Comp. & Maths. with Appls, Vol. 1, pp. 167–173. Pergamon Press, 1975. Printed in Great Britain.

THE NUMERICAL INTEGRATION OF LAMINAR BOUNDARY LAYER EQUATIONS

B. FRAEIJS DE VEUBEKE* and C. DELCOURT-BON†

Department of Aerospace Engineering, University of Liège, Liège, Belgium
Communicated by E. Y. Rodin

(*Received August* 1974)

Abstract—Self-similar solutions of boundary layer equations obey non-linear differential equations, automorphic under certain continuous transformation groups. Changes of variables suggested by the theory of continuous LIE groups may reduce the problem to the integration of a first order non linear differential equation, followed by quadratures, thereby greatly simplifying computer integration.

The famous Blasius equation, governing the asymptotic laminar boundary layer flow over a semi-infinite plate is presented as a typical example.

1. POSITION OF THE PROBLEM

The problem is that of the two-dimensional steady flow of an incompressible Newtonian fluid of density ρ along a semi-infinite plate, whose trace is the $[0, \infty]$ segment of the x axis. At infinity upstream the flow has the uniform velocity $(U, 0)$. Reduced co-ordinates

$$\xi = R_e x = xU/\nu \qquad \eta = yU/\nu \tag{1}$$

where ν is the kinematic viscosity, combined with the use of U as the velocity unit and ρU^2 as the pressure unit, yield the following Navier–Stokes and volume conservation equations

$$u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \eta} = -\frac{\partial p}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \tag{2}$$

$$u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \eta} = -\frac{\partial p}{\partial \eta} + \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} \tag{3}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0. \tag{4}$$

An asymptotic solution (valid for sufficiently high ξ values) is found, following Prandtl[1], by neglecting $\frac{\partial p}{\partial \xi}$ and $\frac{\partial^2 u}{\partial \xi^2}$ in equation (2).

*Professor of Aerospace Engineering.

†Research Assistant.

The problem is then to solve the system of two equations in the unknowns (u, v) formed by (4) and

$$u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} \tag{2'}$$

subject to the boundary conditions

$$\begin{aligned} u = 0, v = 0 \qquad &\text{for } \eta = 0 \\ u = 1 \qquad &\text{for } \eta = \infty. \end{aligned} \tag{5}$$

Solving (2') for v and substituting into (4), furnishes a single partial differential equation for u. For an asymptotic solution of type

$$u = \lambda g(\beta) \qquad \beta = \eta\phi(\xi) \tag{6}$$

it reduces to the form with separated variables (c is the separation constant)

$$\phi^{-3}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = (\lambda g)^{-1}(\ddot{g}/\dot{g})^{\cdot} = -c. \tag{7}$$

This solution is self-similar; that is the velocity profile u against the distance η to the plate is only subject to a change of scale when the distance ξ to the leading edge of the plate is altered. For such a solution (2') gives

$$v = \phi\{\ddot{g}/\dot{g} + \lambda c\beta g\}. \tag{8}$$

It follows from (7) that

$$\phi = \frac{1}{\sqrt{(2c\xi)}} \tag{9}$$

and that the function g obeys the differential equation

$$(\ddot{g}/\dot{g})^{\cdot} + \lambda c g = 0 \tag{10}$$

with the following boundary conditions stemming from (5)

$$g(0) = 0 \qquad \ddot{g}(0) = 0 \qquad g(\infty) = \lambda^{-1}. \tag{11}$$

2. AUTOMORPHISM AND NORMALIZATION

Self-similar solution (6) contains two arbitrary parameters, λ and the separation constant c. By fixing the product λc, the differential equation to be solved (10) is "normalized". Here we make the choice $\lambda c = 2$.

There remains one degree of freedom. Either we may choose c independently and normalize the function $\phi(\xi)$, hence also the variable β in (6). Or we may choose λ independently, which would allow a normalization of the third of the boundary conditions (11). Our choice will be guided here by the elegant modification of the boundary conditions due to Toepfer[2].

3. TRANSFER OF THE THIRD BOUNDARY CONDITION

Numerical integration of differential equation (10) could be achieved by a marching procedure, provided all the boundary conditions were known in $\beta = 0$. This can be obtained precisely, even after normalization of the differential equation, by the existence of its remaining automorphism.

