has only the trivial solution. Premultiplying the first equation by s^T and using the second equation we must have

$$s^T\ F\ s = 0$$

and, since F is non negative,

$$F\ s = 0$$

We cannot adopt the solution $s = s_o$ (the only non trivial one) because

$$W^T\ s_o \neq 0$$

as one of the columns of W, or a linear combination of them, is our former w and $w^T s_o = 1$. Hence we must take s=0. The first equation then reduces to W h = 0 and can only be satisfied by h = 0, as seen under 1).

The matrix has thus an inverse

$$\begin{pmatrix} F^{\#} & V \\ V^T & Q \end{pmatrix} \qquad (F^{\#})^T = F^{\#} \qquad Q^T = Q$$

such that

$$F\,F^{\#} + W\,V^T = I \qquad FV + W^T Q = 0 \qquad F^{\#} W = 0 \qquad W^T V = I \quad (78)$$

and the solution of the variational equations is

$$s = F^{\#}\,S^T\,q \qquad h = V^T\,S^T\,q \qquad (78)$$

The last result furnishes the information about the rotation of the element

Again from equations (78) we obtain the characteristic pseudo-inverse property

$$F^{\#}\,F\,F^{\#} = F^{\#}$$

and $F^{\#}$ (not the same as the former one) is still non negative.

3) As a result of the last property we are again led to investigate the q vectors satisfying

$$S^T\,q = W\,h \qquad (79)$$

as representing the rigid body modes or mechanisms of problem

$$K\,q = S\,F^{\#}\,S^T\,q = 0$$

distinct from the two translational modes. Equation (79) is in fact one of the variational equations for $s = 0$.

The uniform rotation mode is certainly a solution, since $Wh = w$ for a suitable choice of h and the existence conditions of a particular solution are then again implemented by (72). Other solutions are undesirable, and the stiffness matrix will be well behaved if at least one of the existence conditions

$$x_{(i)}^T\,W\,h = 0 \qquad (80)$$

is violated for each other (linearly independent) choice of h. Given n, this will be shown to be true when r has its maximal value n-1 and, a fortiori, if it is taken to be smaller.

Let $n = 2$ and $r = 1$. We must prove that in each of the cases $\omega = x$ and $\omega = y$ (corresponding to choices of h linearly independent of the choice $\omega = 1$) a self-stressing state (A and B vanishing at the boundaries) can be found such that

$$\iint_E \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)\omega \, dxdy = -\iint_E \left(A \frac{\partial \omega}{\partial x} + B \frac{\partial \omega}{\partial y}\right) dxdy \qquad (81)$$

does not vanish.

The polynomial degree of A and B is here at most equal to 3, and we dispose of only two different self-stressings. For $\omega=x$ we select

$$A = L_1 \, L_2 \, L_3$$

and obtain for the right-hand side of (81)

$$-\iint_E L_1 \, L_2 \, L_3 \, dxdy < 0$$

since the bubble function $L_1 \, L_2 \, L_3$ is positive at every interior point of E. The same bubble function is used for B in the case $\omega = y$ with the same result.

Let now $n=3$ and $r=2$. The preceding choices of self-stressings remain valid for the same cases but we must now investigate in addition the cases $\omega = x^2$, xy and y^2. On the other hand the degree of A and B may be raised to 4.

For $\omega = x^2$, select $A = x \, L_1 \, L_2 \, L_3$ and obtain

$$-2 \iint_E x^2 \, L_1 \, L_2 \, L_3 \, dxdy < 0$$

For $\omega = xy$, take $A = y \, L_1 \, L_2 \, L_3$ and $B = x \, L_1 \, L_2 \, L_3$, to find

$$-\iint_E (x^2 + y^2) \, L_1 \, L_2 \, L_3 \, dxdy < 0$$

Finally for $\omega = y^2$ the choice $B = y\, L_1\, L_2\, L_3$ gives

$$- \iint_E y^2\, L_1\, L_2\, L_3\, dxdy < 0$$

This systematic technique can obviously be extended indefinitely to higher values of n and furnishes the proof that the stiffness matrix remains well behaved for $r \leq n-1$.

