DISCRETIZATION OF ROTATIONAL EQUILIBRIUM IN THE FINITE ELEMENT METHOD

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Summary

The theory of equilibrium elements [1,2] shows that their stiffness matrices may present a singular behavior due to the presence of mechanisms (deformation modes without strain energy). The origin of such difficulties is easily traced to the rigorous requirement of rotational equilibrium (symmetry of the stress tensor) and equivalently, if the discretization is performed on the basis of stress functions, to the C_1 continuity requirement involved. Moreover loss of diffusivity (reciprocity of surface traction distributions at interfaces) is incurred in an isoparametric coordinate transformation to curved boundaries, whenever preservation of C_1 continuity is at stake.

Both difficulties are resolved by enforcing rotational equilibrium only in weak form. First order stress functions are used to preserve rigorous translational equilibrium and diffusivity. They need only be C_o continuous, a property that remains invariant under isoparametric coordinate transformations.

The theory of discretized rotational equilibrium has been investigated in detail for membrane elements [3]. The paper is devoted to the more difficult case of axisymmetric elements.

1. AXISYMMETRIC EQUILIBRIUM EQUATIONS

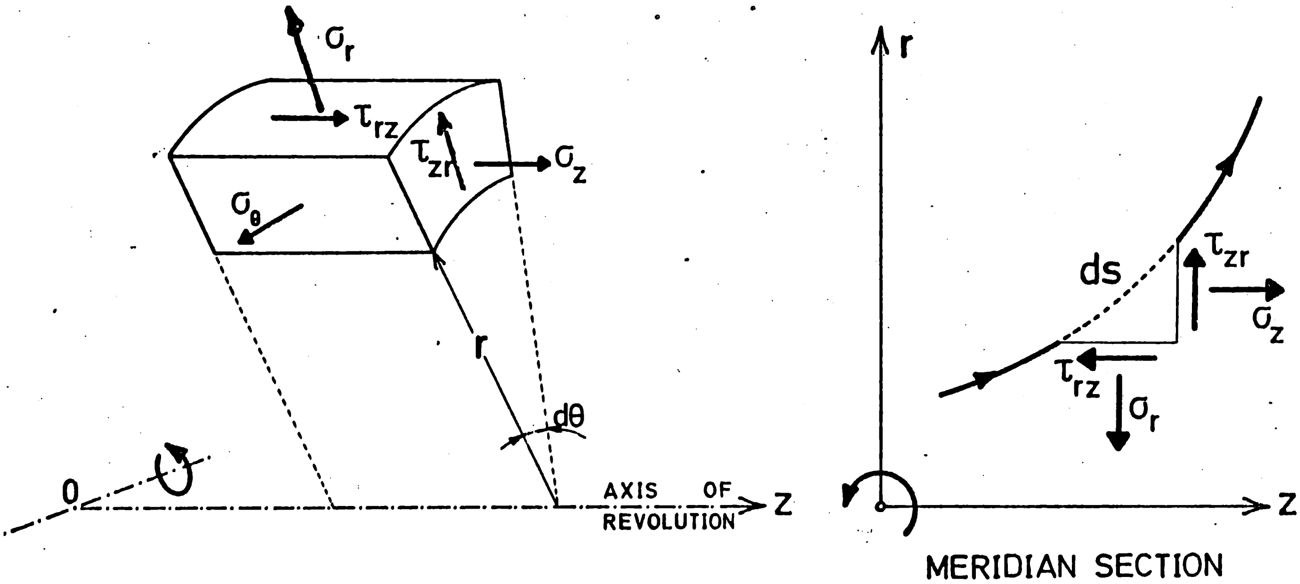

FIGURE 1

The translational equilibrium equations of the axisymmetric state of stress are conveniently presented in the following form

axial direction $$\frac{\partial}{\partial r}(r\,\tau_{rz}) + \frac{\partial}{\partial z}(r\,\sigma_z) = 0 \qquad (1)$$

radial direction $$\frac{\partial}{\partial r}(r\,\sigma_r) + \frac{\partial}{\partial z}(r\,\tau_{zr}) = \sigma_\theta \qquad (2)$$

a moment equilibrium of a slice dθ about an axis perpendicular to the mean meridian plane requires

$$d\theta\left[\oint r\,(z\,t_r - r\,t_z)ds - \iint z\,\sigma_\theta\,dr\,dz\right] = 0 \qquad (3)$$

where the curvilinear integral is around the boundary of the meridian cross section and the meridian surface tractions are given by

$$t_r \, ds = \tau_{zr} \, dr - \sigma_r \, dz \tag{4}$$

$$t_z \, ds = \sigma_z \, dr - \tau_{rz} \, dz \tag{5}$$

The hoopstress σ_θ gives a downward component due to the curvature that is responsible for the last term. Substitution of (4) and (5) into (3) and transformation of the curvilinear to a double integral yield

$$d\theta \iint \left[z \left\{ \frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial}{\partial z}(r\tau_{zr}) - \sigma_\theta \right\} - r \left\{ \frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) \right\} \right] drdz$$

$$+ \, d\theta \iint (\tau_{zr} - \tau_{rz}) \, rdrdz = 0$$

Taking (1) and (2) into account this reduces to

$$\iint (\tau_{zr} - \tau_{rz}) \, rdrdz = 0 \tag{6}$$

and, for an elementary surface of the meridian cross section, to the local rotational equilibrium condition

$$\tau_{zr} - \tau_{rz} = 0 \tag{7}$$

It should be observed that, even if this condition is not fulfilled, the axisymmetric ring of same meridian cross section is, by reason of symmetry, in rotational equilibrium about all axes.

