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STRUCTURAL OPTIMIZATION

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A NEW HEURISTIC METHOD FOR GENERAL MIXED INTEGER LINEAR AND QUADRATIC PROGRAMS

W.Oettli

THE PRINCIPLE OF FEASIBLE DIRECTIONS

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STRUCTURAL OPTIMIZATION

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PART I

I. INTRODUCTION

Numerical methods of structural analysis have reached a high standard of efficiency. As a consequence they tend to overgrow their usefulness as numerical checks of stress distribution, amplitudes of displacement, natural frequencies and elastic stability to become adjuvants to design procedures.

The objectives of design vary according to the purpose of the structure. In aerospace engineering the weight is the prominent factor and is often the only goal of numerical optimization studies. In other cases the functional that is subject to minimization is more complex, economical factors of various kind being incorporated with their relative weights into the cost function.

Until recently the minimization was carried out by trial and error, the preliminary design and the modifications introduced after evaluation of a numerical structural analysis being largely based on engineering judgment. Presently there is a tendancy to a more scientific approach in which the changes in design parameters are evaluated on the basis of algorithms. Efficient algorithms are those that tend to bring the functional to its minimum with the smallest number of iterations requiring a subsequent structural reanalysis. Moreover they have to satisfy many kinds of side constraints such as :

- remain within the elastic limits of the material in each structural member under a given set of load distributions;
- keep displacement-type limitations;
- avoid elastic instability;
- keep natural frequencies within prescribed limits;
- keep member sizes above minimum values.

The starting point of such automated Structural Optimization Programs is a given preliminary design. It is therefore difficult to evaluate the cost of optimization procedures, since the computer time devoted to reach a near-optimal stage will heavily depend on the quality of the preliminary design.
In case where the unicity of the optimal solution is not guaranteed a poor preliminary design can even lead to a local and not to the global minimum.
For this reason, while optimization programs will probably remain essential tools, the objectives of optimality will also tend to incorporate the computer in the preliminary design stage. This more direct approach towards an optimal structure is the aim of "Computer aided Design".
It is also a much more ambitious goal and, fortunately perhaps, will never obliterate the exercize of engineering art. It is indeed difficult to conceive a selection by the computer of the best "topology" of structural members to carry the loads according to the purpose of the structure, taking immediatly the effect of side constraints into account.
On the other hand, once the topology has been fixed by engineering judgment and experience, we will probably reach the stage where the computer will carry out from there the sizing of the members and even such other alterations in their geometry, permissible under the given topology and external constraints. Whether such ambitious programs will ever become operational within economical limits is a question that only experience will answer.

2. DESIGN VARIABLES

One can divide design variables in groups according to their relative importance. For aerospace structures, with a finite element method idealization, the following groups are proposed :

2.1. Element sizes

They comprize cross-sectional areas of beam, membrane and plate thicknesses ...
The optimization of those variables alone leaves the topology (system of element interconnexions) and other geometrical characteristics (height, length, taper of beams, planforms of membranes and plates ...) unchanged. Figure 1-a illustrates this in the simple case of a 3-bar truss where only the cross-sections of the bars are subject to optimization with side constraints consisting in upper bounds to tensile or compressive stresses and possibly limitations to nodal displacements.

2.2. Geometric variables

The choice of geometrical variables may alter the configuration of the structure but not its topology. In the finite element method they correspond to modifications in the nodal coordinates as shown on figure 1-b.

2.3. Material properties

The efficiency of the structure can be improved by a change of nature of the material selected for some of its members. For example Young's modulus, Poisson's ratio, the elastic limits vary with the density of the material and eventually the temperature to which it is subjected. There are here seldom changes in the decisions taken in the preliminary design, and if they prove advantageous, they introduce discrete parameter modifications as opposed to the continuous variations possible in the previous design variables (Figure 1-c).

2.4. Topology

A change in topology is also, and more fundamentally so, a discrete modification to the structure.
A set of members may be replaced by a new one with different elements, differently connected. Figure 1-d shows the substitution of a triangular membrane in place of the previous truss.