In conclusion it may be said that the existence of a new familly of two-dimensional membrane elements has been established. They share with the pure equilibrium elements the properties of rigorous translational equilibrium and diffusivity at the boundaries. They have the advantage of being formulated in terms of first order stress functions only, with simple C_o continuity requirements and the attendant possibilities of isoparametric coordinate transformations to curved boundaries. They distinguish themselves from pure equilibrium models by the fact that they satisfy only average rotational equilibrium equations. The degree of enforcement of the rotational equilibrium may follow one step behind the degree of approximation of the stress distribution or may be lower, leaving a trade-off possibility between rotational equilibrium ans strain compatibility.

14. AN ALTERNATIVE APPROACH TO THE CONSTRUCTION OF THE STIFFNESS MATRIX

Take the case n = 1. As the stresses are known to vary linearly, they are uniquely determined by the 3 corner values of the unsymmetrical stress tensor

$$\begin{pmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{pmatrix} .$$

Consider then the stress vector σ containing this information

$$\sigma^T = (\sigma_{x1}\ \sigma_{x2}\ \sigma_{x3}\ \tau_{yx1}\ \tau_{yx2}\ \tau_{yx3}\ \tau_{xy1}\ \tau_{xy2}\ \tau_{xy3}\ \sigma_{y1}\ \sigma_{y2}\ \sigma_{y3})$$

The information enables the surface tractions and consequently the generalized boundary (interface) loads to be computed. It furnishes a 12 × 12 matrix G such that

$$g = G\ \sigma$$

Up to now equilibrium was disregarded entirely. From the rigid body modes

$$q = R\ r \qquad R \text{ a } 12 \times 3 \qquad r \text{ arbitrary}$$

of the element, the global equilibrium conditions are

$$R^T g = R^T G \sigma = 0$$

They are enforced by means of a Lagrange vector multiplier λ in the complementary energy principle

$$\frac{1}{2}\ \sigma^T\ \Phi\ \sigma + \lambda^T\ R^T\ G\ \sigma - q^T\ G\ \sigma \qquad \text{minimum}$$

The minimizing equations are

$$\begin{pmatrix} \Phi & G^T R \\ R^T G & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ \lambda \end{pmatrix} = \begin{pmatrix} G^T q \\ 0 \end{pmatrix}$$

and, from the previous theory, the matrix is invertible, yielding

$$\sigma = \Phi^{\#}\ G^T q$$

and finally

$$g = K\ q \qquad \text{with} \qquad K = G\ \Phi^{\#}\ G^T\ .$$

No consideration was given here to the existence of first order stress functions, but it is easily established that the translational equilibrium equations are rigorously satisfied. Starting from the divergence theorem

$$\int (\sigma_x \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} + \sigma_y \frac{\partial v}{\partial y}) \, dx \, dy$$

$$= \oint (u t_x + v t_y) \, ds - \int \left[u(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}) + v \, (\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}) \right] dx \, dy$$

We note that the volume forces equilibrating the stresses

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = X \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = Y$$

are at most constants. Thus if u and v are arbitrary constants

$$0 = u \oint t_x \, ds + v \oint t_y \, ds - (u \, X + v \, Y) \int dx \, dy$$

However, since global equilibrium was enforced,

$$\oint t_x \, ds = 0 \qquad \oint t_y \, ds = 0$$

and the result holds for arbitrary u and v if and only if $X = 0 \quad Y = 0$. This proves that local equilibrium holds for translation. For rotation we start from the identity

$$\int \left[\frac{\partial}{\partial x} (x \, \tau_{xy} - y \, \sigma_x) + \frac{\partial}{\partial y} (x \, \sigma_y - y \, \tau_{yx}) \right] dx \, dy = \oint (x \, t_y - y \, t_x) ds$$

The right hand side is zero when global rotational equilibrium is enforced. Evaluating the left hand side we then find

$$\int (\tau_{xy} - \tau_{yx}) \, dx \, dy + \int (x \, Y - y \, X) \, dx \, dy = 0$$

and, since $X = 0$ and $Y = 0$, find, as before, that rotational equilibrium is only satisfied in the average.

