

DUAL ANALYSIS FOR HEAT CONDUCTION PROBLEMS BY FINITE ELEMENTS

B. M. FRAEIJIS DE VEUBEKE† AND M. A. HOGGE‡

Aerospace Engineering Laboratory, University of Liège, Liège, Belgium

SUMMARY

An alternative approach to the usual finite element treatment of steady-state temperature problems is presented, using approximations for the field of the dual variables.

The appropriate extremum principle is established and its minimization is discussed in connection with a plane triangular finite element process. Original heat flow elements are derived: in conjunction with temperature elements, they enable dual analysis of a given structure and an important estimate of the convergence to the true solution by upper and lower bounds to the dissipation function, as illustrated by means of several examples.

INTRODUCTION

Steady-state heat conduction is one of the many field problems of engineering that can receive a variational formulation. The basic formulation¹ is in terms of the temperature field alone and the discretization in finite elements makes use of parametric approximations of the temperature within each element with continuity secured across the interfaces. In the present paper an alternative formulation is presented in terms of the dual (vector) variable: the heat flow. In the corresponding finite element models the heat flow satisfies thermal equilibrium within each element and flux continuity is maintained at the interfaces. In this manner a dual analysis of a steady-state heat conduction problem extends to the benefit of the dissipation function a numerical estimate of the convergence by upper bounds (temperature models) and lower bounds (heat flow models) when no heat sources are prescribed. When the body is in contact with a uniform temperature bath taken as zero level, the role of bounds is reversed. The general case where both sources and non-uniform outside temperatures prevail can be treated by superposition if convergence estimates are required.

BASIC TEMPERATURE FORMULATION

The temperature functional in a simply connected domain D

$$I(T) = \int_D (\frac{1}{2} k_{ij} \partial_i T \partial_j T - \bar{Q} T) dD + \int_{\partial_1 D} \bar{q} T dS + \int_{\partial_2 D} \frac{1}{2} h (T - \bar{T}_e)^2 dS \quad (1)$$

where $\partial = \partial_1 D + \partial_2 D + \partial_3 D$, denotes its boundary, $\partial_i T$ the temperature gradient vector, k_{ij} a symmetrical heat conductivity tensor, \bar{Q} a prescribed volume heat source distribution, \bar{q} a prescribed surface heat sink, h a convection coefficient and \bar{T}_e the local prescribed external temperature, is stationary under small perturbations δT of the temperature field subject to the boundary conditions

$$T = \bar{T}_e, \quad \delta T = 0 \quad \text{on } \partial_1 D \quad (2)$$

† Professor of Continuum Mechanics.

‡ Research Assistant.

when the Euler equation

$$\partial_i(k_{ij}\partial_j T) + \bar{Q} = 0 \quad (3)$$

and the natural boundary conditions (n_i direction cosines of the outward normal)

$$n_i k_{ij}\partial_j T + \bar{q} = 0 \quad \text{on } \partial_2 D \quad (4)$$

$$n_i k_{ij}\partial_j T + h(T - \bar{T}_e) = 0 \quad \text{on } \partial_3 D \quad (5)$$

are satisfied. The Euler equation is a statement of thermal equilibrium for the unit volume; under Fourier's law for the heat flow vector

$$q_i = -k_{ij}\partial_j T \quad (6)$$

it becomes

$$-\partial_i q_i + \bar{Q} = 0 \quad (7)$$

Three types of boundary conditions are distinguished. The surface heat sink condition (4) on part $\partial_2 D$ of the boundary, that can also be written

$$-n_i q_i + \bar{q} = 0 \quad \text{on } \partial_2 D \quad (8)$$

the convection loss condition (5) or

$$-n_i q_i + h(T - \bar{T}_e) = 0 \quad \text{on } \partial_3 D \quad (9)$$

and the prescribed surface temperature condition (2) that can be regarded as a limiting case $h \rightarrow \infty$ of equation (9).

The functional of this 'principle of variation of temperatures' is the sum of a 'dissipation functional'

$$F(T) = \int_D \frac{1}{2} k_{ij} \partial_i T \partial_j T dD + \int_{\partial_2 D} \frac{1}{2} h (T - \bar{T}_e)^2 dS \quad (10)$$

of the whole system body plus surrounding,² i.e. including the boundary heat convection effect, and a 'potential energy of prescribed thermal loads'

$$P(T) = - \int_D \bar{Q} T dD + \int_{\partial_2 D} \bar{q} T dS \quad (11)$$

EXTENSION BY RECIPROCAL TRANSFORMATION

As was done in the case of elasticity theory³ the basic formulation will now be extended through a reciprocal transformation.⁴ Fourier's law is used to eliminate the temperature gradients in the dissipation function that becomes

$$F(\vec{q}, T) = \int_D \frac{1}{2} r_{im} q_i q_m dD + \int_{\partial_2 D} \frac{1}{2} h (T - \bar{T}_e)^2 dS \quad (12)$$

The heat resistivity tensor r_{im} is reciprocal to the heat conductivity one

$$r_{im} k_{ij} = \delta_{mj} \quad (13)$$

so that Fourier's law, solved for the temperature gradients, reads

$$r_{im} q_i = -r_{im} k_{ij} \partial_j T = -\partial_m T$$

Considered as differential constraints

$$\partial_m T + r_{im} q_i = 0 \quad (14)$$

between the temperature and heat flow fields those relations are incorporated into a 'dislocation functional'

involving Lagrange multipliers λ_m

$$D(\vec{q}, T, \vec{\lambda}, \mu) = - \int_D \lambda_m (\partial_m T + r_{im} q_i) dD + \int_{\partial_1 D} \mu (T - \bar{T}_e) dS \quad (15)$$

the second part of which also removes the necessity of satisfying *a priori* the boundary constraint (2).

The 'three-field' variational principle

$$F(\vec{q}, T) + D(\vec{q}, T, \vec{\lambda}, \mu) + P(T), \quad \text{stationary} \quad (16)$$

has the following variational equations and natural boundary conditions:

$$r_{im}(q_m - \lambda_m) = 0 \quad \text{or} \quad q_m = \lambda_m \quad (17)$$

as variational equation for the heat flow field,

$$\partial_m \lambda_m - \bar{Q} = 0 \quad (18)$$

$$-n_m \lambda_m + \mu = 0 \quad \text{on} \quad \partial_1 D \quad (19)$$

$$-n_m \lambda_m + \bar{q} = 0 \quad \text{on} \quad \partial_2 D \quad (20)$$

$$-n_m \lambda_m + h(T - \bar{T}_e) = 0 \quad \text{on} \quad \partial_3 D \quad (21)$$

as variational equation and natural boundary conditions for the temperature field, and finally (14) and (2) for the variations on the Lagrange multipliers. Equations (17) and (19) identify those multipliers with the heat flow field and its normal component on $\partial_1 D$. When incorporated into the other equations they express all the thermal equilibrium conditions and boundary conditions of the problem. When incorporated *ab initio* into (16) they reduce this principle to the thermal analogue of the Hellinger-Reissner principle of elasticity:

$$- \int_D (q_m \partial_m T + \frac{1}{2} r_{im} q_i q_m + \bar{Q} T) dD + \int_{\partial_1 D} n_m q_m (T - \bar{T}_e) dS + \int_{\partial_2 D} \bar{q} T dS + \int_{\partial_3 D} \frac{1}{2} h (T - \bar{T}_e)^2 dS, \quad \text{stationary} \quad (22)$$

It contains only two independent fields, the temperature and the heat flow. Its corresponding variational equations are respectively (7) and (14), while (8), (9) and (2) all appear as natural boundary conditions. An integration by parts on the first term of (22) puts it into the equivalent form

$$\int_D T (\partial_m q_m - \bar{Q}) dD - \int_D \frac{1}{2} r_{im} q_i q_m dD - \int_{\partial_1 D} n_m q_m \bar{T}_e dS + \int_{\partial_2 D} (\bar{q} - n_m q_m) T dS + \int_{\partial_3 D} \{ \frac{1}{2} h (T - \bar{T}_e)^2 - n_m q_m T \} dS, \quad \text{stationary.} \quad (23)$$

THE DUAL OR HEAT FLOW FUNCTIONAL

The presence of the temperature field can be completely eliminated from the functional (23) by assuming that we satisfy *a priori* the volume heat equilibrium equation (7) and the corresponding surface equation (8), whereby the first and fourth terms disappear.

