DISCRETIZATION OF STRESS FIELDS IN THE FINITE ELEMENT METHOD

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ABSTRACT

Several aspects of the discretization of stress fields, as opposed to displacement fields, are reviewed. The most classical satisfies rigorous equilibrium, both translational and rotational, in the interior domain of each element and reciprocity of surface tractions at interelement boundaries (strong diffusivity).
The difficulties associated with kinematical deformation modes are analysed and resolved by different procedures :

- the composite element technique,
- quasi-diffusivity controlled by the dual patch test,
- discretization of the displacement connectors, or hybridation,
- discretization of rotational equilibrium.

This last and recent approach is discussed in some detail. It involves direct or indirect use of first order stress functions, whose C_0 continuity is sufficient for strong diffusivity. One of its advantages is the possibility of curving the boundaries by a geometric isoparametric coordinate transformation. Some numerical convergence tests are presented to conclude.

1. THE CLASSICAL THEORY OF DIFFUSIVE EQUILIBRIUM ELEMENTS

A diffusive equilibrium element is based on the discretization of the field of stresses, together with organization of reciprocity of surface tractions at the interelement boundaries. Its variational support is the so-called complementary energy principle with prescribed boundary displacements $\bar{u}_j$.

$$\int_E \Phi(\tau)\, dV - \int_{\partial E} n_i\, \tau_{ij}\, \bar{u}_j\, dS \qquad \min_{\tau} \tag{1}$$

The minimum is a constrained one; the stresses must satisfy the volume equilibrium equations

$$D_i\, \tau_{ij} + \bar{X}_j = 0 \tag{2}$$

The complementary energy density Φ is a positive definite function of its arguments τ_{ij} (homogeneous quadratic for the usual linear stress-strain relations). If prescrib body forces $\bar{X}_j$ are present, the equilibrium constraints (2) require the determination of a particular set of stresses $\bar{\tau}_{ij}$ satisfying the non homogeneous equations, to which may be added a general solution σ_{ij} of the homogeneous ones. In practice it is necessary to follow a converse procedure. The stresses are discretized as complete polynomials of a certain degree and the equilibrium equations determine the pattern of body loads that is acceptable.

In the following two-dimensional example

$$\begin{aligned}
\sigma_x &= \alpha_1 + \alpha_2\, x + \alpha_3\, y + \alpha_4\, x^2 + 2\, \alpha_5\, xy + \alpha_6\, y^2 \\
\sigma_y &= \beta_1 + \beta_2\, x + \beta_3\, y + \beta_4\, x^2 + 2\, \beta_5\, xy + \beta_6\, y^2 \\
\tau_{xy} &= \gamma_1 + \gamma_2\, x + \gamma_3\, y + \gamma_4\, x^2 + 2\, \gamma_5\, xy + \gamma_6\, y^2
\end{aligned} \tag{3}$$

we find that

$$\begin{aligned}
\bar{X} &= -\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) = \lambda_1 + \lambda_2 x + \lambda_3 y \\
\bar{Y} &= -\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) = \mu_1 + \mu_2 x + \mu_3 y
\end{aligned} \tag{4}$$

and we are allowed to introduce linear distributions of body forces, whose intensitie are connected to the stress parameters by the algebraic equations

$$- \lambda_1 = \alpha_2 + \gamma_3 \qquad - \lambda_2 = 2(\alpha_4 + \gamma_5) \qquad - \lambda_3 = 2(\alpha_5 + \gamma_6)$$
$$- \mu_1 = \gamma_2 + \beta_3 \qquad - \mu_2 = 2(\gamma_4 + \beta_5) \qquad - \mu_3 = 2(\gamma_5 + \beta_6) \tag{5}$$

More generally (3) will be represented by a discretization such as

$$\tau_{ij} = \sigma_m S_{mij} (x) \tag{6}$$

where $S_{mij}(x)$ are assumed functions and σ_m the unknown stress parameters; (4) will then become

$$\overline{X}_j = h_m X_{mj} (x) \tag{7}$$

with h_m as intensities of the body load distributions $X_{mj}(x)$ and the algebraic relations between the h_m and σ_m will follows as in (5). We write them in matrix form as

$$h = W^T s \tag{8}$$

where the row vector h contains the h_m in a conventional sequence and s the σ_m. As the h_m are independent the matrix W^T will have linearly independent rows.

We can transform the constrained minimum problem (1) to an unconstrained one by the LAGRANGE multiplier technique and consider the augmented functional

$$\int_E [\Phi(\tau) + u_j (D_i \tau_{ij} + \overline{X}_j)] \ dV - \int_{\partial E} n_i \tau_{ij} \overline{u}_j \ dS$$

The variational derivatives with respect to the now unconstrained τ_{ij} are

$$\frac{\partial \Phi}{\partial \tau_{ij}} = \frac{1}{2} (D_i u_j + D_j u_i)$$

showing that, since $\partial\Phi / \partial\tau_{ij} = \varepsilon_{ij}$, the strains, the multipliers u_j are, as the notation already betrayed, the displacements themselves. Similarly, when in the discretized problem we wish to remove the algebraic constraints (8) by augmenting the functional with the term

$$b^T (h-W^T s) \tag{9}$$

the column b of multipliers represents generalized displacements conjugate in the virtual work sense to the body load intensities :

$$\int_E \overline{X}_j u_j \, dV = h_m \int_E X_{mj} (x) u_j \, dV = h_m b_m = b^T h \tag{10}$$

so that the element b_m conjugate to h_m

$$b_m = \int_E X_{mj} (x) u_j \, dV \tag{11}$$

is a weighted average of the unknown displacement field.
Other weighted averages of displacements will be provided by the boundary term of the functional. Different parts of the boundary, identified by a Greek subscript, will be distinguished by the fact that they belong to different parts of the global structural boundary or are connected to different adjacent elements. Along one of those partial boundaries the surface tractions are obtained from (6) and the direction cosines n_i of the outward normal as

$$t_j = n_i \tau_{ij} = \sigma_m n_i S_{mij} (x) \qquad x \in \partial_\alpha E \tag{12}$$

We need however another description in terms of generalized boundary loads $g_{r(\alpha)}$, each of which is the intensity factor of an assumed surface traction distribution :

$$t_j = g_{r(\alpha)} T_{rj(\alpha)} (x) \qquad x \in \partial_\alpha E \tag{13}$$

By identifying (12) and (13) we obtain the algebraic relationship between the $g_{r(\alpha)}$ and the σ_m ; in matrix form

$$g_{(\alpha)} = S_{(\alpha)} s$$

Or, after collecting the contributions from all partial boundaries,

$$g = S\ s \tag{14}$$

Figure 1 shows a possible choice of surface traction modes and associated boundary loads for the preceding two-dimensional example.
Again the virtual work computation

$$\int_{\partial_\alpha E} t_j\ u_j\ dS = g_{r(\alpha)} \int_{\partial_\alpha E} T_{rj(\alpha)}\ (x)\ u_j\ dS = g_{r(\alpha)}\ q_{r(\alpha)} \tag{15}$$

defines the conjugate generalized boundary displacements

$$q_{r(\alpha)} = \int_{\partial_\alpha E} T_{rj(\alpha)}\ (x)\ u_j\ dS \tag{16}$$

Collected in a single vector q, in the same conventional sequence as in g, they give to the last term of the functional in (1) the discretized form

$$\int_{\partial E} t_j\ u_j\ dS = q^T\ g \tag{17}$$

The bar, indicating that the boundary displacements are prescribed, was omitted for the sake of notational simplicity.
The first term of the functional, assuming linear stress-strain relations :

$$\varepsilon_{ij} = F_{ij}^{pq}\ \tau_{pq}$$

will become

$$\int_E \Phi(\tau)\ dV = \frac{1}{2} \int_E \varepsilon_{ij}\ \tau_{ij}\ dV = \frac{1}{2} \int_E F_{ij}^{pq}\ \tau_{pq}\ \tau_{ij}\ dV = \frac{1}{2}\ s^T\ F\ s \tag{18}$$

where F is a positive definite matrix of generalized compliances.
The unconstrained discretized functional can thus be written

$$\frac{1}{2}\ s^T\ F\ s + b^T\ (h - W^T\ s) - q^T\ S\ s \qquad \min_s \tag{19}$$

and the set of minimizing equations is composed of (8) and

$$F\,s - W\,b = S^T q \tag{20}$$

When it is satisfied we can deduce from it that

$$s^T F\,s = (W^T s)^T b + (Ss)^T q = h^T b + g^T q$$

a statement, in discretized form, of the virtual work theorem.
Given the body loads h and boundary displacements q, the minimizing equations and the original constraints constitute a system that determines uniquely the stress parameters and internal displacements

$$\begin{pmatrix} F & W \\ W^T & 0 \end{pmatrix} \begin{pmatrix} s \\ -b \end{pmatrix} = \begin{pmatrix} S^T q \\ h \end{pmatrix} \tag{21}$$

