VARIATIONAL PRINCIPLES IN FLUID MECHANICS and FINITE ELEMENT APPLICATIONS

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TABLE OF CONTENTS

1. Eulerian and Lagrangian coordinates
2. Lagrangian and Eulerian variations
3. The Hamilton principle for an inviscid fluid
4. Derivation of an Eulerian principle
5. A self-supporting Eulerian variational principle
6. Elimination of the Lagrangian variation of position
7. The pressure integral
8. The particular case of incompressible flow
9. The vector potential
10. Orthogonality aspects of isochoric and irrotational flow
11. Bounding of the kinetic energy
12. Finite element implementation of the Rayleigh-Ritz processes
13. Variational principles for viscous flow
14. References

1. EULERIAN AND LAGRANGIAN COORDINATES

The fundamental theorems of classical mechanics deal with closed systems, that is fixed sets of material particles which are followed in their motion. In most problems of fluid mechanics, one is rather interested in the phenomena that occur in a fixed region of space traversed by the particles. This modification of point of view requires some essential transformations to the classical variational formulations of Hamiltonian dynamics.

By $a = (a_1, a_2, a_3)$ we denote generalized lagrangian or "material" coordinates identifying a particle. They may, but need not necessarily, represent the space coordinates (x_1, x_2, x_3) occupied by the particle at a conventional epoch, usually denoted by $t = 0$. Their general définition is that of an independent set of integration constants of the differential equations of trajectories

$$\frac{dx_1}{u_1(x,t)} = \frac{dx_2}{u_2(x,t)} = \frac{dx_3}{u_3(x,t)} = dt \qquad (1)$$

In this differential system, $u_i(x,t)$ is the velocity field of the particles expressed as a function of space location and time. Any set of three independent first integrals of (1)

$$a_i = a_i(x,t) \qquad (2)$$

provides an implicit description at a regular point of the parametric equations of the trajectories of particles

$$x_j = X_j(a,t) \qquad (3)$$

At a regular point in the field, the Jacobian determinant

$$J = \frac{D(x_1, x_2, x_3)}{D(a_1, a_2, a_3)}$$

is different from zero and the volume element containing the particles with material coordinates $\hat{a}_i$ in the set $a_i \leq \hat{a}_i \leq a_i + da_i$ is given by

$$d\Omega = J\, da_1 da_2 da_3 \tag{4}$$

The symbol D_t is used to denote the material time derivative ; D_i to denote the partial derivative with respect to the material coordinate a_i under fixed time. Both are partial derivatives of any intensive variable of the field when expressed in the form of $f(a,t)$ and we have the commutative property

$$D_t D_i f = D_i D_t f.$$

Similarly ∂_t will be the symbol of local time derivative; ∂_j that of partial derivative with respect to x_j. Both are partial derivatives of intensive variable f when it is expressed in Eulerian form $f(x,t)$ and we have the commutative property

$$\partial_t \partial_j f = \partial_j \partial_t f.$$

In general, all transformations will be performed on the Eulerian description of variables and, for intensive variables, there follows from the definitions

$$D_t f = \partial_t f + \partial_j f\, D_t X_j$$

$$D_t X_j = \frac{\partial}{\partial t} X_j(a,t) = U_j(a,t) = u_j(x,t) \tag{5}$$

so that

$$D_t f = \partial_t f + u_j \partial_j f \tag{6}$$

In particular, since a material coordinate is attached to a particle throughout its motion

$$D_t a_i = \partial_t a_i + u_j \partial_j a_i = 0 . \tag{7}$$

and (2) may also be considered to be a solution of the partial differential equation (7).

To compute the material time derivative of the volume element, consider the Laplace expansion of the Jacobian determinant

$$J = e_{mnp}\, D_m X_1\, D_n X_2\, D_p X_3$$

where use is made of the permutation symbol e_{mnp}.

$$D_t J = e_{mnp}\,(D_t D_m X_1\, D_n X_2\, D_p X_3 + D_m X_1\, D_t D_n X_2\, D_p X_3 + D_m X_1\, D_n X_2\, D_t D_p X_3)$$

However,

$$D_t D_m X_1 = D_m D_t X_1 = D_m U_1 = \partial_i u_1 D_m X_i$$

hence

$$e_{mnp} D_t D_m X_1\, D_n X_2\, D_p X_3 = \partial_i u_1 e_{mnp}\, D_m X_i\, D_n X_2\, D_p X_3$$

$$= \partial_i u_1\, e_{i23} J = (\partial_1 u_1)\, J$$

With a similar treatment of the other terms, ther finally comes

$$D_t J = (\partial_j u_j)\, J$$

or, in view of (4)

$$D_t\, d\Omega = (\partial_j u_j)\, d\Omega \qquad (8)$$

Formulas (6) and (8) funish a justification of the general statement about the material time derivative of any extensive quantity

$$D_t \int_\Omega f\, d\Omega = \int_\Omega D_t (f\, d\Omega) = \int_\Omega (\partial_t f + u_j \partial_j f + f\, \partial_j u_j)\, d\Omega$$

$$= \int_\Omega \{ \partial_t f + \partial_j (u_j f) \} \, d\Omega$$

or, after application of the divergence theorem

$$D_t \int_\Omega f d\Omega = \int_\Omega \partial_t f \, d\Omega + \int_{\partial\Omega} f(n_j u_j) \, dS \qquad (9)$$

where n_j denote the direction cosines of the outward normal to the surface $\partial\Omega$ bounding the set of particles.

2. LAGRANGIAN AND EULERIAN VARIATIONS

The preceding reminder of well known results is useful in order to establish a complete analogy with similar conclusions relating material and local variations. In Hamiltonian mechanics the real motion of each particle is compared to perturbed or "varied" motions. We may, for instance, consider a family of virtual motions

$$x_j = X_j(a,t;\varepsilon) \qquad (10)$$

of which the real motion (3) would correspond to the value zero for the parameter . The material or Lagrangian variation of position of a particle can then be defined as

$$\Delta X_j = \frac{\partial}{\partial \varepsilon} X_j \Big|_{\varepsilon=0} d\varepsilon \qquad (11)$$

Inversion of (10)

$$a_i = a_i(x,t;\varepsilon) \qquad (12)$$

leads naturally, for the material coordinates, to a concept of local or eulerian variation, in which the space coordinates are kept fixed

$$\delta a_i = \frac{\partial}{\partial \varepsilon} a_i \Big|_{\varepsilon=0} d\varepsilon \qquad (13)$$

Since by definition $\delta x_j = 0$, we obtain from (10), considering a back-substitution of the material coordinates through (12),

$$\Delta X_j + \delta a_i \, D_i X_j = 0 \tag{14}$$

Conversely, becouse $\Delta a_i = 0$ by definition,

$$\delta a_i + \Delta X_j \partial_j a_i = 0 \tag{15}$$

which is the analogue to (7). Thus there is an equivalence by one to one correspondence between ΔX_j and δa_i; (14) and (15) are indeed the inverse relationships to one another, since for $\varepsilon = 0$

$$D_i X_j \partial_j a_m = \delta_{im} \qquad \partial_j a_i D_i X_m = \delta_{jm} \tag{16}$$

More generally, the variations of an intensive variable $f(x,t;\varepsilon)$ are related by

$$\Delta f = \delta f + \Delta X_j \, \partial_j f \tag{17}$$

which is the analogue of (6), or by

$$\delta f = \Delta f + \delta a_m \, D_m f$$

that follows immediately from its alternative representation as a function of $(a,t;\varepsilon)$.