However, since the last term still contains T on $\partial_3 D$, equation (9) is solved for T and back substituted. This produces, after a change of sign the 'principle of variation of heat flow':

$$J(\vec{q}) = \int_D \frac{1}{2} r_{im} q_i q_m dD + \int_{\partial_1 D} n_m q_m \bar{T}_e dS + \int_{\partial_3 D} \{ (n_m q_m)^2 / 2h + n_m q_m \bar{T}_e \} dS, \quad \text{stationary.} \quad (24)$$

It is the thermal analogue of the complementary energy principle of elasticity. It is again the sum of the

dissipation functional (10) expressed however in terms of the heat flow through Fourier's law and (9)

$$F(\vec{q}) = \int_D \frac{1}{2} r_{im} q_i q_m dD + \int_{\partial_s D} \frac{1}{2h} (n_m q_m)^2 dS \quad (25)$$

and a 'dual or complementary potential energy'

$$Q(\vec{q}) = \int_{\partial_i D + \partial_s D} \bar{T}_e n_m q_m dS \quad (26)$$

The upper and lower bound character to the dissipation functional of approximations based on either

$$I(T) = F(T) + P(T), \quad \text{minimum} \quad (27)$$

or

$$J(\vec{q}) = F(\vec{q}) + Q(\vec{q}), \quad \text{minimum} \quad (28)$$

is briefly discussed in Appendix I.

FINITE ELEMENT ANALYSIS

The temperature models of finite elements have been fully described elsewhere¹ as based on a continuous, piecewise differentiable temperature field with a finite number of degrees of freedom. Note, however, an essential difference in the treatment of the heat convection boundary condition (5). This case is no longer treated by additional generalized fluxes but results in essentially different elements because of the addition of the second term in the dissipation functional (10). This modification is necessary to obtain bounds to this functional. With temperature models, temperature boundary specifications \bar{T}_e can be accounted for exactly if expressible in terms of the approximating shaping functions of the models. Heat flux boundary specifications \bar{q} are translated into equivalent generalized fluxes, and boundary conditions such as (4) are only satisfied on the average. The same is true for thermal equilibrium in volume, equation (3) and there are discontinuities in heat flux transmission between adjacent elements. This behaviour is analogous to the behaviour of approximate stresses in conforming displacement models for structural analysis.

The heat flow models based on the dual principle (28) must, by contrast, satisfy in detail the thermal equilibrium conditions (7) and (8). In general, the finite degree of freedom approximations on the heat flow do not satisfy the integrability conditions for the temperature field required by equations (14). Similarly, temperature boundary conditions such as (2) and (9) are only averaged. This averaging process results automatically from the definitions of generalized temperature co-ordinates as weighted averages along the boundaries of the element or in its interior, so that no restrictive conditions exist on the prescribed temperatures input. This situation is again completely analogous to the one governing the theory of equilibrium models in elasticity.³ Particular emphasis will be given here on the heat flow models and, for brevity, only internally unloaded ($\bar{Q} = 0$) elements will be considered.

HEAT FLOW ELEMENTS GENERATION

Let E denote the simply connected domain occupied by the element and ∂E its boundary. The boundary consists of several parts $\partial_\sigma E$ ($\sigma = 1, 2, \dots, s$) corresponding to various adjacent elements or to pieces of the boundary of the complete structure.

The heat flow vector field in E is taken as a linear superposition of 'modes' $M_{ij}(x)$, each in thermal equilibrium without internal sources:

$$q_i = \alpha_j M_{ij}(x) \quad (29)$$

$$\partial_i M_{ij} = 0 \quad (j = 1, 2, \dots, n) \quad (30)$$

Consider now the normal heat flux generated along part $\partial_\sigma E$ of the boundary by each of the heat flow modes

$$f_j(x) = n_i(x) M_{ij}(x), \quad x \in \partial_\sigma E \quad (31)$$

Determine a maximal set of independent vectors γ_{jm} ($m = 1, 2, \dots, M_\sigma$) such that

$$\gamma_{jm} f_j(x) = 0, \quad x \in \partial_\sigma E \quad (32)$$

Then, any complementary set of vectors η_{jp} ($p = 1, 2, \dots, P_\sigma$; $M_\sigma + P_\sigma = n$) such that the $n \times n$ matrix, j indexing the columns,

$$\mathbf{G}_\sigma = \begin{pmatrix} \gamma_{jm} \\ \eta_{jp} \end{pmatrix}$$

is non-singular, will generate a set of non-zero and independent boundary flux modes along $\partial_\sigma E$

$$\eta_{jp} f_j(x) = b_p(x), \quad x \in \partial_\sigma E \quad (p = 1, 2, \dots, P_\sigma) \quad (33)$$

Introducing the matrix notations

$$\mathbf{f}_\sigma^T(x) = (f_1(x) \dots f_n(x)), \quad x \in \partial_\sigma E$$

$$\mathbf{b}_\sigma^T(x) = (b_1(x) \dots b_{P_\sigma}(x)), \quad x \in \partial_\sigma E$$

(32) and (33) are equivalent to the matrix equation

$$\mathbf{G}_\sigma \mathbf{f}_\sigma(x) = \begin{bmatrix} 0 \\ \mathbf{b}_\sigma(x) \end{bmatrix}, \quad x \in \partial_\sigma E \quad (34)$$

Inversion of this produces

$$\mathbf{f}_\sigma(x) = \mathbf{B}_\sigma^T \mathbf{b}_\sigma(x), \quad x \in \partial_\sigma E \quad (35)$$

where \mathbf{B}_σ^T , which is a part of \mathbf{G}_σ^{-1} , is a $n \times P_\sigma$ matrix of rank P_σ . Thus, along $\partial_\sigma E$, the general parametric heat flow will generate a boundary flux of type

$$\mathbf{a}^T \mathbf{f}_\sigma(x) = (\mathbf{B}_\sigma \mathbf{a})^T \mathbf{b}_\sigma(x), \quad x \in \partial_\sigma E \quad (36)$$

The set of 'generalized flux co-ordinates' for $\partial_\sigma E$ is defined by

$$\mathbf{g}_\sigma = \mathbf{B}_\sigma \mathbf{a} \quad (37)$$

so that (36) becomes equivalent to

$$\alpha_j f_j(x) = g_p b_p(x), \quad x \in \partial_\sigma E \quad (38)$$

The generalized flux co-ordinates are thus intensities of the boundary heat flux modes; their knowledge determines the complete heat flux distribution along $\partial_\sigma E$ so that the continuity of heat fluxes at interfaces can be secured simply by reciprocity of corresponding generalized flux co-ordinates. Any predetermined set of boundary heat flux modes may be replaced by a new set of P_σ linearly independent combinations of the elements of the old set. Advantage can be taken of this freedom to give simple physical significance to the generalized flux co-ordinates, either as local values or moments of various order.