2. A VARIATIONAL PRINCIPLE

We satisfy the axial equilibrium condition (1) by a first order stress function

$$\tau_{rz} = -\frac{1}{r}\frac{\partial \phi}{\partial z} \qquad \sigma_z = \frac{1}{r}\frac{\partial \phi}{\partial r} \tag{8}$$

that brings the axial component of surface traction to the simple form

$$t_z \, ds = \frac{1}{r} \left(\frac{\partial\phi}{\partial r} dr + \frac{\partial\phi}{\partial z} dz\right) = \frac{1}{r} d\phi \tag{9}$$

Because we first consider σ_θ to be directly determined through the radial equilibrium condition (2) we do not, at this stage, introduce another stress function. We also consider τ_{zr} separately from τ_{rz} but enforce the equilibrium condition (7) by means of a Lagrangian multiplier ω. The stress energy density is then considered to be a positive definite function of the arguments (σ_r , σ_z , σ_θ , $\frac{1}{2}$ (τ_{zr} + τ_{rz})) with the stress-strain properties

$$\varepsilon_r = \frac{\partial\Phi}{\partial\sigma_r} \qquad \varepsilon_z = \frac{\partial\Phi}{\partial\sigma_z} \qquad \varepsilon_\theta = \frac{\partial\Phi}{\partial\sigma_\theta}$$

$$\varepsilon_{rz} = \frac{\partial\Phi}{\partial\tau_{rz}} = \frac{\partial\Phi}{\partial\tau_{zr}} = \varepsilon_{zr} \tag{10}$$

The fact that Φ is a symmetrical function with respect to both shearing stresses ensures the symmetry of the corresponding shear strains; moreover translational equilibrium is assumed to hold. Thus the arguments τ_{rz} and σ_z must be expressed in terms of the stress function ϕ as in (8) and the hoopstress is expressed as in (2). The complementary energy principle then takes the following form

$$\iint \left[\Phi+\omega\left(\tau_{zr}+\frac{1}{r}\frac{\partial\phi}{\partial z}\right)\right] r dr dz - \oint \bar{u}\, r(\tau_{zr} dr-\sigma_r dz)+\bar{w} d\phi$$

$$\text{stationary} \tag{11}$$

the displacements being assumed to be given on the boundary of the meridian cross section.

The Euler equations resulting from unconstrained variations on σ_r , τ_{zr} and ϕ are respectively

$$\varepsilon_r = \frac{\partial}{\partial r} (r \ \varepsilon_\theta) \tag{12}$$

$$\omega + \varepsilon_{rz} = \frac{\partial}{\partial z} (r \ \varepsilon_\theta) \tag{13}$$

$$\frac{\partial}{\partial z} (\varepsilon_{rz} - \omega) = \frac{\partial}{\partial r} \varepsilon_z \tag{14}$$

Both σ_r and τ_{zr} give the same natural boundary condition

$$\bar{u} = r \ \varepsilon_\theta \tag{15}$$

while for ϕ we obtain

$$d\bar{w} = (\varepsilon_{rz} - \omega) \, dr + \varepsilon_z \, dz \tag{16}$$

3. SOLUTION OF THE VARIATIONAL EQUATIONS

At this stage, in preparation of the imposition of constraints on rotational equilibrium, we consider $\omega(r,z)$ as a given function. Setting

$$u = r \ \varepsilon_\theta \tag{17}$$

The Euler equation (12) becomes

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{18}$$

and u is recognized to be the radial displacement. Euler equation (14) is solved by introducing a function $w(r,z)$ such that

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{19}$$

$$\varepsilon_{rz} - \omega = \frac{\partial w}{\partial r} \tag{20}$$

this new function is thus the axial displacement and, combining (20) with the last Euler equation (13)

$$\varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{21}$$

$$\omega = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \tag{22}$$

so that the Lagrangian multiplier is, as expected, the material rotation about an axis normal to the meridian plane.

We conclude that for unconstrained variations on σ_r , τ_{zr} and ϕ , the compatibility equations are satisfied and the displacement field obtained, satisfies the given boundary data and has the given $\omega(r,z)$ function as its rotation field. This however requires an obvious global compatibility condition on the data :

$$\oint \bar{u}\, dr + \bar{w}\, dz = 2 \iint \omega\, dr\, dz \tag{23}$$

Equation (22) appears in this context as one of the differential equations governing the displacement field.

To obtain the second, we must express the equilibrium equations that are satisfied in terms of displacements.

For simplicity take the isotropic linear stress-strain laws in the form

$\sigma_r = 2\, G\, (\varepsilon_r + \eta\, \varepsilon)$ where G is the shear modulus

$\sigma_\theta = 2\, G\, (\varepsilon_\theta + \eta\, \varepsilon)$ $\qquad \eta = \nu/(1-2\nu)$

$\sigma_z = 2\, G\, (\varepsilon_z + \eta\, \varepsilon)$ $\qquad \varepsilon = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$

$\frac{1}{2} (\tau_{rz} + \tau_{zr}) = 2\, G\, \varepsilon_{rz}$

The last is the only one generated by the complementary energy density as a symmetrical function. To separate the two shear stresses we introduce a shear strain unbalance function ζ

$$\tau_{rz} = 2\,G\,(\varepsilon_{rz} + \zeta) \qquad \tau_{zr} = 2\,G\,(\varepsilon_{rz} - \zeta)$$

Replacing the strains in terms of displacements through equations (17),(18) and (21) the equilibrium equations (1) and (2) can now be placed in the form

$$(1-2\nu)\frac{\partial}{\partial r}\{\,r(\zeta-\omega)\,\} + (1-\nu)\frac{\partial}{\partial z}(r\,\varepsilon) = 0$$

$$(1-\nu)\,r\,\frac{\partial\varepsilon}{\partial r} - (1-2\nu)\frac{\partial}{\partial z}\{\,r(\zeta-\omega)\,\} = 0 \qquad (24)$$

The elimination of $(\zeta-\omega)$ produces the second differential equation governing with (22), and the boundary conditions $u = \bar{u}$, $v = \bar{v}$, the displacement field

$$\frac{\partial^2}{\partial z^2}(r\varepsilon) + \frac{\partial}{\partial r}\left(r\,\frac{\partial\varepsilon}{\partial r}\right) = 0 \qquad (25)$$

$$\varepsilon = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \qquad (26)$$

If, on the contrary, ε is eliminated between equations (24), we obtain a differential equation satisfied by the shear strain unbalance when the rotation ω is imposed

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\,r(\zeta-\omega)\,\right]\right\} + \frac{\partial^2}{\partial z^2}(\zeta-\omega) = 0 \qquad (27)$$

When ω is not given, but an unconstrained Lagrangian multiplier, its variational equation requires the shear unbalance to vanish

$$r\,\tau_{zr} + \frac{\partial\phi}{\partial z} = r\,(\tau_{zr} - \tau_{rz}) = -\,4\,Gr\zeta = 0$$

and (27) becomes the differential equation governing the distribution of the material rotation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(\omega\ r)\right]+\frac{\partial^2}{\partial z^2}\omega=0 \tag{28}$$