Obviously, Δ commutes with D_i and D_t, while δ commutes with ∂_j and ∂_t. From a computation similar to that of $D_t J$, we obtain the analogue to (8)

$$\Delta J = (\partial_j \, \Delta X_j) \, J \qquad \text{or} \qquad \Delta d\Omega = (\partial_j \, \Delta X_j) \, d\Omega \tag{18}$$

and finally, the analogue to (9)

$$\Delta \int_\Omega f d\Omega = \int_\Omega \delta f d\Omega + \int_{\partial\Omega} f(n_j \cdot \Delta x_j) \, dS \tag{19}$$

The operators ∂_t and δ keeping the space coordinates fixed, we have also

$$\partial_t d\Omega = 0 \qquad \delta d\Omega = 0 \tag{20}$$

3. THE HAMILTON PRINCIPLE FOR AN INVISCID FLUID

In Hamilton's principle the set of particles is kept fixed and must be followed throughout its motions, a fact that will be stressed in the formulas by writing $\Omega(t)$ and $\partial\Omega(t)$ for its volume and bounding surface. The Lagrangian per unit mass will be

$$L = \frac{1}{2} D_t X_i \, D_t X_i - U - G \tag{21}$$

where G is a gravitational potential, assumed to be a function of the space coordinates only G(x), so that

$$g_j = -\partial_j G \tag{22}$$

is the local gravitational acceleration acting on a particle, and

$$\Delta G = -\partial_j G \Delta X_j \qquad \delta G = 0 \tag{23}$$

The specific internal energy U is in general a function of ρ, the mass per unit volume, and of the specific entropy S. To obtain a true variational principle, it will be necessary to make the assumption that there are no heat exchanges between the particles, nor momentum exchanges. Thus, neglecting conductivity and viscosity the entropy of each particle remains the same at any time. In addition we assume, for simplicity, that the entropy of each particle is the same (homentropic flow).
It is then possible to ignore entirely the dependence of U on S; the thermodynamical pressure becomes defined by

$$p(\rho) = \rho^2 \frac{dU}{d\rho} \tag{24}$$

and the fluid is barotropic.

Hamilton's principle asserts that

$$\Delta \int_{t_1}^{t_2} \int_{\Omega(t)} \rho L d\Omega dt - \int_{t_1}^{t_2} \int_{\partial_2 \Omega(t)} \bar{p}\, n_j\, \Delta X_j\, dS\, dt = 0 \tag{25}$$

provided $\Delta X_j = 0$ at $t = t_1$ and $t = t_2$. The second term represents the virtual work of a prescribed $\bar{p}$ in part $\partial_2\Omega$ of the boungary.
On the complementary part $\partial_1\Omega$ the fluid may be assumed to glide along a fixed or moving wall so that the constraint

$$n_j\, \Delta x_j = 0 \qquad \text{on} \qquad \partial_1\Omega \tag{26}$$

must be applied independently.

As it is stated the principle does not take care of conservation of mass that is also to be entered as a side constraint

$$\Delta(\rho d\Omega) = 0 \qquad \text{or} \qquad \Delta\rho + (\partial_j\, \Delta x_j)\rho = 0 \tag{27}$$

This gives

$$\int_{t_1}^{t_2} \int_{\Omega(t)} \rho\, \Delta L\, d\Omega\, dt - \int_{t_1}^{t_2} \int_{\partial_2 \Omega(t)} \bar{p}\, n_j \Delta x_j\, dS\, dt = 0$$

Now, in view of (23) and (24)

$$\Delta L = D_t X_j\, D_t \Delta X_j - \frac{p}{\rho^2} \Delta\rho - \partial_j\, G\, \Delta X_j$$

or writing u_j for $D_t X_j$ and using (27) again

$$\rho\ \Delta L\ \ d\Omega = \rho\ u_j\ d\Omega\ D_t\ \ \Delta X_j\ \ + p(\partial_j\ \Delta X_j)\ \ d\Omega - \rho\ \partial_j\ G\ \Delta X_j\ \ d\Omega$$

Because of the motion of the region occupied by the particles it is necessary to apply the operator D_t on $d\Omega$ included for any integration by parts with respect to time. This, and the application of the divergence theorem, are prepared by transforming as follows :

$$\rho\ \ \Delta L\ d\Omega = D_t(\rho\ \ u_j\ \ \Delta X_j\ d\Omega) - \Delta X_j\ D_t\ (\rho\ u_j\ d\Omega)$$

$$+\ \ \partial_j(p\ \Delta X_j)\ d\Omega - (\partial_j\ p + \ \partial_j\ G)\ \Delta X_j\ d\Omega$$

We can now carry out the required integrations

$$\int_{\Omega\,(t_2)} \rho\ u_j\ \Delta X_j\ d\Omega - \int_{\Omega\,(t_1)} \rho\ u_j\ \Delta X_j\ d\Omega\ \ + \int_{t_1}^{t_2} \int_{\partial\Omega\,(t)} p\ n_j\ \Delta X_j\ dS\ dt$$

$$-\int_{t_1}^{t_2} \int_{\partial_2\,\Omega\,(t)} \bar{p}\ n_j\ \Delta X_j\ dS\ dt -$$

$$-\int_{t_1}^{t_2} \int_{\Omega\,(t)} \Delta X_j\ (D_t(\rho u_j d\Omega)+(\partial_j p+\rho\partial\ _j G)d\Omega)\ dt = 0$$

The two first terms vanish on account of the vanishing of ΔX_j at time limits. The Euler equation provided by the arbitrariness of ΔX_j in the last term turns out to be the correct Newtonian of motion

$$\rho\ D_t\ u_j + \partial_j p + \rho\partial_j\ G = 0 \qquad (28)$$

after, again, a separate consideration of time conservation of mass

$$D_t(\rho d\Omega) = 0 \qquad \text{or} \qquad D_t\rho + \rho\partial_i u_i = 0 \tag{29}$$

If due account is taken of (26), the arbitrariness of ΔX_j on $\partial_2 \Omega(t)$, yields the natural boundary condition

$$p = \bar{p} \qquad \text{on} \qquad \partial_2 \Omega(t) \tag{30}$$

4. DERIVATION OF AN EULIRIAN PRINCIPLE

A step towards the derivation of a corresponding Eulerian principle consists in an application of formula (19) to the Hamiltonian principle. Thus we must have

$$\int_{t_1}^{t_2} \int_{\Omega(t)} \delta(\rho L) d\Omega dt = \Delta \int_{t_1}^{t_2} \int_{\Omega(t)} \rho L d\Omega dt - \int_{t_1}^{t_2} \int_{\partial\Omega(t)} \rho L (n_j \Delta x_j) dS dt$$

and in view of (25)

$$\int_{t_1}^{t_2} \int_{\Omega(t)} \delta(\rho L) d\Omega dt + \int_{t_1}^{t_2} \int_{\partial\Omega(t)} \rho L (n_j \Delta x_j) dS dt$$

$$- \int_{t_1}^{t_2} \int_{\partial_2\Omega(t)} \bar{p}\,(n_j \Delta x_j)\, dS dt = 0$$

The correctness of this can be verified. However, the principle is not exactly Eulerian in the sense that the time integrals still imply that the motion of the volume occupied by the particles by the particles be accounted for.

