THE DUAL PRINCIPLES OF ELASTODYNAMICS

FINITE ELEMENT APPLICATIONS

B. FRAEIJS de VEUBEKE (*)

SUMMARY

Up to the present time, most of the finite element applications to linear structural dynamics have been based on the displacement approach and its corresponding Hamiltonian variational principle. It is sufficient to this effect to add to the static stiffness matrix lagrangian, or so-called coherent, mass matrices.
The first really satisfactory formulation of a dual principle, in which the kinetic energy is transformed through satisfaction of the dynamic equilibrium equations into a functional expressed in terms of space derivatives of an impulse field, is due to Toupin [5]. Similar approaches were followed by Crandall [9], Yu Chen [10] and Gladwell and Zimmermann [11]. Finite element applications to the eigenvalue problems of beams and plates are due to Tabarrok, Sakaguchi and Karnopp [8], [12] and to Geradin [7].
The paper gives a logical derivation of the dual dynamic principle through the canonical form in the spirit of Friedrichs transformations [1]. It also discusses the general procedures for assembling equilibrium type finite elements in order to implement the dual principle. Finally it is show that there is no advantage in using the dual principle together with the requirement of forcing orthogonality with respect to all zero frequency modes. Instead, experience shows [7], [8] that by ignoring this unnecessary requirement eigenvalues generally converge to their exact values by lower bounds, hence giving precious accuracy estimates by comparison with the displacement approach.

(*) Head Aerospace Laboratory University of Liège, Belgium.

1. Hamilton's principle

Hamilton's principle for conservative systems is a statement of stationarity of a functional of the displacement field

$$f\,[u] = \int_{t_1}^{t_2} (T - V)\, dt \tag{1}$$

where T denotes the kinetic energy

$$T = \int_R \frac{1}{2} \rho\, \dot{u}_j\, \dot{u}_j\, dR \tag{2}$$

and V the total potential energy of the system. V can have several distinct parts; one is the strain energy

$$V_1 = \int_R W\,(Du)\, dR \tag{3}$$

where the energy density is expressed in terms of strains, i.e

$$\varepsilon_{ij} = \frac{1}{2} (D_i\, u_j + D_j\, u_i) \tag{4}$$

if we restrict ourselves to the linearized case; another one can be a potential energy associated to the displacements on a part $\partial_2 R$ of the boundary

$$V_2 = - \int_{\partial_2 R} F\,(u,t)\, dS \tag{5}.$$

On the remaining part $\partial_1 R$ of the boundary we shall assume that the displacements are prescribed time functions

$$u_j = \bar{u}_j\,(t) \qquad \text{on } \partial_1 R \tag{6}.$$

For simplicity we do not take into consideration other potential energies like that associated with conservative external body forces functions of position. In applying the principle

$$\delta f\,[u] = 0$$

with Hamilton's rule

$$\delta u_j \equiv 0 \qquad \text{for } t = t_1 \text{ and } t = t_2 \tag{7}$$

and $\delta u_j = 0$ at all times on $\partial_1 R$, which is a consequence of (6), the following variational derivatives are obtained

$$- \rho \ddot{u}_j + D_i \tau_{ij} = 0 \qquad (8)$$

where $$\tau_{ij} = \frac{\partial \mathcal{N}}{\partial \varepsilon_{ij}} \qquad (9)$$

are the stresses. They are the dynamic equilibrium conditions in d'Alembert's form. The natural boundary conditions supplementing (6) are found to be

$$- n_i \tau_{ij} + \frac{\partial F}{\partial u_j} = 0 \qquad \text{on } \partial_2 R \qquad (10)$$

with the usual notation

$$t_j = n_i \tau_{ij} \qquad (11)$$

for the surface tractions, they state that these surface tractions are prescribed functions of displacement and time :

$$t_j = \frac{\partial F}{\partial u_j} = \bar{t}_j (u,t) \qquad \text{on } \partial_2 R \qquad (12).$$

An important special case is that of a linear function

$$F (u,t) = u_j \bar{t}_j (t) \qquad (13)$$

in which the natural boundary conditions reduce to

$$t_j = \bar{t}_j (t) \qquad \text{on } \partial_2 R \qquad (14).$$

2. Friedrichs transformation. The canonical variational principle.

We follow the way indicated by Friedrichs [1] to transform Hamilton's principle into canonical and, later, dual form. To this purpose we introduce a dislocation potential Δ into the functional (1) that becomes

$$\chi \quad \hat{f} [u,v,\varepsilon,\lambda,\alpha,\mu] = \int_{t_1}^{t_2} (T - V + \Delta) \, dt \qquad (15)$$