In other words the symmetrical matrix of the system is invertible. This can be seen setting q=0 and h=0 and verifying that the homogeneous set of equations only possesses the trivial solution s=0 and b=0. Firstly $W^T s = 0$ induces $s^T Wb = (b^T W^T s)^T = 0$ From this and Fs − Wb = 0 we deduce $s^T Fs = 0$ and thus s=0 since F is positive definit Finally we must have Wb=0 and the conclusion b=0 follows from the fact that W has linearly independent columns. Let

$$\begin{pmatrix} F^{\dagger} & V \\ V^T & -Q \end{pmatrix} \tag{22}$$

be the inverse matrix (obtained in practice by numerical inversion), so that

$$s = F^{\dagger}\, S^T q + Vh \tag{23}$$

$$b = -\, V^T S^T q + Qh \tag{24}$$

This last set furnishes the internal displacements conjugate to the body loads; they are simply linked to the boundary displacements when the body loads vanish. From the set (23) we deduce in conjunction with (14) the stiffness relations of the element

$$g = K\,q + SVh \tag{25}$$

where $$K = SF^{\dagger}S^{T} \tag{26}$$

is the stiffness matrix. Setting q=0 in (25) it can be seen how the body loads are to be balanced by boundary reaction loads.
Substitution of (24) and (25) into the CLAPEYRON form of the energy of the element shows it to be the sum of two energies, one due to the boundary displacements only, the other due to the body loads only

$$\frac{1}{2}\,(b^{T}h + q^{T}g) = \frac{1}{2}\,q^{T}\,K\,q + \frac{1}{2}\,h^{T}\,Qh \tag{27}$$

K is merely non-negative and its singularities will be analyzed in the next section, Q must be positive definite as, with fixed boundaries, the imposition of body loads must always generate stress-energy. Indeed, (22) being a left inverse of the matrix in (21) we have the relationships

$$\left.\begin{aligned} F^{\dagger}F + V\,W^{T} &= (s/s) \\ F^{\dagger}W &= 0 \\ V^{T}F - Q\,W^{T} &= 0 \\ V^{T}W &= (h/h) \end{aligned}\right\} \tag{28}$$

where (s/s) and (h/h) are identity matrices of the dimensions of s and h respectivel
From the last two follows

$$V^{T}FV = Q\,W^{T}V = Q(V^{T}W)^{T} = Q \tag{29}$$

and, since F is positive definite, so will be Q provided V has linearly independent columns.

This however is obvious considering the last relation in transpose

$$W^T V = (h/h)$$

from which it is seen that $V\,h_o = 0$ implies $W^T V\,h_o = h_o = 0$. From the first two of relations (28) we can also deduce

$$F^{\dagger} F F^{\dagger} = F^{\dagger}.$$

Showing $F^{\dagger}$ to be non negative.

Another method consists in the generally easy preliminary process of solving the constraints (8), expressing the stress parameters in terms of a particular solution and the general solution for h = 0

$$s = V\,h + F^{\dagger}\,\hat{s} \qquad (W^T V = (h/h) \quad W^T F^{\dagger} = 0)$$

(as the notation suggests, V and $F^{\dagger}$ satisfy the same equations as the second and last of (28) but are not otherwise the same matrices). In the example, this operation could simply consist in the elimination of the parameters

$$\alpha_2 = -(\lambda_1 + \gamma_3) \quad \alpha_4 = -\frac{1}{2}\lambda_2 - \gamma_5 \quad \alpha_5 = -\frac{1}{2}\lambda_3 - \gamma_6$$

$$\beta_3 = -(\mu_1 + \gamma_2) \quad \beta_5 = -\frac{1}{2}\mu_2 - \gamma_4 \quad \beta_6 = -\frac{1}{2}\mu_3 - \gamma_5$$

as justified by (5). We would then retain as independent set $\hat{s}$

$$\hat{s}^T = (\alpha_1\ \alpha_3\ \alpha_6\ \beta_1\ \beta_2\ \beta_4\ \gamma_1\ \gamma_2\ \gamma_3\ \gamma_4\ \gamma_5\ \gamma_6)$$

and use

$$h^T = (\lambda_1\ \lambda_2\ \lambda_3\ \mu_1\ \mu_2\ \mu_3)$$

as load intensity parameters.

The functional will now read

$$\frac{1}{2}\,\hat{s}^T F_{ss}\,\hat{s} + \hat{s}^T F_{sh}\,h + \frac{1}{2}\,h^T F_{hh}\,h - q^T g \qquad \min_{s}$$

with $g = (SV)h + (SF^{\dagger})\hat{s} = Hh + \hat{S}\hat{s}$ (31)

and the minimizing equations

$$F_{ss}\,\hat{s} + F_{sh}\,h = \hat{S}^T q$$

Because F_{ss} is positive definite

$$\hat{s} = F_{ss}^{-1}\,(\hat{S}^T q - F_{sh}\,h)$$

and the stiffness relations are

$$g = Kq + (H - \hat{S}\,F_{ss}^{-1}\,F_{sh})\,h \tag{32}$$

$$K = \hat{S}\,F_{ss}^{-1}\,\hat{S}^T \tag{33}$$

If desired, the internal displacements can be recuperated as CASTIGLIANO derivatives of the stress energy with respect to the elements of h

$$b = F_{hs}\,\hat{s} + F_{hh}\,h \qquad (F_{hs} = F_{sh}^T) \tag{34}$$

In any case the final form of the stiffness relations shows that the equilibrium element can be assembled by the same direct stiffness method as conforming kinematical elements, the only difference being in the weak character of the displacement connectors q. The limitation in the knowledge of displacements to weak quantities (weighted averages) is unavoidable, as the integrability conditions of strains will, as a rule, be violated. This may seem to contrast with the case of the kinematical models in which the strong information on displacements can be differentiated to produce also strong stress information. The reliable information on stresses is however the one provided by the conjugates to nodal and internal displacements which, being in the form of weighted averages, are weak.

2. THE KINEMATICAL DEFORMATION MODES

Since the earliest conception of equilibrium models [1,2] it is known that, while the stiffness matrix always exists, it is not necessarily well behaved. By this we mean that the solutions of the problem

$$K\, q = 0 \tag{35}$$

may comprize more than the boundary displacements associated with a displacement of the element as a rigid body (the so-called rigid body modes) but also boundary deformation modes. If we take K as given by (33), the problem (35) is fully equivalent to the simpler problem

$$\hat{S}^T\, q = 0 \tag{36}$$

that depends only on the kinematics and statics of the element. The $\hat{S}$ matrix has $n(\hat{s})$ columns and $n(g)$ rows; according to its definition by (31) it governs the boundary loads generated by stresses in equilibrium without body loads. If the complete polynomials adopted to represent the state of stress are of sufficiently high degree, self-stressing states can exist, such that

$$\hat{S}\hat{s} = 0 \tag{37}$$

They correspond to stresses in equilibrium without either body loads or surface tractions applied at the boundary. We denote by X a matrix, whose linearly independe columns constitute a fundamental set of solutions of (37), so that any linear combination

$$s = X\, x \tag{38}$$

is a self-stressing. The number of independent self-stressings will be denoted by $n(x)$, the number of elements of vector x. The rank of $\hat{S}$ is then $n(\hat{s}) - n(x)$. Since $\hat{S}^T$ has the same rank we will have

$$n(\hat{s}) - n(x) = n(g) - n(r) \tag{39}$$

where $n(r)$ is the number of independant solutions of (36) , the general solution of which is written in the form

$$q = Rr \tag{40}$$

an arbitrary linear combination of a fundamental set contained as columns of R. By definition then we shall have

$$\hat{S}\, X = 0 \qquad \text{and} \qquad \hat{S}^T R = 0 \tag{41}$$

Using the transpose of the last property we find from (32)

$$R^T g = R^T Hh \tag{42}$$

the existence conditions for solving (32) with respect to q. They are automatically satisfied for such columns of R that represent rigid body modes being in this case a virtual work statement of global equilibrium and, as such, a consequence of our imposition of local equilibrium.
For each column of R representing a kinematical deformation mode, (42) imposes an additional and unwelcome restriction on the loading possibilities of the element necessary in order to avoid the excitation of a deformation to which the element offers no elastic reaction.

The origin of kinematical deformation modes can be traced to our desire to have a diffusive, as well as equilibrated, element, whereby it also becomes endowed with the interesting property of providing a strain-energy bound opposite to that of kinematical and conforming elements [1,3,4].
Diffusivity requires that the generalized boundary loads be defined, together with their associated weak displacement connectors, on each partial boundary separately. This often results in a n(g) that is too large. In the example, assuming a triangular element (the quadrilateral is of course still worse) we have n(x) = 0 and $n(r) = n(g) - n(\hat{s}) = 18 - 12 = 6$ leading to 3 kinematical modes in addition to the 3 rigid body modes.