The difference between the true Eulerian variation

$$\delta \int_{t_1}^{t_2} \int_{\Omega} \rho L d\Omega dt \qquad \text{and} \qquad \int_{t_1}^{t_2} \int_{\Omega(t)} \delta(\rho L) d\Omega dt$$

is in the time integrations by parts which require the ∂_t operator in the first case, the D_t operator in the second. The terms requiring integration by parts in time in $\delta(\rho L)$ are

$$\rho\, u_i\, \delta\,(D_t X_i)\quad d\Omega$$

We have from (17)

$$\delta\, D_t X_i = \Delta D_t X_i - \Delta X_j\, \partial_j u_i = D_t \Delta X_i - \Delta X_j\, \partial_j u_i$$

whence

$$\rho\, u_i\, \delta\,(D_t X_i)\, d\Omega = D_t(f\, d\Omega) - E_j\, \Delta X_j\, d\Omega \qquad (31)$$

where $f = \rho u_i\, \Delta X_i$

and

$$E_j = \rho\, D_t u_j + \rho u_i \partial_j u_i \qquad (32)$$

Equation (31) is in the form required by the second case. It will generate two terms that vanish at the time limits and a contribution $-E_j$ to the Euler equation. If we now transform in (31)

$$D_t\,(f\, d\Omega) = (\partial_t\, f)\, d\Omega + \partial_j\,(u_j\, f)\, d\Omega$$

it follows that for the first case we generate again terms that vanish at the time limits, a surface term

$$\int_{t_1}^{t_2} \int_{\partial\Omega} (n_j u_j) \rho\, u_i \Delta X_i\, dS dt$$

and the same contribution to the Euler equation. As a consequence we do not alter the Euler equation by switching from the quasi-Eulerian functional to the really Eulerian one, that involves a fixed region Ω of space; but in so doing, we introduce a complicated combination of Lagrangian and Eulerian surfaceintegrals. This is not surprising as the boundary conditions represented by free surfaces or moving walls are essentially Lagrangian by nature. In the sequel we must be satisfied with taking Eulerian variations

of the Eulerian functional

$$\int_{t_1}^{t_2} \int_{\Omega} \rho L d\Omega dt$$

and fit whatever boundary conditions that are "natural" to the situation.

5. A SELF-SUPPORTING EULERIAN VARIATIONAL PRINCIPLE

To avoid the necessity of dealing separately with conservation of mass, the functional can be augmented by means of a Lagrangian multiplier to include the satisfaction of eq. (27) . This step was taken originally by Herivel (Ref. 4) but is known to be insufficient when the variations on particle displacement are transferred to material variations of the velocity field; it restricts the flow to the irrotational case.
A logical step to remove this restriction was taken by the author (Ref. 7) in 1965, it consists in augmenting the functional further by incorporating the constraints :

$$D_t X_i - u_i = 0 \qquad (33)$$

by means of a vector Lagrangian multiplier ψ_i. We thus examine the Eulerian variations of the new functional

$$\int_{t_1}^{t_2} \int_{\Omega} \left[\rho L + \theta(\partial_t \rho + \partial_i(\rho u_i)) + \rho\psi_i(D_t X_i - u_i)\right] d\Omega dt \qquad (34)$$

where L may now be written as

$$L = \frac{i}{2} u_i u_i - U(\rho) - G(x) \qquad (35)$$

and $\delta\rho$, $\delta\theta$, $\delta\psi_i$, δu_i, ΔX_i are independent variations.

The variations on the Lagrangian multipliers raise (27) and (33) to the status of Euler equations. If we collect the terms due to the variation of

$$(\frac{1}{2}\, u_i u_i - \frac{d}{d\rho}\,(\rho U) - G)\ \delta\rho + \theta \partial_t \delta\rho + \theta \partial_i (u_i\, \delta\rho) + \delta\rho \psi_i (D_t X_i - u_i),$$

take into consideration that (33) is satisfied and prepare the required integrations by parts

$$\partial_t(\theta\delta\rho) + \partial_i(\theta\, u_i \delta\rho) - \delta\rho\ (\frac{d}{d\rho}\,(\rho U) + \partial_t\theta + u_i\partial_i\theta + G - \frac{1}{2}\, u_i u_i)$$

we obtain as Euler equation

$$\delta\rho \rightarrow \qquad D_t\,\theta = \frac{1}{2}\ u_i u_i - G - \frac{d}{d\rho}\ (\rho U) \tag{36}$$

The variation on u_i alone generate the terms

$$\rho u_i\ \delta u_i + \theta \partial_i(\rho \delta u_i) - \rho\psi_i\ \delta u_i = \partial_i(\theta\rho\delta\, u_i) + \rho(u_i - \partial_i\theta - \psi_i)\ \delta u_i$$

and the corresponding Euler equations

$$\delta u_i \rightarrow \qquad u_i = \partial_i\theta + \psi_i \tag{37}$$

For ΔX_i we must manipulate

$$\delta D_t X_i = \Delta D_t X_i - (\partial_j u_i)\ \Delta X_j = \partial_t \Delta X_i + u_j \partial_j \Delta\, X_i - (\partial_j u_i)\Delta X_j$$

generating the terms

$$\rho\psi_i \delta D_t X_i = \partial_t(\rho\psi_i \Delta X_i) - \partial_t(\rho\psi_i)\Delta X_i + \partial_j(\rho\psi_i u_j \Delta X_i) - \partial_j(\rho\psi_i u_j)\Delta X_i$$

$$-\rho\psi_j \partial_i u_j \Delta X_i$$

and obtain the Euler equation

$$\Delta X_i \rightarrow \qquad \partial_t(\rho\psi_i) + \partial_j(\rho\psi_i u_j) + \rho\psi_j \partial_i u_j = 0$$

In view of (29) this simplifies to

$$\Delta X_i \rightarrow D_t \psi_i + \psi_j \partial_i u_j = 0 \qquad (38)$$

From the various contributions we can also derive the following integral at the time limits

$$\int_\Omega (\theta \delta\rho + \rho \psi_i \Delta X_i) d\Omega \Big|_{t_1}^{t_2}$$

and the surface terms related to possible boundary conditions

$$\int_{t_1}^{t_2} \int_{\partial\Omega} (\theta (u_i n_i) \delta\rho + \rho\theta (n_i \delta u_i) + \rho (n_j u_j) \psi_i \Delta X_i) \, dS \, dt$$

They will be discussed later, after elimination of the only remaining Lagrangian variation ΔX_i.