The dislocation poetntial consists of three parts :

$$\chi \quad \Delta_1 = \int_R \lambda_{ij} (\varepsilon_{ij} - \frac{1}{2} D_i u_j - \frac{1}{2} D_j u_i) \, dR \qquad (16)$$

that incorporates the constraints (4) as natural variational equations for the stationarity of (15) by means of a tensor of lagrangian multipliers λ_{ij}. Correspondingly the potential energy V_1 is now expressed as

$$V_1 = \int_R \mathcal{N} (\varepsilon) \, dR \qquad (3').$$

A second part $\Delta_2 = \int_{\partial_1 R} \alpha_j \{ u_j - \bar{u}_j (t) \} dS$ (17)

incorporates the boundary constraints (6) as natural boundary conditions through the vector of lagrangian multipliers α_j. Finally

$$\Delta_3 = \int_R \mu_j (\dot{u}_j - v_j) dR \tag{18}$$

introduces the velocity vector field v_j as an independent field and the kinetic energy is now expressed as

$$T = \int_R \frac{1}{2} \rho v_j v_j dR \tag{2'}$$

The dislocation potentials Δ_1 and Δ_2 have been used in the static problem [2,3] to provide a logical approach to the two-field variational principles, one of which is the Reissner-Hellinger principle. The introduction of the dislocation potential Δ_3 is an essential step in the logical transformation from Hamilton's principle to the eulerian variational principles of fluid mechanics[4].

The first step in the simplification of the functional (15) consists in identifying the multipliers λ_{ij} and μ_j. This results immediatly from setting the variational derivatives of ε_{ij} and v_j equal to zero :

$$\lambda_{ij} - \frac{\partial W}{\partial \varepsilon_{ij}} = 0 \tag{19}$$

$$\rho v_j - \mu_j = 0 \tag{20}$$

Hence the λ_{ij} tensor is identified with the stress tensor τ_{ij} related to the ε_{ij} by the elastic constitutive equations (9); the μ_j vector is identified with the momentum per unit mass. When (19) is substituted into the functional, the expression

$$\varepsilon_{ij} \frac{\partial W}{\partial \varepsilon_{ij}} - W = \phi (\tau) \tag{21}$$

that appears on the left-hand side is treated as a Legendre transformation introducing the complementary energy density ϕ, a function of the elements of the stress tensor. This transformation is possible whenever the strains can be expressed in terms of the stresses, in which case the constitutive equations (9) are equivalent to

$$\varepsilon_{ij} = \frac{\partial \phi}{\partial \tau_{ij}} \tag{22}$$

Thus one obtains

$$\Delta_1 - V_1 = U_1 - \Lambda_1$$

where $$U_1 = \int_R \Phi(\tau)\, dR \qquad (23)$$

the complementary strain energy, and

$$\Lambda_1 = \int_R \tau_{ij}\, D_i\, u_j\, dR \qquad (24)$$

where account was taken of the symmetry of the stress tensor. Also the substitution of (20) yields

$$T + \Delta_3 = - T + \Lambda_3$$

where $$\Lambda_3 = \int_R \rho\, v_j\, \dot{u}_j\, dR \qquad (25)$$

The last set of multipliers is identified through the natural boundary conditions stemming from variations on the displacements on $\partial_1 R$

$$\alpha_j - n_i\, \tau_{ij} = 0 \qquad \text{on } \partial_1 R \qquad (26)$$

This is simply used to transform (17) into

$$\Delta_2 = \int_{\partial_1 R} n_i\, \tau_{ij}\, \{ u_j - \bar{u}_j(t) \}\, dS \qquad (17')$$

The final result is a canonical variational principle in the Friedrich sense, requiring the stationarity of the functional

$$c[u,v,\tau] = \int_{t_1}^{t_2} (U_1 - T - V_2 + \Delta_2 - \Lambda_1 + \Lambda_3)\, dt \qquad (27)$$

in which the various integrands are given respectively by (23), (2'), (5), (17'), (24) and (25). It can be used to develop approximate solutions based on independent discretizations of displacements, velocities and stresses. Its variational equations are respectively

$$D_i\, \tau_{ij} - \rho\, \dot{v}_j = 0 \qquad (23)$$

for the variations on displacements (the dynamic equilibrium equations);

$$\rho(-v_j + \dot{u}_j) = 0 \qquad (29)$$

for the variations on velocities, restoring the constraints between velocity and displacement field;

$$\frac{\partial \Phi}{\partial \tau_{ij}} - \frac{1}{2}(D_i\, u_j + D_j\, u_i) = 0 \qquad (30)$$