The general situation for the triangular membrane elements is easily analyzed. Taking for the elements of $\hat{s}$ the coefficients of a complete polynomial approximation to an AIRY function $\Phi(x,y)$, amputated of its linear improductive terms, we have, if n is the polynomial degree of the stresses and n+2 that of the AIRY function

$$n(\hat{s}) = \frac{1}{2}(n+3)(n+4) - 3 = \frac{1}{2}(n+1)(n+6)$$

The forces generated on a piece ds of the boundary are

$$ds\, t_x = \sigma_x\, dy - \tau_{yx}\, dx = \frac{\partial^2 \phi}{\partial y^2} dy + \frac{\partial^2 \phi}{\partial x \partial y} dx = d\, \frac{\partial \phi}{\partial y} \qquad (43)$$

$$ds\, t_y = \tau_{xy}\, dy - \sigma_y\, dx = -\frac{\partial^2 \phi}{\partial y \partial x} dy - \frac{\partial^2 \phi}{\partial x^2} dx = -\, d\, \frac{\partial \phi}{\partial x}$$

Hence, if the surface tractions t_x and t_y are to be zero on the boundary, the first partial derivatives of ϕ must be constant and we may take them to be zero by adjustin the coefficients β and γ of the improductive terms $\alpha+\beta x+\gamma y$. Then, as we shall have $\partial\phi/\partial s = 0$ on the boundary, we may also take $\phi = 0$ on the boundary by proper adjustment of α. This allows us to count n(x) by observing that a polynomial AIRY function vanishing together with its normal derivative on the boundary of a triangle is necessarily of the form

$$\phi = (L_1\, L_2\, L_3)^2\, P(x,y)$$

where $L_i = 0$ is the equation (linear in x and y) of one side of the triangle. P(x,y) is thus a polynomial of degree $n + 2 - 6 = n - 4$ and the number of its coefficients is

$$n(x) = \frac{1}{2}(n-3)(n-2) \qquad (n(x) = 0 \text{ if } n \leq 3)$$

On each side t_x and t_y are polynomials of degree n in the arc length, determined by (n+1) local values, or generalized loads, so that

$$n(g) = 6(n+1)$$

Equation (39) then furnishes for $n(k) = n(r)-3$ the number of kinematical deformation modes

$$n(k) = \begin{cases} 0 & \text{if } n=0 \\ 2 & \text{if } n=1 \\ 3 & \text{if } n\geq 2 \end{cases}$$

While the stiffness matrix is well-behaved for the constant stress element $n=0$, the kinematical modes may appear at the assembling level[1].
For $n \geq 1$ they appear at the element level and in order to get rid of them a geometrical interpretation is useful. Their maximum number of 3 immediatly suggests that they must be due to the generalized reciprocity principle of stresses at the 3 corners that relates the surface traction distributions of two edges.
Let n'_i and n''_i denote the direction cosines of the outward normal on two adjacent edges. The generalized reciprocity principle in terms of the surface tractions t'_i and t''_i at the corner will read

$$n'_j\, t''_j = n'_j\, n''_i\, \tau_{ij} = n''_i\, n'_j\, \tau_{ji} = n''_i\, t'_i \qquad (44)$$

Suppose then that two pairs of the generalized forces defined along an edge result from the linear distributions of surface tractions associated to their cornér values, as illustrated for the pair X_{12} and X_{21} in figure 1.

In the reciprocity principle applied at corner 1

$$n_{x12}\, t_{x13} + n_{y12}\, t_{y13} = n_{x13}\, t_{x12} + n_{y13}\, t_{y12}$$

we can substitute (c_{12} and c_{13} are the length of edges 12 and 13)

$$c_{12}\, n_{x12} = y_2 - y_1 \quad c_{12}\, n_{y12} = x_1 - x_2 \quad c_{13}\, n_{x13} = y_1 - y_3 \quad c_{13} n_{y13} = x_3 - x_1$$

$$c_{12}\, t_{x12} = 2\, X_{12} \quad c_{12} t_{y12} = 2\, Y_{12} \quad c_{13} t_{x13} = 2\, X_{13} \quad c_{13} t_{y13} = 2 Y_{13}$$

and obtain a constraint between the 4 boundary loads near corner 1

$$(y_2 - y_1) X_{13} - (x_2 - x_1) Y_{13} = (y_1 - y_3) X_{12} - (x_1 - x_3) Y_{12}$$

This can now be interpreted, in view of the point of application of the loads at one third edge length from the corner, as the vanishing of their total moment with respect to the barycenter $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$. This is true for any values of the stress parameters and the virtual work interpretation indicates that this constraint would be enforced if a triangular piece connecting the two points of application were pin-jointed at the barycenter. This "floating corner" concept corresponds in general to three separate mechanisms, one for each corner (figure 2). For n=1 there are no other points of application of boundary loads, and a common rotation of the three pieces is not a kinematical deformation mode any more, it is a rigid body mode. Hence the number of kinematical modes is exceptionally reduced to 2 in that case. For n=0, the generalized reciprocity constraints at the corners become identical to the global equilibrium constraints of the loads.

3. THE COMPOSITE ELEMENT TECHNIQUE

Assembling a small number of elements in a special pattern to form a composite is one of the oldest answers to the nuisance of kinematical deformation modes. Figure 3 shows two typical patterns for membrane analysis.
The first consists in subdivising a triangle by an arbitrary internal point in three triangular subelements. In order to show how the kinematical modes are inhibited, it was sufficient to illustrate the connexions corresponding to the case n=1. The second pattern is a quadrilateral subdivided by its diagonals. The geometry is then such that the deformation mode of the internal four bar linkag can take place with A, B, C, D as fixed pivots and does not displace the outer boundary of the composite [1]. The analogy between AIRY stress function and transvers deflection of KIRCHHOFF plate bending theory [4,8] is another justification of thes patterns that, as a matter of fact, were initially invented to solve the C_1 conformity problem of plate deflexion[3,6]. It appears clearly from (43) that the analogous C_1 continuity of the AIRY function solves the diffusivity problem. Costly as they may seem in degrees of freedom, the composite elements enjoy the, often precious , advantage of dual energy bounders'.

4. THE VIRTUAL WORK TEST AND QUASI DIFFUSIVE ELEMENTS

Another efficient remedy against kinematical modes, leading to simpler equilibrium elements, is to apply the dual version of the patch test invented by IRONS and reformulated by STRANG [10,11]. In a complete cubic AIRY function there is a special symmetrical term

$$\phi = \alpha L_1 L_2 L_3 \qquad (45)$$

that vanishes along all sides of the triangle. As a result $\partial^2\phi / \partial s^2 = 0$ on the boundary and only tangential surface tractions are generated. Their resultant moment with respect to a corner point reduces to that of the opposite edge distribution and must vanish because there is global equilibrium. Consequently along each edge the linear tangential surface traction distribution is statically equivalent to zero; it is antisymmetrical and vanishes at the mid-edge point. It will produce no virtual work on any displacement distribution along the edge that is linear, such as would be generated by an arbitrary state of constant strain within the element. Assuming that the dual patch test, or its equivalent the zero interface virtual work on linear edge displacement [13], is a sufficient condition for energy convergence, which seems pretty well substantiated on many numerical examples (but not the necessity), we may reduce diffusivity to quasi-diffusivity by renouncing to transmit such an antisymmetrical distribution across an interface. What can happen then is that the two tangential surface tractions across an interface will not be exactly reciprocal (except at mid-edge) but will differ by a linear antisymmetrical distribution. A stress parameter like α in (45) will be considered as an internal degree of freedom and computed from energy minimization at each element level. If we thus reduce by 1 the number of stress parameters $\hat{s}$, we also reduce by 3 number of boundary loads required for quasi-diffusivity. On each side we need only relate to the six remaining stress parameters, the total normal load, the total tangential load and the total moment of surface tractions at mid edge. Those edge resultants will be exactly transmitted to the neighbour by identifying the conjugate weak displacements : average normal displacement, average tangential and average rotation of the edge.
The 9 boundary loads being related by 3 global equilibrium equations, 6 of them are independent and match the 6 remaining stress parameters; there are no kinematical deformation modes.
Such a quasi diffusive equilibrium element, or its generalization for $n > 1$, has unfortunately never been tested up to now.

5. WEAK DIFFUSIVITY

A variational justification of strong diffusivity [13] is as follows. On the faces F^+ and F^- of an interface F we require coherence of the displacement data

$$\bar{u}_j^{\,+} = \bar{u}_j^{\,-} = \bar{u}_j$$

and obtain as constrained variational principle extended now to the union of all element with boundary ∂U

$$\int_{UE} \Phi(\tau)\, dV - \sum_F \int_F \bar{u}_j\, (t_j^+ + t_j^-) dS - \int_{\partial U} \bar{u}_j\, t_j\, dS \qquad \min_\tau$$

We may consider $\bar{u}_j$ to be prescribed on ∂U in the global problem, but on each F it is now essentially an internal unknown. If undiscretized, its free variation requires the strong diffusivity proprety (reciprocity of surface tractions)

$$t_j^+ + t_j^- = 0$$

Conversely, if this property is satisfied by the proper piecing together of the stress fields, the connectors $\bar{u}_j$ disappear from the global variational principle that reduces to the form originally used at element level. As shown before, the numerical information received for the connectors is in weak form.
Considering that strong diffusivity can lead to kinematical deformation modes a natural step consists in constraining the connectors. Following T.H.H. PIAN [9] we consider a coherent connector displacement field to be defined at the interfaces in terms of a limited number of degrees of freedom

$$\bar{u}_j = \bar{q}_m\, Q_{mj}\,(x) \qquad x \in F$$

and the variations taken on the $\bar{q}_m$ will now require weak diffusivity

$$\int_F Q_{mj}\,(x)\,(t_j^+ + t_j^-)\, dS = 0$$

The theory at element level of such hybrid elements is not fundamentally different from that of the strongly diffusive ones. The only difference is in the computation of the boundary term of the functional (1).
The boundary loads, instead of being defined on each partial boundary to represent intensities of surface traction distributions, become virtual work conjugates to the nodal displacements defined on the boundary. The result is a different kinematical matrix $\hat{S}$. Given a polynomial stress field, the larger the number of degrees of freedom allocated to the displacements, the closerto strong diffusivity but also to the danger of kinematical deformation modes. Figure 4 shows two recentl tested membrane hybrids based on a complete cubic AIRY function generating the interior stress field (n=1) with 9 strong displacement connectors at the boundary. The first has a linear normal displacement along each boundary and a quadratic tangential component. The reverse is true for the second. None has kinematical deformation modes. They are the exact duals of recently proposed KIRCHHOFF plate bending hybrids based on a cubic deflexion field and strong SOUTHWELL stress functions connectors [13] . The equivalent connexions of the AIRY function and its first derivatives is shown on the right. They do not satisfy exactly the C_1 continuity requirement (otherwise the element would be strongly diffusive) but satisfy the dual patch test requirements.