It should be observed that none of the Euler equations correspond directly to the Newtonian equations of motion. They are, however, contained as combinations. First of all, we find from (24)

$$\frac{d}{d\rho} (\rho U) = U + \rho \frac{dU}{d\rho} = U + \frac{P}{\rho} = I(\rho) \qquad (39)$$

the specific enthalpy considered as a function of ρ. If we then take the material derivative of (37) and eliminate ψ_i by using (38), there comes

$$D_t (u_i - \partial_i \theta) + (u_j - \partial_j \theta) \partial_i u_j = D_t u_i - (\partial_t + u_j \partial_j) \partial_i \theta - \partial_j \theta \partial_i u_j$$

$$+ \partial_i \frac{u_j u_j}{2} = D_t u_i - \partial_i D_t \theta + \partial_i \frac{u_j u_j}{2} = 0$$

or, finally, using (36)

$$D_t u_i + \partial_i (I + G) = 0 \qquad (40)$$

which is a classical form of the equations of motion.

6. ELIMINATION OF THE LAGRANGIAN VARIATION OF POSITION

To eliminate the use of ΔX_i we must solve the corresponding Euler equations (38) and substitute solution into the functional. Equation (38) states that the ψ_i field is a constant circulation one; on any small segment dx_i carried by the particles the circulation of ψ_i does depend on time :

$$D_t\,(\psi_i dx_i) = 0 \tag{41}$$

Indead, this is equivalent to

$$D_t\psi_i dx_i + \psi_j D_t dx_j = D_t\psi_i dx_i + \psi_j dD_t X_j = D_t\psi_i dx_i + \psi_j du_j$$

$$= dx_i(D_t\psi_i + \psi_j \partial_i u_j) = 0$$

Equation (41) indicates that the Pfaffian form $\psi_i dx_i$ depends only on material coordinates. In any of its canonical representations

$$\psi_i dx_i = d\gamma + \alpha d\beta \tag{42}$$

the variables α, β, γ are material variables :

$$D_t\gamma = 0 \qquad D_t\alpha = 0 \qquad D_t\beta = 0 \tag{43}$$

From (42) follows then the general solution of (38)

$$\psi_i = \partial_i\gamma + \alpha\partial_i\beta \tag{44}$$

and this, substitued into (37) yields the general Clebsch representation (Ref. 1) of the rotational flow of an inviscid fluid :

$$u_i = \partial_i\phi + \alpha\partial_i\beta \tag{45}$$

where $\phi = \theta+\gamma$ is a velocity potential and α and β are material variables. If the potential ϕ is single-valued the velocity field retains the constant circulation property for closed contours carried by the flow. Moreover

$$\text{rot } \vec{u} = \text{grad}\alpha \times \text{grad}\beta \tag{46}$$

and we have the statement that the vortex lines which are the intersections of the surface families α = constant and β= constant, are carried by the flow. The constant circulation property indicates the existence of an acceleration potential, indeed,

$$D_t u_i = \partial_t(\partial_i\phi+\alpha\partial_i\beta) + u_j\partial_j(\partial_i\phi+\alpha\partial_i\beta)$$

$$= \partial_i(\partial_t\phi+\alpha\partial_t\beta) + \partial_t\alpha\partial_i\beta - \partial_i\alpha\partial_t\beta + u_j\partial_i(\partial_j\phi+\alpha\partial_j\beta)$$

$$+ u_j\partial_j\alpha\partial_i\beta - u_j\partial_i\alpha\partial_j\beta$$

$$= \partial_i\left(\partial_t\phi+\alpha\partial_t\beta + \frac{u_j u_j}{2}\right) + \partial_i\beta D_t\alpha - \partial_i\alpha D_t\beta$$

and the two last terms vanish by (43). With this result and (40), we obtain the generalized Bernoulli integral of the equations of motion

$$\partial_t\phi+\alpha\partial_t\beta + \frac{u_j u_j}{2} + I + G = h(t) \tag{47}$$

Since the potential ϕ contains an arbitrary additive function of time only, there is no restriction in making $h(t) = 0$.

To eliminate ΔX_i replace in (34)

$$\psi_i D_t X_i = (\partial_i\gamma+\alpha\partial_i\beta)D_t X_i = D_t\gamma - \partial_t\gamma + \alpha(D_t\beta - \partial_t\beta) = -(\partial_t\gamma+\alpha\partial_t\beta)$$

Then

$$\rho\psi_i(D_t X_i - u_i) = -\rho(\partial_t\gamma+\alpha\partial_t\beta) - \rho u_i(\partial_i\gamma+\alpha\partial_i\beta)$$

If in addition we transform

$$\theta\{\partial_t\rho+\partial_i(\rho u_i)\} = \partial_t(\rho\theta) + \partial_i(\theta\rho u_i) - \rho\partial_t\theta - \rho u_i\partial_i\theta$$

the two contributions combine into

$$\theta\{\partial_t\rho+\partial_i(\rho u_i)\} + \rho\psi_i(D_t X_i - u_i) = \partial_t(\rho\theta) + \partial_i(\theta\rho u_i) - \rho D_t\phi - \rho\alpha D_t\beta$$

Thus discarding the term that goes to time limits and the divergence term, we obtain a functional

$$\int_{t_1}^{t_2}\int_{\Omega} \rho K d\Omega dt$$

$$K = \frac{1}{2} u_i u_i - U - G - D_t\phi - \alpha D_t\beta \qquad (D_t = \partial_t + u_i\partial_i) \tag{48}$$

The Eulerian variation of this functional yields the following Euler equations :

$$\delta\rho \rightarrow \frac{u_i u_i}{2} - I - G - D_t\phi - \alpha D_t\beta = 0 \tag{49}$$

$$\delta u_i \rightarrow \rho(u_i - \partial_i\phi - \alpha\partial_i\beta) = 0 \tag{50}$$

$$\delta\phi \rightarrow \partial_t\rho + \partial_i(\rho u_i) = 0 \tag{51}$$

$$\delta\alpha \rightarrow D_t\beta = 0 \tag{52}$$

$$\delta\beta \rightarrow \partial_t(\rho\alpha) + \partial_i(\rho\alpha u_i) = 0 \tag{53}$$

The last being equivalent to $D_t\alpha = 0$ in view of (51).
The time limits term is

$$-\int_{\Omega} \rho(\delta\phi + \alpha\delta\beta) d\Omega \Big|_{t_1}^{t_2}$$

and can be deleted under the convention that the variations on α and β

vanish for $t = t_1$ and $t = t_2$.

The surface terms are

$$- \int_{t_1}^{t_2} \int_{\partial\Omega} \rho(\delta\phi + \alpha\delta\beta)(n_i u_i)\, dS\, dt$$

The simpler boundary condition is that of a fixed wall. Freedom in the variations of either ϕ or β at such a boundary wall will give the corresponding requirement $n_i u_i = 0$ as a natural boundary condition. If a normal velocity $(n_i \overline{u_i})$ is imposed at a part $\partial_1\Omega$ of the boundary we add to the functional the term

$$\int_{t_1}^{t_2} \int_{\partial_1\Omega} \rho\phi(n_i u_i)\, dS\, dt$$

and the free variation of ϕ will impose $n_i u_i = n_i \overline{u_i}$ on $\partial_1\Omega$. The free variation on ρ in this additional term will cause $\phi = 0$ on the same boundary.