The order in which the groups of design variables have been listed is roughly that of increasing complexity in an optimization program and attendant increasing cost.
This consideration has led numerous research teams to limit themselves to the first category. There is also some justification for it in the fact that the general layout of a structure is often dictated by other considerations than a certain definition of optimality.
Aerodynamic shape, headroom, access facilities, failsafe design are characteristic examples.
The relative simplicity of dealing with element sizes only is enhanced by the choice of a finite element method for the discretization of the structure. As nodes are kept in place and element interconnexions are invariant, the statics and kinematics of the structure are not modified by alterations in element sizes.
This can make a large part of the optimization program a fixed subroutine.
In the sequel we shall deal only with this restricted aspect of optimization.

3. NUMERICAL METHODS OF STRUCTURAL OPTIMIZATION.

This section describes briefly two main approaches encountered in structural optimization and discusses their relative capabilities.

3.1. Mathematical programming.

In this relatively recent approach, minimum weight design is treated as the mathematical problem of extremizing a cost-function in design space. Each dimension of this space is related to one design variable, so that each point corresponds to a possible design. The side-constraints consist of limits to the design variables (element sizes) themselves and to stresses or displacements, the latter constraints being generally functions of the design variables. Symbolically, denoting by A_i $(i=1...n)$ the design variables

$$W = W\,(A_1 ... A_n) \quad \min.$$

$$\underline{A_i} \leq A_i \leq \overline{A_i} \qquad i=1...n$$

$$g_j\,(A_1 ... A_n) \leq \bar{g}_j \qquad j=1...p$$

The cost function and the nature of the second type of constraints determine whether the problem can be treated by linear or non-linear programming. The second case usually prevails for structural optimization. The iterative search procedure may be summarized as follows

$$\vec{A}_{\nu+1} = \vec{A}_\nu + \Delta_\nu\,\vec{D}_\nu$$

The vector $\vec{D}_\nu$ determines the direction of search and the scalar Δ_ν the step length made in this direction. They are determined on the basis of the stress analysis made at the ν-th step. The step length depends on the position of the representative design point with respect to the restraint surface delimiting in design space the feasible and non-feasible designs. More recently unconstrained formulations have been proposed wherein the cost function is augmented by penalty functions. The analytical form of the cost and constraints are then of less importance and more complex problems can be handled.

Drawbacks inherent to the mathematical programming approach appear with large numbers of design variables as the number of cycles required to get close to the optimum rapidly rises. Each cycle involves a costly stress reanalysis and the computational expenditure rapidly becomes prohibitive. On the other hand the method is very general and reliable. If a solution converges to a local minimum instead of the global minimum recuired, this can always be checked and the necessary steps be taken to reinitiate the procedure.

3.2. <u>Optimality criteria</u>.

Intuitive considerations as to the nature of the optimal design may lead to adopt optimality criteria that are not directly related to the minimization of the given cost function but sometimes constitute a satisfactory approximation to it. They can then provide a basis for the search techniques and lead to simple recursions formulas for redesign. The best known and widely used example of such a procedure is the "fully stressed design" concept.

According to it, each component of an optimal structure is stressed to its limit in at least one of the loading conditions. It leads to the so-called "stress-ratio redesign" method in which the element sizes are cycled by the recursion formula