It is worth noticing that this equation is not satisfied by $\omega_\lambda=\omega_{\lambda o}$ a constant but that the simple solutions independent of z are

$$\omega=\frac{\omega_{-1}}{r}+\omega_1 r \tag{29}$$

4. THE ZERO ENERGY STATE

Our separation of the two shear stresses creates a well-defined state of stress for which the complementary energy vanishes. The energy density being a positive definite function, the conditions for zero energy are

$\sigma_z \equiv 0$, whence from (1)

$$r\ \tau_{rz}=f(z)$$

$\sigma_r \equiv 0$ and $\sigma_\theta \equiv 0$, whence from (2)

$$r\ \tau_{zr}=g(r)$$

$\tau_{rz}+\tau_{zr}\equiv 0$ whence

$$-f(z)=g(r)=\gamma \quad \text{a constant}$$

The zero energy stress distribution, in translational but not rotational equilibrium is thus characterized by the shear stresses distribution

$$\tau_{rz}=-\frac{\gamma}{r} \qquad \tau_{zr}=+\frac{\gamma}{r}$$

Any imposition of a global rotational equilibrium condition

$$\iint \omega\,(\tau_{zr} - \tau_{rz})\, r dz d\omega = 0$$

where ω is one of the simple solutions (29) will eliminate the possibility of such a situation to prevail.

5. STRESS FUNCTIONS DISCRETIZATION

In the discretization we consider the presence of σ_θ in the equilibrium equation (2) as analogous to a body load. We subdivide the stress distribution of σ_r and τ_{zr} in a particular solution taking into account a non zero hoop stress and a general solution without hoop stress. This can be done conveniently by introducing two new first order stress functions as follows :

$$\sigma_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} - \frac{\partial\lambda}{\partial z} \qquad \tau_{zr} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial\lambda}{\partial r} \qquad (30)$$

from which we find from (2)

$$\sigma_\theta = -\frac{\partial\lambda}{\partial z} \qquad (31)$$

and for the radial surface traction

$$t_r\, ds = \frac{1}{r}\, d\psi + d\lambda \qquad (32)$$

C_o continuity of ϕ will thus ensure reciprocity of the axial surface tractions and C_o continuity of ψ and λ , reciprocity of the radial surface tractions.
The stress functions will now be discretized as polynomials in r and z and, to obtain similar surface traction distributions for ψ and λ, the degree of λ will have to be one unit less than that of ψ .
The first model corresponds to

$$\phi = \phi_o + \phi_1 r + \phi_2 z + \phi_3 r^2 + 2\phi_4 rz + \phi_5 z^2$$

$$\psi = \psi_o + \psi_1 r + \psi_2 z + \psi_3 r^2 + 2\psi_4 rz + \psi_5 z^2 \qquad (33)$$

$$\lambda = \lambda_o + \lambda_1 r + \lambda_2 z$$

The constants ϕ_o and ψ_o are not producive of stresses, they will only play a role in organizing diffusivity in a finite element of triangular meridian section by expressing, in the usual way, the functions ϕ and ψ by interpolation functions related to the local values at vertices and mid-edges. When such local values are taken as nodal values at interfaces, C_o continuity follows. In the case of λ , we see that neither λ_o, nor λ_1, produce any hoop stress. As the general equilibrium state without hoop stress is already accounted for by ψ , these terms can be dropped. However they are needed again when organizing diffusivity, this time by expressing λ in terms of interpolation functions related to the three vertex values.

We now define the generalized boundary loads associated with the linear distributions of rt_r and rt_z on a slice $d\theta$.

Let $2c_{ij}$ denote the length of side ij of the triangle and the distance s be measured in anticlockwise sense from i to j with origin at the mid point. The non dimensional distance

$$\sigma = \frac{s}{c_{ij}}$$

will vary in the interval $[-1 , +1]$. We then introduce

$$R_{ij} = \int_i^j rt_r \, ds \quad \text{and} \quad Z_{ij} = \int_i^j rt_z \, ds \qquad (34)$$

the total, respectively radial and axial, loads associated with the surface tractions on a slice per unit angle θ. Correspondingly we introduce the total reduced moments

$$\rho_{ij} = \int_i^j rt_r \, \sigma \, ds \quad \text{and} \quad \zeta_{ij} = \int_i^j rt_z \, \sigma \, ds \quad (35)$$

We then find easily that along ij

$$rt_r = \frac{1}{2c_{ij}} R_{ij} + \frac{3\sigma}{2c_{ij}} \rho_{ij}$$

$$rt_z = \frac{1}{2c_{ij}} Z_{ij} + \frac{3\sigma}{2c_{ij}} \zeta_{ij} \qquad (36)$$

and furthermore

$$r = \frac{1}{2} r_i \ (1-\sigma) + \frac{1}{2} r_j \ (1+\sigma) \qquad dr = \frac{1}{2} (r_j - r_i) d\sigma$$

$$z = \frac{1}{2} z_i \ (1-\sigma) + \frac{1}{2} z_j \ (1+\sigma) \qquad dz = \frac{1}{2} (z_j - z_i) d\sigma$$

From this it becomes possible to compute the matrix S relating the generalized boundary loads to the <u>active</u> stress parameters listed in the vector s :

$$s^T = (\phi_1 \ \phi_2 \ \phi_3 \ \phi_4 \ \phi_5 \ \psi_1 \ \psi_2 \ \psi_3 \ \psi_4 \ \psi_5 \ \lambda_2)$$

through the relation

$$g = S \ s \qquad (37)$$

where in g the generalized loads are conventionally sequenced

$$g^T = (R_{12} \ R_{23} \ R_{31} \ Z_{12} \ Z_{23} \ Z_{31} \ \rho_{12} \ \rho_{23} \ \rho_{31} \ \zeta_{12} \ \zeta_{23} \ \zeta_{31}) \qquad (38)$$

The first row of S is obtained by replacing in the definition of R_{12}, the expression of $rt_r ds$ in terms of the stress parameters

$$R_{12} = \int_1^2 d\psi + r d\lambda = \psi_1 \int_1^2 dr + 2\psi_3 \int_1^2 r dr + 2\psi_4 \int_1^2 z \ dr$$

$$+ \psi_2 \int_1^2 dz + 2\psi_4 \int_1^2 r dz + 2\psi_5 \int_1^2 z dz + \lambda_2 \int_1^2 r dz$$

There follows for this first row

$$\{\ 0\ 0\ 0\ 0\ 0 \int_1^2 dr \int_1^2 dz \int_1^2 d(r^2)\ 2\int_1^2 d(rz) \int_1^2 d(z^2) \int_1^2 rdz\ \}$$

The only geometrical integral that is not immediatly expressible in terms of the vertex coordinates is the last

$$\int_1^2 rdz = \frac{1}{2}(z_2-z_1)\{\frac{1}{2}r_1 \int_{-1}^{1}(1-\sigma)d\sigma+\frac{1}{2}r_2 \int_1^1 (1+\sigma)d\sigma\}=\frac{1}{2}(z_2-z_1)(r_1+r_2)$$

The other rows follow by similar procedures.