7. THE PRESSURE INTEGRAL

Consider the energy per unit volume

$$f(\rho) = \rho U(\rho)$$

The variable conjugate to ρ is by definition

$$\frac{df}{d\rho} = U + \frac{p}{\rho} = I$$

A co-energy is then defined as in elasticity theory by the Legendre transformation

$$\rho I - f = \rho(I-U) = p(I) \qquad (54)$$

and turns out to be the pressure to be considered as a function of the enthalpy. Differentiation of (54) produces then the involutory of conjugate

variables

$$\rho dI = dp \tag{55}$$

Precisely, when (49) is used to eliminate the consideration of as a variable, the Kernel of the functional reduces to

$$\rho K = \rho(I-U) = p$$

We obtain one of Bateman's Eulerian variational principles (Ref. 3), the so-called pressure integral

$$\int_{t_1}^{t_2} \int_{\Omega} p(I)\, d\Omega\, dt \tag{56}$$

in which the enthalpy must be considered to be expressed through (49) as

$$I = \frac{u_i u_i}{2} - G - D_t\phi - \alpha D_t\beta \tag{57}$$

From (55) and (57) there comes

$$\delta p = \frac{dp}{dI}\,\delta I = \rho(u_i \delta u_i - \delta D_t\phi - \delta\alpha D_t\beta - \alpha\delta D_t\beta)$$

and the Eulerian equations are still given by (50), (51), (52) and (53); nothing is changed concerning the time limit and surface terms.

The pressure integral can be further simplified by accepting a priori the Euler equation (50). Hence substituting the Clebsch representation

$$u_i = \partial_i\phi + \alpha\partial_i\beta$$

we obtain the pressure integral (56) with the enthalpy given this time by

$$I = -(\frac{1}{2}(\partial_i\phi+\alpha\partial_i\beta)(\partial_i\phi+\alpha\partial_i\beta) + G + \partial_t\phi + \alpha\partial_t\beta) \tag{58}$$

as would result from the generalized Bernoulli integral (47).

This principle depends only on the potential ϕ and the Lagrangian variables of the Clebsch representation. Their variations produce the Euler equations (51), (52) and (53); again the time limits and surface terms are unaltered.

The principle has received a good deal of attention in the problem of the perturbation of a uniform flow of a compressible fluid by aerodynamic bodies. Using a pressure coefficient

$$P = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2}$$

in place of the pressure itself; introducing the Mach number

$$M = \frac{U}{a_\infty} \qquad \text{with } a_\infty^2 = \gamma \frac{p_\infty}{\rho_\infty}$$

and the variable

$$Z = \frac{I_\infty - I}{U^2}$$

instead of the enthalpy itself, the relationships

$$I - I_\infty = c_p(T - T_\infty) \qquad \text{and} \qquad \frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

of gas dynamics yield the explicit law

$$P = \frac{2}{\gamma M^2}\left[(1 - (\gamma-1)M^2 Z)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{59}$$

While, if the flow depends only on the potential ϕ, and a perturbation potential η of the uniform flow along x_1 be introduced by

$$\phi = U(x_1 + \eta)$$

$$Z = \partial_1\eta + \frac{1}{U}\partial_t\eta + \frac{1}{2}\partial_i\eta\partial_i\eta \tag{60}$$

An approximate determination of the perturbation potential is possible by application of the Rayleigh-Ritz method via the variational principle

$$\delta \int_{t_1}^{t_2} \int_{\Omega} \left[(1-(\gamma-1)M^2 Z)^{\frac{\gamma}{\gamma-1}} - 1 \right] d\Omega\, dt = 0$$

with Z given by (60). The principle is also useful as a theoretical tool for delivering coherent approximations to the field equation governing the perturbation potential and its boundary conditions, by assuming small pressure and velocity perturbations and expanding the Kernel by the binomial theorem.

8. THE PARTICULAR CASE OF INCOMPRESSIBLE FLOW

Incompressible flow is an idealized case where the pressure is no more of thermodynamical origin but constitutes a purely mechanical reaction against changes of volume. One can, however, consider it as a limiting case of the enthalpy formula through (55) since then

$$I = \int \frac{dp}{\rho} = \frac{p}{\rho}$$

and $\frac{p}{\rho}$ is sometimes called the specific pressure energy.
Thus the general pressure formulation applies here in the form (56) with

$$p = \rho\left(\frac{u_i u_i}{2} - G - D_t\phi - \alpha D_t\beta\right) \qquad (61)$$

There is no loss in generality in dropping the constant factor ρ in the Kernel of the principle. It will also be observed that G plays no role in the variational equations and may be dropped in the Kernel that becomes :

$$\delta \int_{t_1}^{t_2} \int_{\Omega} \left(\frac{u_i u_i}{2} - D_t\phi - \alpha D_t\beta \right) d\Omega dt = 0 \qquad (62)$$

However, G retains its role as additional hydrostatic pressure when the pressure is computed from (61) after the potential and Lagrangian functions have been determined.

Consider now the very special case of stationary potential flow of an incompressible inviscid fluid. The assumptions can be summarized in

$$\partial_t \vec{u} = 0 \qquad \text{rot } \vec{u} = 0 \qquad \text{div } \vec{u} = 0$$

and the problem is almost purely grometrical in nature.
Because of the stationarity assumption, the time integral may be dropped in the variational principle. The second assumption allows to retain only ϕ, and (62) degenerates into

$$\int_\Omega (u_i \partial_i \phi - \frac{u_i u_i}{2}) d\Omega - \int_{\partial_2 \Omega} \phi \overline{u_\nu} \; dS - \int_{\partial_1 \Omega} (n_i u_i)(\phi - \overline{\phi}) dS$$

$$\star \qquad \begin{vmatrix} \min \\ \phi \end{vmatrix} \quad \begin{vmatrix} \max \\ u_i \end{vmatrix} \qquad (63)$$

The sign of the functional has been changed and surface terms added to provide for natural boundary conditions throughout.
Those and the Euler equations are in fact

$$\delta u_i \quad \text{in } \Omega \quad \rightarrow \quad u_i = \partial_i \phi$$

$$\delta u_i \quad \text{on } \partial_1 \Omega \quad \rightarrow \quad \phi = \overline{\phi}$$

which is equivalent to the imposition of the velocity components tangent to this boundary,

$$\delta \phi \quad \text{in } \Omega \quad \rightarrow \quad \partial_i u_i = 0$$

$$\delta \phi \quad \text{on } \partial_2 \Omega \quad \rightarrow \quad n_i u_i = \overline{u}_\nu$$

the imposition of the velocity component normal to this boundary.

Principle (63) is in the so-called canonical (in the sense of Hamilton) or involutory form advocated by Friedrichs and whose analogue in elasticity theory is better known under the name of Reissner. It is a saddle point principle in which, after looking for a maximizing choice of the velocity field under a given potential, one looks after the minimum of all those

maxima for the choice of the potential.

From it, two simpler single-field principles may be derived, whose dual character enables the kinetic energy estimates of the flow, obtained from Rayleigh-Ritz approximations, to be bounded from below and from above respectively.