$$A_{i,\nu+1} = A_{i,\nu} \left(\frac{\sigma_{i,\nu}}{\sigma_{i,all}} \right)$$

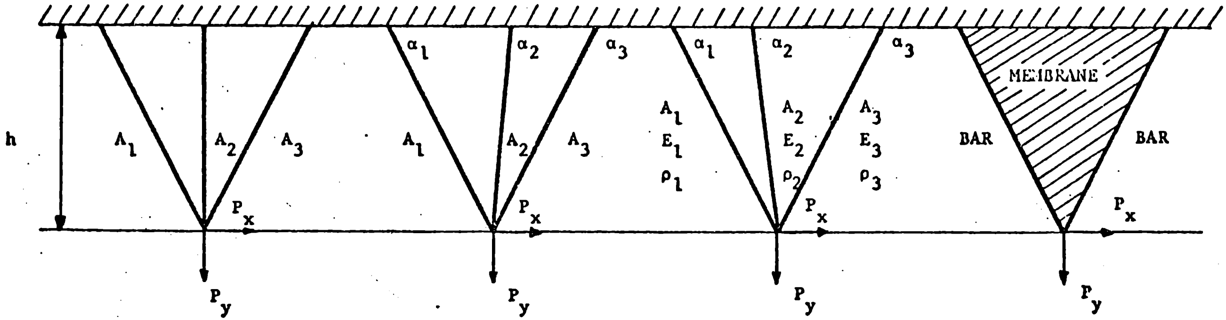
where $\sigma_{i,\nu}$ is the actual stress parameter at the ν-th cycle, representative of the stressing state within the element and $\sigma_{i,all}$ its allowable upper limit. As several loading cases are involved, $\sigma_{i,\nu}$ must be taken to be the largest stress value encountered.

Convergence to the optimal solution, according to the fully stressed design criterion, is obtained in one iteration for statically determinate structures.

In statically determinate cases the internal loads are indeed independent of the design variables and optimality based on fully stressed design coincides with the exact minimal weight criterion if no limitations are put on displacements. In the statically indeterminate case each redesign modifies the internal loading distribution and fully stressed design does not yield the minimum weight but may be considered to approach it satisfactory.

An attractive feature of fully stressed design that explains its relative success is its tendancy to converge in a number cycles independent of the number of design variables, in contrast to the more rigorous mathematical programming method.

Moreover each redesign cycle is fairly simple.



(a) Member sizes
Design variables :
A_1 , A_2 , A_3

(b) Configuration
Design variables :
+ α_1 , α_2 , α_3

(c) Material properties
Design variables :
+ { E_1 , E_2 , E_3 ; ρ_1 , ρ_2 , ρ_3

(d) Topology

FIGURE 1 : HIERARCHY OF DESIGN VARIABLES

(h is a prescribed parameter).

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PART II

AN ALGORITHM FOR MINIMUM WEIGHT DESIGN UNDER A SET OF LOADING MODES WITH CONSTRAINTS ON STRESSES AND DISPLACEMENTS.

1. PROBLEM DEFINITION.

The structure is in the linear elastic regime and idealized by finite elements. Under all the specified loading distributions certain constraints on stresses and nodal displacements must be satisfied. The geometry and the material properties are predetermined.
The functional to be minimized

$$W = \sum_i \rho_i L_i A_i \qquad i = 1 \ldots n_e \qquad (1)$$

is the structural weight, proportional in each element to the material density ρ_i, to the design variable A_i (cross-section of bars, thickness of membrane,...) and a geometrical parameter L_i (length of bar, area of membrane...).

2. DESCRIPTION OF THE CONSTRAINTS.

2.1. Production constraints.

They place a lower and sometimes an upper limit to the design variables

$$\underline{A_i} \leq A_i \leq \overline{A_i} \qquad (2)$$

2.2. Stress limitations.

In bar-type elements the tensile stress limit is determined by the elastic properties of the material, the compressive limit may be reduced to take into consideration, in a simple manner , a safeguard against buckling.
If σ_i is the actual stress in the bar

$$\underline{\sigma_i} \leq \sigma_i \leq \overline{\sigma_i} \qquad (3)$$

In shear panels one assumes a maximum allowable shear stress, usually governed by buckling considerations

$$\tau_i \leq \overline{\tau_i} \qquad (4)$$

In more general membrane elements, where the three stress components σ_x, σ_y and τ_{xy} play equally important roles, a reference stress related to an elastic limit criterion may be introduced. For the VON MISES criterion

$$\sigma_{ref} = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)^{\frac{1}{2}} \leq \overline{\sigma_i} \tag{5}$$

2.3. Displacement constraints.

They assign upper bounds to generalized displacements. To determine analytical expressions for them in terms of the design variables, the virtual work theorem is used.