15. RECTANGULAR ELEMENTS WITH DISCRETIZED ROTATIONAL EQUILIBRIUM

The theory established in sections 8 to 13 applies as well to elements of basic rectangular geometry. For definiteness we take the example of a rectangle with boundaries $x = \pm \ell$, $y = \pm h$ in which the unsymmetrical stess tensor is generated by bi-quadratic first order stress functions

$$A = \alpha_o + \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy^2 + \alpha_7 x^2 y + \alpha_8 x^2 y^2$$

$$B = \beta_o + \beta_1 x + \beta_2 y + \beta_3 xy + \beta_4 x^2 + \beta_5 y^2 + \beta_6 xy^2 + \beta_7 x^2 y + \beta_8 x^2 y^2$$

The stress tensor

$$\sigma_x = \frac{\partial A}{\partial y} = \alpha_2 + \alpha_3 x + 2\alpha_5 y + 2\alpha_6 xy + \alpha_7 x^2 + 2\alpha_8 x^2 y$$

$$- \tau_{xy} = \frac{\partial A}{\partial x} = \alpha_1 + \alpha_3 y + 2\alpha_4 x + 2\alpha_6 y^2 + 2\alpha_7 xy + 2\alpha_8 xy^2$$

$$\tau_{xy} = \frac{\partial B}{\partial y} = \beta_2 + \beta_3 x + 2\beta_5 y + 2\beta_6 xy + \beta_7 x^2 + 2\beta_8 x^2 y$$

$$- \sigma_y = \frac{\partial B}{\partial x} = \beta_1 + \beta_3 y + 2\beta_4 x + 2\beta_6 y^2 + 2\beta_7 xy + 2\beta_8 xy^2$$

presents linear variations only along the boundaries $x = \pm \ell$, $y = \pm h$.

The active stress parameters are sequenced in the vector s

$$s^T = (\alpha_1 \ldots \alpha_8 \mid \beta_1 \ldots \beta_8) \qquad n(s) = 16$$

The stress energy is again a quadratic form $\frac{1}{2}\, s^T F s$ with a non negative flexibility matrix possessing as unique non trivial solution to Fs = 0, the vector

$$s_o^T = (\gamma\ 0 \ldots 0 \mid 0\ \gamma \ldots 0)$$

Along each boundary we have 4 generalized loads, the 16 generalized loads vector g being related to the active stress parameters by

$$g = S\ s \qquad\qquad S \text{ a } 16 \times 16 \text{ matrix.}$$

However there are two self-stressing states

$$A = \lambda(x^2 - \ell^2)(y^2 - h^2)$$

$$B = \mu(x^2 - \ell^2)(y^2 - h^2)$$

and the generalized loads depend really on 16 - 2 = 14 parameters only. In fact the parameters α_8 and β_8 could be replaced by λ and μ and, g being independent of λ and μ , the S matrix would have corresponding zero columns.
Thus of the 16 generalized loads, 14 only are independent, a situation coherent with the fact that they already automatically satisfy 2 global equilibrium conditions in translation. Any additional constraint imposed on the parameters, like the one resulting from

$$\int (\tau_{xy} - \tau_{yx})\ dx\ dy = 0$$

will also be reflected as a constraint on the generalized loads, here the global rotational equilibrium constraint.
Furthermore, as there are two self-stressings we may hope to improve, if we so desire, the rotational equilibrium. The situation is not fundamentally different from the previous one. We must verify whether our choice of $\omega(x,y)$ is such that, according to (81)

$$\iint_E (A \frac{\partial \omega}{\partial x} + B \frac{\partial \omega}{\partial y})\ dx\ dy = 0$$

is violated for at least one of the self-stressings.