The first is obtained by accepting a priori the potential character of the flow. If we add to this the a priori satisfaction of $\phi = \bar{\phi}$ on $\partial_1\Omega$, we obtain

$$\int_\Omega \frac{1}{2}\, \partial_i\phi\partial_i\phi \, d\Omega - \int_{\partial_2\Omega} \phi\, \bar{u}_\nu \, dS \qquad \min_{\phi} \tag{64}$$

and the principle accounts simply for the incompressibility condition

$$\partial_i u_i = \partial_i\partial_i\phi = 0$$

and for the boundary conditions on $\partial_2\Omega$.

If, on the contraty, we want to simplify (63) by a priori satisfaction of the incompressibility condition plus the boundary condition in $\partial_2\Omega$, it becomes necessary to transform the functional by an integration by parts :

$$-\int_\Omega \left(\phi\partial_i u_i + \frac{u_i u_i}{2}\right) d\Omega + \int_{\partial_2\Omega} \phi(n_i u_i - \bar{u}_\nu)\, dS + \int_{\partial_1\Omega} n_i u_i \phi\, dS$$

Accepting now a priori the constraints

$$\partial_i u_i = 0 \qquad \text{and } n_i u_i = \bar{u}_\nu \qquad \text{on } \partial_2\Omega$$

we obtain the dual single-field principle (the sign has again be changed)

$$\int_\Omega \frac{1}{2}\, u_i u_i \, d\Omega - \int_{\partial_1\Omega} n_i u_i \bar{\phi}\, dS \qquad \min_{u_i \text{ constrained}} \tag{65}$$

9. THE VECTOR POTENTIAL

The implementation of the incompressibility constraint on the u_i field calls naturally for the use of a vector potential $\vec{A}$:

$$\vec{u} = \text{rot } \vec{A} \rightarrow \text{div } \vec{u} = 0 \qquad (66)$$

but introduces interpretation difficulties for the boundary terms. For this reason we carry out the required transformations on (63) instead of (65). Since the use of a vector potential

$$u_i = e_{ipq}\, \partial_p\, A_q$$

automatically entails $\partial_i u_i = 0$, the functional in (63) may already be transformed to

$$-\int_\Omega \frac{u_i u_i}{2}\, d\Omega + \int_{\partial_2\Omega} \phi(n_i u_i - \bar{u}_\nu)\, dS + \int_{\partial_1\Omega} \bar{\phi} n_i u_i\, dS$$

Or, with the understanding that $\phi = \bar{\phi}$ on $\partial_1\Omega$,

$$-\int_\Omega \frac{u_i u_i}{2}\, d\Omega + \int_{\partial\Omega} \phi n_i u_i\, dS - \int_{\partial_2\Omega} \phi \bar{u}_\nu\, dS$$

The free variation $\delta\phi$ on $\partial_2\Omega$ produces the natural boundary condition

$$\vec{n}.\vec{u} = \bar{u}_\nu \qquad \text{on } \partial_2\Omega,$$

The variation $\delta u_i = e_{ipq}\, \partial_p \delta\, A_q$ gives

$$-\int_\Omega \delta\vec{A}.\text{rot } \vec{u}\, d\Omega - \int_{\partial\Omega} n u_i e_{ipq} \delta A_q\, dS + \int_{\partial\Omega} \phi n_i e_{ipq}\, \partial_p \delta A_q\, dS = 0$$

The Euler equation is obviously rot $\vec{u} = 0$, as was to be expected. The last term, that contains derivatives of the variations of the vector potential, is transformed as follows :

$$\int_{\partial\Omega} \phi \vec{n}.\text{rot } \delta\vec{A}\, dS = \int_{\partial\Omega} \vec{n}.\text{rot}(\phi\delta\vec{A})\, dS - \int_{\partial\Omega} \vec{n}.(\text{grad}\phi \times \delta A)\, dS$$

where, on account of

$$\vec{n}\,.(\mathrm{grad}\phi \times \delta\vec{A}) = \delta\vec{A}.(\vec{n}\times\mathrm{grad}\phi)$$

the scalar potential has only to be defined on the surface.

As $\int_{\partial\Omega} \vec{n}.\mathrm{rot}\ (\phi\,\delta\vec{A})\ dS = 0$, we finally obtain for the surface terms

$$\int_{\partial\Omega} \delta\vec{A}.\left[\ \vec{n}\times(\vec{e}-\mathrm{grad}\phi)\ \right] dS = 0 \qquad \text{with } \phi = \bar{\phi} \text{ on } \partial_1\Omega$$

The boundary conditions are thus finally obtained in the form

$$\vec{n}\times\vec{u} = \vec{n}\times\mathrm{grad}\bar{\phi} \qquad \text{on} \qquad \partial_1\Omega$$

$$\vec{n}.\vec{u} = \bar{u}_\nu \qquad \text{on} \qquad \partial_2\Omega$$

10. ORTHOGONALITY ASPECTS OF ISOCHORIC AND IRROTATIONAL FLOW

Consider on the one hand an irrotational flow described by a scalar potential :

$$u_i = \partial_i\phi \quad \rightarrow \quad \mathrm{rot}\ \vec{u} = 0$$

on the other hand, and isochoric flow, described by a vector potential :

$$v_i = e_{ipq}\,\partial_p A_q \quad \rightarrow \quad \mathrm{div}\ \vec{v} = \partial_i v_i = 0$$

and define a scalar product between the two as

$$(u,v) = \int_\Omega u_i v_i d\Omega$$

From a first type of integration by parts

$$(u,v) = \int_\Omega v_i\partial_i\phi\, d\Omega = \int_{\partial\Omega}\phi n_i v_i dS - \int_\Omega \phi\partial_i v_i d\Omega = \int_{\partial\Omega}\phi n_i v_i\ dS \qquad (67)$$

It is apparent that this scalar product vanishes if the bounding surface is subdivised in parts :

$\partial_1\Omega$ over which $\phi = 0$

$\partial_2\Omega$ over which $n_i v_i = \vec{n}.\mathrm{rot}\,\vec{A} = 0$

The second type of integration by parts

$$(u,v) = \int_\Omega u_i e_{ipq} \partial_p A_q d\Omega = \int_{\partial\Omega} n_p u_i e_{ipq} A_q dS + \int_\Omega \vec{A}.\mathrm{rot}\,\vec{u}\, d\Omega \ , \quad \text{or}$$

$$(u,v) = -\int_{\partial\Omega} \vec{A}.(\vec{n}\times\vec{u})\, dS = +\int_{\partial\Omega} \vec{u}.(\vec{n}\times\vec{A})\, dS \tag{68}$$

leads to the same conclusions. On $\partial_1\Omega$ the imposition of $\phi = 0$ is equivalent to the requirement $\vec{n}\times\vec{u} = 0$. On $\partial_2\Omega$ the requirement $\vec{n}.\mathrm{rot}\,\vec{A} = 0$ is satisfied by the somewhat stronger one $\vec{n}\times\vec{A} = 0$.