We now define the (weak) generalized boundary displacements, conjugate to the loads, by expressing the virtual work at each partial boundary in canonical form.

$$\int_1^2 (u\ rt_r+w\ rt_z)ds = R_{12}U_{12}+Z_{12}W_{12}+\rho_{12}\alpha_{12} + \zeta_{12}\ \beta_{12} \quad (39)$$

Substituting the surface traction distributions in terms of the generalized loads, as in (36) and comparing, there follows

$$U_{12} = \frac{1}{2c_{12}} \int_1^2 u\ ds \qquad W_{12} = \frac{1}{2c_{12}} \int_1^2 ds \quad (40)$$

the ordinary averages of displacements, and

$$\alpha_{12} = \frac{3}{2c_{12}} \int_1^2 u\sigma ds \qquad \beta_{12} = \frac{3}{2c_{12}} \int_1^2 w\sigma ds \quad (41)$$

which are "moments" of the displacement distribution. Similar definitions ensue for the two other partial boundaries.

The generalized boundary displacements are sequenced in the corresponding order as that choosen for g

$$q^T = (U_{12}\ U_{23}\ U_{31}\ W_{12}\ W_{23}\ W_{31}\ \alpha_{12}\ \alpha_{23}\ \alpha_{31}\ \beta_{12}\ \beta_{23}\ \beta_{31}) \tag{42}$$

so that the scalar product $q^T g$ reproduces the complete canonical expansion of virtual work :

$$\oint (u\ rt_r + w\ rt_z)\ ds = q^T g = q^T\ S\ s \tag{43}$$

We have now available the discretized form of the last term in the variational principle (11).
For linear homogeneous stress-strain relations, Φ will be a quadratic form of its arguments, and, after discretization of the stress distribution by means of the stress functions, the complementary energy becomes a quadratic form

$$\iint \Phi\ rdrdz = \frac{1}{2}\ s^T\ Fc \tag{44}$$

in the active stress parameters. This quadratic form is merely non negative, because of the existence of the zero-energy state.
Indeed this state is included in our approximation as corresponding to the choice of parameters :

$$\psi_1 = \phi_2 = \gamma \qquad \text{all other parameters zero.}$$

This means that the stress parameter vector

$$s_o^T = \gamma\ (0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0) \tag{45}$$

corresponds to $s_o^T\ F\ s_o = 0$, and, the flexibility matrix F being non negative, to

$$F\ s_o = 0 \tag{46}$$

6. ROTATION DISCRETIZATION

There remains to discretize the part corresponding to the Lagrangian multiplier

$$\iint \omega \left(\frac{\partial\psi}{\partial r} + \frac{\partial\phi}{\partial z} + r\,\frac{\partial\lambda}{\partial r}\right) dr\, dz = s^T R\, h \qquad (47)$$

a bilinear form in the active stress parameters and whatever coordinates h_i are used in an expansion of ω in interpolation functions.
The linear independence of the columns of the matrix R of the bilinear form will appear later as a necessary condition for a solution to the discretized problem.
This imposes limitations on the choice of a discretized ω.
It easily established that if n+1 is the polynomial degree of ψ and ϕ (and n that of λ), the columns of R are linearly independent when the polynomial degree of ω is not higher than n. The proof can be based on the fact that under the opposite assumption : columns of R linearly dependent, we reach a contradiction.
If the columns of R are linearly dependent there exists a non zero vector h^* such that

$$R\, h^* = 0 \rightarrow s^T R\, h^* = 0 \quad \text{for all } s.$$

Thus there would exist a non identically zero polynomial ω^* of degree not higher than n, such that

$$\iint \omega^* \left(\frac{\partial\psi}{\partial r} + \frac{\partial\phi}{\partial z} + r\,\frac{\partial\lambda}{\partial r}\right) dr\, dz = 0$$

for arbitrary polynomials ψ and ϕ of degree n+1, λ of degree n.
In particular we would have

$$\iint \omega^* \frac{\partial\psi}{\partial r}\, dr\, dz = 0$$

for an arbitrary polynomial ψ of degree n+1. However, as we may choose ψ as a particular integral of

$$\frac{\partial \psi}{\partial r} = \omega^*$$

we reach the contradiction

$$\iint \omega^{*2} \, dr \, dz = 0 \quad \text{for } \omega^* \text{ not identically zero.}$$

On the basis of the discretization (33) of the stress functions, we may thus take

$$\omega = \omega_0 + \omega_1 r + \omega_2 z \tag{48}$$

$$h^T = (\omega_0 \quad \omega_1 \quad \omega_2)$$

7. SOLUTION OF THE DISCRETIZED VARIATIONAL EQUATIONS

The discrete form

$$\frac{1}{2} s^T F s + s^T R h - q^T S s \quad \text{stationary} \tag{49}$$

of the variational principle (11), where the generalized displacements q are assumed to be given, yields as variational equations

$$\begin{aligned} F s + R h &= S^T q \\ R^T s &= 0 \end{aligned} \tag{50}$$

The first system of equations is generated by variations on ε, the second by the variations on h. Although F has been seen to be singular because of (46), the matrix

$$\begin{pmatrix} F & R \\ R^T & 0 \end{pmatrix}$$

of the system is not. The proof of this assertion consists in showing that the homogeneous system

$$F\,s + R\,h = 0 \qquad R^T s = 0$$

has only the trivial solution. Premultiply the first equation by s^T and use the second equation in transposed form to obtain

$$s^T F\,s = 0 \;\rightarrow\; s = \gamma\, s_0$$

The proof is then achieved if we succeed in showing that $R^T s_0 \neq 0$, because then

$$\gamma\, R^T s_0 = 0 \;\rightarrow\; \gamma = 0 \;\rightarrow\; s = 0$$

and then the first equation requires

$$R\,h = 0 \;\rightarrow\; h = 0 \quad \text{because } R \text{ has linearly independent columns.}$$