Imagine the conditions (11) be replaced by

$$g(0) = 0 \qquad \dot{g}(0) = \frac{1}{2} \qquad \ddot{g}(0) = 0 \tag{12}$$

under which the normalized differential equation

$$(\ddot{g}/\dot{g})^{\cdot} + 2g = 0 \tag{13}$$

would yield an asymptotic value

$$g(\infty) = \frac{m}{2}.$$

By comparison, the previous boundary conditions (11) are now satisfied by the choice $\lambda = 2/m$, giving explicitly

$$u = \frac{2}{m} g(\beta) \qquad \beta = \frac{n}{\sqrt{(2m\xi)}} \tag{15}$$

$$v = \frac{1}{\sqrt{(2m\xi)}}(\ddot{g}/\dot{g} + 2\beta g). \tag{16}$$

4. THE BLASIUS EQUATION

From (13), integrating from $\beta = 0$ and noting that $\ddot{g}(0) = 0$

$$\ddot{g}/\dot{g} + 2\int_0^\beta g(\beta')\mathrm{d}\beta' = 0.$$

Hence introducing the new function

$$f(\beta) = 2\int_0^\beta g(\beta')\mathrm{d}\beta'$$

a normalized form of the Blasius equation

$$\dddot{f} + f\ddot{f} = 0 \tag{17}$$

with normalized boundary conditions

$$f(0) = 0 \qquad \dot{f}(0) = 0 \qquad \ddot{f}(0) = 1 \tag{18}$$

and asymptotic value

$$\dot{f}(\infty) = m. \tag{19}$$

This equivalent mathematical form, due to Blasius[3], is directly related to his use of a stream function ψ

$$u = \partial\psi/\partial\eta \qquad v = -\partial\psi/\partial\xi$$

to satisfy immediately the incompressibility condition (4). In terms of our self-similar solution, there comes

$$\psi = \sqrt{\left(\frac{2\xi}{m}\right)} f(\beta) \qquad u = \frac{1}{m}\dot{f} \qquad v = \frac{1}{\sqrt{(2m\xi)}}(\beta\dot{f} - f). \tag{20}$$

5. FIRST REDUCTION OF THE DIFFERENTIAL EQUATION

New independent variable: $mu = 2g = w$.
New unknown function: $2\dot{g} = \mathrm{d}w/\mathrm{d}\beta = p$.
As

$$\frac{\mathrm{d}p}{\mathrm{d}w} = \frac{\mathrm{d}p}{\mathrm{d}\beta}\frac{\mathrm{d}\beta}{\mathrm{d}w} = \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\beta}$$

$$\frac{\mathrm{d}^2p}{\mathrm{d}w^2} = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\beta}\right)\frac{\mathrm{d}\beta}{\mathrm{d}w} = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\beta}\right)$$

equation (13), can also be written,

$$\left(\frac{2\ddot{g}}{2\dot{g}}\right)^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\beta}\right) = -2g = -w$$

and is split into the pair

$$p\frac{\mathrm{d}^2p}{\mathrm{d}w^2} + w = 0; \qquad \frac{\mathrm{d}\beta}{\mathrm{d}w} = \frac{1}{p}. \tag{21}$$

From the definition of w

$$w = 0 \quad \text{for } \beta = 0; \qquad w = m \quad \text{for } \beta = \infty. \tag{22}$$

While from the definition of p

$$p(0) = 2\dot{g}(0) = 1 \tag{23}$$

$$p'(0) = \frac{1}{p(0)}\left(\frac{\mathrm{d}p}{\mathrm{d}\beta}\right)_0 = 2\ddot{g}(0) = 0. \tag{24}$$

These results establish the initial values and the domain of integration of the differential system. The existence of an asymptotic value of g when $\beta \to \infty$, leads to

$$\dot{g}(\infty) = 0; \qquad \text{hence } p(m) = 0. \tag{25}$$

6. SECOND REDUCTION OF THE DIFFERENTIAL EQUATION

The first of differential equations (21) has itself an automorphism. It remains invariant under the continuous group of transformations

$$\hat{p} = \gamma^{3/2} p; \qquad \hat{w} = \gamma w. \tag{26}$$

Setting $r = \mathrm{d}p/\mathrm{d}w$, the extended group of infinitesimal transformations is easily found to be

$$\frac{\delta w}{w} = \frac{\delta p}{\frac{3}{2}p} = \frac{\delta r}{\frac{1}{2}r} = \delta\gamma,$$

and the following first integrals are available:

$$wp^{-2/3} = c_1; \qquad rp^{-1/3} = c_2.$$

This suggests a solution of the form

$$\frac{\mathrm{d}p}{\mathrm{d}w} = -p^{1/3}F(\omega); \qquad \omega = wp^{-2/3} \tag{27}$$

which is equivalent to $c_2 + F(c_1) = 0$.

From (27) the required computations can be conducted as follows:

$$\frac{\mathrm{d}^2p}{\mathrm{d}w^2} = -\tfrac{1}{3}p^{-2/3}\frac{\mathrm{d}p}{\mathrm{d}w}F - p^{1/3}\frac{\mathrm{d}F}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}w};$$

but

$$\frac{\mathrm{d}\omega}{\mathrm{d}w} = p^{-2/3} - \tfrac{2}{3}p^{-5/3}w\frac{\mathrm{d}p}{\mathrm{d}w} = p^{-2/3}(1 + \tfrac{2}{3}\omega F);$$

whence

$$\frac{\mathrm{d}^2p}{\mathrm{d}w^2} = \tfrac{1}{3}p^{-1/3}F^2 - p^{-1/3}\frac{\mathrm{d}F}{\mathrm{d}\omega}(1 + \tfrac{2}{3}\omega F)$$

and the first of differential equations (21) splits into

$$\frac{\mathrm{d}F}{\mathrm{d}\omega} = \frac{F^2 + 3\omega}{3 + 2\omega F} \tag{28}$$