If F denotes a vector (column matrix) of externally applied loads,
u the conjugate vector of generalized displacements,
σ the stress vector,
ε the conjugate strain vector,
the virtual work is given by :

$$\Delta = F^T_{(v)} \; u_{(r)} = \int_V q^T_{(r)} \; \varepsilon_{(v)} \; dV = \int_V \sigma^T_{(v)} \; \varepsilon_{(r)} \; dV \tag{6}$$

The subscripts between brackets refer to either a virtual or a real vector, the superscript T denotes transposition. Splitting the integral into the sum of contributions of each finite element

$$\Delta = \sum_i \int_{V_i} (\sigma^T_{(r)} \; \varepsilon_{(v)})_i \; dV_i \tag{7}$$

According to the finite element theory we have

$$\int_{V_i} (\sigma^T_{(r)} \varepsilon_{(v)})_i \; dV_i = q^T_{(r)i} \; K_i \; q_{(v)i} = q^T_{(r)i} g_{(v)i} \tag{8}$$

where q_i is the vector of generalized displacements of element i and g_i its conjugate of generalized loads. K_i is the stiffness matrix of the element.
Let now u_j denote a displacement component of the nodal displacement vector of the structure. Applying a corresponding virtual unit load to the structure, (7) and (8) give

$$u_j = \sum_i q^T_{(r)i} \, g_{i(j)} \tag{9}$$

where the $g_{i(j)}$ are the corresponding virtual loads generated at each element level. In statically determinate structures the loads $g_{i(j)}$ are uniquely determined by the unit load and (9) turns out to be given in terms of the design variables by

$$u_j = \sum_i \frac{c_{ij}}{A_i} \tag{10}$$

where the c_{ij} are constants. In redundant structures those coefficients are themselves implicit functions of the design variables.

Remark

It is sometimes of interest to consider more general displacement constraints than local limitations. A good example is the requirement of a straight-line configuration under load for a set of nodes. This is expressible as a constraint on a linear combination of local displacements and needs only consideration of the corresponding linear combination of unit loads.

3. FORMULATION.

Let us begin with the statically determinate case

3.1. Analysis stage.

The structure is analyzed under

- the n_r real loading systems of the design specification
- the n_v virtual loading cases connected with each displacement constraint.

3.2. Redesign stage.

If the stress constraints are

$$\sigma_{i\ell} \leq \bar{\sigma}_{i\ell} \qquad i = 1 \ldots n_e \qquad \ell = 1 \ldots n_r \tag{11}$$

where $\sigma_{i\ell}$ is the actual stress in element i under the loading case ℓ, the redesign is effected in a single step by

$$A_i^* = A_i \quad \max_{\ell} \left\{ \frac{\sigma_{i\ell}}{\bar{\sigma}_{i\ell}} \right\} \tag{12}$$

with, in addition, the minimum size requirement

$$A_i^* \geq \underline{A_i} \tag{13}$$

This method provides a "fully stressed" design, each element reaching its limiting stress (or having its minimal size) under at least one of the loading cases.
If we have displacement constraints, the analysis stage will provide the matrix of c_{ij} coefficients appearing in (10).
Taking the A_i^* values appearing in (12) as minimal, the problem with the addition of displacement constraints can be stated as follows

$$W = \sum_i \rho_i L_i A_i \qquad \min$$

under

$$\sum_i \frac{c_{ij}}{A_i} \leq \bar{u}_j \qquad j = 1 \ldots n_t \tag{14}$$

$$\bar{A}_i \geq A_i \geq A_i^* \qquad i = 1 \ldots n_e \tag{15}$$

where $\quad n_t = n_r \times n_v$

Because of the assumption of statical determinacy the formulation is rigorous, the solution unique and only one stress analysis is required.

Real structures, however, are rarely statically determinate. If subjected to different load distributions they are in fact both stiffer and even lighter if proper use is made of the stress cooperation provided by redundancy. But in this case both the c_{ij} and the A_i^* become implicit functions of the design variables. Each change in those will produce new c_{ij} and A_i^* that can only be known exactly through a costly stress reanalysis.
The following approach is suggested. The problem as defined by equations (14) and (15) is solved by considering the c_{ij} and A_i^* as constants. The evolved solution for the design variables is inserted in a new stress analysis to provide new c_{ij} and A_i^* values with which to reinitiate problem (14) , (15) untill close to convergence.