The set of relations (37) for all parts of the boundary ∂E results in a global relation

$$\mathbf{g} = \mathbf{B} \mathbf{a} \quad (39)$$

where the boundary flux co-ordinates are listed in \mathbf{g} in some conventional order and \mathbf{B} correspondingly partitioned. As \mathbf{a} determines \mathbf{g} , we should have conversely that a compatible set \mathbf{g} determines \mathbf{a} uniquely. The condition therefore is that $\mathbf{B} \mathbf{a} = \mathbf{0}$ possesses only the trivial solution $\mathbf{a} = \mathbf{0}$. If this is not true there are 'bubble heat flow modes' that were not taken into consideration. Bubble heat flow modes are linear combinations of the modes $M_{ij}(x)$ that produce no heat flux along any part of the boundary ∂E and are therefore not accounted for by any of the generalized boundary heat flux co-ordinates. This happens only for models involving a large number of parameters; if the $M_{ij}(x)$ form a complete set of polynomials of degree $\leq N$ satisfying (30), the bubble modes begin to appear for $N = 2$. In the examples only $N = 0$ and $N = 1$ have been considered.

It is sufficient to state here that an internal generalized heat flow co-ordinate can be assigned to each bubble mode and relation (39) completed so that \mathbf{B} becomes really of rank n and $\mathbf{B}\mathbf{a} = \mathbf{0}$ has no other than the trivial solution. The internal co-ordinates can be expressed later in terms of the boundary co-ordinates by minimizing the functional (28) at the element level with respect to the internal co-ordinates.

The boundary co-ordinates themselves are not independent: the homogeneous adjoint relation to (39)

$$\mathbf{B}^T \mathbf{z} = \mathbf{0} \quad (40)$$

has at least one non-trivial solution $\mathbf{z} = \mathbf{v}$. The corresponding necessary condition for (39) to be solvable for \mathbf{a}

$$\mathbf{v}^T \mathbf{g} = \mathbf{0} \quad (41)$$

expresses the vanishing of the total outgoing heat flux; it always exists because of the detailed equilibrium conditions (30) and is the analogue of the over-all equilibrium conditions between generalized forces in elasticity. If equation (40) possesses no other non-trivial solution, equation (41) is also sufficient for the existence of \mathbf{a} , the uniqueness of which was already discussed. This is the case for the heat flow models presented as examples in this paper.

If equation (40) has other non-trivial solutions we have a situation analogous to that of 'spurious kinematic freedoms';³ techniques to deal with such situations are similar to those of the equilibrium models in elasticity but will not be presented here.

Consider now the discretization of the functional (28) resulting from (29) or its equivalent matrix form

$$\mathbf{q} = \{q_i\} = \mathbf{M}\mathbf{a} \quad (42)$$

where \mathbf{M} is the matrix of the modal functions $M_{ij}(x)$. The boundary fluxes needed to evaluate the second part of the dissipation functional are given by combination of equations (29), (32) and (36) as

$$n_m q_m = \mathbf{a}^T \mathbf{B}_\sigma^T \mathbf{b}_\sigma(x), \quad x \in \partial_\sigma E \quad (43)$$

The dissipation functional (25) becomes a positive definite quadratic form in the parameters \mathbf{a} ,

$$F(\mathbf{a}) = \frac{1}{2} \mathbf{a}^T (\mathbf{R} + \mathbf{R}_c) \mathbf{a} \quad (44)$$

with generalized resistivity matrices

$$\mathbf{R} = \int_E \mathbf{M}^T \mathbf{C}^{-1} \mathbf{M} dE, \quad \mathbf{C} = (k_{ij}) \quad (45)$$

$$\mathbf{R}_c = \sum_\sigma \mathbf{B}_\sigma^T \left(\int_{\partial_\sigma E} \frac{1}{h} \mathbf{b}_\sigma \mathbf{b}_\sigma^T dS \right) \mathbf{B}_\sigma \quad (46)$$

Advantage is taken of the subdivision of the structure in elements to incorporate eventual heat convection losses at interfaces where bodies of different physical nature are in contact and not only along the external boundary $\partial_3 D$ of the complete domain D . Each element E is therefore provided with a (variable if the case may be) transfer coefficient h along its boundaries ∂E .

At an interface without external heat supply it is readily seen that the effective transfer coefficient is given by

$$\frac{1}{\bar{h}} = \frac{1}{h_+} + \frac{1}{h_-}$$

where h_+ and h_- have been allocated to each face. Moreover an interface can be kept at a prescribed temperature T_e by supplying or removing heat at a surface rate that will result from the computation, or this supply rate will be prescribed and the temperature T_e will result from the computation. In both cases it is necessary to distinguish h_+ and h_- .

For faces without convection losses h can be set equal to infinity. The complementary potential energy (26) is then calculated by assuming the external temperature T_e to be defined along the whole boundary ∂E ; it then becomes formally

$$Q(\vec{q}) = \mathbf{a}^T \sum_{\sigma} \left(\mathbf{B}_{\sigma}^T \int_{\partial_{\sigma} E} T_e \mathbf{b}_{\sigma} dS \right) \quad (47)$$

We now define generalized temperature co-ordinates along each boundary by

$$t_p = \int_{\partial_{\sigma} E} T_e b_p(x) dS, \quad x \in \partial_{\sigma} E \quad (p = 1, 2, \dots, P_{\sigma}) \quad (48)$$

Collected in a column matrix in the same order as for \mathbf{g}_{σ} they yield

$$\mathbf{t}_{\sigma} = \int_{\partial_{\sigma} E} T_e \mathbf{b}_{\sigma} dS \quad (49)$$

so that we can write

$$Q(\vec{q}) = \mathbf{a}^T \sum_{\sigma} \mathbf{B}_{\sigma}^T \mathbf{t}_{\sigma} \quad (50)$$

Again if the co-ordinates of each boundary are collected in a single column matrix \mathbf{t} , in the same order as the \mathbf{g}_{σ} into \mathbf{g} ,

$$\sum_{\sigma} \mathbf{B}_{\sigma}^T \mathbf{t}_{\sigma} = \mathbf{B}^T \mathbf{t} \quad (51)$$

and finally

$$Q(\vec{q}) = \mathbf{a}^T \mathbf{B}^T \mathbf{t} = \mathbf{g}^T \mathbf{t} \quad (52)$$

This shows that the generalized temperature co-ordinates are conjugate variables to the generalized heat fluxes. Definition (48) shows that they are weighted averages of the external temperature distribution, the weighting functions being the boundary heat flux modes.