for the variations on stresses, restoring the constraints between stresses and displacements. All boundary conditions are natural. On $\partial_1 R$ they follow from the variations on the surface tractions (11) and correspond to (6); on $\partial_2 R$ they follow from the variation of displacements and correspond to (12).
For finite element applications it is important to recognize the nature of the transition conditions at interfaces. Because the only space derivatives contained in the functional are those affecting displacements, we must consider that physical continuity

$$(u_j)_+ = (u_j)_- \tag{31}$$

is an a priori requirement; the natural transition conditions provided by the principle and stemming from the common variations $(\delta u_j)_+ = (\delta u_j)_-$ at interfaces are then

$$(t_j)_+ + (t_j)_- = 0 \tag{32}.$$

From that stand point the canonical principle (27) behaves as the classical principle of variation of displacements. Obviously the interface constraints (31) could be incorporated into the principle through an additional dislocation potential and the corresponding multipliers identified. Then, just as was found to be the case for the boundary conditions, all transitional conditions can be cared for by the principle.

3. The complementary energy principle of elastodynamics

It can be deduced from the canonical principle by specializations. To this effect the functional (27) is integrated by parts with respect to the displacements in both the terms Λ_1 and Λ_3 of the integrand

$$\Lambda_1 = \int_{\partial R} n_i \tau_{ij} u_j \, dS - \int_R u_j D_i \tau_{ij} \, dR$$

$$\int_{t_1}^{t_2} \Lambda_3 \, dt = \left| \int_R \rho v_j u_j \, dR \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_R \rho \dot{v}_j u_j \, dR \, dt$$

This operation cancels part of the contribution of Δ_2. Furthermore the term at the time limits is dropped on the basis of the requirements (7) and introduction of the new requirements

$$\delta v_j \equiv 0 \qquad \text{for} \quad t = t_1 \quad \text{and} \quad t = t_2 \tag{33}$$

Thus, for the following portion of the canonical functional,

$$\int_{t_1}^{t_2} (-V_2 + \Delta_2 - \Lambda_1 + \Lambda_3) dt = \int_{t_1}^{t_2} \int_R u_j (D_i \tau_{ij} - \rho \dot{v}_j) dR \, dt$$

$$- \int_{t_1}^{t_2} \int_{\partial_1 R} n_i \tau_{ij} \bar{u}_j (t) \, dS \, dt + \int_{t_1}^{t_2} \int_{\partial_2 R} \{F(u,t) - n_i t_{ij} u_j\} dS dt \tag{34}$$

The left-hand side is now simplified by making the assumption that the dynamic equilibrium equations (28) can be satisfied a priori; this causes the first term to vanish. In view of (11) and (12), the integrand of the last term is obviously related to another Legendre transformation

$$u_j \frac{\partial F}{\partial u_j} - F(u,t) = G(t_j, t) \tag{35}$$

provided equations (12) can be solved for the boundary displacements. This is the case in principle whenever the Hessian

$$\left| \frac{\partial^2 F}{\partial u_i \, \partial u_j} \right|$$ does not vanish. The complementary function G is

then such that

$$\frac{\partial G}{\partial t_j} = u_j \tag{36}$$

With the notations

$$B_1 = \int_{\partial_1 R} t_j \bar{u}_j (t) \, dS \qquad B_2 = \int_{\partial_2 R} G(t_j, t) \, dS \tag{37}$$

the canonical functional is now reduced to one that depends only on the stress field

$$g[\tau] = \int_{t_1}^{t_2} (U_1 - T - B_1 - B_2) \, dt \tag{38}$$

provided the kinetic energy can be expressed in terms of it.
The dynamical equilibrium equations (28) that must be satisfied a priori allow precisely to express the velocity field in terms of the stresses :

$$v_j = v_j \Big|_{t_1} + \frac{1}{\rho} D_i \int_{t_1}^{t} \tau_{ij} \, dt \tag{39}$$

This suggest the introduction of an impulse field [5] T_{ij} such that

$$v_j = \frac{1}{\rho} D_i T_{ij} \tag{40},$$

hence

$$T_{ij} = T_{ij} \Big|_{t_1} + \int_{t_1}^{t} \tau_{ij} \, dt \tag{41}$$

and $\dot{T}_{ij} = \tau_{ij}$ (42).

