Berge-acyclic multilinear 0–1 optimization problems

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Abstract

We investigate the problem of optimizing a multilinear polynomial f in 0–1 variables and characterize instances for which the classical standard linearization procedure guarantees integer optimal solutions. We show that the standard linearization polytope P_H is integer exactly when the hypergraph H defined by the higher-degree monomials of f is Berge-acyclic, or equivalently, when the matrix defining P_H is balanced. This characterization follows from more general conditions that guarantee integral optimal vertices for a relaxed formulation depending on the sign pattern of the monomials of f.

Keywords— multilinear 0–1 optimization, standard linearization, Berge cycle, balanced matrix, signed hypergraph

1 Introduction

We consider the problem of maximizing a given multilinear polynomial f in 0–1 variables $x_1, \dots, x_n$,

$$\max_{x \in \{0,1\}^n} f(x) = \sum_{i \in V} c_i x_i + \sum_{e \in E} a_e \prod_{j \in e} x_j, \tag{1}$$

where $V = \{1, \dots, n\}$, $E \subseteq \{e \in 2^V : |e| \geq 2\}$, $c \in \mathbb{R}^V$, and $a \in \mathbb{R}^E$. Problem (1) is known to be $\mathcal{NP}$-hard, even when the objective function is quadratic (in which case it is equivalent with max-cut; see [4, 5, 15]). Several approaches have been

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proposed to solve the multilinear binary optimization problem, such as reductions to the linear or to the quadratic case, algebraic methods, enumerative methods like branch-and-bound and its variants, or cutting-plane methods (see, for example, [5, 7, 13, 27]). The efficacy of these techniques strongly depends on the structure of the problem, and it is unclear whether one approach is generally better than the others. In this paper we focus on linearization, an approach that attempts to draw benefit from the extensive literature on integer linear programming.

More precisely, we concentrate on the *standard linearization*, a classical linearization procedure that consists in substituting each nonlinear monomial $\prod_{j\in e} x_j$ by a new auxiliary variable y_e and imposing $y_e = \prod_{j\in e} x_j$ for all $e \in E$. These polynomial constraints are then substituted by the set of linear inequalities

$$y_e \leq x_j, \qquad \forall j \in e \tag{2}$$

$$y_e \geq \sum_{j\in e} x_j - (|e| - 1), \tag{3}$$

$$y_e \geq 0. \tag{4}$$

The standard linearization was proposed by several authors independently ([21, 22, 30, 31]), in a slightly different form from (2)–(4) and with integrality constraints on the variables y_e. The initial formulation was later improved by Glover and Woolsey, in a first contribution by adding fewer constraints and variables in the reformulation [23], and in a second contribution by introducing continuous auxiliary variables rather than integer ones [24].

When all variables x_j with $j \in e$ are binary, the feasible solutions of the constraints (2)–(4) are exactly the solutions of the polynomial equation $y_e = \prod_{j\in e} x_j$. However, when the integrality constraints are dropped, the linear relaxation provided by (2)–(4) leads to very weak bounds in general. The multilinear polytope P_H defined by the constraints (2)–(4), for all $e \in E$, and by the constraints $0 \leq x_i \leq 1$, $i \in V$, has been investigated in many papers; among recent contributions, let us mention [6, 14, 16, 18, 19].

In this article, we aim at characterizing the instances for which the polytope P_H has only integer vertices. One of our main results characterizes integrality of P_H in terms of the properties of the hypergraph $H = (V, E)$, defined by the nonlinear monomials of the multilinear expression (1). More precisely, we will show that P_H has integer vertices if and only if H is Berge-acyclic. Furthermore, we show that Berge-acyclicity is equivalent to the constraint matrix of P_H being balanced; for this, we rely on a fundamental result by Conforti and Cornuéjols [10]. We derive these characterizations from a more general result, characterizing the integrality of an optimal vertex of P_H in terms of the signs of the coefficients of the nonlinear terms of f. As a byproduct, we deduce the existence of an efficient algorithm for testing whether a given sign pattern of the nonlinear terms guarantees integrality or not. These results generalize those obtained by Padberg [29] and by Hansen and Simeone [25] for the quadratic case, and by Crama [12] for the general case. They unify and clarify the relation between these earlier characterizations.

2 Definitions and statement of the main results

This section formally introduces relevant definitions and states the main results of this article. Let $H = (V, E)$ be a finite hypergraph. We assume throughout that E does not contain singletons. We denote by $\mathcal{P}(H)$ the set of multilinear expressions f of type (1), obtained by defining the coefficients $c \in \mathbb{R}^V$ and $a \in \mathbb{R}^E$. For the sake of simplicity, we assume $a_e \neq 0$ for all $e \in E$.