The minimization of the discretized functional

$$\frac{1}{2} \mathbf{a}^T (\mathbf{R} + \mathbf{R}_c) \mathbf{a} + \mathbf{a}^T \mathbf{B}^T \mathbf{t}$$

with respect to the parameters furnishes

$$(\mathbf{R} + \mathbf{R}_c) \mathbf{a} = -\mathbf{B}^T \mathbf{t} \quad (53a)$$

or

$$\mathbf{a} = -(\mathbf{R} + \mathbf{R}_c)^{-1} \mathbf{B}^T \mathbf{t} \quad (53b)$$

Combined with (39) this result allows to relate the generalized boundary heat fluxes to the generalized temperatures through a generalized conductivity matrix \mathbf{K}

$$\mathbf{g} = \mathbf{K} \mathbf{t} \quad (54)$$

with

$$\mathbf{K} = -\mathbf{B}(\mathbf{R} + \mathbf{R}_c)^{-1} \mathbf{B}^T \quad (55)$$

The complete structural conductivity matrix is built up by properly addressing the elemental matrices \mathbf{K} just as for stiffness matrices in structural analysis. The principle used here is the identification of the interface external temperature co-ordinates, so that corresponding heat flux co-ordinates become either reciprocal or balance external interface heat inputs. It should be observed that the structure (54) of the generalized conductivity matrix is such that the compatibility condition (41) is automatically satisfied by $\mathbf{v}^T \mathbf{B} = 0$.

Appendices II and III present the detailed matrices for constant and linearly varying two-dimensional heat flow elements, with orthotropic conductivity characteristics.

EXAMPLES

The following examples illustrate the main features described above in the case of constant and linear heat flow elements.

Example 1

Let us consider a large composite wall with an isothermal surface at T_0 , the other suffering convection losses (h, T_f). The unidimensional heat flow is analysed through a mesh of twelve linear heat flow elements. In this case the temperature gradient field is itself integrable so that the generalized temperatures obtained yield the local ones (Figure 1).

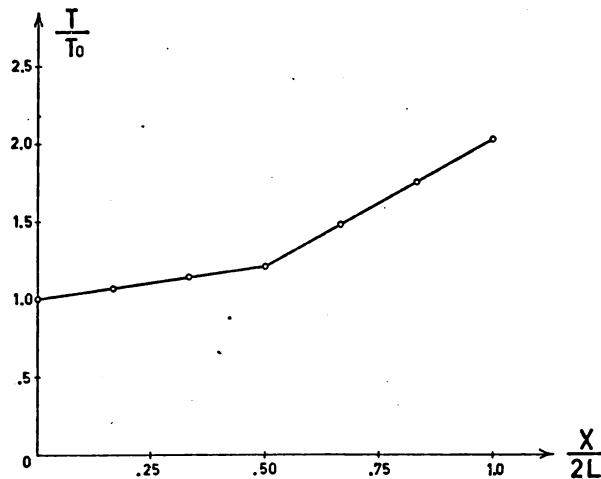
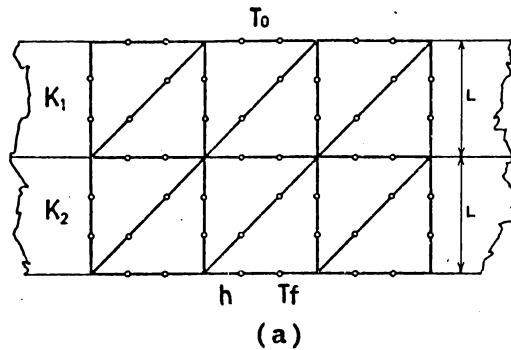


Figure 1. An infinite composite wall. (a) Mesh idealization with boundary conditions; forty-six degrees of freedom. (b) Unidimensional temperature plot showing perfect accuracy towards the exact solution (analytical)

Example 2

This attempts to show that the generalized temperatures give a good idea of local values even in case of non-integrable gradient fields. Figure 2 illustrates a conduction-cooled turbine blade having its root $x=0$ maintained at a uniform temperature T_0 and operating in an effective gas with temperature T_f (exact solution under unidimensional heat flow assumption, Reference 5).

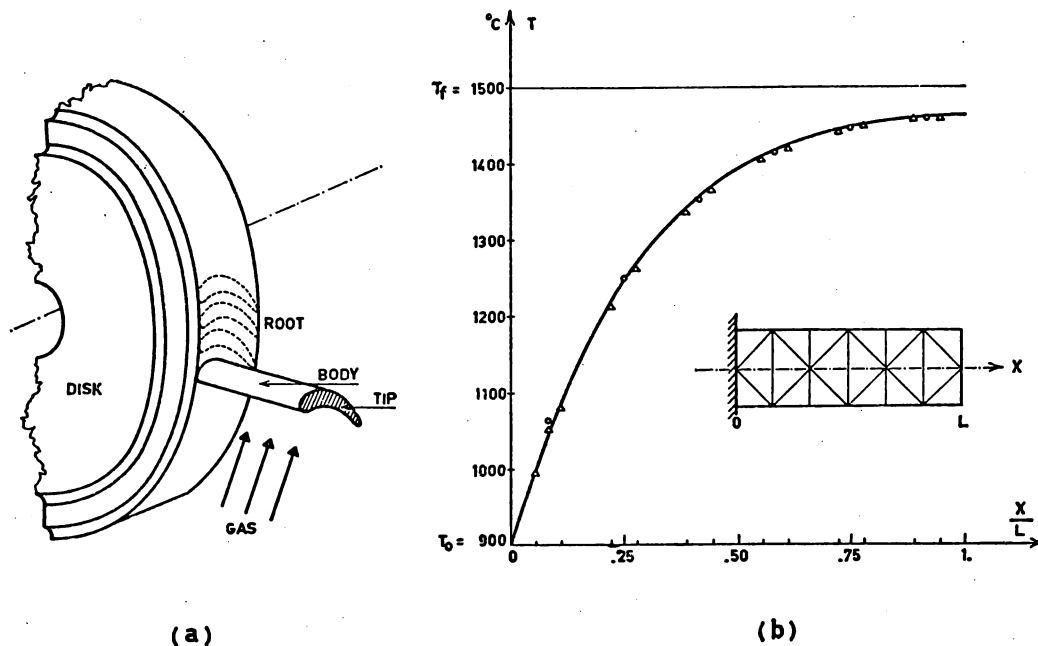


Figure 2. Turbine blade. (a) Blade on cooled turbine disk, with insulated tip, body in ambient gas at T_f and isothermal root disk at T_0 . (b) Finite element idealization and temperature plots for blade core:—, exact; \circ , constant heat flow model; Δ , linear heat flow model

A dual analysis of the structure yields upper and lower bounds (Table I). The larger dissipation of the exact treatment is due to the assumption of unidimensionality.