6. DISCRETIZATION OF ROTATIONAL EQUILIBRIUM

The most recent proposal for removing kinematical deformation modes is to relax the rigid enforcement of local rotational equilibrium, as expressed by the symmetry of the stress tensor. It has a firm variational background, initially developped to aproach a complementary energy formulation for finite displacements and later applied in linearized form to its present purpose [12,15]. It is formulated here for the membrane case only in order to unify the presentation.

The equilibrium equations without body loads (this simplification is not essential to the theory)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

are solved by using a first order stress functions vector (A,B)

$$\sigma_x = \frac{\partial B}{\partial y} \quad \tau_{yx} = -\frac{\partial B}{\partial x} \quad \sigma_y = \frac{\partial A}{\partial x} \quad \tau_{xy} = -\frac{\partial A}{\partial y} \tag{46}$$

and the surface traction vector, computed like in (43), becomes

$$t_x\, ds = dB \qquad t_y\, ds = -\, dA \tag{47}$$

The rotational equilibrium condition

$$\tau_{xy} - \tau_{yx} = \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = 0 \tag{48}$$

is incorporated as a constraint in the complementary energy principle by means of a Lagrangian multiplier $\omega(x,y)$

$$\int_E \{ \Phi + \omega(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})\} \; dxdy - \oint \bar{u}\, d\, B - \bar{v}\, d\, A \qquad \min_{A,B} \tag{49}$$

The complementary energy is now considered as a (positive definite) function of the arguments

$$\sigma_x = \frac{\partial B}{\partial y} \qquad \frac{1}{2}(\tau_{xy} + \tau_{yx}) = -\frac{1}{2}\left(\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y}\right) \qquad \sigma_y = \frac{\partial A}{\partial x} \tag{50}$$

The symmetry of the strain tensor is maintained by the fact that the energy remains a symmetric function of the stresses without requiring that (48) be satisfied. The variational equations and natural boundary conditions for unconstrained ω, A and B are respectively : equation (48),

$$\frac{\partial \omega}{\partial x} = -\frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial x} \qquad \frac{\partial \omega}{\partial y} = -\frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} \tag{51}$$

$$\begin{aligned} d\bar{u} + \Big(\varepsilon_2 \sin\theta + (\omega - \varepsilon_{xy})\cos\theta\Big)\, ds = 0 \\ d\bar{v} + \Big((\varepsilon_{xy} + \omega)\sin\theta - \varepsilon_y \cos\theta\Big)\, ds = 0 \end{aligned} \tag{52}$$

where θ is the angle between ox and the outward normal to the boundary.
Equations (51) are recognized to be the BELTRAMI equations for the integration of the material rotation ω .
The complementary energy of an element will vanish if and only if the field of its arguments (50) vanishes identically. This will be the case when

$$B = B(x) \qquad A = A(y) \qquad \frac{dB}{dx} + \frac{dA}{dy} = 0$$

and the only possibility is

$$\frac{dB}{dx} = - \frac{dA}{dy} = \gamma \qquad \text{that is} \qquad \tau_{xy} = - \tau_{yx} = \gamma \tag{53}$$

an arbitrary constant. It is the zero energy stress field, representing in its purest form the violation of rotational equilibrium. It is interesting to oberve that the field

$$A = \alpha_o - \gamma y \qquad B = \beta_o + \gamma x \tag{54}$$

analogous to a rigid body displacement vector, represents the energy improductive part of the stress functions vector.

The idea is now to replace this vector together with the rotation multiplier ω by complete polynomial approximations. So doeing it may be expected that since the reciprocity principle at corners need no more apply, the kinematical deformation modes may disappear. This proves to be correct provided the degree of ω be at most n-1 when the degree of A and B is n+1 (n is still the polynomial degree of the stress field).
The set s of stress parameters will be taken to be coefficients of the monomials in A and B, excluding the improductive constant α_o and β_o

$$n(s) = 2 \left(\frac{(n+2)(n+3)}{2} - 1 \right) = (n+1)(n+4) \tag{55}$$

The energy of the element will be of the form

$$\frac{1}{2} s^T F s$$

with F non negative and possessing one non trivial singular solution

$$F\, s_o = 0 \tag{56}$$

due to γ the zero energy state.

The coefficients of the ω polynomial will be listed in vector h so that

$$\int_E \omega \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = s^T W h \tag{57}$$

bilinear form that needs some preliminary discussion.

Arbitrary variations on h will produce the discretized rotational equilibrium constraints

$$W^T s = 0 \tag{58}$$

and the bilinear form will always be zero.

1. If ω is arbitrary of degree n we can, for any A and B of degree n+1, select it so that $\omega = \lambda\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$

and for this mode of rotation we will have

$$\int_E \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)^2 dxdy = 0$$

Hence in this case we have always

$$\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$$

and rotational equilibrium is rigorously enforced, we are back to a classical equilibrium model.

2. If ω is arbitrary of degree $\leq$ n we establish that W has linearly independent columns. The proof is by contradiction. If we had a non zero vector h_o such that $W\, h_o = 0$ then $s^T\, W\, h_o = 0$ for any s vector.

This means that a non zero polynomial ω_o of degree $\leq n$ would exist such that

$$\int_E \omega_o\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = 0$$

for any polynomials A and B of degree n+1.This should be true in particular of $A = 0$ and B a primitive of $\frac{\partial B}{\partial x} = \omega_o$.

But then $\int_E \omega_o^2 \, dxdy = 0$ and $\omega_o \equiv 0$.

To finish the discretization, the boundary term of the functional is treated as in the classical theory of equilibrium elements, defining a set g of boundary loads attached to each partial boundary, establishing their relationship

$$g = S\,s \tag{59}$$

to the stress parameters and their conjugate weak displacements q.
The minimizing equations of

$$\frac{1}{2} s^T F s + s^T W h - q^T S s \qquad \min_{s,h}$$

are then presented in a form

$$\begin{pmatrix} F & W \\ W^T & 0 \end{pmatrix} \begin{pmatrix} s \\ h \end{pmatrix} = \begin{pmatrix} S^T q \\ 0 \end{pmatrix} \tag{60}$$

reminiscent of system (21). While F is now singular because of (56), the matrix is still invertible; the homogeneous system has only the trivial solution :

$$W^T s = 0 \rightarrow s^T W h = 0 \rightarrow s^T F s = 0 \rightarrow s = \lambda\, s_o$$

The proof is achieved if we can show that $W^T s_o \neq 0$, for then

$$\lambda W^T s_o = 0 \rightarrow \lambda = 0 \rightarrow s = 0 \rightarrow Wh = 0 \rightarrow h = 0.$$

Let us achieve the proof by contradiction again. If $W^T s_o = 0$, then $s_o^T W h = 0$ for any h. This means that for any polynomial ω we should have, when $A = -\gamma y$ and $B = \gamma x$

$$\int_E \omega\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = 2\gamma \int_2 \omega \; dxdy = 0$$

and this already fails to hold for the simplest case ω = constant that would insure the minimum requirement of global rotational equilibrium of the element. Using the notation (22) for the inverse matrix

$$s = F^{\dagger} S^T q \qquad h = V^T S^T q \tag{61}$$

and the stiffness relation of the element is

$$g = (SF^{\dagger}S^T)q \tag{62}$$

It remains to show that if the degree of ω is $\leq$ n-1 the stiffness matrix is well behaved.