This is again the case for the choices $\omega = x$ or $\omega = y$, since

$$\iint_E (x^2 - \ell^2)(y^2 - h^2)\, dx\, dy < 0$$

16. ISOPARAMETRIC COORDINATE TRANSFORMATIONS

Let (ξ,η,ζ) be global cartesian coordinates of a "primal" mesh of finite elements with plane boundaries. In polynomial approximations, the first order stress functions are piecewise analytic functions of (ξ,η,ζ), C_o continuous at the interfaces. Consider now any isoparametric coordinate transformation from (ξ,η,ζ) to (x,y,z), using displacement type interpolation functions $P_i(\xi,\eta,\zeta)$

$$x = \xi + \Sigma\, \hat{u}_i\, P_i\, (\xi,\eta,\zeta)$$

$$y = \eta + \Sigma\, \hat{v}_i\, P_i\, (\xi,\eta,\zeta)$$

$$z = \zeta + \Sigma\, \hat{w}_i\, P_i\, (\xi,\eta,\zeta)$$

to preserve continuity of the continuum by conformity of the fixed displacements

$$(\hat{u}_i)_+ = (\hat{u}_i)_- \qquad (\hat{v}_i)_+ = (\hat{v}_i)_- \qquad (\hat{w}_i)_+ = (\hat{w}_i)_-$$

across interfaces.

Considering (x,y,z) as new global cartesian coordinates, we obtain the "dual" mesh with curved boundaries and interfaces. The coordinate transformation is continuous at the interfaces and, considered as functions of (x,y,z), the first order stress functions remain C_o continuous so that diffusivity is maintained. Thus there is no need whatsoever to impose to the interpolation functions $P_i(\xi,\eta,\zeta)$ the need to be the same as the interpolation functions used in the primal mesh to obtain C_o continuity of the first order stress functions.

As already observed earlier, the need of C_1 continuity preservation when second order stress functions are involved, may warrant a closer investigation of whether this is at all possible by relating the interpolation functions.

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APPENDIX A

A property of non negative matrices

In section 10 and later use is made of the property that for a non negative matrix F

$$s^T F s = 0 \rightarrow F s = 0$$

A simple proof is as follows. Suppose that a non zero vector s satisfies

$$s^T F s = 0 \tag{A.1}$$

and set $F s = x$ (A.2)

Then, since F is non negative,

$$(x + \lambda s)^T F (x + \lambda s) \geqslant 0 \quad \text{for any scalar } \lambda .$$

Expanding and using (A.1) and (A.2)