Definition 1. *The* standard linearization polytope P_H *associated with a hypergraph* $H = (V, E)$ *is the polytope defined by the constraints*

$$-y_e + x_i \geq 0 \qquad \forall i \in e, \forall e \in E, \tag{5}$$

$$y_e - \sum_{i \in e} x_i \geq 1 - |e| \qquad \forall e \in E, \tag{6}$$

and by the bounds $0 \leq x_i \leq 1$ *for all* $i \in V$*, and* $0 \leq y_e \leq 1$ *for all* $e \in E$*. We denote by* M_H *the matrix of coefficients of the left-hand-sides of* (5) *and* (6).

Definition 2. *Given a multilinear expression* $f \in \mathcal{P}(H)$*, its* linearized form *is defined as*

$$L_f(x, y) = \sum_{i \in V} c_i x_i + \sum_{e \in E} a_e y_e,$$

where the coefficients c_i *and* a_e *are exactly the same as in* f*.*

As already mentioned in the introduction, all integer points $(x, y) \in P_H$ are such that $y_e = \prod_{j \in e} x_j$ for all $e \in E$. As a consequence, maximizing the linearized form L_f over the integer points of P_H is equivalent to maximizing $f(x)$ over $\{0, 1\}^n$.

Notice that when maximizing a linearized form L_f over P_H, constraints (5) are not binding when the coefficient a_e is negative, and constraints (6) are not binding when a_e is positive. This observation motivates the following definitions.

Definition 3. *A* signed hypergraph $H(s)$ *is a hypergraph* $H = (V, E)$ *together with a* sign pattern $s \in \{-1, 1\}^E$*. The set of* positive edges *of* $H(s)$ *is* $E^+ = \{e \in E \colon s_e = 1\}$ *and the set of* negative edges *is* $E^- = \{e \in E \colon s_e = -1\}$*.*

Clearly, every element $f \in \mathcal{P}(H)$ (or the associated linearized form L_f) defines a sign pattern by setting $s_e := \mathrm{sgn}(a_e)$ and hence, induces a signed hypergraph $H(s)$. Sign patterns can thus be considered as equivalence classes of $\mathcal{P}(H)$ with respect to the signs of the coefficients.

Definition 4. *The* signed standard linearization polytope $P_{H(s)}$ *associated with a signed hypergraph* $H(s)$ *is the polytope defined by the constraints*

$$-y_e + x_i \geq 0 \qquad \forall i \in e, \forall e \in E^+, \tag{7}$$

$$y_e - \sum_{i \in e} x_i \geq 1 - |e| \qquad \forall e \in E^-, \tag{8}$$

and by the bounds $0 \leq x_i \leq 1$ *for all* $i \in V$*, and* $0 \leq y_e \leq 1$ *for all* $e \in E$*. We denote the matrix of coefficients of the left-hand-sides of* (7) *and* (8) *by* $M_{H(s)}$*.*

The notion of cycles in hypergraphs will frequently be used in this paper. Several definitions of cycles in hypergraphs have been given in the literature, such as Berge-cycles [3], α-cycles ([2]), special cycles ([1]) (also called weak β-cycles [20]), or γ-cycles ([20]). In our context, we use the following definitions.

Definition 5. *Given a hypergraph $H = (V, E)$, a* Berge-cycle *C of length p is a sequence $(i_1, e_1, i_2, e_2, \dots, i_p, e_p, i_{p+1} = i_1)$ where*

1. *$p \geq 2$,*
2. *$i_k, i_{k+1} \in e_k$ for $k = 1, \dots p-1$ and $i_p, i_1 \in e_p$,*
3. *$i_1, \dots, i_p$ are pairwise distinct elements of V, and*
4. *$e_1, \dots, e_p$ are pairwise distinct elements of E.*

If, additionally, $e_k \cap \{i_1, \dots, i_p\} = 2$ for all $k = 1, \dots, p$, we call C a special cycle *of H.*

(In the definition of special cycles, it is usually assumed that $p \geq 3$. We only impose here that $p \geq 2$.) Given a Berge-cycle C, we denote by $V_C = \{i_1, \dots, i_p\}$ its set of vertices and by $E_C = \{e_1, \dots, e_p\}$ its set of edges.

Lemma 1. *Any hypergraph containing a Berge-cycle also contains a special cycle.*

Proof. Let $C = (i_1, e_1, i_2, e_2, \dots, i_p, e_p, i_1)$ be a Berge-cycle of minimal length in $H = (V, E)$. We claim that C is special. Assume on contrary that C is not special and that, without loss of generality, $|e_1 \cap V_C| > 2$. Choose $i_k \in (e_1 \cap V_C) \setminus \{i_1, i_2\}$. Then $(i_1, e_1, i_k, e_{k+1}, \dots, e_p, x_1)$ is a Berge-cycle strictly shorter than C, contradicting the choice of C. □

We next extend the classical definition of negative cycles in signed graphs [26].

Definition 6. *A* negative (special) cycle *in a signed hypergraph $H(s)$ is a (special) cycle containing an odd number of negative edges.*

Finally, we recall the definition of balanced matrices [3, 10].