The requirement (33) can now obviously be replaced by

$$\delta T_{ij} \equiv 0 \qquad \text{for} \qquad t = t_1 \quad \text{and} \quad t = t_2 \qquad (43)$$

A convenient formulation of the complementary energy principle of elastodynamics is thus the stationarity of the functional

$$g\,[\,T_{ij}\,] = \int_{t_1}^{t_2} \{ \int_R (\phi(\dot{T}_{ij}) - \frac{1}{2\rho} D_i\, T_{ij}\, D_m\, T_{mj}) dR$$

$$- \int_{\partial_1 R} \dot{T}_j\, \bar{u}_j\,(t)\, dS - \int_{\partial_2 R} G\,(\dot{T}_j, t)\, dS \,\}\, dt \qquad (44)$$

where $T_j = n_i\, T_{ij}$ $\qquad$ $t_j = \dot{T}_j$ (45)

The variational derivatives of this principle are

$$-\frac{d}{dt}\,\frac{\partial \phi}{\partial \dot{T}_{ij}} + \frac{1}{2}\,(D_i\, D_m\, T_{mj} + D_j\, D_m\, T_{mi}) = 0 \qquad (46)$$

that, in view of (42), (22) and (40), are easily interpreted as the time derivatives of compatibility conditions :

$$-\dot{\epsilon}_{ij} + \frac{1}{2}\,(D_i\, v_j + D_j\, v_i) = 0$$

The natural boundary conditions are seen to be

$$-\frac{1}{\rho} D_i\, T_{ij} + \frac{d}{dt}\,\bar{u}_j\,(t) = 0 \qquad \text{on } \partial_1 R \qquad (47)$$

$$-\frac{1}{\rho} D_i\, T_{ij} + \frac{d}{dt}\,\frac{\partial G}{\partial \dot{T}_j} = 0 \qquad \text{on } \partial_2 R \qquad (48)$$

Both are easily interpreted through (40) and (36).
In the special case (13), the Legendre transformation (35) yields a function G that is identically zero; the boundary conditions (14) become a priori boundary conditions in the complementary energy formulation, thus

$$\dot{T}_j = \bar{t}_j\,(t) \qquad \text{on } \partial_2 R \qquad (14').$$

Similarly, because the displacement field vanished completely, the transitional conditions at interfaces (32) have now to be satisfied a priori.

4. Finite element discretization using the complementary energy principle

The approach to discretization by finite elements can follow a very similar path to that used in statics [6]. At the level of a single finite element it is convenient to consider boundary conditions to consist only on imposed displacements.
The tensorial impulse field is discretized in the same manner as the stress field was in statics, namely

$$\tau(x,t) = R(x)\, c(t) + S(x)\, s(t) \tag{49}$$

where $\tau(x,t)$ here denotes a column matrix of the elements of the impulse tensor, ranged in the same order as the stresses in reference 6, and where the parameters ranged in the column matrices c and s are now functions of time. As was assumed in reference 6, the approximating functions in S(x) are such that

$$D'S(x) = 0$$

where D' is the differential operator generating the quantities $D_i\, T_{ij}$.
Thus the s(t) coordinates do not contribute to the kinetic energy.
On the other hand each column of

$$D'R(x) = U(x)$$

furnishes the components of an inertia force distribution accounted for in the discretization. It should at least contain the inertia force distributions related to the motions of the finite element as a rigid body. The discretization (49) furnishes, when used in the evaluation of the kinetic energy, a consistent "inverse mass" or "mobility matrix" N :

$$\int_E \frac{1}{2\rho} D_i\, T_{ij}\, D_m\, T_{mj}\, dE = \frac{1}{2}\int_E \frac{1}{\rho}\{ D'R(x)c\}'\{D'R(x)c\}\, dE = \frac{1}{2}\, c'N\, c$$

(50)

$$N = \int_E \frac{1}{\rho}\, U'\, U\, dE \qquad \text{positive definite} \tag{51}$$

It also furnishes a stress energy

$$\int_E \phi(\dot{T}_{ij})\, dE = \frac{1}{2}\int_E \dot{\tau}'\, H^{-1}\, \dot{\tau}\, dE$$

$$= \frac{1}{2}\dot{c}'\, F_{cc}\, \dot{c} + \dot{c}'\, F_{cs}\, \dot{s} + \frac{1}{2}\dot{s}'\, F_{ss}\, \dot{s} \tag{52}$$

with the same flexibility matrices that were already defined in reference 6.
Again, generalized boundary impulses can be defined by examination of the distributions N'R(x) and N'S(x) of the T_j impulses appearing along each boundary of the element; and they are finally expressible in terms of the parameters of the impulse field through the same load connection matrices G and C

$$g = G\, c + C s \tag{53}$$

The generalized boundary displacements q^* conjugate to the generalized body loads $\dot{g}$ are defined exactly as in section 4.2 of reference 6.
Thus the discretization of (44), assuming imposed displacements along the whole boundary, leads to a requirement of stationarity of