Let us examine in succession the influence of the different terms of the polynomial expansion (48) on the condition $R^T s_0 \neq 0$.
For $h_1^T = (1 \quad 0 \quad 0)$, that is, using only the constant term of ω, the first column $r_1 = Rh_1$ of R is found to be

$$r_1^T = \iint drdz\,(0 \quad 1 \quad 0 \quad 2\hat{r} \quad 2\hat{z} \quad 1 \quad 0 \quad 2\hat{r} \quad 2\hat{z} \quad 0 \quad 0)$$

where $\hat{r} = \frac{1}{3}(r_1 + r_2 + r_3)$ $\quad \hat{z} = \frac{1}{3}(z_1 + z_2 + z_3)$

are the coordinates of the center of area of the meridian section. Whence, by reference to (45)

$$r_1^T s_0 = 2\gamma \iint dr\,dz = 1 \qquad (51)$$

if the zero energy vector is "normed" by the condition

$$\gamma = (2 \iint dr\ dz)^{-1} \quad .$$

Hence the condition $R^T s_0 \neq 0$ will be satisfied if the ω_0 term is retained.

For $h_2^T = (0 \;\; 1 \;\; 0)$, selecting the term $\omega_1 r$, we find

$$r_2^T = \iint dr\ dz\ (0 \;\; \hat{r} \;\; 0 \;\; 2\overline{rr} \;\; 2\overline{rz} \;\; \hat{r} \;\; 0 \;\; 2\overline{rr} \;\; 2\overline{rz} \;\; 0 \;\; 0)$$

where

$$\overline{rr} \iint dr\ dz = \iint r^2\ dr\ dz$$

$$\overline{rz} \iint dr\ dz = \iint rz\ dr\ dz$$

and

$$r_2^T\ s_0 = \hat{r} > 0 \quad \text{with the same norm of } s_0 \qquad (52)$$

Hence again, it is sufficient to retain the $\omega_1 r$ term in ω to satisfy the condition.

However for the last term $\omega_3 z$, $h^T = (0 \;\; 0 \;\; 1)$,

$$r_3^T = \iint dr\ dz\ (0 \;\; \hat{z} \;\; 0 \;\; 2\overline{rz} \;\; 2zz \;\; \hat{z} \;\; 0 \;\; 2\overline{rz} \;\; 2\overline{zz} \;\; 0 \;\; 0)$$

and $\quad r_3^T\ s_0 = \hat{z}$

which depends on the origin of axes and can be made to vanish by letting the r axis pass through the center of area of the element.

In conclusion, the variational equations will be invertible if the discretization of the rotation contains either the ω_0 term, or the $\omega_1 r$ term.

The structure of the inverted matrix is

$$\begin{pmatrix} F^{\#} & R^{\#} \\ R^{\#T} & G^{\#} \end{pmatrix} \qquad F^{\#} = (F^{\#})^T \qquad G^{\#} = (G^{\#})^T$$

Postmultiplying by the original matrix, we find the relations

$$F^{\#} F + R^{\#} R^T = (s/s) \qquad F^{\#} R = 0$$
$$(R^{\#})^T F + G^{\#} R^T = 0 \qquad R^{\#T} R = (h/h) \tag{53}$$

where (s/s) and (h/h) denote identity matrices of respectively the size of s and of h. It is seen that $F^{\#}$ typically satisfies a pseudo-inverse relationship with F

$$F^{\#} F F^{\#} = F^{\#}$$

from which it can be concluded that it is also a non negative matrix.
In practice the inversion

$$s = F^{\#} S^T q \tag{54}$$

$$h = R^{\#T} S^T q \tag{55}$$

is obtained numerically. It gives simultaneously the values of the active stress parameters, thus the state of stress, and the rotation field of the element, when the boundary displacements are given.
The stiffness matrix of the element is obtained as a consequence of (37) and (54) in the form

$$g = K q \qquad K = SF^{\#} S^T \tag{56}$$

The determination of the stiffness matrix allows the use of the same assembling software as in the case of elements based on a discretized displacement field. The nodal displacement identification is here replaced by the identification of the weak generalized displacements at the interfaces and insures diffusivity instead of conformity.

8. THE AXIAL RIGID BODY MODE

In principle the stiffness matrix of an axisymmetric element, being representative of a complete "ring", should contain only one rigid body mode, the axial translation mode.

Any radial translation of the meridian section should generate hoop stresses and deformation energy. Likewise, rotation of the meridian section should generate twisting energy. It is easily verified that the axial translation mode is correctly built into the model. If we input

$$w = w_0 \qquad \text{a constant}$$

into the definitions (40) and (41) of the generalized displacements, we find a rigid body mode vector

$$q_0^T = w_0 \, (0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0) \qquad (57)$$

that should generate no loads and consequently satisfy

$$K \, q_0 = 0$$

In fact it does so because it already satisfies $S^T q_0 = 0$. We may prove it by showing that $q_0^T \, S \, s = 0$ for all s vectors or, in other terms, by reverting to the discretization (43) of virtual work at the boundary, that

$$\oint (u \; rt_r + w \; rt_z) ds = \oint u(d\psi + r d\lambda) + w \; d\phi = 0$$

for any state of discretized stress, when $u \equiv 0$ and $w = w_0$. This follows obviously for any discretized model where the stress function ϕ is single-valued

$$\oint d\phi = 0 \quad .$$

9. SELF STRESSINGS

To see whether this axial rigid body mode is the only solution of problem

$$S^T q = 0 \tag{58}$$

we can use the algebraic property

$$n(s) + n(r) = n(g) + n(x) \tag{59}$$

linking the number $n(s)$ of columns of S, $n(r)$ of linearly independent solutions of our problem, $n(g)$ number of rows of S and $n(x)$, number of linearly independent solutions of the homogeneous adjoint problem

$$S\, x = 0 \tag{60}$$

This last problem is that of the so-called self-stressing states of the element, we look for the non zero stress states that produce no boundary loads, that is no surface tractions at all.

In the model proposed under section 5 it is easily shown that no self-stressings exist. For if there are no boundary tractions, we must have by integration of (2)

$$\iint \sigma_\theta \, dr\, dz = \oint r t_{zr}\, dr - r\sigma_r dz = \oint r\, t_r\, ds = 0 \tag{61}$$

and, consequently,

$$\sigma_\theta = - \lambda_2 = 0$$

As λ_0 and λ_1 are improductive, we may take $\lambda \equiv 0$. Then, the vanishing of boundary tractions requires

$$d\phi = 0 \qquad d\psi = 0 \qquad \text{on boundary}$$

so that both stress functions must reduce to their improductive constant terms.
Since for the present model n(s) = 11 and n(g) = 12, we have n(r) = 1 and q_0 will be the only non trivial solution to problem (58).