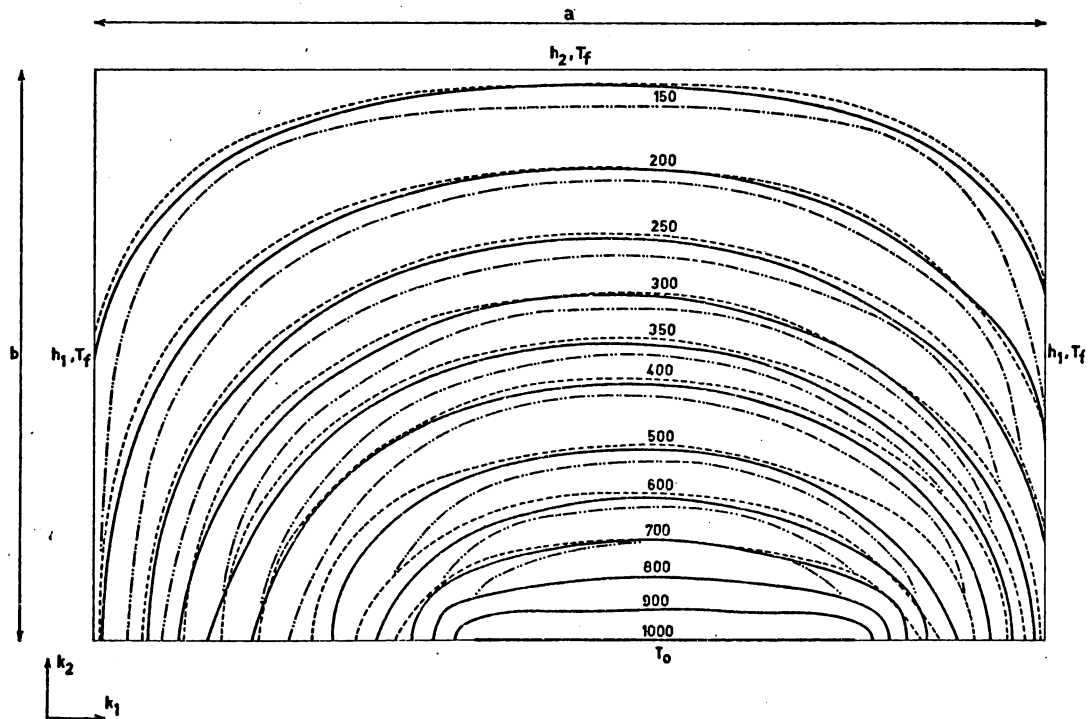
Table I. Dissipation bounds for several models of the blade

Finite element model	Degrees of freedom	$10^{-4} E$
Linear temperature field	23	20.91
Parabolic temperature field	67	20.53
Linear heat flow field	88	20.47
Constant heat flow field	44	19.55
Exact dissipation under unidimensional heat flow assumption		20.76

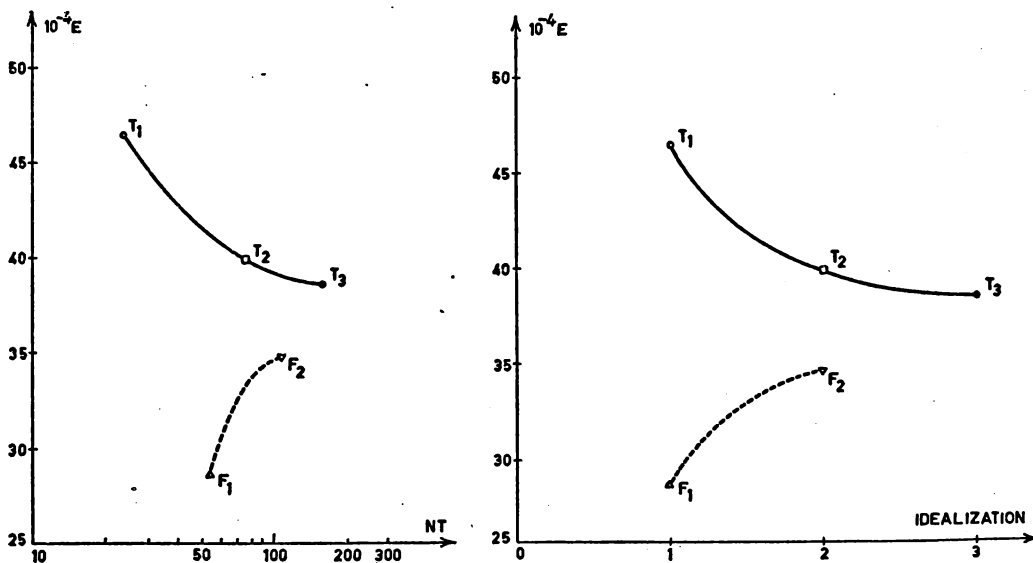
Example 3

In this example, a dual analysis of a laminated slab is presented (Figure 3).

No exact solution is known, so that temperature mapping is compared to a cubic temperature model treatment, which is nearly exact with respect to energy bounds. Heat flow models tend to converge faster than temperature ones; however, they need more degrees of freedom for a given grid and equal degrees in the polynomial discretization. Table III shows remarkable accuracy for heat flux equilibrium at the boundary.



(a)



(b)

Figure 3. Laminated slab. (a) Slab with boundary conditions ($k_1 = 0.8 - k_2 = 0.4$ W/cm °C; $h_1 = 0.9 - h_2 = 0.5$ W/cm² °C; $T_0 = 1,000 - T_f = 100$ °C; $a = 10 - b = 6$ cm) and temperature maps: —, cubic temperature model; --, parabolic temperature model; -·-, linear heat flow model. (b) Dissipation bounds showing convergence versus degrees of freedom or degrees of polynomial idealization

Table II. Notation and number of degrees of freedom for Figure 3

Elements	Symbol	Degrees of freedom
Linear temperature field	○	27
Parabolic temperature field	□	80
Cubic temperature field	⊙	163
Constant heat flow field	△	53
Linear heat flow field	▽	106

Table III. Computed heat flux equilibrium for the slab (watt/cm)

Idealization	Total outgoing heat flux through the convection boundary	Isothermal boundary (part of lower edge)
T_1	1031.607	-1031.607
T_2	887.318	-887.318
T_3	858.147	-858.147
F_2	772.272	-772.272
F_1	636.289	-636.288

Example 4

This final example considers a heat-exchanger tube with a non-uniform surface conductance on its outer surface (Figure 4).

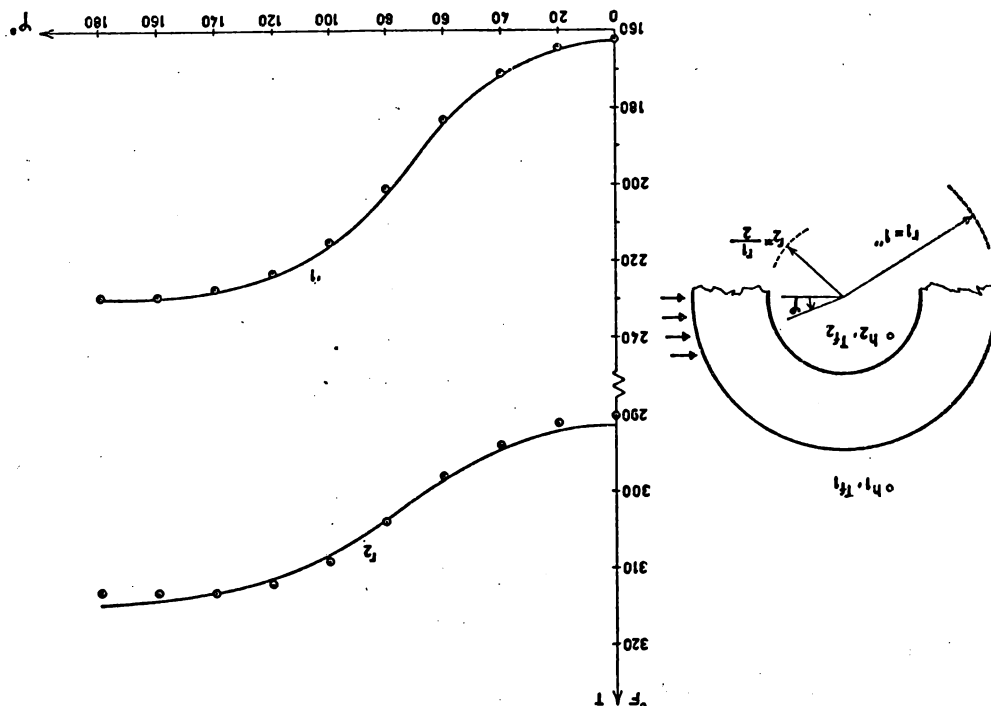


Figure 4. Temperature distribution in a cross-flow heat-exchanger tube of unit thermal conductivity, with high temperature gas at $T_{T_2} = 370^\circ\text{F}$ and cooling gas at $T_{T_1} = 70^\circ\text{F}$, flowing normal to the tube (⊙, relaxation; —, finite elements)

Table IV. Distribution of surface conductances

ϕ^0	h_1 (Btu/h ft ² . °F)	h_2
0-10	25	55
10-30	24	55
30-50	22	55
50-70	18	55
70-90	12	55
90-110	10	55
110-180	9	55

Upper and lower bounds (Table V) allow to consider the T_2 -solution as exact. The corresponding solution is plotted versus a finite difference relaxation solution (Reference 5).