$$\int_{t_1}^{t_2} \{ \frac{1}{2} \dot{c}' F_{cc} \dot{c} + \dot{c}' F_{cs} \dot{s} + \frac{1}{2} \dot{s}' F_{ss} \dot{s} - \frac{1}{2} c' N c - (\dot{c}' G' + \dot{s}' C') q^* \} dt$$

with δc, δs vanishing at the time limits. Hence

$$- F_{cc} \ddot{c} - F_{cs} \ddot{s} - N c + G' \dot{q}^* = 0 \qquad (54)$$

$$- F_{sc} \ddot{c} - F_{ss} \ddot{s} + C' \dot{q}^* = 0 \qquad (55)$$

When the s coordinates are taken from the second set of equations and substituted into the first, there comes

$$F \ddot{c} + N c = (G' - F_{cs} F_{ss}^{-1} C') \dot{q}^* \qquad (56)$$

with
$$F = F_{cc} - F_{cs} F_{ss}^{-1} F_{sc} \qquad (57)$$

Equations (56) can be regarded as determining the dynamics of the impulse coordinates to which an inertia is attached as excited by prescribed boundary displacements.
The finite elements are now assembled with the same procedure as in statics, allocating each generalized boundary displacement of an element E to a set w^* of nodal displacements

$$q_E^* = L_E w^* \qquad (58)$$

with the help of Boolean matrices. Then, for each finite element, we write the set of equations (56) in the form :

$$F_E \ddot{c}_E + N_E c_E = J'_E \dot{w}^* \qquad (E = 1, 2, \ldots N) \qquad (59)$$

with
$$J'_E = (G' - F_{cs} F_{ss}^{-1} C')_E L_E \qquad (60)$$

There are as many equations in the set (59) as there are coordinates in the set of all c_E. We refer to them as the "internal coordinates" and note that their dynamics are described by (59), provided we consider them excited by known time histories of the nodal displacements. Of course we need a complementary set of dynamic equations to obtain those time histories. They are obtained by using the property of internal connexion loads $\dot{g}_E$ to be in equilibrium with the external loads applied at interfaces or at the boundaries of the assembled structure. Thus, by reference to equation (105) of reference 6,

$$\sum_E L'_E \dot{g}_E = \dot{y}_{(w)} \qquad (61)$$

where $y_{(w)}$ denotes the external impulse vector conjugate to the nodal displacements. The connexion loads, or rather their time derivatives, can be obtained in terms of the internal coordinates and nodal velocities by differentiating (53) twice with respect to time and substituting $\ddot{s}$ from (55) again. Whence, for each element

$$\ddot{g}_E = (G - C\, F_{ss}^{-1}\, F_{sc})_E\, \ddot{c}_E + K_E\, L_E\, \dot{w}^* \tag{62}$$

$$K_E = (C\, F_{ss}^{-1}\, C')_E \tag{63}$$

We recognize in K_E the stiffness matrix of the equilibrium element as used in statics. Differentiating (61) with respect to time and substituting (62), we obtain the complementary set of dynamic equations

$$\sum_E J_E\, \ddot{c}_E + K\, \dot{w}^* = \ddot{y}_{(w)} \tag{64}$$

where in

$$K = \sum_E L'_E\, K_E\, L_E \tag{65}$$

we recognize the master stiffness matrix of the assembled structure. The equations (59) and (63) constitute a complete set of dynamic equations for the unknowns. There is no difficulty in taking into account lumped inertias associated to the nodal displacements, that is applied either at interfaces or along the structural boundary, we merely have to set up the corresponding consistent mass matrix M and write

$$\dot{y}_{(w)} = -\, M\, \ddot{w}^* + f_{(w)}\,(t) \tag{66}$$

modifying equation (64) that becomes now

$$\sum_E J_E\, \ddot{c}_E + M\, \ddot{w}^* + K\, \dot{w}^* = f_{(w)}\,(t) \tag{67}$$

The real excitation of the system takes place either from elements of $f_{(w)}\,(t)$ as prescribed external forces, or from elements of w^* as prescribed displacements in which case the conjugate elements in $f_{(w)}$ become unknown reactions.

5. Free vibrations

Consider the problem of a modal analysis of the free vibrations. If the structure is floating, that is not subjected to homogeneous displacement boundary conditions, we can simply set $f_{(w)} = 0$ and consider all c_E and w^* elements as unknowns. Setting

$$c_E = \hat{c}_E \sin \omega t \qquad w^* = \hat{w}^* \cos \omega t$$

the eigenvalue problem is given in the form

$$(K_E - \omega^2 F_E)\hat{c}_E + \omega\, J'_E\, \hat{w}^* = 0 \qquad E = 1, 2 \ldots N \tag{68}$$

$$(K - \omega^2 M)\hat{w}^* + \omega \sum_E J_E\, \hat{c}_E = 0 \tag{69}$$

or, with a modal amplitude vector

$$x' = (\hat{c}'_1\ \hat{c}'_2\ \dots\ \hat{c}'_N\ \hat{w}^{*\prime})$$

$$\begin{bmatrix} N_1 - \omega^2 F_1 & 0 & 0 & 0 & \omega J'_1 \\ 0 & N_2 - \omega^2 F_2 & 0 & 0 & \omega J'_2 \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & N_N - \omega^2 F_N & \omega J'_N \\ \omega J_1 & \omega J_2 & \cdots & \omega J_N & K - \omega^2 M \end{bmatrix} x = 0 \qquad (70)$$