Rigid body and kinematical deformation modes must be such values of q and h that may exist without stresses. Hence, setting s=0 in (60) they must satisfy

$$S^T q = Wh \tag{63}$$

For h=0 (no rotation of the element) we expect to find the solutions corresponding to translational modes q_x and q_y. That those are indeed solutions is obvious from the fact that A and B were devised to satisfy local translational equilibrium. Consequently global translational equilibrium is satisfied. In virtual work form this means that

$$g^T q_x = s^T S^T q_x = 0 \qquad g^T q_y = s^T S^T q_y = 0 \qquad \text{for any } s$$

and establishes the proof. That there are no other solutions for h=0 may be established by counting as by (39)

$$n(r) = n(g) - n(s) + n(x)$$

$n(s)$ is given by (55) , $n(g) = 6(n+1)$ as for pure equilibrium elements.
To count the sef-stressing states we observe on (47) that the surface tractions vanish if A and B remain constant on the boundary. We may then add to each an improductive constant to obtain $A = 0$ and $B = 0$ on the boundary. In a triangular element each must then be of the form

$$L_1 L_2 L_3 P(x,y)$$

Where P is a polynomial of degree $n+1 - 3 = n-2$ with $\frac{1}{2}(n-1)n$ coefficients.
Thus $n(x) = n(n-1)$ and we obtain as expected

$$n(r) = 2$$

Let h_o be the vector corresponding to a uniform rotation $\omega = 1$; it will be shown that a solution q_o (a rigid body rotation mode) exists to the problem

$$S^T q = W h_o$$

The existence conditions : orthogonality of the right-hand side with respect to all solutions of the homogeneous adjoint $Ss = 0$

$$s^T W h_o = 0 \qquad \text{s an arbitrary self-stressing}$$

are satisfied. Indeed this amounts to verify that for $\omega = 1$ and A and B zero on the boundary

$$\int_E \omega\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = \int_{\partial E} A\,dx + B\,dy = 0$$

The element has thus the 3 rigid body modes and it remains to find the rules under which, for h vectors independent of h_o, the existence conditions for a solution of q

$$s^T W h = 0 \qquad \text{s an arbitrary self-stressing} \qquad (64)$$

can be violated by a proper choice of self-stressing, so that the element is free

from kinematical deformation modes.
A sufficient rule is that the polynomial degree of ω be $\leq$ n-1. A proof consists in testing all monomials of ω of the form

$$\omega = x^a\, y^b \qquad \text{a and b non negative and not both zero}$$

with respect to selfstressings in which the polynomial degree of A and B, being at least two units above that ω, is allowed to be

$$n+1 \geq a+b+2$$

We thus examine for A and B vanishing at the boundary the value of

$$s^T Wh = \int_E x^a\, y^b \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = \int_E (A\, bx^a\, y^{b-1} - Bax^{a-1}\, y^b)\, dxdy$$

For a and b odd we choose the admissible selfstressing

$$A = 0 \qquad B = -y\, L_1\, L_2\, L_3$$

For a odd and b even

$$A = 0 \qquad B = -L_1\, L_2\, L_3$$

For a even and b odd

$$A = L_1\, L_2\, L_3 \quad B = 0$$

For a and b even

$$A = y\, L_1\, L_2\, L_3 \quad B = -x\, L_1\, L_2\, L_3$$

In any case we find that the existence condition (64) is violated by the result $s^T W\, h > 0$ since $L_1\, L_2\, L_3$ is positive at any interior point of E and multiplied by even powers of x and y.

Numerical experience obtained by the junior author and reported in the last section establishes that this type of element gives good convergence provided ω be not given a lower degree than its maximum n-1 compatible with the disappearence of kinematical modes.
It should also be observed that there are local points where the discrepancy between τ_{yx} and τ_{xy} disappears, the barycenter for example and other points when $n > 1$.

7. ISOPARAMETRIC TRANSFORMATION OF DIFFUSIVE ELEMENTS

One of the most interesting features of rotationally discretized elements is the possibility of obtaining curved boundaries by isoparametric coordinate transformations without loosing the strong diffusivity property. Let (ξ,η) denote the cartesian coordinates for elements with straight boundaries, that become curvilinear coordinates in the physical coordinates

$$x = \xi + U_m W_m (\xi,\eta) \qquad\qquad y = \eta + V_m W_m(\xi,\eta)$$

The $W_m(\xi,\eta)$ are interpolation (shaping) functions related to nodal boundary shifts (U_m,V_m) defining the new geometry of the element. The identification of nodal shifts at interfaces allows in principle to keep the curved elements contiguous so that if A and B are C_o continuous in the (ξ,η) system, they remain C_o continuous as functions of x and y. This is of course sufficient to preserve strong diffusivity However, not any type of geometrical transformation is permissible, if some other basic properties are to be maintained. Let C_r (ξ,η) be interpolation functions related to nodal values (A_r,B_r) of the first order stress functions

$$A = A_r C_r(\xi,\eta) \qquad\qquad B = B_r C_r (\xi,\eta)$$

and such that the identification of nodal values at interfaces insures C_o continuity of A and B. We need essentially that (A_r,B_r) values exist such that we may have in physical coordinates

$$A = \alpha_o + \alpha_1 x + \alpha_2 y \qquad\qquad B = \beta_o + \beta_1 x + \beta_2 y$$

for <u>any</u> values of the (α_i, β_i) parameters. This means that we keep the existence in the physical plane of the energy improductive terms and of an arbitrary state of uniform stress. In the (ξ,η) plane this requires the existence of a solution (A_r, B_r) to the equations

$$A_r\, C_r\, (\xi,\eta) = (\alpha_o + \alpha_1\xi + \alpha_2\eta) + (\alpha_1\, U_m + \alpha_2\, V_m)\, W_m\, (\xi,\eta)$$

$$B_r\, C_r\, (\xi,\eta) = (\beta_o + \beta_1\xi + \beta_2\eta) + (\beta_1\, U_m + \beta_2\, V_m)\, W_m\, (\xi;\eta)$$

A first part of the solution

$$A_r^{(1)} C_r\, (\xi,\eta) = \alpha_o + \alpha_1\xi + \alpha_2\eta \qquad B_r^{(1)}\, C_r\, (\xi,\eta) = \beta_o + \beta_1\xi + \beta_2\eta$$

certainly exists, because in the (ξ,η) plane the stress functions are at least complete linear polynomials. A sufficient condition for the existence of the second part

$$A_r^{(2)} C_r\, (\xi,\eta) = (\alpha_1\, U_m + \alpha_2\, V_m)\, W_m\, (\xi,\eta) \qquad B_r^{(2)} C_r (\xi,\eta) = (\beta_1 U_m + \beta_2 V_m) W_m (\xi,\eta)$$

is clearly that the geometrical interpolation functions $W_m\, (\xi,\eta)$ form a subset of the $C_r\, (\xi,\eta)$.

8. NUMERICAL EXPERIENCE WITH THE DISCRETIZATION OF ROTATIONAL EQUILIBRIUM

The results presented were obtained with a triangular plane stress equilibrium membrane element, the polynomial degree n of the stresses varying from 0 to 3, and the degree m of rotation being inferior to n, except for n=0, which is in fact a pure equilibrium model free of kinematical modes at the element level. The connectors are presented on fig. 5.

8.1. FIRST EXAMPLE

The first application was to a rectangular cantilever beam. The geometry, the numerical characteristics and the adopted mesh are shown on fig. 6.
The theoretical total potential energy is Ep = 509.366.
The results for all the different combinations of degrees are :

n	m	Ep
1	0	453.624
	0	344.293
2	1	499.260
	0	327.246
3	1	495.316
	2	505.956

It must be observed that the calculated energy is always inferior to the theoretical value, that is to say the model seems, with regards to energy bounds, to behave as a conforming displacement model.
More comments will be given in example 2.
Figure 7 gives some idea about the quality of the element when compared to others, among which the hybrids presented in section 5.
DQ1, DQ2 denote respectively : a conforming displacement triangular model of degree 1 , and the analogous of degree 2. The element is indicated by the couple of values (n,m). Notice that (1,o) falls between DQ1 and DQ2.

8.2. SECOND EXAMPLE

The convergence tests have been established for a trapezoïdal cantilever beam, illustrated on figure 8. Four different meshes have been used, dividing the edges of the beam in 1, 2, 4, 8 equal segments; each quadrilateral so obtained being then subdivided by its diagonals into 4 triangular elements.
The convergence of the total potential energy is plotted in terms of the mesh sizes in fig. 9. Two boundary values, calculated from elementary strengh of materials are : 319.4232 and 359.1122.
The computed results for the energy are summarized hereunder :

(n,m)	1x1	2x2	4x4	8x8
(1,0)	173.7009	251.7284	306.2896	329.4888
(2,0)	143.1882	241.6242	303.6846	329.0178
(2,1)	329.6447	338.5844	339.6372	339.7740
(3,0)	141.5422	240.6105	303.3672	328.9484
(3,1)	325.6752	338.0253	339.5080	339.7298
(3,2)	338.8439	339.6576	339.7708	339.7902

Comments on the results obtained with the discretization of rotational equilibrium

Example 2 confirms the feature of lower bound energy convergence of the element. An explanation could be the following : for a given stress degree n , a better solution is expected when the degree of the rotation is increased. The rotation being of displacement nature , the element will be more flexible and the energy higher, which is confirmed by the numerical results. Conversely, when m is fixed, the solution deteriorates for a higher stress degree, since the model becomes stiffer. It is interesting to note the energy leap for the three n cases, when the rotation passes from degree 0 to 1.
This brings out the decisive part played by the rotation, specially in this geometrically severe example.
For the two examples, the classification established for an increasing quality is

(3,0), (2,0), (1,0), (3,1), (2,1), (3,2).

As a conclusion, the proposed element gives a good convergence provided the degree of the rotation is increased in parallel with the stress degree, that is to say $m = n - 1$. Otherwise, a lack of balance exists between the rigorous satisfaction of translational equilibrium and the minimum acceptable state of rotational equilibrium , which is detrimental to the element.