Table V. Dissipation bounds for the exchanger tube

Finite element model	Degrees of freedom	$2 \times 10^{-4} E$
Linear temperature field	35	13·970
Parabolic temperature field	107	13·875
Linear heat flow field	144	13·871
Constant heat flow field	72	13·677

APPENDIX I

Upper and lower bounds

We now sketch briefly the theory of functional bounding by the dual approach. Let q denote a vector of a linear space F , representing a field of heat flow in each element of the structure. Define a scalar product in F by

$$(q, \hat{q}) = \int_D r_{ij} q_i \hat{q}_j dD + \int_{\Sigma \partial E} \frac{1}{h} (n_i q_i) (n_m \hat{q}_m) dS = (\hat{q}, q) \quad (56)$$

The squared norm (q, q) is equal to twice the dissipation functional for the whole structure and its surrounding.

$$(q, q) = 0 \quad \text{implies} \quad q = 0$$

i.e. the complete vanishing of all heat flows.

A field will be called integrable if a temperature field exists in each element such that

$$\partial_m T = -r_{im} q_i \quad (14')$$

and such that the external face temperature

$$T_e = T - (n_i q_i / h) \quad (57)$$

be continuous at interfaces. If i denotes an integrable field, we have from equation (56)

$$(i, \hat{q}) = - \int_D \hat{q}_j \partial_j T dD + \int_{\Sigma \partial E} (T - T_e) n_m \hat{q}_m dS$$

or, after integration by parts,

$$(i, \hat{q}) = \int_D T \partial_j \hat{q}_j dD - \int_{\Sigma \partial E} T_e n_m \hat{q}_m dS \quad (58)$$

A temperature compatible field is an integrable field with the additional conditions

$$T_e = \bar{T}_e \quad \text{along } \Sigma_1 \partial E \quad (59)$$

where $\Sigma_1 \partial E$ comprises the set of interfaces and boundary faces of ∂D where the external temperature is prescribed. Any particular temperature compatible field is denoted by c_0 . The set of all temperature compatible fields satisfying the homogeneous conditions

$$T_e = 0 \quad \text{along } \Sigma_1 \partial E \quad (60)$$

where $T_e = 0$ is some uniform reference temperature, is a subspace of F denoted by C . An arbitrary member of C is denoted by c . Thus $c_0 + c$ is an arbitrary temperature compatible field for equation (59) and belongs to the linear manifold $c_0 + C$.

A balanced heat flow field is one for which

$$\partial_j \hat{q}_j = \bar{Q} \quad \text{within each element} \quad (61)$$

and

$$n_m \hat{q}_m = \bar{q} \quad \text{along } \Sigma_2 \partial E \quad (62)$$

where $\Sigma_2 \partial E$ comprises the boundary faces of ∂D where surface sources \bar{q} are prescribed and the set of interfaces where surface sources ($\bar{q}_+ + \bar{q}_-$) are prescribed. Note that a positive \bar{q} is really a heat sink. The set $\Sigma_2 \partial E$ is complementary to $\Sigma_1 \partial E$. Any particular balanced heat flow field is denoted by h_0 . The set of all balanced heat flow fields satisfying the corresponding homogeneous conditions

$$\partial_j \hat{q}_j = 0 \quad \text{in } D, \quad n_m \hat{q}_m = 0 \quad \text{along } \Sigma_2 \partial E \quad (63)$$

is a subspace of F denoted by H . An arbitrary member of H is denoted by h . Thus $h_0 + h$ is an arbitrary heat flow field balanced with respect to conditions (61) and (62) and belongs to the linear manifold $h_0 + H$.

From equations (58)–(63) we readily find

$$(c_0, h) = - \int_{\Sigma_1 \partial E} \bar{T}_e n_m \hat{q}_m dS \quad (64)$$

$$(c, h_0) = \int_D T \bar{Q} dD - \int_{\Sigma_2 \partial E} T_e \bar{q} dS \quad (65)$$

$$(c_0, h_0) = (c_0, h) + (c, h_0) \quad (66)$$

$$(c, h) = 0 \quad (67)$$

This last result shows that subspaces C and H are orthogonal.

The exact solution s must satisfy simultaneously equations (14), (57), (59), (61) and (62); it is the intersection of the linear manifolds $c_0 + C$ and $h_0 + H$. In approaching the solution from two sides either by temperature compatible fields or by heat balanced flows, it is thus natural to require that the squared distance between each approximation

$$\rho^2 = (c_0 + c - h_0 - h, c_0 + c - h_0 - h)$$

be minimum. Using property (67) it is easily verified that

$$\rho^2 = (c_0 + c - h_0, c_0 + c - h_0) + (h_0 + h - c_0, h_0 + h - c_0) - (h_0 - c_0, h_0 - c_0)$$

The last term is constant; the first term is positive and depends only on the choice of the arbitrary element c ; the second is also positive and depends only on the choice of the arbitrary element h . Thus we can minimize separately the first and second term. In the first we can drop the constant (h_0, h_0) and, dividing by 2, obtain a statement equivalent to the minimization of the temperature functional:

$$\frac{1}{2}(c_0 + c, c_0 + c) - (c_0 + c, h_0), \quad \text{minimum} \quad (68)$$

In the second we drop the constant (c_0, c_0) , divide by 2, and obtain a statement equivalent to the minimization of the dual functional:†

$$\frac{1}{2}(h_0 + h, h_0 + h) - (h_0 + h, c_0), \quad \text{minimum} \quad (69)$$

Any approximate solution to (68) by discretization in n degrees of freedom consists in taking $c = \alpha_i c_i$, where the c_i are chosen members of C , the summation index running from 1 to n , and determining the best coefficients α_i^* . Then (68) becomes

$$\frac{1}{2}\alpha_i \alpha_j (c_i, c_j) + \alpha_i (c_i, c_0 - h_0) + \frac{1}{2}(c_0, c_0) - (h_0, c_0), \quad \text{minimum}$$

and furnishes the set of linear equations in the unknowns α_i^*

$$\alpha_j^* (c_i, c_j) + (c_i, c_0 - h_0) = 0 \quad (i = 1, 2, \dots, n) \quad (70)$$

With $c_0^* = c_0 + \alpha_i^* c_i$ denoting the best approximation, equation (70) can also be placed in the form

$$(c_i, c_0^* - h_0) = 0 \quad (i = 1, 2, \dots, n)$$

Then, multiplying each equation by α_i^* and adding

$$(c_0^* - c_0, c_0^* - h_0) = 0 \quad (71)$$

A similar treatment of problem (69) yields

$$(h_0^* - c_0, h_0^* - h_0) = 0 \quad (72)$$

Obviously the exact solution obeys the same relation

$$(s - c_0, s - h_0) = 0 \quad (73)$$

This can in fact be considered as an application of (67) since $s - c_0$ belongs to C and $s - h_0$ belongs to H .