In many applications there are no lumped inertias attached to the nodal points and consequently no matrix M. This makes it possible to reduce drastically the size of the eigenvalue problem by a static condensation of most or all of the nodal displacements, leaving essentially the few internal coordinates as unknowns. In the case of the floating structure this means that we should use

$$K\,\hat{w}^* = -\omega \sum_E J_E\,\hat{c}_E \qquad (71)$$

to eliminate the nodal amplitudes. Because K is singular on account of the presence of structural rigid body modes or mechanisms, it will be necessary to keep enough displacement coordinates to represent those. This is in general achieved by suppressing just enough rows and corresponding columns in K to obtain a non singular sub matrix. Let P and R denote column selection operators of same size as K

$$P + R = I \qquad \text{identity matrix,}$$

and decompose $\hat{w}^*$ into

$$\hat{w}^* = P\,p + R\,r$$

Then, with the notations

$$P'K\,P = K_{pp} \qquad P'K\,R = K_{pr} \qquad R'K\,P = K_{rp} \qquad R'K\,R = K_{rr}$$

equation (71) becomes after premultiplication by P' and R' respectively

$$K_{pp}\,p + K_{pr}\,r = -\omega\,P' \sum_E J_E\,\hat{c}_E \qquad (72)$$

$$K_{rp}\,p + K_{rr}\,r = -\omega\,R' \sum_E J_E\,\hat{c}_E \qquad (73)$$

Let K_{pp} be non singular, r representing kinematical freedoms of the assembled structure; p can be extracted from (72) and substituted into (68) and (73) to produce an eigenvalue problem in the unknowns $\hat{c}_E$ and r only. Clearly similar procedures apply to structures that are isostatically or hyperstatically grounded

by setting some of the elements of $\hat{w}$ equal to zero. However elegant, the formulation (68) does not lend itself to the classical algorithms for eigenvalue search in many degrees of freedom. The reason is of course the use of mixed displacement and stress type variables tnat induces a special coupling through the J_E matrices. It might prove interesting to generalize the iterative type algorithms to this type of formulation; but it is also possible, as shown by GERADIN [7], to obtain a classical formulation by a change of variables.

6. Classical form of the eigenvalue problem

The change of variables consists in setting

$$- N c = \dot{p}^* \tag{74}$$

and, in view of (54), consider p^* as a displacement vector conjugate to c. Equations (54) and (55) are then brought together in the form

$$\hat{F} \begin{pmatrix} \ddot{c} \\ \ddot{s} \end{pmatrix} = \hat{G}' \begin{pmatrix} \dot{q}^* \\ \dot{p}^* \end{pmatrix} \tag{75}$$

with extended flexibility and load connexion matrices

$$\hat{F} = \begin{pmatrix} F_{cc} & F_{cs} \\ F_{sc} & F_{ss} \end{pmatrix} \qquad \hat{G} = \begin{pmatrix} G & C \\ I & 0 \end{pmatrix} \tag{76}$$

Equation (75) is solved for the $\ddot{c}$ and $\ddot{s}$ and substituted into

$$\hat{G} \begin{pmatrix} \ddot{c} \\ \ddot{s} \end{pmatrix} = \hat{G} \hat{F}^{-1} \hat{G}' \begin{pmatrix} \dot{q}^* \\ \dot{p}^* \end{pmatrix} \tag{77}$$

However, using (53), the left-hand side is also equal to

$$\begin{pmatrix} G \ddot{c} + C \ddot{s} \\ \ddot{c} \end{pmatrix} = \begin{pmatrix} \ddot{g} \\ \ddot{c} \end{pmatrix} \tag{78}$$