4. <u>SOLUTION OF THE LINEARIZED PROBLEM (14),(15)</u>.

In order to solve this problem, in which the c_{ij} and A_i^* are assumed to be given, it is beneficial to take the reciprocals of sizing variables as design variables :

$$x_i = \frac{1}{A_i} \tag{16}$$

The problem can now be recast as follows

Minimize the <u>non linear</u> objective function

$$W = \sum_i \frac{\rho_i L_i}{x_i}$$

Subject to the <u>linear</u> constraints

$$\begin{aligned} &\sum_i c_{ij}\, x_i \leq \bar{u}_j && j = 1 \ldots n_t \\ &x_i \leq \bar{x}_i && i = 1, n_e \\ &-x_i \leq -\underline{x}_i && i = 1, n_e \end{aligned} \tag{17}$$

where $\bar{x}_i = \dfrac{1}{A_i^*}$

and $\underline{x_i} = 1/\overline{A}_i$

Even if there are no upper bounds on the element sizes, the latter constraints have to be taken into account (in the form $-x_i \leq -\varepsilon$, ε being a very small positive number) in order to insure that the new design space, corresponding to the x_i variables, be a <u>bounded</u> convex region.
The recast problem (17) may then be solved by means of the gradient projection method for linear constraints (ref. [6]) adapted to the problem under consideration.
This method uses as feasible directions projections of the objective function gradient into the subspace satisfying constraints which are currently active.
This linear intersection subspace may be altered only in the two following cases : either to add a new active constraint or to drop an idle constraint. Thus each point succesively obtained satisfies exactly some of the constraints without violating any of the other prescribed constraints . Moreover a continuous decrease of the objective function is obtained after each step.

As required by the gradient projection method, the initial point must be a feasible point, a point lying in the convex region formed by the prescribed constraints.
In any given case, such a point can readily be found by linear scaling of all member sizes, so that a feasible bounded design is generated (one constraint at its critical value, others subcritical).
This scaling of <u>all</u> the design variables does not introduce stress redistribution : each stress and each displacement are simply divided by the same scaling factor.

5. <u>APPLICATION OF THE METHOD</u>.

Amongst known optimization programs we can mention:
- GELLATLY and BERKE (ref. [2])
- TAIG and KERR (ref. [3]).

As mentioned previously a structural optimization program performs iterative cycling between a structural analysis stage and a redesign stage. The programs under study at the Aerospace Laboratory of the University of Liège are coupled to the extensively developped ASEF code as the analysis module. The structural idealization consists up to now of axial force members and triangular or quadrilateral membrane elements. The degree of displacement polynomials within the elements is allowed to vary from 1 to 3.
A symmetry option has been introduced, that constrains members of any specified group to be identical ; in this case, the number of design variables is reduced to the number of element groups.

6. EXAMPLES.

The method proposed in section 3 has been tested against solutions to classical problems found in the literature.
The two first examples show clearly that the rate of convergence of the redesign procedure is not directly related to the size of the problem under consideration.

6.1. Four-Bar Pyramid (fig. 1.).

This very simple structure is subjected to the single loading case given table I(a). Constraints are placed on maximum stress (25000 psi), minimum area (0,1 in^2) and node displacement in z-direction (0,3 in).
The present results (table II-b,c) duplicate those of Taig and Kerr (ref. [3]) and of Gellatly and Berke (ref. [2]). Fig. 4 shows the strange pattern followed by the iteration procedure : the design seems to converge after 2 cycles and only after several more cycles does the rate of weight reduction accelerate till the final design (after 20 iterations) is generated.

6.2. - 72- Bar Four-Level Tower (fig.2).

This doubly symmetric tower is subjected to two loading cases (table II-a) . Symmetry is achieved by use of the input option, which reduces the number of design variables from 72 to 16. The stress limits are again 25000 psi with 0,1 in^2 minimum area. The displacements of the four uppermost nodes are limited to 0,25 in. in the x and y directions.
In spite of the larger size of the problem, convergence is very rapid and optimal design is reached in only five iterations (see fig. 4.b). The results (table II-b-c) are the same as those of Taig and Kerr (ref. [3])
For comparison, results obtained by Gellatly and Berke (ref. [2]) and Venkayya (ref. [1]) are also presented.