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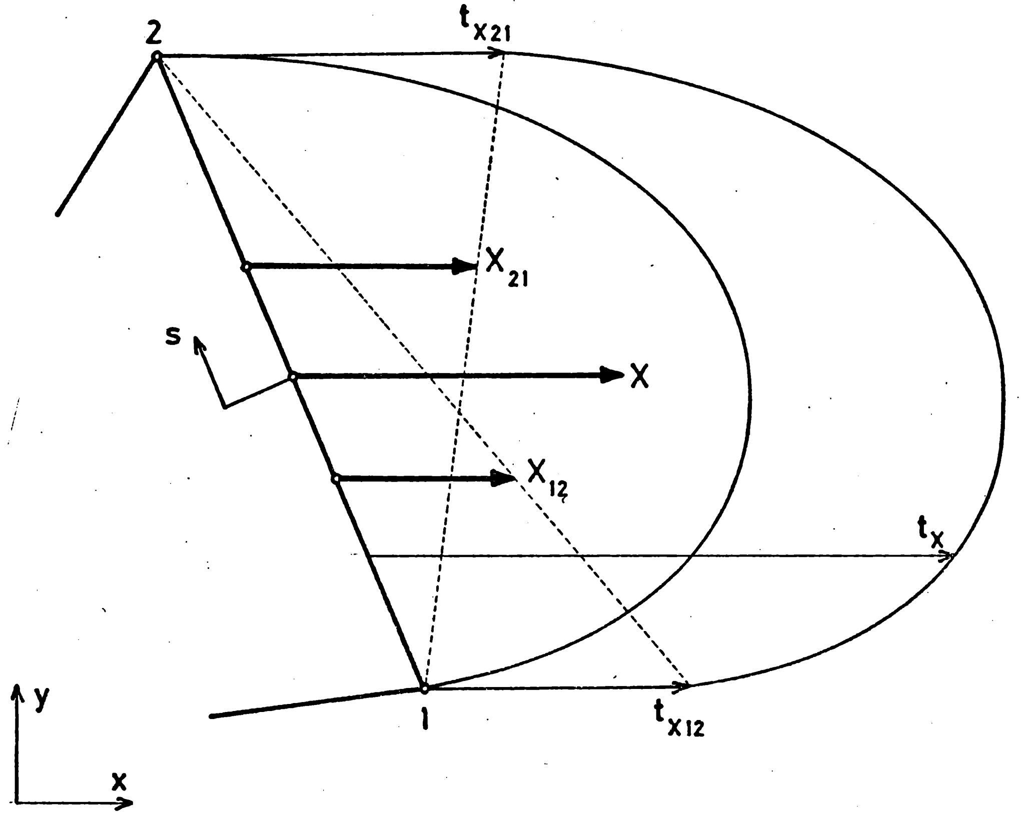
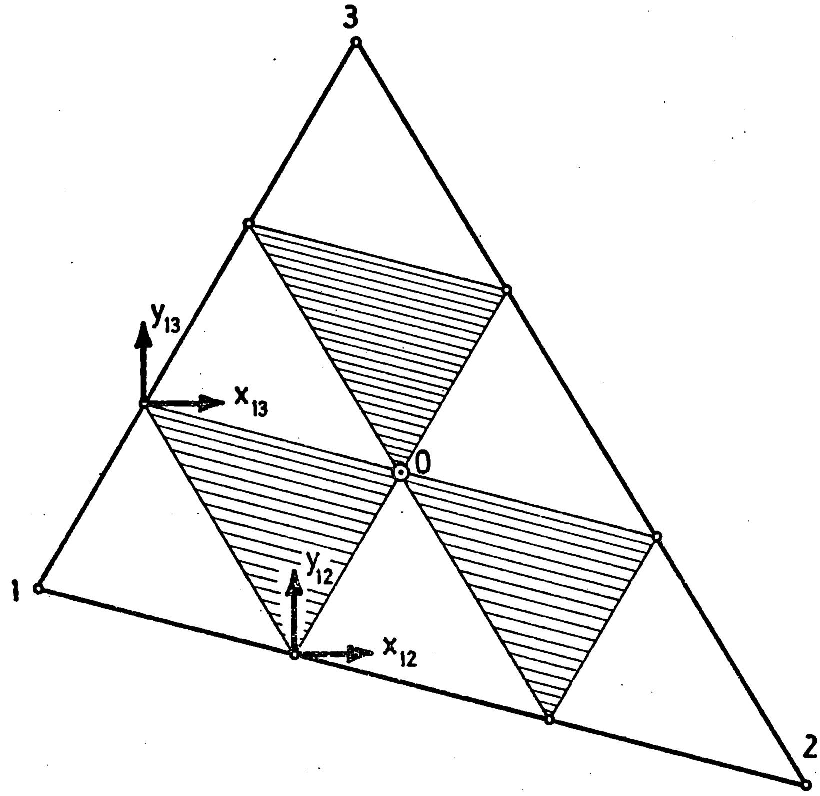


DECOMPOSITION OF A PARABOLIC SURFACE TRACTION DISTRIBUTION t_X IN THREE PARTS

$$t_X = t_{X12}\,\frac{1}{2}(1-\sigma) + t_{X21}\,\frac{1}{2}(1+\sigma) + \lambda(1-\sigma^2) \qquad \sigma = \frac{s}{c_{12}}$$

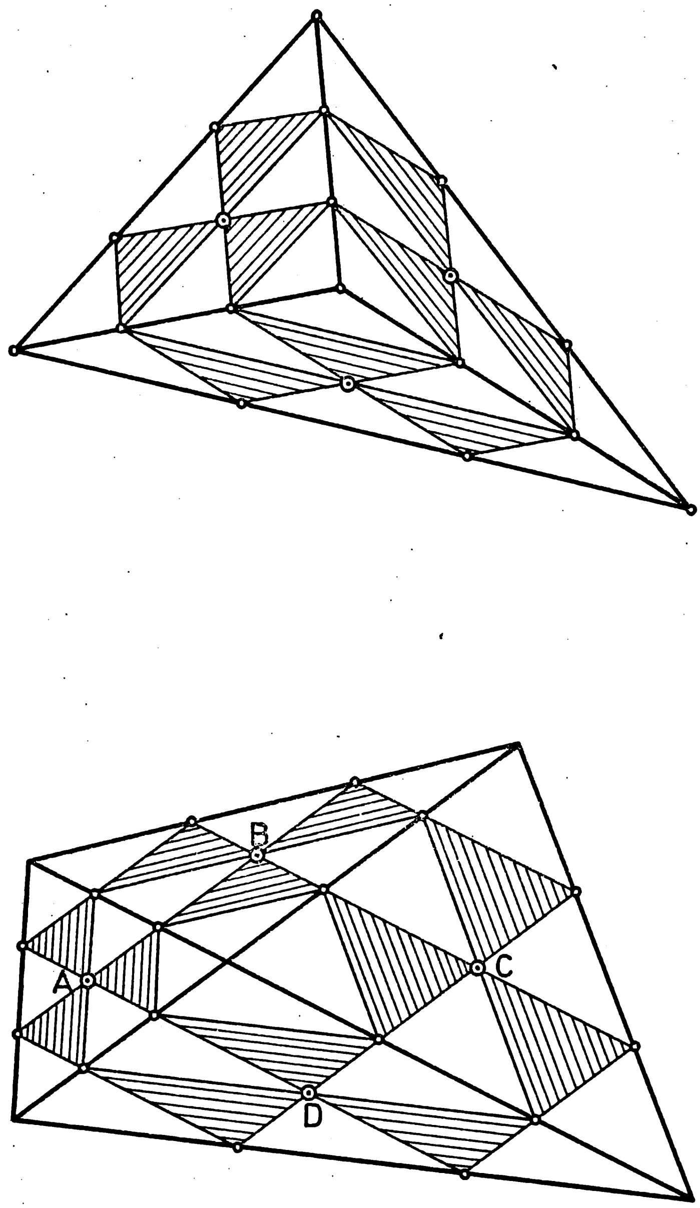
EACH PART HAS A RESULTANT LOAD X_{12}, X_{21} AND X