Consequently, partitionning the extended stiffness matrix

$$\hat{G} \hat{F}^{-1} \hat{G}' = \begin{pmatrix} \hat{K}_{qq} & \hat{K}_{qp} \\ \hat{K}_{pq} & \hat{K}_{pp} \end{pmatrix} \tag{79}$$

we obtain

$$\ddot{g} = \hat{K}_{qq} \dot{q}^* + \hat{K}_{qp} \dot{p}^* \tag{80}$$

$$\ddot{c} = \hat{K}_{pq} \dot{q}^* + K_{pp} \dot{p}^* \tag{81}$$

These results are applicable to each finite element. Thus, using (58) and (74) again in (81), we have a first set of dynamic equations

$$- N_E^{-1} \dot{p}_E^* = (\hat{K}_{pq})_E L_E \dot{w}^* + (\hat{K}_{pp})_E \dot{p}_E^* \qquad (E = 1, 2 \ldots N) \quad (82)$$

Also, substituting the equations of type (80) into the time derivative of (61) and using (58) and (66)

$$\sum_E L'_E (\hat{K}_{qq})_E L_E \dot{w}^* + \sum_E L'_E (\hat{K}_{qp})_E \dot{p}_E^* = - M \ddot{w} + \dot{f}_{(w)} \quad (83)$$

Equations (82) and (83) form again a complete dynamical system for the variables w^* and p_E^*.

The corresponding eigenvalue problem has now the classical form

$$(\mathring{K} - \omega^2 \mathring{M})\, \mathring{x} = 0 \quad (84)$$

with as modal column the amplitudes of the p_E^* and w^*, as stiffness matrix

$$\mathring{K} = \begin{bmatrix} (\hat{K}_{pp})_1 & 0 & 0 & 0 & (\hat{K}_{pq})_1 L_1 \\ 0 & (\hat{K}_{pp})_2 & 0 & 0 & (\hat{K}_{pq})_2 L_2 \\ 0 & 0 & \ddots & 0 & \ldots \\ 0 & 0 & 0 & (\hat{K}_{pp})_N & (\hat{K}_{pq})_N L_N \\ L'_1 (\hat{K}_{qp})_1 & L'_2 (\hat{K}_{qp})_2 & \ldots & L'_N (\hat{K}_{qp})_N & \sum_E L'_E (\hat{K}_{qq})_E L_E \end{bmatrix} \quad (85)$$

and as mass matrix

$$\mathring{M} = \begin{bmatrix} N_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & N_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & N_N^{-1} & 0 \\ 0 & 0 & 0 & & M \end{bmatrix} \quad (86)$$

Again, if M disappears, static condensation of the nodal displacements becomes possible.

7. The Rayleigh quotient

Turning back to the dual variational principle for a continuum, the results obtained in section 3 show that the eigenvalue problem can be stated in the form of the stationarity of the following Rayleigh quotient

$$\omega^2 = \frac{\int_R \frac{1}{2\rho} D_i A_{ij} D_m A_{mj} \, dR}{\int_R \phi (A_{ij}) \, dR} \tag{87}$$

where A_{mj} is either the tensor of impulse amplitudes or more simply that of stress amplitudes.

Moreover $$n_i A_{ij} = 0 \qquad \text{on } \partial_2 R \tag{88}$$

are stress type boundary conditions to be satisfied a priori, while the complementary boundary conditions, of kinematical type,

$$D_i A_{ij} = 0 \qquad \text{on } \partial_1 R \tag{89}$$

are natural; they are taken care of by the principle itself.
This is the converse of the displacement formulation, where the kinematical conditions are those to be satisfied ab initio. Another contrast is the nature of the zero frequency modes; they are obviously given by stress distributions in static equilibrium with themselves or self-stressings obeying

$$D_i A_{ij} = 0 \qquad \text{in } R \qquad\qquad n_i A_{ij} = 0 \qquad \text{on } \partial_2 R$$

For a continuum their number is infinite, while in the displacement formulation the zero frequency modes are related to a finite number of rigid body or kinematical degrees of freedom. Otherwise the Rayleigh quotient (87) has characteristics very similar to those of the displacement type quotient

$$\omega^2 = \frac{\int_R W (D u) \, dR}{\int_R \frac{1}{2} \rho u_i u_i \, dR} \tag{90}$$

In both cases the functional in the numerator is constructed with space derivatives of the variables involved in the functional of the denominator. All minimax properties of (90), as established by Courant [1], can be translated to (87), as shown for instance by Tabarrok [8].
Thus in theory (87) will furnish an upper bound to the eigenvalue ω_1^2, provided $(A_{ij})_1$ is orthogonal to all the self-stressing modes of non zero frequency.
It is of interest to note that it is quite possible to construct all the stress distributions that are orthogonal to the self-stressing modes. Orthogonality means that

$$\int_R A_{ij} \hat{\varepsilon}_{ij} \, dR = 0 \tag{91}$$

where A_{ij} is any self-stressing and $\hat{\varepsilon}_{ij}$ is the distribution of strains associated with the orthogonal stress distribution through the constitutive equations. A sufficient condition to implement (91) is to make the strain distribution compatible :

$$\hat{\varepsilon}_{ij} = \frac{1}{2} (D_i u_j + D_j u_i) \quad \text{in } R \qquad u_i = 0 \quad \text{on } \partial_1 R \qquad (92)$$