6.3. Cantilever Frame (fig. 3).

The 10 bar-truss is subjected to the single loading case indicated on fig. 3. The stress limit in all members is 25000 psi, with 0,1 in^2. minimum area. The node displacements in y - direction are prescribed to be less than 0,2 in. Table III shows the results obtained from the present method and, for comparison, from other methods (ref. [1], [2] , [3]).
In addition, stress constraints have been formulated in a similar way than for displacement constraints : each member stress is linearly expressed in terms of the inverses of the design variables. Corresponding results are given in table III under the title " Experimental Method". This method generates a design weighting 5060,8 lb. which exhibits the following particular characteristics. Member 6 is fully stressed, while being at its minimum area. Furthermore only one displacement constraint is exactly satisfied (node 1) while another prescribed displacement is close to its limiting value (node 3).

For the other designs, shown on table III, these two displacements reach simultaneously their limiting values.

7. CONCLUSION.

While using a mathematical programming algorithm, the method that was presented has the convergence characteristics of an optimality criterion approach. Except for the last example (cantilever frame), the same results as those of Taig and Kerr (ref. [3]) have been obtained for each analysis and redesign step. In addition, when there is only one active displacement constraint, the results of Gellatly and Berke (ref. [2]) are also identical to ours. In fact, all these methods are based on the same technique : by means of the virtual work theorem each limited displacement (or linear combination of displacements) is expressed in terms of the design variables. The resulting relations remain exact only in the case of a statically determinate structure; for a redundant structure, they become approximated. The redesign procedures are characterized by the algorithm used in order to resolve the ensuing linearized problem (14)(15).
The present method has the advantage of using a particularly suggestive algorithm : each path up to an "approached" optimum readily shows whether a constraint becomes active or not. Futhermore each point of this path is a feasible bounded point. That important feature allows the algorithm to be eventually stopped before reaching the optimum, in order to avoid a final divergence due to a too strong internal redundancy.

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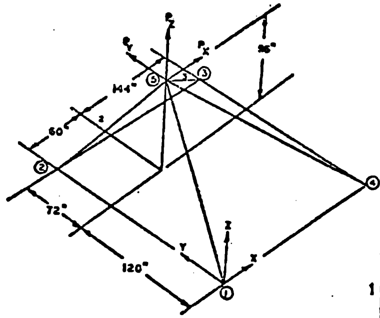
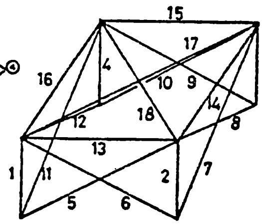
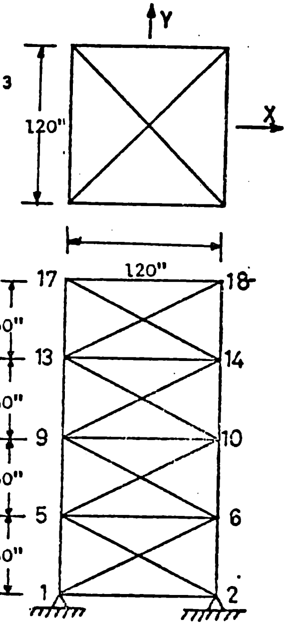


FIGURE 1 : FOUR-BAR PYRAMID



Typical Numbering



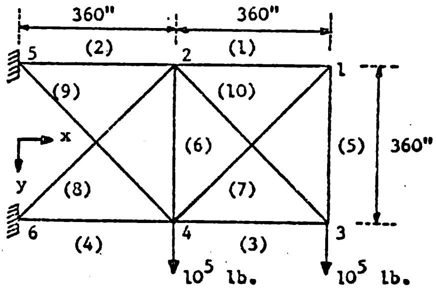
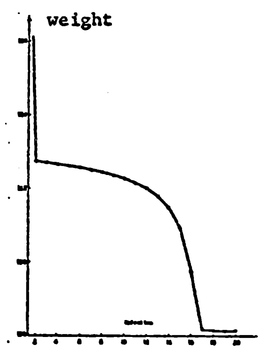
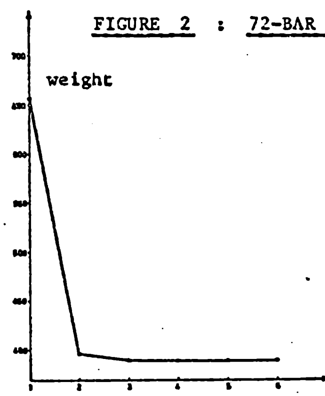