The equivalence between the two surface integrals to which the scalar product reduces follows also from the general statement

$$\int_\Omega \mathrm{div}\ \mathrm{rot}\,\vec{B}\, d\Omega = \int_{\partial\Omega} \vec{n}.\mathrm{rot}\,\vec{B}\, dS = 0$$

with $\vec{B} = \phi A$

This gives indeed, from

$$\mathrm{rot}(\phi\vec{A}) = \phi\ \mathrm{rot}\,\vec{A} + \mathrm{grad}\phi\times\vec{A}$$

the result

$$\int_\Omega \phi\vec{n}.\mathrm{rot}\,\vec{A}\, dS = \int_{\partial\Omega} \mathrm{grad}\,\phi.(\vec{n}\times\vec{A})\ dS \tag{69}$$

The orthogonality property is thus found to hold between an irrotational flow, whose tangential velocity component vanishes on $\partial_1\Omega$ and isochoric flow whose normal velocity component vanishes on the complementary part $\partial_2\Omega$.

11. BOUNDING OF THE KINETIC ENERGY

Consider a flow that is both irrotational and isochoric and satisfies non homogeneous boundary conditions on $\partial_1\Omega$, where the tangential velocity component is specified, and $\partial_2\Omega$ where the normal component is specified.

In keeping with the preceding section, denote by u an irrotational flow that satisfies the homogeneous condition on $\partial_1\Omega$ (no tangential velocity) but is left unspecified on $\partial_2\Omega$, while u_o denotes any particular irrotational flow complying with the non homogeneous data on $\partial_1\Omega$.

Similarly, v will denote any isochoric flow satisfying the homogeneous condition on $\partial_2\Omega$ (no normal velocity component) and without specification on $\partial_1\Omega$, while v_o will denote any particular isochoric flow satisfying the non homogeneous data on $\partial_2\Omega$. We then find

$$(u_o,v) = \int_{\partial_1\Omega} \overline{\phi} \; (\vec{n}.\mathrm{rot}\,\vec{A})\, dS = + \int_{\partial_1\Omega} \overline{\vec{A}.(\vec{n}\times\mathrm{grad}\phi)}\, dS \qquad (70)$$

$$(u,v) = \int_{\partial_2\Omega} \phi \; \overline{u_\nu}\, dS = - \int_{\partial_2\Omega} \overline{\mathrm{grad}\phi.(\vec{n}\times\vec{A})}\, dS \qquad (71)$$

In approximating the flow by a numerical analysis of Rayleigh-Ritz type, we may either consider the irrotational flow to contain adjustable parameters in u to satisfy in some best sense the incompressibility condition and the boundary data on $\partial_2\Omega$, or the isochoric flow to contain adjustable parameters in v to satisfy in some best sense the irrotationality condition and the boundary data on $\partial_1\Omega$.

Both viewpoints are combined in the requirement that the squared "distance" between the two adjustable fields by minimized :

$$(u+u_o-v-v_o,\; u+u_o-v-v_o) = (u+u_o,u+u_o)+(v+v_o,v+v_o)-2(u+u_o,v+v_o)$$
$$\text{minimum}$$

Because of the orthogonality property $(u,v) = 0$, this condition naturally splits into the two independant requirements

$$(u+u_o,u+u_o) - 2(u,v_o) \qquad \underset{u}{\text{minimum}} \qquad (72)$$

$$(v+v_o,v+v_o) - 2(v,u_o) \qquad \underset{v}{\text{minimum}} \qquad (73)$$

The term $-2(u_o,v_o)$ has been dropped as constant.

Now $u + u_o$ is the potential flow $u_i = \partial_i \phi$ where $\phi = \overline{\phi}$ as specified on $\partial_1 \Omega$, and the first requirement is identical to our previous variational principle (64) :

$$\frac{1}{2} \int_\Omega \partial_i \phi \partial_i \phi d\Omega - \int_{\partial_2 \Omega} \phi \, \overline{u_\nu} \, dS \qquad \text{minimum}$$

Similarly, $v + v_o$ is the isochoric flow $\vec{v} = \text{rot } \vec{A}$ where $\vec{n} \cdot \text{rot } \vec{A} = \overline{u_\nu}$ is specified on $\partial_2 \Omega$, and the second requirement is identical to the irrotational principle (65) implemented by a vector potential

$$\frac{1}{2} \int_\Omega \text{rot } \vec{A} \, \text{rot } \vec{A} \, d\Omega - \int_{\partial_1 \Omega} \overline{\phi} \, (\vec{n} \cdot \text{rot } \vec{A}) \, dS \qquad \text{minimum}$$

We obtain a bounding of the kinetic energy of the flow by considering the separate problems :

Problem 1 : The data specified on $\partial_1 \Omega$ are non homogeneous ($u_o \neq 0$) but homogeneous on $\partial_2 \Omega$ ($v_o = 0$)

Problem 2 : The complementary problem : $u_o = 0$, $v_o \neq 0$.

We may note that the general problem can always be handled by linear superposition of problems 1 and 2.

In problem 1 we must find the best approximations to

$$(u + u_o, u + u_o) \qquad \text{minimum} \tag{74}$$

$$(v,v) - 2(v,u_o) \qquad \text{minimum} \tag{75}$$

Set $u = \sum_1^n \alpha_j u_j$

where each u_j field is generated by an assumed potential that is zero on $\partial_1 \Omega$. The best coefficients $\hat{\alpha}_j$ are given by equating to zero the partial derivaties of the quadratic form

$$(u,u) + 2(u,u_o) = \Sigma\Sigma \alpha_j \alpha_k (u_j, u_k) + 2\Sigma \, \alpha_j (u_j, u_o)$$

Thus

$$\Sigma\ \hat{\alpha}_k(u_j,u_k) + (u_j,u_o) = 0 \qquad j = 1,2, \ldots n$$

Denoting by

$$\hat{u} = u_o + \Sigma\ \hat{\alpha}_k\ u_k$$

The best approximation, those equations are equivalent to

$$(\hat{u}_j,\hat{u}) = 0 \qquad j = 1, 2, \ldots, n$$

Multiplying each by its coefficient $\hat{\alpha}_j$ and summing

$$(\hat{u}-u_o,\hat{u}) = 0 \qquad \text{or } (\hat{u},\hat{u}) = (\hat{u},u_o) \tag{76}$$

The exact solution s, which is both irrotational and isochoric satisfies the similar equation

$$(s,s) = (s,u_o) \tag{77}$$

Indeed, s is both a v-type field ($v_o = 0$) and simultaneously $s - u_o$ is a u-type field, so that $(s-u_o,s) = 0$ by orthogonality.
Furthermore, since the minimum in (74) is not necessarily reached,

$$(\hat{u},\hat{u}) \geq (s,s) \tag{78}$$

We can give a similar treatment to

$$v = \sum_{1}^{n}\ \beta_j\ v_j$$

each v_j field being generated by a vector potential such that $\vec{n}.\mathrm{rot}\vec{A} = 0$ on $\partial_2\Omega$.