Indeed one obtains then

$$\int_R A_{ij} \hat{\varepsilon}_{ij} \, dR = \int_R A_{ij} D_i u_j \, dR = \int_{\partial R} n_i A_{ij} u_j \, dS - \int_R u_j D_i A_{ij} dR$$

and the right-hand side vanishes in view of (88), (89) and (92).
The condition is also necessary as can be shown by satisfying (89) with the help of stress functions, integrating (91) by parts and using the arbitrariness in the stress functions to show that the strains have to satisfy the integrability conditions for the existence of displacements. Finally the most general stress distribution orthogonal to all the self-stressing states is given by

$$\hat{A}_{ij} = C_{ij}^{pq} D_p u_q \qquad (93)$$

where the u_q are arbitrary save for the boundary condition

$$u_q = 0 \qquad \text{on } \partial_1 R \qquad (94)$$

But then, the Rayleigh quotient (87) becomes expressible entirely in terms of the auxiliary displacement field u_q

$$\omega^2 = \frac{\int_R \frac{1}{2\rho} C_{ij}^{pq} C_{mj}^{rs} D_i D_p u_q D_m D_r u_s \, dR}{\int_R W(Du) \, dR} \qquad (95)$$

This new Rayleigh quotient, that leads immediatly to the non zero eigenvalues of the problem, has the same kinematical boundary conditions to be satisfied ab initio as (90) but in addition the transformed boundary conditions (88)

$$n_i C_{ij}^{pq} D_p u_q = 0 \qquad \text{on } \partial_2 R \qquad (96)$$

There is also an interesting inequality relating (90) and (95).
To prepare its proof consider the following obvious in equality

$$\int_R \left(\sqrt{\rho} a_j - \frac{\lambda b_j}{\sqrt{\rho}} \right) \left(\sqrt{\rho} a_j - \frac{\lambda b_j}{\sqrt{\rho}} \right) dR \geq 0$$

valid for any value of the scalar λ; expanding it

$$\int_R \rho a_j a_j \, dR - 2\lambda \int_R a_j b_j \, dR + \lambda^2 \int_R \frac{1}{\rho} b_j b_j \, dR \geq 0$$

The minimum of the left-hand side is reached for

$$\lambda = \int_R a_j \, b_j \, dR \Big/ \int_R \frac{1}{\rho} \, b_j \, b_j \, dR$$

hence, after substitution

$$\int_R \frac{1}{2} \rho \, a_j \, a_j \, dR - \{ \int_R \frac{1}{2} a_j \, b_j \, dR \}^2 \Big/ \int_R \frac{1}{2\rho} \, b_j \, b_j \, dR \geq 0 \qquad (97)$$

Setting $a_j = u_j$ and $b_j = C_{ij}^{pq} \, D_i \, D_p \, u_q$

we already identify two of the integrals as the denominator of (90) and the numerator of (96) respectively. For the third integral

$$\int_R \frac{1}{2} a_j \, b_j \, dR = \int_R \frac{1}{2} u_j \, C_{ij}^{pq} \, D_i \, D_p \, u_q \, dR$$

$$= \int_{\partial R} \frac{1}{2} u_j \, n_i \, C_{ij}^{pq} \, D_p \, u_q \, dS - \int_R \frac{1}{2} C_{ij}^{pq} \, D_i \, u_j \, D_p \, u_q \, dR$$

$$= - \int_R W \, (D \, u) \, dR < 0$$

since the surface integral vanishes on account of (94) and (96).
Manipulating the inequality (97), there comes

$$\omega_R^2 \leq \omega_R^{2'} \qquad (98)$$

where ω_R^2 denotes the classical Rayleigh quotient (90) and $\omega_R^{2'}$, the quotient (95). This means that whenever a displacement field is chosen that satisfies both the kinematical and the stress boundary conditions, the classical Rayleigh quotient is always a better approximation than the one derived from the stress approach with orthogonality to all zero frequency modes. However, as experience shows, direct application of the quotient (87) is beneficial in that the frequencies are usually underestimated and converge through lower bounds when the degrees of freedom are increased [7], [8]. This is of course due to the fact that the assumed stress modes do not satisfy orthogonality with respect to all the self-stressing modes. Unfortunately there is no guarantee of this property of lower boundedness and further theoretical research is necessary to be able to incorporate this into the formalism.

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