Consider now the inequalities

$$\frac{1}{2}(c_0^*, c_0^*) - (c_0^*, h_0) \geq \frac{1}{2}(s, s) - (s, h_0) \quad (74)$$

$$\frac{1}{2}(h_0^*, h_0^*) - (h_0^*, c_0) \geq \frac{1}{2}(s, s) - (s, c_0) \quad (75)$$

resulting from the fact that the minima of (68) and (69) are reached by the exact solution. Eliminating in them (s, s) through (73) and combining with (71) and (72) there comes

$$(c_0^*, h_0 - c_0) \leq (s, h_0 - c_0) \leq (h_0^*, h_0 - c_0) \quad (76)$$

This upper and lower bound characteristic for the linear functional

$$\begin{aligned} (s, h_0 - c_0) &= (s, h_0) - (s, c_0) = (c + c_0, h_0) - (h + h_0, c_0) \\ &= (c, h_0) - (h, c_0) \\ &= \int_D T\bar{Q} \, dD + \int_{\Sigma, \partial E} \bar{T}_e n_m q_m \, dS - \int_{\Sigma, \partial E} T_e \bar{q} \, dS \end{aligned}$$

can be used directly in solving a mixed boundary value problem but does not guarantee convergence in norm.

† Complete analogous statements to equations (27) and (28) are in fact:

1. Principle of variation of temperatures:

$$\frac{1}{2}(c_0 + c - h_0, c_0 + c - h_0) - \frac{1}{2}(h_0 - c_0, h_0 - c_0), \quad \text{minimum}$$

or

$$\frac{1}{2}(c_0 + c, c_0 + c) - (c, h_0), \quad \text{if we drop the constant } (c_0, c_0)$$

2. Principle of variation of heat flow:

$$\frac{1}{2}(h_0 + h - c_0, h_0 + h - c_0) - \frac{1}{2}(h_0 - c_0, h_0 - c_0), \quad \text{minimum}$$

or

$$\frac{1}{2}(h_0 + h, h_0 + h) - (h, c_0), \quad \text{if we drop the constant } (h_0, h_0)$$

To obtain a measure of convergence in norm it is necessary to solve the problem in two steps by superposition.

First step. $c_0 = 0$, the prescribed temperatures are set equal to zero, only the prescribed heat sources are taken into account. Then $c_0^* = c^*$ and the previous result becomes

$$(c^*, h_0) \leq (s, h_0) \leq (h_0^*, h_0)$$

However from expressions (71)–(73) also

$$(c^*, c^*) = (c^*, h_0) \quad (s, s) = (s, h_0) \quad (h_0^*, h_0^*) = (h_0^*, h_0)$$

so that we obtain upper and lower bounds for the quadratic dissipation functional

$$(c^*, c^*) \leq (s, s) \leq (h_0^*, h_0^*)$$

Second step. $h_0 = 0$, the flow rate of the prescribed heat sources is set equal to zero but account is taken of the prescribed temperatures. Then $h_0^* = h^*$ and (76) becomes

$$(c_0^*, c_0) \geq (s, c_0) \geq (h^*, c_0)$$

Furthermore from expressions (71)–(73)

$$(c_0^*, c_0) = (c_0^*, c_0^*) \quad (s, c_0) = (s, s) \quad (h^*, c_0) = (h^*, h^*)$$

and the dissipation functional of that case is now bracketed in the reverse way

$$(c_0^*, c_0^*) \geq (s, s) \geq (h^*, h^*)$$

APPENDIX II

Constant heat flow element

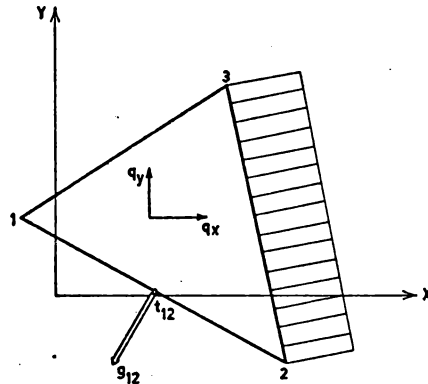


Figure 5. Constant heat flow element

$$\begin{aligned} q_x &= \alpha_1, & n &= 2, & \mathbf{M} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{E}, & \mathbf{a}^T &= (\alpha_1 \ \alpha_2) \\ q_y &= \alpha_2, \end{aligned}$$

$$\begin{aligned} M_\sigma &= 1, & \mathbf{f}_\sigma^T &= \frac{1}{L_{ij}} (y_{ij} \ x_{ij}), & \mathbf{G}_\sigma &= \begin{pmatrix} x_{ij} & -y_{ji} \\ y_{ji}/L_{ij}^2 & x_{ij}/L_{ij}^2 \end{pmatrix} \\ P_\sigma &= 1, & \mathbf{b}_\sigma^T &= \frac{1}{L_{ij}} \end{aligned}$$

$$\mathbf{B}_\sigma = (y_{ji} \ x_{ij}), \quad \mathbf{g}_\sigma^T = \alpha_1 y_{ji} + \alpha_2 x_{ij}, \quad \mathbf{t}_\sigma^T = \frac{1}{L_{ij}} \int_{\partial_\sigma E = ij} T_e dS$$

where

$$L_{ij} = \text{Length of edge } i-j, \quad x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j$$

$$\mathbf{g}^T = (g_{12} \ g_{23} \ g_{31}), \quad \mathbf{t}^T = (t_{12} \ t_{23} \ t_{31})$$

$$\mathbf{B} = \begin{pmatrix} y_{21} & x_{12} \\ y_{32} & x_{23} \\ y_{13} & x_{31} \end{pmatrix}, \quad \mathbf{R} = \Delta \begin{pmatrix} 1/k_x & 0 \\ 0 & 1/k_y \end{pmatrix}, \quad \mathbf{R}_c = \begin{pmatrix} \sum_{\sigma} \frac{y_{ji}^2}{h_{ij} L_{ij}} & \text{SYM} \\ \sum_{\sigma} \frac{x_{ij} y_{ji}}{h_{ij} L_{ij}} & \sum_{\sigma} \frac{x_{ij}^2}{h_{ij} L_{ij}} \end{pmatrix}$$

where

$$\Delta = \text{Area of the element}, \quad h_{ij} = \text{Uniform transfer coefficient for edge } i-j \\ = \infty \text{ if } (T)_{ij} = (T_e)_{ij}$$