10. MECHANISMS

The other possible solutions to the homogeneous problem

$$K\ q = S\ F^{\#}\ S^T\ q = 0$$

may be termed kinematical deformation modes or "mechanisms". They consist in boundary displacements that would normally deform the ring and create strain energy but do, in fact, produce no virtual work because of what may be considered as a deficiency in the model. Since $F^{\#}$ is non negative, such modes are in fact solutions of

$$F^{\#}\ S^T\ q = 0$$

distinct from (58). We must therefore look after solutions of problem

$$F^{\#}\ m = 0 \tag{62}$$

and, having found then, look after the solutions of the inhomogeneous problem

$$S^T\ q = m \tag{63}$$

From the first of equations (53) in transpose we obtain that if m satisfies (62), it satisfies also

$$R\, R^{*T}\, m = m$$

so that any solution m is necessarily a linear combination of the columns of R. Furthermore, from the second of equations (53), we see that all columns of R are solutions and, those columns being linearly independent, we have all possible mechanisms by looking after the solutions of

$$S^T\, q = R\, h \qquad h \text{ arbitrary} \qquad (64).$$

The necessary and sufficient condition for the existence of solutions, is that the right-hand side be orthogonal to all the solutions of the homogeneous adjoint problem (60)

$$x^T\, R\, h = 0 \qquad \text{all self-stressings } x \qquad (65)$$

In the present model, there is no self-stressing and equation (64) has a solution, a mechanism, for any choice of h. Thus any weak enforcement of the rotational equilibrium condition (7) will create a mechanism. On the other hand at least one enforcement based on either the constant rotation field ω_0, or on the field $\omega = \omega_1 r$, is necessary to prevent the zero energy state. This is a characteristic weakness of the present model, that has however no counterpart in the simple two-dimensional membrane case. It remains to be seen whether this inconvenience will disappear after assembling at least two elements together.
The following remarks are pertinent to this last aspect :

1. If one uses the complete rotation field (48), the linear function

$$r(\tau_{zr} - \tau_{rz}) = \frac{\partial\psi}{\partial r} + \frac{\partial\phi}{\partial z} + r\,\frac{\partial\lambda}{\partial r} = L(r,z)$$

$$= (\psi_1 + \phi_2) + (2\psi_3 + 2\phi_4 + \lambda_1)r + 2(\psi_4 + \phi_5)z$$

submitted to the constraints

$$\iint L \, dr \, dz = 0 \qquad \iint r \, L \, dr \, dz = 0 \qquad \iint z \, L \, dr \, dz = 0$$

must vanish completely and rotational equilibrium is enforced <u>exactly</u>. We thus retrieve a pure equilibrium model with three mechanisms, an interpretation of which can be obtained as follows. Introduce the barycentric coordinates L_i, defined by

$$\begin{aligned} 1 &= L_1 + L_2 + L_3 \\ z &= z_1 L_1 + z_2 L_2 + z_3 L_3 \\ r &= r_1 L_1 + r_2 L_2 + r_3 L_3 \end{aligned} \qquad (66)$$

and express the stress functions symmetrically as

$$\phi = \phi_1 L_1^2 + \phi_2 L_2^2 + \phi_3 L_3^2 + 2\phi_{12} L_1 L_2 + 2\phi_{23} L_2 L_3 + 2\phi_{31} L_3 L_1$$

$$\psi = \psi_1 L_1^2 + \psi_2 L_2^2 + \psi_3 L_3^2 + 2\psi_{12} L_1 L_2 + 2\psi_{23} L_2 L_3 + 2\psi_{31} L_3 L_1 \qquad (67)$$

$$\lambda = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3$$

(the coefficients ϕ_i , ψ_i , λ_i bear no direct relationship with the preceding ones). If $A = \iint dr \, dz$ denotes the area of the triangle, it is easily found that

$$2 A \frac{\partial L_1}{\partial z} = r_2 - r_3 \qquad 2 A \frac{\partial L_1}{\partial r} = z_3 - z_2 \qquad (68)$$

and the other derivatives follow by cyclic permutation. The quantity

$$\frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial r} + r \frac{\partial \lambda}{\partial r} \qquad (69)$$

is then easily expressed as a linear homogeneous function of the L_i and its complete vanishing requires the vanishing of the coefficient of L_1

$$\phi_1(r_2-r_3)+\phi_{12}(r_3-r_1)+\phi_{31}(r_1-r_2)+\psi_1(z_3-z_2)+\psi_{12}(z_1-z_3)+\psi_{31}(z_2-z_1)$$

$$+\frac{r_1}{2}\{\lambda_1(z_3-z_2)+\lambda_2(z_1-z_3)+\lambda_3(z_2-z_1)\}=0 \qquad (70)$$

and those of L_2 and L_3 that follow by cyclic subscript permutations.

Equation (70) is now reinterpreted as a constraint between boundary loads in the vicinity of vertex 1.

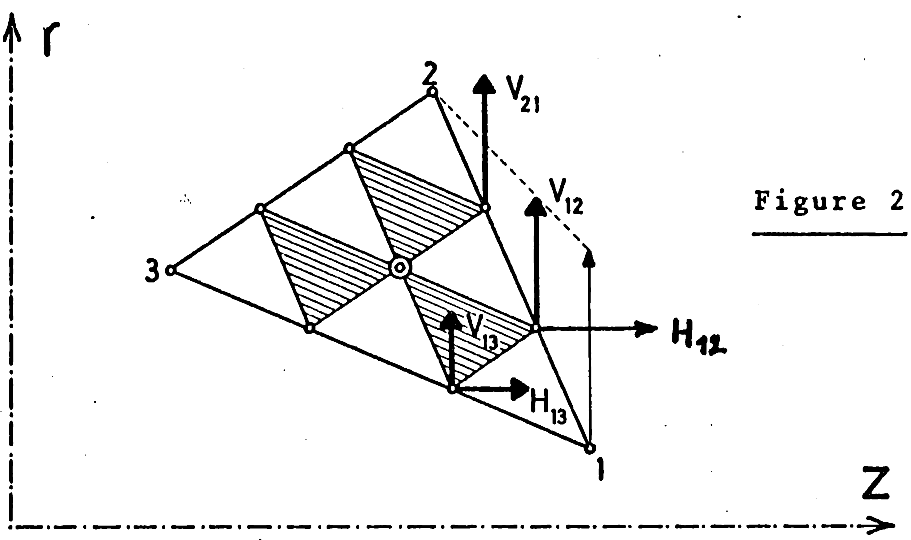

Figure 2

Along boundary 12, where $L_3 = 0$, $dL_1 = -dL_2$, $ds = 2c_{12}dL_2$, we have

$$rt_r ds = 2\psi_1 L_1 dL_1 + 2\psi_2 L_2 dL_2 + 2\psi_{12}(L_1 dL_2 + L_2 dL_1)$$