FIG. 1



VIRTUAL WORK INTEGRATION OF KINEMATICAL DEFORMATION MODE

FIG. 2



COMPOSITE ELEMENTS INHIBITING KINEMATICAL MODES

FIG. 3

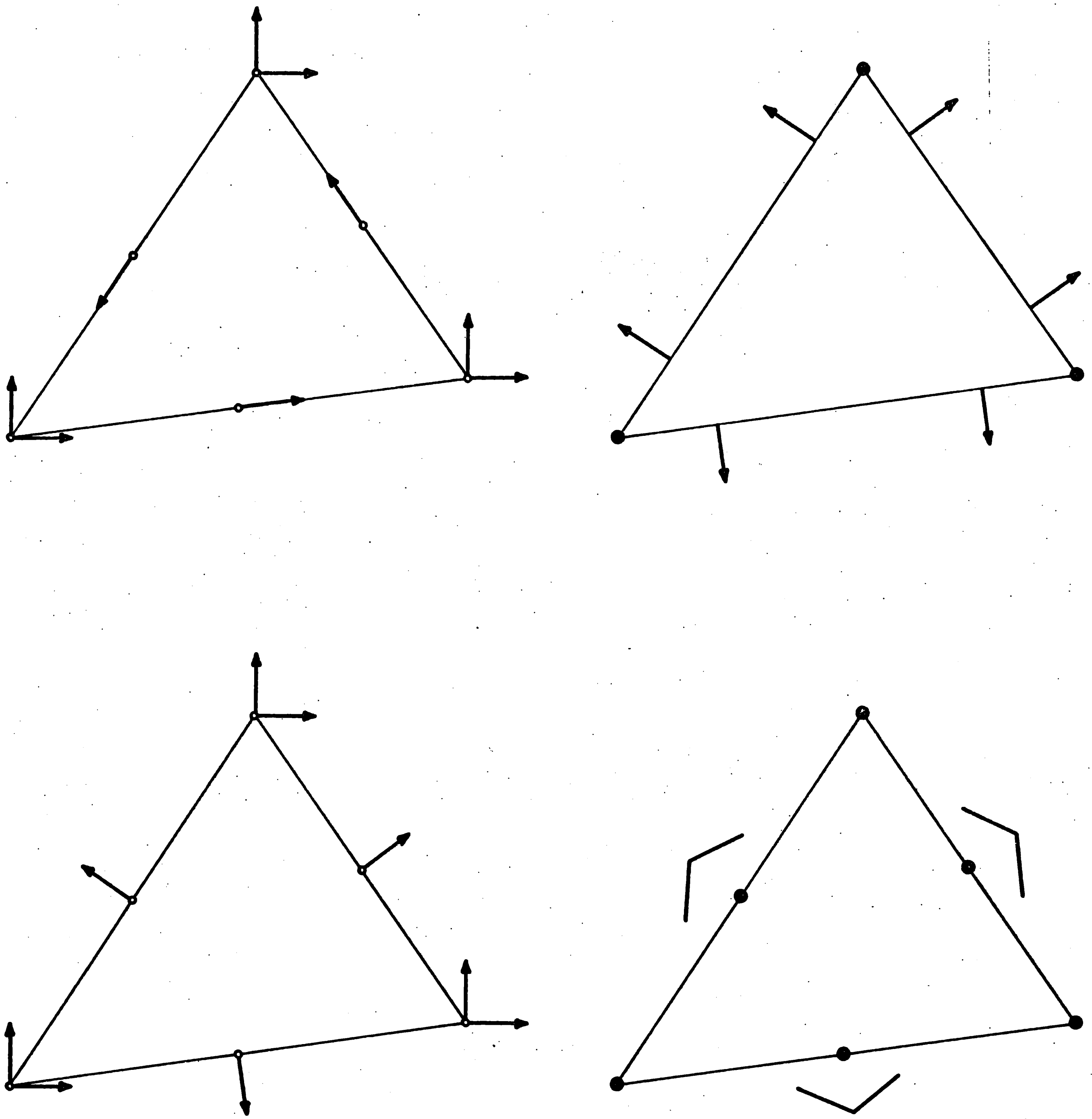


FIG. 4

Caption of Fig. 4

Two hybrid elements based on equilibrated linear stress fields

Left : local displacement connectors

Right: equivalent Airy stress function connectors

● local value of Airy function

⟶ normal slope of Airy function at Gauss points

^ average normal slope.

In each case a 10 th internal stress parameter to be determined by energy minimization at element level.

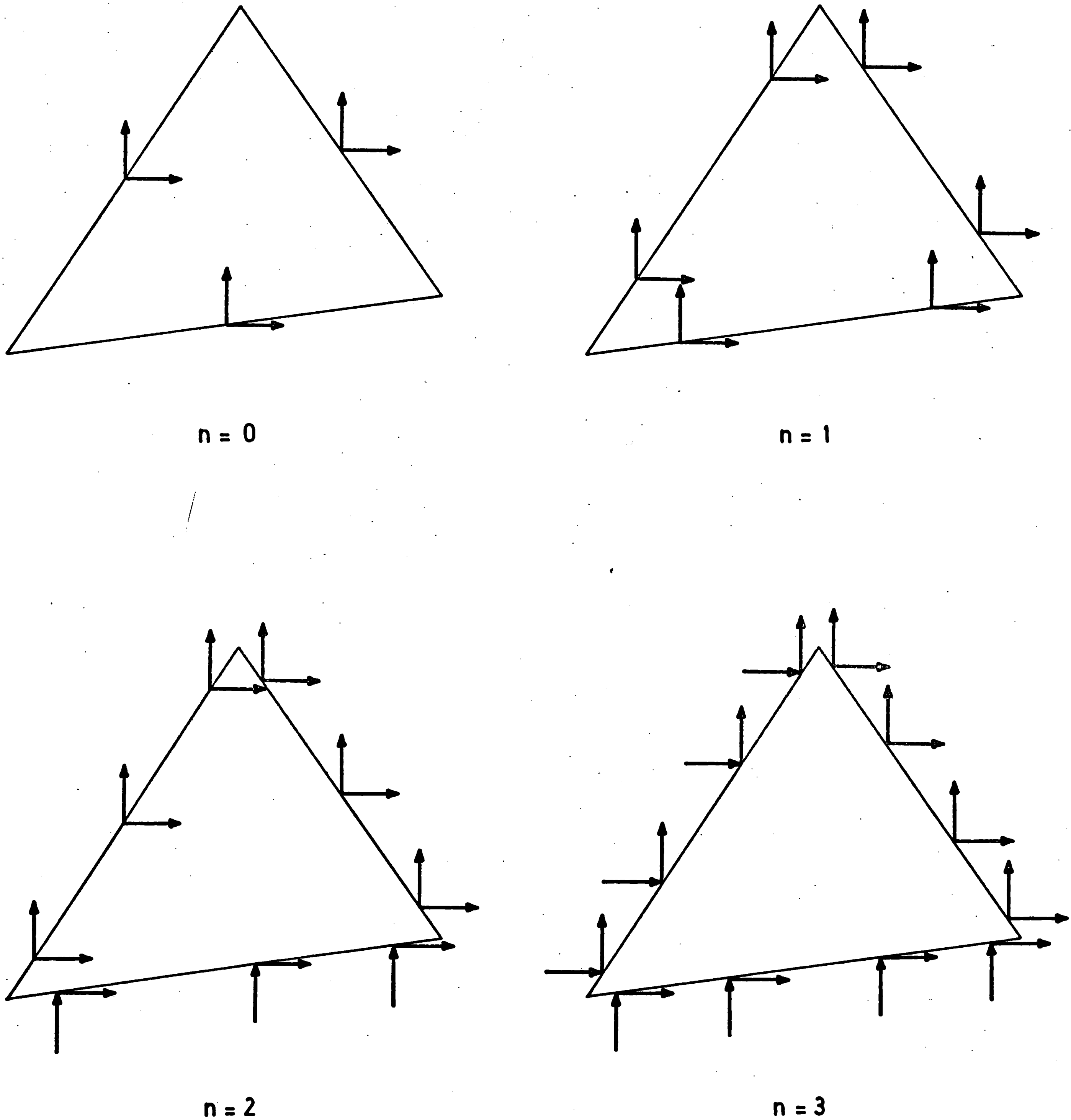
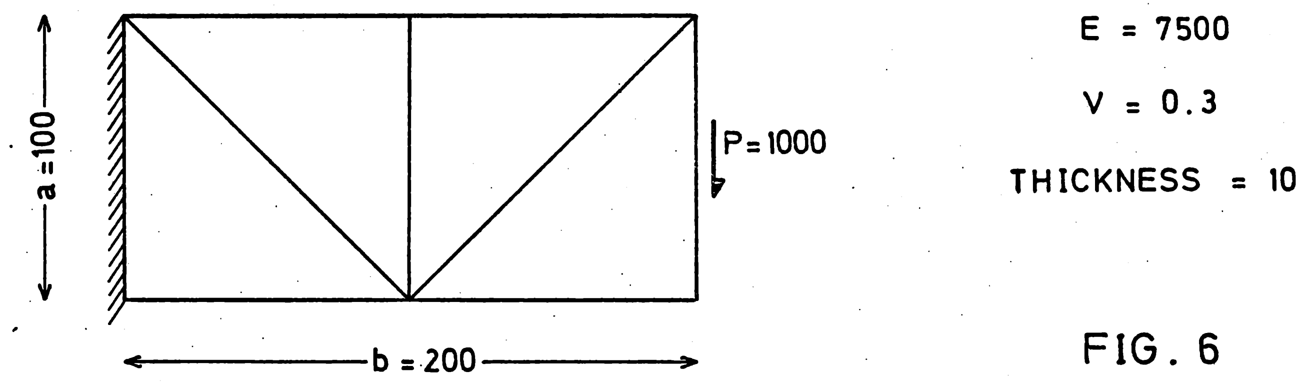


FIG. 5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b = 200
P = 1000
E = 7500
V = 0.3
THICKNESS = 10