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\omega} = -\frac{3F}{3 + 2\omega F} \tag{29}$$

with as boundary conditions,

$$\left.\begin{array}{llll} \text{for } \beta = 0, & w = 0 & \text{and} \quad p = 1, & \text{hence } \omega = 0 \\ \text{for } \beta = \infty, & w = m & \text{and} \quad p = 0, & \text{hence } \omega = \infty \end{array}\right\} \tag{9}$$

$$\left.\begin{array}{ll} \dfrac{\mathrm{d}p}{\mathrm{d}w} = 0, \quad \text{for } w = 0, & \text{hence } F(0) = 0 \\ & \text{and} \quad p(0) = 1 \end{array}\right\} \tag{31}$$

The differential equation (28) itself shows that $F'(0) = 0$ and an extremely accurate starting solution is obtained by (alternating) power series

$$F = \omega^2\left(\frac{1}{2} - \frac{1}{20}\omega^3 + \frac{1}{80}\omega^6 - \frac{59}{13.200}\omega^9 + \frac{151}{92.400}\omega^{12} - \frac{16.539}{25.132.800}\omega^{15}\ldots\right).$$

After the numerical integration of F, we have from (29) and the boundary conditions a quadrature for the computation of p:

$$p = \exp\left(-\int_0^\omega \frac{3F\mathrm{d}\omega'}{3 + 2\omega' F}\right). \tag{32}$$

Similarly, from the second of equations (27) and the previous result, a quadrature for the computation of the horizontal velocity

$$w = 2g = mu = \omega p^{2/3} = \omega \exp\left(-\int_0^\omega \frac{2F\mathrm{d}\omega'}{3 + 2\omega' F}\right). \tag{33}$$

Finally a second quadrature is required to obtain the co-ordinate β. Using the second of equations (21) and the second of equations (27).

$$\frac{\mathrm{d}\beta}{\mathrm{d}\omega} = \frac{\mathrm{d}\beta}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}\omega} = p^{-1/3}(1 + \tfrac{2}{3}\omega F)^{-1} \tag{34}$$

the starting value of which is $\beta(0) = 0$.

The asymptotic value m of w is one of the essential numerical results. As equation (33) yields in the limit $\omega \to \infty$, an indeterminate product, the following transformation is indicated

$$\omega = \exp \ln \omega = \exp \int_1^\omega \frac{\mathrm{d}\omega'}{\omega'}$$

and (33) is modified for $\omega > 1$ into

$$w = \exp\left(-\int_0^1 \frac{2F\mathrm{d}\omega}{3 + 2\omega F}\right) . \exp\left(\int_1^\omega \frac{\mathrm{d}\omega'}{\omega'} - \frac{2F\mathrm{d}\omega'}{3 + 2\omega' F}\right).$$

After reduction of the second integral, that becomes a convergent one

$$w = w(1) \exp \int_1^\omega \frac{3\mathrm{d}\omega'}{\omega'(3 + 2\omega' F)} \tag{35}$$

$$w(1) = \exp\left(-\int_0^1 \frac{3F\mathrm{d}\omega}{3 + 2\omega F}\right). \tag{36}$$

The asymptotic behavior of F for large ω is obtainable from the approximate differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}\omega} = \frac{F}{2\omega} + \frac{3}{2F}$$

the two contributions to the derivative having the same order of magnitude if F is of the order of $\sqrt{\omega}$.

Setting

$$F = \sqrt{(\omega)}H$$

the resulting approximate differential equation

$$2H\,\mathrm{d}H = 3\frac{\mathrm{d}\omega}{\omega}$$

has the exact solution

$$H^2 = K + 3\ln\omega$$

and an asymptotic value of F is given by

$$F = \sqrt{\omega}\sqrt{(K + 3\ln\omega)}$$

The numerical integrations were carried out on the IBM 370–155 computer of the University by the junior author. They are in complete agreement with the numerical results obtained by Smith[5]; in particular for the asymptotic value

$$m^3 = 4{\cdot}53465$$

whence the friction coefficient $m^{-3/2}$ in the tangent stress formula

$$\tau = \rho U^2 \sqrt{\left(\frac{\nu}{2Ux}\right)}(m^{-3/2})$$

receives the already widely accepted value of 0·664.

REFERENCES

1. L. Prandtl, Ueber Flüssigkeitsabwegung bei sehr kleiner Reibung, *Vier Abhandlungen zur Hydrodynamik und Aerodynamik*, pp. 1–8, Göttingen (1972).
2. C. Toepfer, *Z. Math. Physik* **60**, 397 (1912).
3. H. Blasius, Grenzschlichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Physik* **56**, 1 (1908).
4. E. L. Ince, *Ordinary Differential Equations*. Dover Press, New York (1943).
5. A. M. O. Smith, *S.M.F. Fund Paper* FF–10, Institute of Aeronautical Sciences (1954).