$$+ (r_1 L_1 + r_2 L_2)(\lambda_1 dL_1 + \lambda_2 dL_2)$$

or $$c_{12} r t_r = -\psi_1 L_1 + \psi_2 L_2 + \psi_{12}(L_1 - L_2) + \frac{1}{2}(r_1 L_1 + r_2 L_2)(\lambda_2 - \lambda_1)$$

By setting $L_1=1$, $L_2=0$, in this relation we obtain the resultant load V_{12}, applied at one third of the edge from 1 and due to the linear rt_r distribution sketched on figure 2.

$$V_{12} = \psi_{12} - \psi_1 + \frac{1}{2} r_1 (\lambda_2 - \lambda_1)$$

The complementary distribution has the resultant

$$V_{21} = \psi_2 - \psi_{12} + \frac{1}{2} r_2 (\lambda_2 - \lambda_1)$$

obtained by setting $L_2 = 1$ and $L_1 = 0$. By cyclic permutation of this last result we obtain also

$$V_{13} = \psi_1 - \psi_{13} + \frac{1}{2} r_1 (\lambda_1 - \lambda_3)$$

In a similar fashion we can obtain from the rt_z distribution

$$H_{12} = \phi_{12} - \phi_1 \qquad \text{and} \qquad H_{13} = \phi_1 - \phi_{13}$$

We can then observe that the condition for the resultant moment of V_{12}, V_{13}, H_{12} and H_{13} with respect to the barycenter of the element to vanish

$$\frac{1}{3} \{ V_{12}(z_1-z_3)+V_{13}(z_1-z_2)-H_{12}(r_1-r_3)-H_{13}(r_1-r_2) \} = 0$$

turns out to be identical to the requirement (70). Hence, as sketched on the figure, the element behaves as if made of three parts articulated at the barycenter. The situation is exactly similar to that of the pure equilibrium membrane element of same degree. In that case however, the rotation of the element as a whole about its barycenter is a rigid body mode on its own right and the relative rotations of the parts only represent two mechanisms. Here this global rotation is also a mechanism as it represents an energyless torsion of the ring.

The interpretation of the mechanisms yields at the same time the answer to the problem of their inhibition by the composite element technique

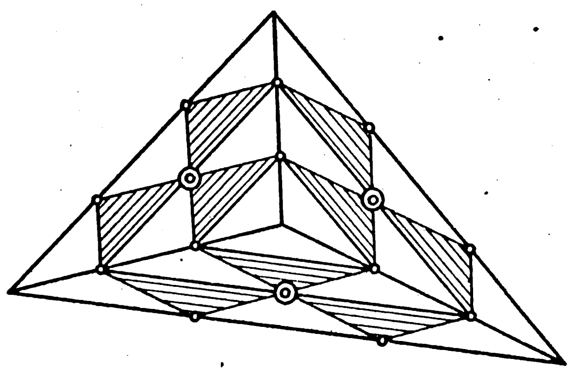

Figure 3

Locking of mechanisms by composite element technique.

2. If we really discretize rotational equilibrium by restricting the rotation field to one of the terms ω_o or $\omega_1 r$, necessary to prevent the zero energy state, the element will present a single mechanism.
In the case of $\omega=\omega_o$, only the average value of expression (69) must vanish.
The corresponding requirement follows by taking the arithmetic mean of the 3 equations of type (70) and is reinterpreted as a rigid body rotation of the meridian section about the barycenter; this represents a pure twisting mechanism of the ring. It is inhibited as soon as we assemble two elements with barycenters of different z coordinate.

11. HIGHER ORDER APPROXIMATIONS. THE LINEAR HOOP STRESS MODEL

The stress functions are of higher degree; complete cubics (in barycentric coordinates)

$$\phi = \phi_1 L_1^3 + \phi_2 L_2^3 + \phi_3 L_3^3$$
$$+ \phi_{12} L_1^2 L_2 + \phi_{21} L_2^2 L_1 + \phi_{23} L_2^2 L_3 + \phi_{32} L_3^2 L_2 + \phi_{31} L_3^2 L_1 + \phi_{13} L_1^2 L_3$$
$$+ \phi_{123} L_1 L_2 L_3$$

$$\psi = \psi_1 L_1^3 + \ldots$$

for ϕ and ψ , complete quadratic for λ

$$\lambda = \lambda_1 L_1^2 + \lambda_2 L_2^2 + \lambda_3 L_3^2 + 2\lambda_{12} L_1 L_2 + 2\lambda_{23} L_2 L_3 + 2\lambda_{31} L_3 L_1$$

this corresponds in cartesian coordinates to

$$\lambda = \gamma_o + \gamma_1 r + \gamma_2 z + \gamma_3 r^2 + 2\gamma_4 rz + \gamma_5 z^2$$

and the hoop stress

$$\sigma_\theta = -\frac{\partial \lambda}{\partial z} = -(\gamma_2 + 2\gamma_4 r + 2\gamma_5 z)$$

can have a linear distribution. The coefficients $(\gamma_o, \gamma_1, \gamma_3)$ are in fact improductive and may be cancelled at will. The number of active stress parameters is thus, discounting one improductive in ϕ , one in ψ , and three in λ , n(s)=21. The parabolic distributions of rt_r and rt_z require a total of 6 generalized loads per side, a total of n(g) = 18. Let us now make a count of the independent self-stressings. The absence of loads requires that at the boundary

$$d\phi = 0 \qquad \text{and} \qquad d\psi + r d\lambda = 0 \quad .$$

The first condition is equivalent to $\phi = 0$, by adjustement of the improductive additive constant, and is satisfies by the last term

$$\phi = \phi_{123} L_1 L_2 L_3$$

that represents a self-stressing of the axial traction loads t_z alone. For the self-stressings between the radial traction loads, it is preferable, for reasons of symmetry, to treat ψ and λ together and revert to $r\sigma_r$ and $r\tau_{zr}$. Those are quadratic polynomials, and we can describe them as

$$r\tau_{zr} = \alpha_1 L_1^2+\alpha_2 L_2^2+\alpha_3 L_3^2+2\alpha_{12}L_1L_2+2\alpha_{23}L_2L_3+2\alpha_{31}L_3L_1$$

$$r\sigma_r = \beta_1 L_1^2+\beta_2 L_2^2+\beta_3 L_3^2+2\beta_{12}L_1L_2+2\beta_{23}L_2L_3+2\beta_{31}L_3L_1$$