$$x^T F x + 2 \lambda x^T x \geqslant 0$$

This can hold for arbitrary λ if and only if

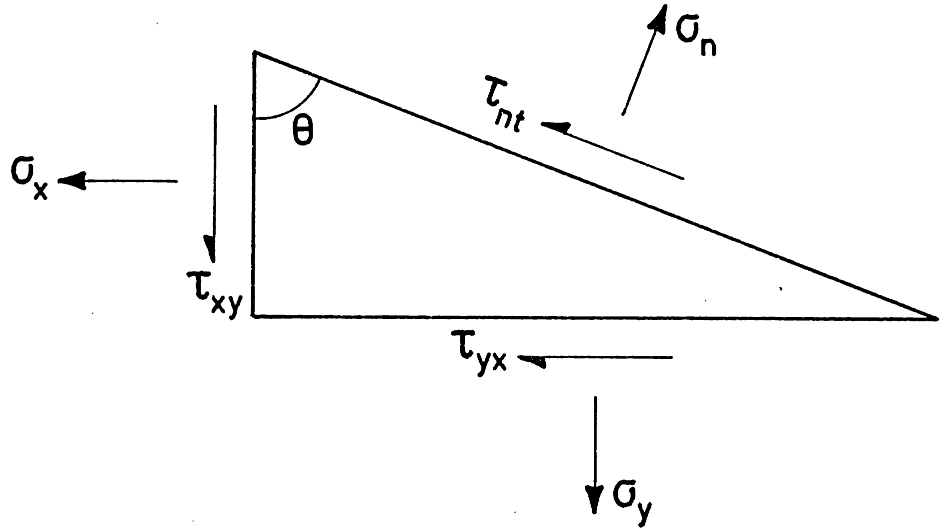
$$x^T x = 0 \rightarrow x = 0$$

and the property is demonstrated.

APPENDIX B

Invariance of local rotational unbalance

The measure $\tau_{xy} - \tau_{yx}$ by which local rotational equilibrium is violated is not related to a particular orientation of cartesian axes. It is in fact invariant under a rotation of the axes.



Equilibrium of the elementary triangular prism yields

$$\sigma_n = \cos^2\theta \ \sigma_x + \sin^2\theta \ \sigma_y + \sin\theta \ \cos\theta \ (\tau_{yx} + \tau_{xy}) \tag{B.1}$$

$$\tau_{nt} = \sin\theta \ \cos\theta \ (\sigma_y - \sigma_x) + \cos^2\theta \ \tau_{xy} - \sin^2\theta \ \tau_{yx} \tag{B.2}$$

Increasing θ by $\pi/2$

$$-\tau_{tn} = \sin\theta \ \cos\theta \ (\sigma_x - \sigma_y) + \sin^2\theta \ \tau_{xy} - \cos^2\theta \ \tau_{yx} \tag{B.3}$$

From these relations follow the classical tensorial transformations when the average shear stress is used

$$\sigma_n = \cos^2\theta \ \sigma_x + \sin^2\theta \ \sigma_y + 2\sin\theta\cos\theta \ \frac{\tau_{yx} + \tau_{xy}}{2} \tag{B.4}$$

$$\frac{\tau_{nt} + \tau_{tn}}{2} = \sin\theta\cos\theta \ (\sigma_y - \sigma_x) + (\cos^2\theta - \sin^2\theta) \ \frac{\tau_{yx} + \tau_{xy}}{2}$$

Also the announced invariance property

$$\tau_{nt} - \tau_{tn} = \tau_{xy} - \tau_{yx} \tag{B.5}$$

In the three-dimensional case we have the tensorial transformation

$$\sigma_{\alpha\beta} = T_{\alpha i} T_{\beta j} \sigma_{ij} \tag{B.6}$$

for the unsymmetrical stress tensor. The symmetrical part

$$\frac{1}{2} (\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) = T_{\alpha i} T_{\alpha j} \frac{1}{2} (\sigma_{ij} + \sigma_{ji}) \tag{B.7}$$

satisfies the same trnasformation, while also

$$\sigma_{\alpha\beta} - \sigma_{\beta\gamma} = T_{\alpha i} T_{\beta j} (\sigma_{ij} - \sigma_{ji}) \tag{B.8}$$

The pseudo-vector of strict components of this antisymmetrical part is obtained through multiplication by the alternating tensor $\frac{1}{2} e_{\alpha\beta\gamma}$

$$\frac{1}{2} e_{\alpha\beta\gamma} (\sigma_{\alpha\beta} - \sigma_{\beta\alpha}) = a_\gamma$$

or, in detail,

$$a_1 = \sigma_{23} - \sigma_{32} \qquad a_2 = \sigma_{31} - \sigma_{13} \qquad a_3 = \sigma_{12} - \sigma_{21}$$

Noting that

$$e_{\alpha\beta\gamma} T_{\alpha i} T_{\beta j} = (\det T) e_{ijk} T_{\gamma k}$$

relation (B.8) is transformed in

$$a_\gamma = (\det T) T_{\gamma k} a_k \tag{B.9}$$

showing that under proper rotations (det T = 1) of the cartesian axes the rotational unbalance tensor a_k behaves as a polar (invariant) vector.