APPENDIX III

Linear heat flow element

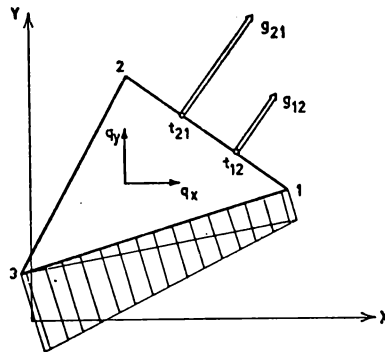


Figure 6. Linear heat flow element

$$q_x = \alpha_1 + \alpha_5 x + \alpha_4 y, \quad n = 5, \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & y & x \\ 0 & 1 & x & 0 & -y \end{pmatrix}, \quad \mathbf{a}^T = (\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5)$$

$$M_{\sigma} = 3, \quad \mathbf{f}_{\sigma}^T = \frac{1}{L_{ij}} (y_{ji} \ x_{ij} \ xx_{ij} \ yy_{ji} \ xy_{ji} - yx_{ij})$$

$$P_{\sigma} = 2, \quad \mathbf{b}_{\sigma}^T = \frac{2}{x_{ij} L_{ij}} (x - x_j \ x_i - x)$$

$$= \frac{2}{y_{ji} L_{ij}} (y_j - y \ y - y_i)$$

$$= \frac{2}{L_{ij}^2} (L_{ij} - s \ s)$$

$$\mathbf{B}_\sigma = \frac{1}{2} \begin{pmatrix} y_{ji} & x_{ij} & x_i x_{ij} & y_i y_{ji} & x_i y_{ji} - y_i x_{ij} \\ y_{ji} & x_{ij} & x_j x_{ij} & y_j y_{ji} & x_j y_{ji} - y_j x_{ij} \end{pmatrix}$$

$$\mathbf{g}_\sigma = \frac{1}{2} \begin{pmatrix} \alpha_1 y_{ji} + \alpha_2 x_{ij} + \alpha_3 x_i x_{ij} + \alpha_4 y_i y_{ji} + \alpha_5 (x_i y_{ji} - y_i x_{ij}) \\ \alpha_1 y_{ji} + \alpha_2 x_{ij} + \alpha_3 x_j x_{ij} + \alpha_4 y_j y_{ji} + \alpha_5 (x_j y_{ji} - y_j x_{ij}) \end{pmatrix}$$

$$\mathbf{t}_\sigma^T = \frac{2}{L_{ij}^2} \left(\int_{\partial\sigma E=ij} T_e(L_{ij}-s) ds \int_{\partial\sigma E=ij} T_e s ds \right)$$

$$\mathbf{g}^T = (g_{12} \ g_{21} \ g_{23} \ g_{32} \ g_{31} \ g_{13})$$

$$\mathbf{t}^T = (t_{12} \ t_{21} \ t_{23} \ t_{32} \ t_{31} \ t_{13})$$

$$\mathbf{B} = \frac{1}{2} \begin{pmatrix} y_{21} & x_{12} & x_1 x_{12} & y_1 y_{21} & x_1 y_{21} - y_1 x_{12} \\ y_{21} & x_{12} & x_2 x_{12} & y_2 y_{21} & x_2 y_{21} - y_2 x_{12} \\ y_{32} & x_{23} & x_2 x_{23} & y_2 y_{32} & x_2 y_{32} - y_2 x_{23} \\ y_{32} & x_{23} & x_3 x_{23} & y_3 y_{32} & x_3 y_{32} - y_3 x_{23} \\ y_{13} & x_{31} & x_3 x_{31} & y_3 y_{13} & x_3 y_{13} - y_3 x_{31} \\ y_{13} & x_{31} & x_1 x_{31} & y_1 y_{13} & x_1 y_{13} - y_1 x_{31} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \Delta/k_x & & & & \\ 0 & \Delta/k_y & & & \text{SYM} \\ 0 & S_x/k_y & I_x/k_y & & \\ S_y/k_x & 0 & 0 & I_y/k_x & \\ S_x/k_x & -S_y/k_y & -J_{xy}/k_y & J_{xy}/k_x & I_x/k_x + I_y/k_y \end{pmatrix}$$

where

$$\Delta = \int_E dx dy, \quad S_x = \int_E x dx dy, \quad I_x = \int_E x^2 dx dy, \quad J_{xy} = \int_E xy dx dy, \quad \text{etc.}$$

$$\mathbf{R}_c = \begin{bmatrix} R_{11} & & & & \\ R_{21} & R_{22} & & & \text{SYM} \\ R_{31} & R_{32} & R_{33} & & \\ R_{41} & R_{42} & R_{43} & R_{44} & \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix}$$

where

$$R_{11} = \sum_\sigma \frac{y_{ji}^2}{h_{ij} L_{ij}}, \quad R_{22} = \sum_\sigma \frac{x_{ij}^2}{h_{ij} L_{ij}}, \quad R_{21} = \sum_\sigma \frac{x_{ij} y_{ji}}{h_{ij} L_{ij}} = R_{12}$$

$$R_{31} = \sum_\sigma \frac{(x_i + x_j) x_{ij} y_{ji}}{2h_{ij} L_{ij}} = R_{13}, \quad R_{32} = \sum_\sigma \frac{(x_i + x_j) x_{ij}^2}{2h_{ij} L_{ij}} = R_{23}$$

$$R_{33} = \sum_\sigma \frac{(x_i^2 + x_i x_j + x_j^2) x_{ij}^2}{3h_{ij} L_{ij}}, \quad R_{44} = \sum_\sigma \frac{(y_i^2 + y_i y_j + y_j^2) y_{ji}^2}{3h_{ij} L_{ij}}$$

$$R_{41} = \sum_\sigma \frac{(y_i + y_j) y_{ji}^2}{2h_{ij} L_{ij}} = R_{14}, \quad R_{42} = \sum_\sigma \frac{(y_i + y_j) x_{ij} y_{ji}}{2h_{ij} L_{ij}} = R_{24}$$

$$R_{43} = \sum_{\sigma} \frac{(2x_i y_i + x_i y_j + x_j y_i + 2x_j y_j) x_{ij} y_{ji}}{6h_{ij} L_{ij}} = R_{34}$$

$$R_{51} = \sum_{\sigma} \frac{(x_i + x_j) y_{ji}^2 - (y_i + y_j) x_{ij} y_{ji}}{2h_{ij} L_{ij}} = R_{15}$$

$$R_{52} = \sum_{\sigma} \frac{(x_i + x_j) x_{ij} y_{ji} - (y_i + y_j) x_{ij}^2}{2h_{ij} L_{ij}} = R_{25}$$

$$R_{53} = \sum_{\sigma} \frac{2(x_i^2 + x_i x_j + x_j^2) x_{ij} y_{ji} - (2x_i y_i + x_i y_j + x_j y_i + 2x_j y_j) x_{ij}^2}{6h_{ij} L_{ij}} = R_{35}$$

$$R_{54} = \sum_{\sigma} \frac{(2x_i y_i + x_i y_j + x_j y_i + 2x_j y_j) y_{ji}^2 - (2y_i^2 + y_i y_j + y_j^2) x_{ij} y_{ji}}{6h_{ij} L_{ij}} = R_{45}$$

$$R_{55} = \sum_{\sigma} \frac{(y_i^2 + y_i y_j + y_j^2) x_{ij}^2 + (x_i^2 + x_i x_j + x_j^2) y_{ji}^2 - (2x_i y_i + x_i y_j + x_j y_i + 2x_j y_j) x_{ij} y_{ji}}{3h_{ij} L_{ij}}$$

REFERENCES

1. O. C. Zienkiewicz and Y. K. Cheung, *The Finite Element Method in Structural and Continuum Mechanics*, McGraw-Hill, London, 1967, Chap. 10.
2. M. A. Biot, 'New methods in heat flow analysis with application to flight structures', *J. aeronaut. Sci.* **24**, 857-873 (1957)
3. B. M. Fraeijis de Veubeke, 'Displacement and equilibrium models', in *Stress Analysis*, Wiley, London, 1965, Chap. 9.
4. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Wiley-Interscience, New York, 1953.
5. P. J. Schneider, *Conduction Heat Transfer*, Addison-Wesley, Reading, 1957.