Along the boundary $L_3 = 0$ we must have

$$r\tau_{zr}\, dr = r\, \sigma_r\, dz$$

$$(\alpha_1 L_1^2+\alpha_2 L_2^2+2\alpha_{12}L_1L_2)(r_1 dL_1+r_2 dL_2) = (\beta_1 L_1^2+\beta_2 L_2^2+2\beta_{12}L_1L_2)(z_1 dL_1+z_2 dL_2)$$

As $dL_1 = - dL_2$ we obtain, equating the coefficients of L_1^2, L_2^2 and L_1L_2

$$\alpha_1(r_1-r_2) = \beta_1(z_1-z_2)$$

$$\alpha_2(r_1-r_2) = \beta_2(z_1-z_2)$$

$$\alpha_{12}(r_1-r_2) = \beta_{12}(z_1-z_2)$$

Proceeding in the same manner for the two other sides, we find that we must have

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \qquad \beta_1 = \beta_2 = \beta_3 = 0$$

but $\alpha_{12} = \sigma_3(z_1-z_2)$ $\qquad \beta_{12} = \sigma_3(r_1-r_2)$

$\alpha_{23} = \sigma_1(z_2-z_3)$ $\qquad \beta_{23} = \sigma_1(r_2-r_3)$ $\qquad \sigma_1,\sigma_2,\sigma_3$

$\alpha_{31} = \sigma_2(z_3-z_1)$ $\qquad \beta_{31} = \sigma_2(r_3-r_1)$ $\qquad$ arbitrary

We thus find 3 self-stressing states, each proportional to a σ_i.

$$r\tau_{zr} = \sigma_3(z_1-z_2)L_1L_2+\sigma_1(z_2-z_3)L_2L_3+\sigma_2(z_3-z_1)L_3L_1$$

$$r\sigma_r = \sigma_3(r_1-r_2)L_1L_2+\sigma_1(r_2-r_3)L_2L_3+\sigma_2(r_3-r_1)L_3L_1$$

From this it is easy to compute the hoop stress distribution

$$\sigma_\theta = \sigma_1(L_2-L_3) + \sigma_2(L_3-L_1) + \sigma_3(L_1-L_2)$$

whose average value is zero in accordance with condition (61).

From the result $n(x) = 4$ we deduce that, again,

$$n(r) = n(g) + n(x) - n(s) = 18 + 4 - 21 = 1$$

so that equation (58) has, again, only one solution, the axial rigid body mode. The mechanisms will be the solutions, if they exist, of equation (64). By the same argument as before, a matrix R with linearly independent columns is found by the choice of a complete quadratic for the rotation (one degree less than the stress functions ϕ and ψ) :

$$\omega = \omega_1L_1^2+\omega_2L_2^2+\omega_3L_3^2+2\omega_{12}L_1L_2+2\omega_{23}L_2L_3+2\omega_{31}L_3L_1 \qquad (74)$$

and the weak enforcement of rotational equilibrium of a slice is equivalent to rigorous enforcement (pure equilibrium model) if we keep the 6 parameters. The existence condition for solutions of (64) will be, as before,

$$x^TRh = \iint \omega\ (\tau_{zr}-\tau_{rz})rdrdz = \iint \omega(r\tau_{zr} + \frac{\partial\phi}{\partial z})drdz = 0$$

(75)

where $r\tau_{zr}$ is replaced by (72) and ϕ by (71), the result holding for arbitrary self-stressing intensities ($\sigma_1,\sigma_2,\sigma_3,\phi_{123}$).

To reach conclusions, the bilinear form is presented in the reduced format

$$y^T M h \quad \text{with} \quad y^T = (\sigma_1(z_2-z_3)A \quad \sigma_2(z_3-z_1)A \quad \sigma_3(z_1-z_2)A \quad \phi_{123})$$

$$h^T = (\omega_1 \quad \omega_2 \quad \omega_3 \quad \omega_{23} \quad \omega_{31} \quad \omega_{12})$$

The integrations required to obtain the matrix M were performed is barycentric coordinates, using the HOLAND-BELL formulas

$$\iint L_1^m L_2^n L_3^p \, dr \, dz = \frac{m!n!p!}{(m+n+p+2)!} 2A$$

$M = \frac{1}{180}$

1	3	3	4	2	2
3	1	3	2	4	2
3	3	1	2	2	4
(r_3-r_2)	(r_1-r_3)	(r_2-r_1)	(r_2-r_3)	(r_3-r_1)	(r_1-r_2)

Mechanisms will be present for a choice of h such that

$$M h = 0 \rightarrow y^T M h = 0 \quad \text{for any } y \; .$$

The rank of the matrix is 3 and there will be 3 independent solutions yielding mechanisms. By simple inspection it is found that

$$h^T = (\alpha \quad \beta \quad \gamma \quad \alpha \quad \beta \quad \gamma)$$

is a solution, provided $\alpha + \beta + \gamma = 0$, which gives two independent solutions.

$$h^T = (8 \quad 8 \quad 8 \quad -7 \quad -7 \quad -7)$$

is a third solution, independent of the others.

However when the rotation field is reduced to its linear part

$$\omega = \hat{\omega}_1 \, L_1 + \hat{\omega}_2 \, L_2 + \hat{\omega}_3 \, L_3$$

in which case we know that the zero energy state is prevented, the existence condition for mechanisms

$$y^T \hat{M} \hat{h} = 0 \qquad \text{for every } y$$

$$h^T = (\hat{\omega}_1 \; \hat{\omega}_2 \; \hat{\omega}_3)$$

has a matrix $\hat{M}$ of maximal rank

$\hat{M} = \frac{1}{120}$

2	4	4
4	2	4
4	4	2
$(r_3 - r_2)$	$(r_1 - r_3)$	$r_2 - r_1)$

and no mechanisms are generated; the stiffness matrix is well behaved.

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