APPENDIX C

Boundary load constraints and stress parameters constraints

We can investigate somewhat differently the relationship between homogeneous boundary loads constraints, that lead to desirable global equilibrium of the element but also to undesirable mechanisms, and stress parameter constraints, that may improve rotational equilibrium of the stress field.
The relation between loads and active parameters

$$g = S\ s \qquad \text{(C.1)}$$

involves the following classical algebraïcal equality

$$m = n(g) - n(y) = n(s) - n(x) \qquad \text{(C.2)}$$

between the number n(g) of boundary loads, n(s) of active stress parameters, n(x) of independent self-stressings, solutions of the homogeneous problem

$$S\ s = 0 \qquad s = x_{(i)} \qquad \text{(C.3)}$$

and n(y) of boundary displacement modes, independent solutions of the homogeneous adjoint problem

$$S^T_q = 0 \qquad q = y_{(j)} \qquad \text{(C.4)}$$

It was used in section 11 to establish that in two dimensional triangular membrane elements with complete polynomial approximations for the stress functions, the solutions of (C.4) were in fact restricted to the translation modes. The same was true for the example of rectangular membrane element and can be shown to be true for three-dimensional tetrahedroms, so that it seems

to be a natural situation for elements generated by first order stress functions.
As

$$y_{(j)}^T \, g = (S^T \, y_{(j)})^T \, s = 0 \tag{C.5}$$

the global equilibrium conditions in translation are the necessary, but also sufficient, conditions for the existence of an inverse to (C.1)

$$s = S^{\#} \, g + \alpha_i \, x_{(i)} \tag{C.6}$$

where $S^{\#}$ is any particular pseudo-inverse to S, generating a particular set of stress parameters corresponding to given equilibrated (in translation only) boundary loads. The self-stressing intensities α_i are arbitrary and form the general solution of (C.1) for $g = 0$.
Any homogeneous constraint

$$q^T \, g = 0 \tag{C.7}$$

imposed on the loading induces through (C.1) a corresponding homogeneous constraint on the stress parameters

$$w^T \, s = 0 \tag{C.8}$$

with

$$w = S^T \, q \tag{C.9}$$

the constraint being effective ($w \neq 0$) if and only if q is linearly independent of the translation modes. Since

$$x_{(i)}^T \, w = (S \, x_{(i)})^T \, q = 0 \tag{C.10}$$

the w vectors generated by (C.9) are automatically orthogonal to all self-stressings. They span a m dimensional subspace of the n(s) dimensional space of all w vectors.
Conversely, any w vector of this subspace, used to induce a stress parameter constraint like (C.8) also induces through C.6

a homogeneous loads constraint of type (C.7) with

$$q = S^{\#T} w \tag{C.11}$$

(C.11) is a particular inverse of (C.9) when the existence conditions for inversion, represented by (C.10), are satisfied. The general inverse is then obviously

$$q = S^{\#T} w + \beta_j y_{(j)} \tag{C.12}$$

To produce a stress parameter constraint that improves rotational equilibrium without inducing a homogeneous loads constraint two conditions are required :

1. The vector w should be one generated by a condition of type

$$e_{mij} \int_E \omega_m \sigma_{ij} dV = \int_E [\omega_1 (\sigma_{23} - \sigma_{32}) + \omega_2 (\sigma_{31} - \sigma_{13}) + \omega_3 (\sigma_{12} - \sigma_{21})] dV = 0 \tag{C.13}$$

2. It must not belong to the subspace of w vectors orthogonal to all self-stressings.

In that case (C.6) shows that the constraint (C.8) is equivalent to a coupling between the loading and the self-stressing intensities

$$w^T S^{\#} g + \alpha_i w^T x_{(i)} = 0 \tag{C.14}$$

Obviously the number of such independent constraints may not exceed the number of independent self-stressings present.