FIG. 6

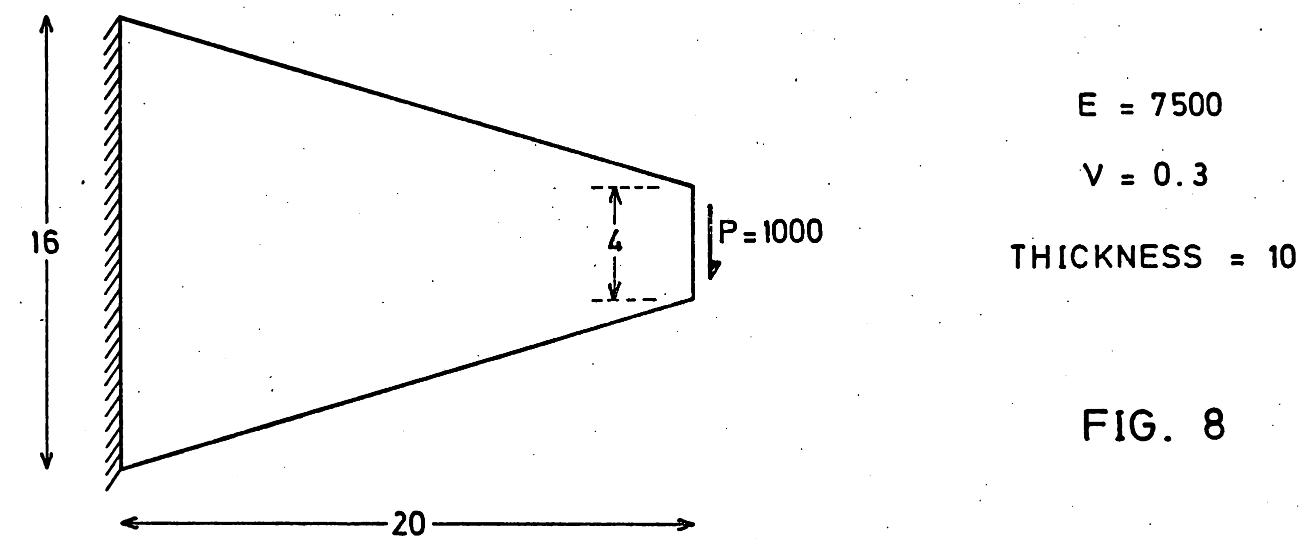
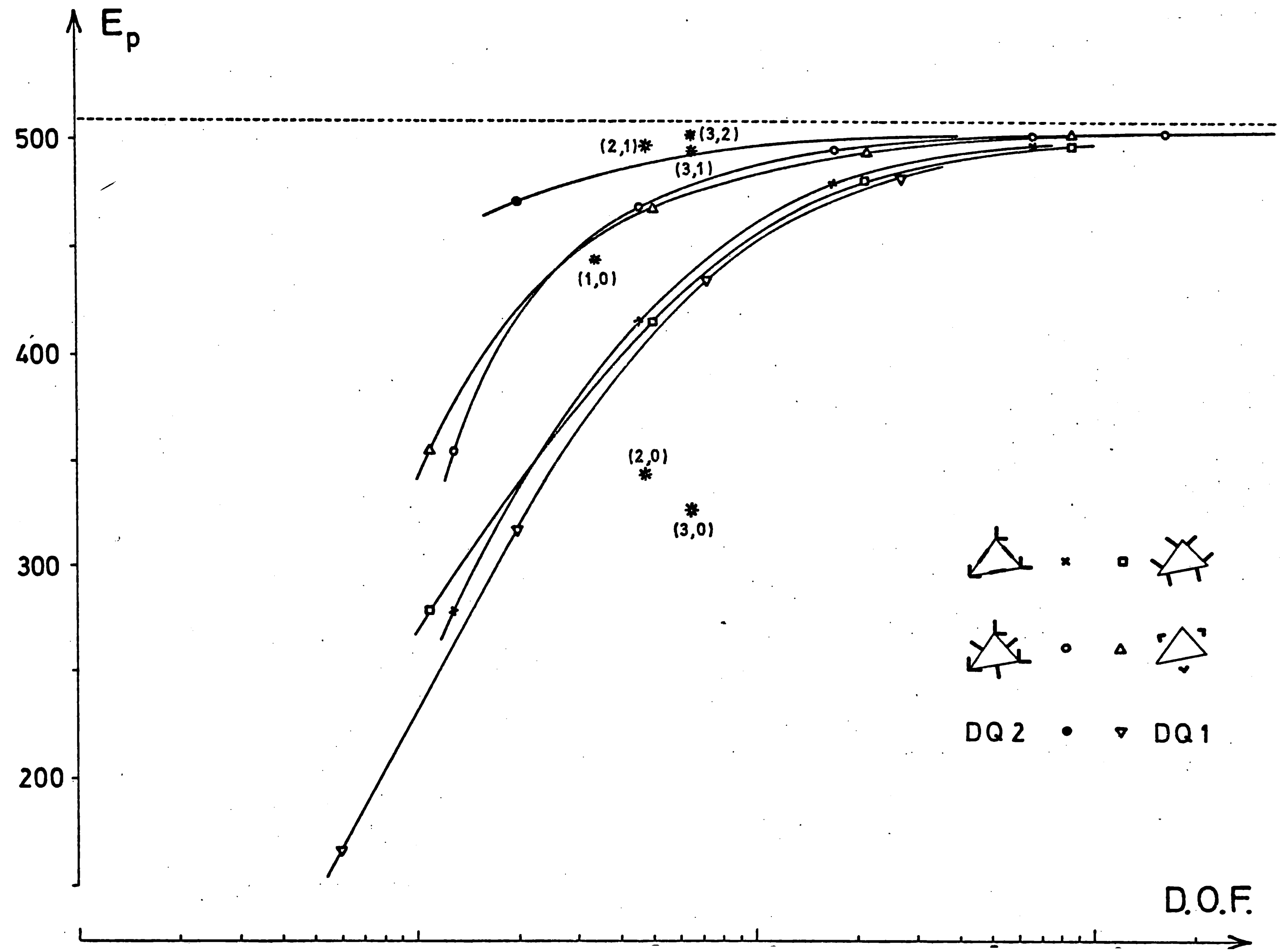
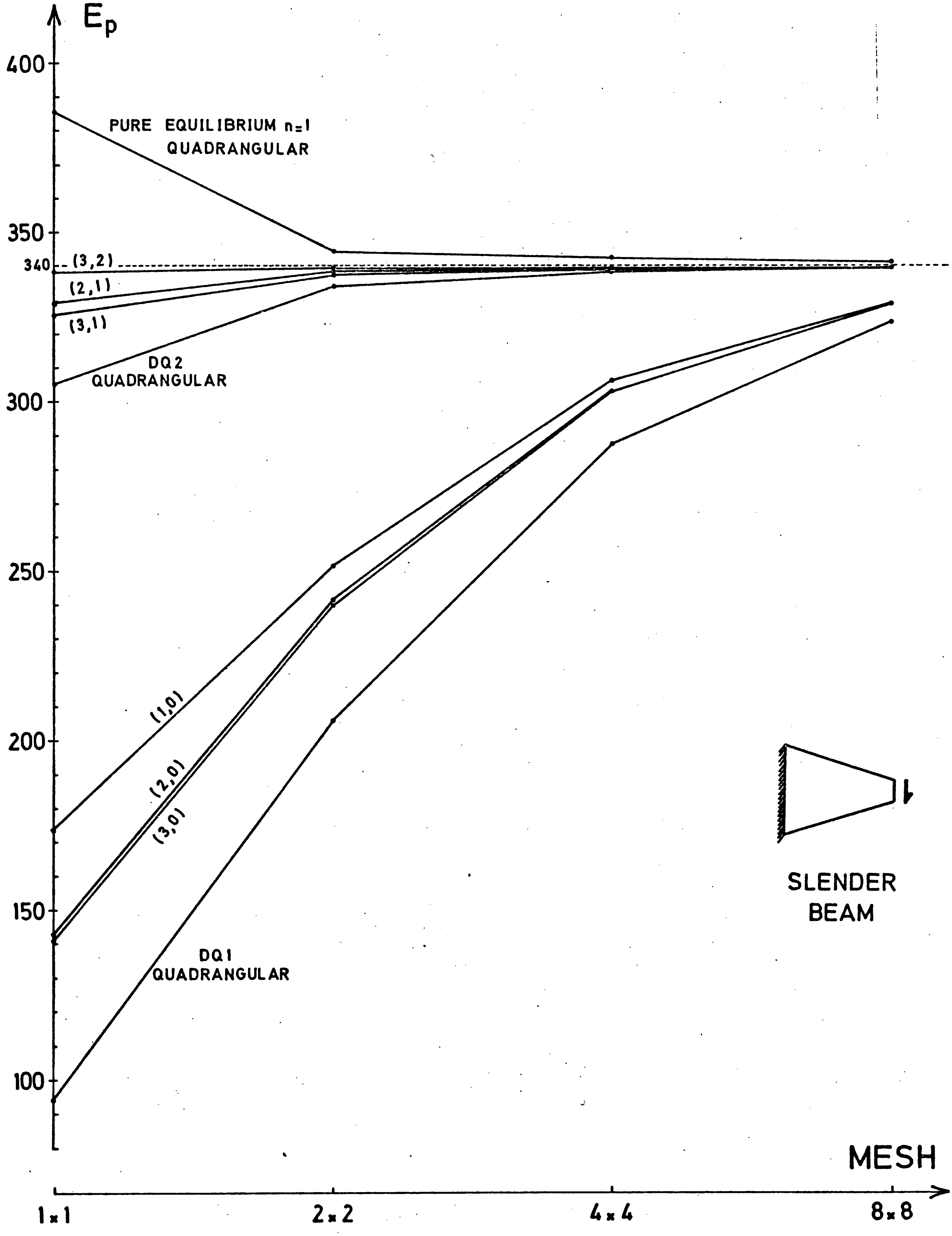

16
4
20
P = 1000
E = 7500
V = 0.3
THICKNESS = 10

FIG. 8



E_p
500
400
300
200
D.O.F.
(2,1)
(3,2)
(3,1)
(1,0)
(2,0)
(3,0)
DQ 2
DQ 1



E_p
400
350
340
300
250
200
150
100
PURE EQUILIBRIUM n=1
QUADRANGULAR
(3,2)
(2,1)
(3,1)
DQ2
QUADRANGULAR
(1,0)
(2,0)
(3,0)
DQ1
QUADRANGULAR
SLENDER
BEAM
MESH
1×1
2×2
4×4
8×8

FIG. 9