$$(v,v) - 2(v,u_o) = \Sigma\Sigma\beta_j\beta_k(v_j,v_k) - 2\Sigma\ \beta_j(v_j,u_o) \text{ minimum}$$

furnishes the linear system

$$\Sigma\ \hat{\beta}_k(v_j,v_k) - (v_j,u_o) = 0$$

or with $\hat{v} = \Sigma\ \hat{\beta}_k v_k$, the best approximation,

$$(v_j,\hat{v}) - (v_j,u_o) = 0 \qquad j = 1, 2, \ldots, n$$

Multiplying by each $\hat{\beta}_j$ and adding

$$(\hat{v},\hat{u}) - (\hat{v},u_o) = 0 \tag{79}$$

Since the minimum of (75) is not necessarily reached

$$(\hat{v},\hat{v}) - 2(\hat{v},u_o) \geq (s,s) - 2(s,u_o)$$

This inequality is transformed by (79) and (77) into

$$-(\hat{v},\hat{v}) \geq -(s,s)$$

and this result combined with (78) gives the kinetic energy bounding

$$(\hat{v},\hat{v}) \leq (s,s) \leq (\hat{u},\hat{u}) \tag{80}$$

A similar treatment of Problem 2 yields the reverse bounding

$$(\hat{u},\hat{u}) \leq (s,s) \leq (\hat{v},\hat{v}) \tag{81}$$

where $\hat{u} = \Sigma\hat{\alpha}_j u_j$ and $\hat{v} = v_o + \Sigma\ \hat{\beta}_j v_j$

12. FINITE ELEMENT IMPLEMENTATION OF THE RAYLEIGH-RITZ PROCESSES

The simply connected region Ω is divided into adjacent subdomains Ω_α, the so-called finite elements. Any integral extended over the whole region is understood to be the sum of integrals over the Ω_α. Whenever an integration by parts is applied, the boundary terms involve the whole set of boundaries $\partial\Omega_\alpha$ of each subdomain and can be regrouped as a sum of integrals covering the external boundary $\partial\Omega$ of Ω and a sum of integrals involving the two faces of interfaces I_β of the subdomains

$$\sum_{\alpha} \int_{\partial\Omega_\alpha} n_i f_i dS = \int_{\partial\Omega} n_i f_i dS + \sum_{\beta} \int_{I_p} n_i (f_i^+ - f_i^-) dS$$

In the last terms the convention is adopted that the normal n_i to the interface is that of one of its faces, denoted by the superscript +. Since for outward normals

$$n_i^- = -n_i^+ = -n_i$$

the minus sign of the contribution of the other face is understood.

Consider a first subdivision into finite elements, in each of which a scaler potential ϕ_α is defined, usually in the form of a complete polynomial of chosen degree with unknown coefficients. The transition conditions to be satisfied at the interfaces can be found by examination of the orthogonality condition, generalizing (67)

$$(u,v) = \Sigma \int_{\Omega_\alpha} v_i \partial_i \phi_\alpha \, d\Omega = - \Sigma \int_{\Omega_\alpha} \phi_\alpha \partial_i v_i d\Omega + \int_{\partial\Omega} \phi_\alpha n_i v_i dS +$$

$$+ \Sigma \int_{I_\beta} n_i (\phi_\alpha^+ v_i^+ - \phi_\alpha^- v_i^-) dS$$

To obtain orthogonality with an isochoric flow ($\partial_i v_i = 0$) satisfying $n_i v_i = 0$ on the part $\partial_2\Omega$ of the outer boundary, while $\phi_\alpha = 0$ on $\partial_1\Omega$, we must still ensure that at the interfaces

$$\Sigma \int_{I_\beta} n_i (\phi_\alpha^+ v_i^+ - \phi_\alpha^- v_i^-) dS = 0$$

This must hold in particular for a continuous isochoric flow, hence $v_i^+ = v_i^-$ at the interfaces and

$$\Sigma \int_{I_\beta} n_i v_i (\phi_\alpha^+ - \phi_\alpha^-) dS = 0$$

It is thus sufficient, although admittedly not necessary, that there be continuity of the scalar potentials at the interfaces. It turns out that in the case of complete polynomials, it is extremely easy to enforce the interface continuity conditions, both in 2 and 3 dimensions. For the two-dimensional case, taking triangular finite elements, the coefficients

of the polynomial defining the scalar potential inside can be determined in terms of local values of the potential at the vertices and along the sides, plus, as turns out ot be the case for degrees higher or equal to 3, at some interior points. If, at an interface, the local values of the potentials ϕ^+ and ϕ^- coincide, the potentials coincide along the whole interface.

Similarly, for the same or another subdivision into finite elements, for each of which a vector potential is assigned containing unknown coefficients, the transition conditions follow from (68)

$$(u,v) = \Sigma \int_{\Omega_\alpha} \vec{u}.\mathrm{rot}\ \vec{A}_\alpha\, d\Omega = \Sigma \int_{\Omega_\alpha} \vec{A}_\alpha.\mathrm{rot}\ \vec{u}\, d\Omega - \int_{\partial\Omega} \vec{A}.(\vec{n}\times\vec{u})\, dS$$
$$+\Sigma \int_{I_\beta} (\vec{A}.(\vec{n}\times\vec{u})^+ + \vec{A}.(\vec{n}\times\vec{u})^-)\, dS$$

To obtain orthogonality with an irrotational flow ($\mathrm{rot}\ \vec{u} = 0$) satisfying $\vec{n}\times\vec{u} = 0$ on part $\partial_1\Omega$ of the outer boundary, while $\vec{A}_\alpha = 0$ on part $\partial_2\Omega$, it is sufficient to have continuity of the vector potential at the interfaces.

Again, this is quite easily implemented for polynomial approximations of the vector potential. Here in the two-dimensional cases of plane flow or axisymetric flow, a scalar stream function replaces the vector potential.

13. VARIATIONAL PRINCIPLES FOR VISCOUS FLOW

There are no true variational principles yielding as Euler equations the Navier-Stokes general equations. There is, however, a principle that governs the dissipation in steady state flow for cases of such low Reynolds numbers that the acceleration terms are negligible. Limiting ourselves to incompressible fluids, the dissipation

$$F = \mu\theta_{ij}\theta_{ij}$$

$$\theta_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and the functional yielding the equations of motion with negligible inertia force terms can be taken as

$$J = \int_{\Omega} (\rho g_i u_i + p(\partial_i u_i) - F)\, d\Omega + \int_{\partial_2\Omega} \bar{t}_i u_i \, dS$$

with $u_i = \bar{u}_i$ on $\partial_1\Omega$.

The pressure p appears here as a Lagrangian multiplier, whose variations enforce the incompressibility condition. With the viscous stresses

$$\sigma_{ij} = \frac{\partial F}{\partial \theta_{ij}}$$

the Euler equations stemming from variations on the velocity field are

$$\rho g_i - \partial_i p + \partial_j \sigma_{ij} = 0$$

and the natural boundary conditions on $\partial_2\Omega$ are

$$n_j(\sigma_{ij} - p\delta_{ij}) = \bar{t}_i$$

As shown by Debongnie (Ref. 9), this principle can be extended by the Friedrichs technique to a canonical form involving simultaneously variations on the viscous stresses themselves.

Applications have been made in several directions. To biomechanics by P. Tong and Y.C. Fung (Ref. 8). To oil bearing problems and flow over deep wells by Debongnie (Ref. 9).
In applying the finite element methods to the two-dimensional cases, advantage may be gained from the remarkable analogy with Kirchhoff plate bending problems. The analogy comprises that between the stream function of the flow and the transverse plate flexure, the viscosity stresses and the bending moments tensor.

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