FIGURE 3 : CANTILEVER FRAME

FIGURE 2 : 72-BAR TOWER



(a) Four-Bar Pyramid



(b) 72-Bar Tower

FIGURE 4 : ITERATION HISTORY

TABLE I FOUR - BAR PYRAMID

(a) LOADING SYSTEM

NODE	X - DIRECTION	Y - DIRECTION	Z - DIRECTION
5	10 000	20 000	- 60 000

(b) ITERATION HISTORY

Iteration No.	Weight	Iteration No.	Weight	Iteration No.	Weight
1	168.48	8	142.21	15	134.75
2	143.63	9	141.82	16	128.78
3	143.46	10	141.34	17	120.80
4	143.27	11	140.75	18	120.75
5	143.02	12	139.97	19	120.73
6	142.82	13	138.92	20	120.73
7	142.54	14	137.37	21	120.73

(c) Final Design

Weight	Member 1	Member 2	Member 3	Member 4
120.73	0.100	3.893	0.747	2.510

TABLE II 72 - BAR TOWER

(a) LOADING SYSTEMS (lb)

LOAD CASE	NODE	X-DIRECTION	Y-DIRECTION	Z-DIRECTION
1	17	5 000	5 000	- 5 000
2	17	0	0	- 5 000
	18	0	0	- 5 000
	19	0	0	- 5 000
	20	0	0	- 5 000

(b) ITERATION HISTORY

ITERATION No.	WEIGHT		
	GELLATLY and BERKE	VENKAYYA	PRESENT METHOD + TAIG and KERR
1	656.77	656.8	656.77
2	416.07	478.6	387.31
3	406.21	455.0	379.77
4	399.06	446.9	379.63
5	396.82	445.5	379.66
8	395.97	391.5	
11		381.2	

(c) FINAL DESIGN

GROUP	MEMBERS	GELLATLY and BERKE	VENKAYYA	PRESENT METHOD + TAIG and KERR
1	1-4	1.464	1.818	1.897
2	5-12	0.521	0.523	0.516
3	13-16	0.100	0.100	0.100
4	17,18	0.100	0.100	0.100
5	19-12	1.024	1.246	1.280
6	23-30	0.542	0.524	0.515
7	31-34	0.100	0.100	0.100
8	35,36	0.100	0.100	0.100
9	37-40	0.552	0.611	0.507
10	41-43	0.608	0.532	0.520
11	49-52	0.100	0.100	0.100
12	53,54	0.100	0.100	0.100
13	55-58	0.149	0.161	0.157
14	59-66,	0.773	0.557	0.536
15	67-70	0.453	0.377	0.410
16	71,72	0.342	0.506	0.654

MEMBER No.	TAIG and KERR	VENKAYYA	GELLATLY and BERKE	PRESENT METHOD (*)	EXPERIMENTAL METHOD
1	0.10	0.13	0.10	0.10	0.10
2	31.98	30.42	31.35	30.73	30.52
3	15.43	14.90	15.60	14.73	15.22
4	22.57	23.41	20.03	23.94	23.20
5	0.57	0.10	0.24	0.10	0.55
6	0.58	0.10	0.14	0.10	0.10
7	0.10	0.19	0.10	0.10	0.10
8	22.76	21.08	22.21	20.95	21.03
9	21.82	21.08	22.06	20.84	21.53
10	6.44	8.70	8.35	8.54	7.46
TOTAL WEIGHT	5167.	5084.90	5112.17	5076.67	5060.85
No. OF ITERATIONS	32	25	19	14	19

(*) Taig and Kerr have also obtained this result by fixing member 5 at its minimum area.