Definition 7. *A matrix M with all entries in $\{-1, 0, 1\}$ is* balanced *if in every submatrix of M having exactly two non-zeros per row and two non-zeros per column, the sum of the entries is a multiple of four.*

The main result of this article is the following characterization. The proof will be given in Section 3.

Theorem 1. *Given a hypergraph $H = (V, E)$ and a sign pattern $s \in \{-1, 1\}^E$, the following statements are equivalent:*

(a) *For all* $f \in \mathcal{P}(H)$ *with sign pattern* s*, every vertex of* P_H *maximizing* L_f *is integer.*

(b) $M_{H(s)}$ *is balanced.*

(c) $H(s)$ *has no negative special cycle.*

(d) $P_{H(s)}$ *is an integer polytope.*

Remark 1. *As shown in [9], it can be checked efficiently whether a matrix with entries in* $\{-1, 0, 1\}$ *is balanced. This implies that all conditions in Theorem 1 can be checked efficiently.*

Remark 2. *Theorem 1 characterizes the sign patterns* s *that guarantee integer optimal solutions for* all $f \in \mathcal{P}(H)$ *with sign pattern* s*. This does not exclude, however, that some functions with a sign pattern different from* s *also lead to integer optimal solutions. This depends on the (relative) values of the coefficients* a_e *and* c_i*. As an example, consider the quadratic function*

$$f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 - x_2x_3 - Mx_1$$

with $M \in \mathbb{R}$*. The corresponding hypergraph* $M(s)$ *contains a negative special cycle, but for large enough values of* M *the optimal vertices of* P_H *with respect to* L_f *are all integer.*

Corollary 1. *Given a hypergraph* H*, the following statements are equivalent:*

(a) P_H *is an integer polytope.*

(b) M_H *is balanced.*

(c) H *is Berge-acyclic.*

Proof. We claim that each of the statements (a), (b) and (c) is equivalent to the respective statement (a), (b) and (c) in Theorem 1 holding for all $s \in \{-1, 1\}^E$. The result then follows from Theorem 1. For (a), this equivalence is obvious.

For (b), it is clear that if M_H is balanced, then $M_{H(s)}$ is balanced for all s, since every submatrix of $M_{H(s)}$ is also a submatrix of M_H. Assume now that M_H is not balanced. Thus, it contains a submatrix B with exactly two nonzeros per row and per column such that the sum of its entries is congruent with 2 modulo 4. As long as there are two rows in B corresponding to constraints of type (5) and (6) for the same edge $e \in E$, the matrix B must contain a submatrix of the form

$$B' = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix},$$

where all other entries of B in the rows and columns of B' are zero by the choice of B. The sum of elements of B' is zero, thus we can recursively delete the corresponding rows and columns from B and finally assume that, for each $e \in E$, B only contains rows associated with constraints of type (5), or only contains a row associated with constraint (6). We may then define a sign pattern s by setting $s_e = 1$ if B contains a row corresponding to (5) for e, $s_e = -1$ if B contains the row corresponding to (6) for e, and $s_e \in \{-1, 1\}$ arbitrarily otherwise. Then, by construction, the matrix B is a submatrix of $M_{H(s)}$, showing that $M_{H(s)}$ is not balanced.

For (c), if H is Berge-acyclic, then clearly $H(s)$ has no negative special cycle, for any sign pattern s. Conversely, assume that H contains a Berge-cycle C. By Lemma 1, we may assume that C is a special cycle. Define a sign pattern s such that $s_e = -1$ for exactly one edge of C and $s_e = 1$ otherwise. Then C is a negative special cycle in $H(s)$. □

For ordinary graphs, i.e., for hypergraphs $H = (V, E)$ where $|e| = 2$ for all edges $e \in E$ (corresponding to quadratic functions f), Padberg [29] proved that P_H has integer vertices if and only if H is an acyclic graph. Corollary 1 generalizes Padberg's result to the case of higher-degree multilinear expressions.

Similarly, Theorem 1 extends results obtained by Hansen and Simeone [25] for the quadratic case (see also Michini [28]). The equivalence of conditions (b) and (d) in Theorem 1 for functions of arbitrary degree was first stated in Crama [11, 12]. However, the proof of this result was omitted from the published version [12]; in the technical report [11], the result was derived from a more general result whose proof was partially erroneous. So, we find it useful to provide here a complete proof of Theorem 1, which also clarifies the link with the integrality properties of P_H and of $P_{H(s)}$.

Theorem 1 and Corollary 1 have been independently derived from Crama's earlier results by Del Pia and Khajavirad [17], and the same authors have established Corollary 1 by different proof techniques in [16].

3 Proof of the main theorem

In this section, we present a proof of Theorem 1 in the form of four separate propositions. We first show that condition (d) implies condition (a) in Theorem 1. This is a direct consequence of the following:

Proposition 1. *Let $f \in \mathcal{P}(\mathcal{H})$ with a sign pattern $s \in \{-1, 1\}^E$. Then the two optimization problems*

$$\max_{(x,y) \in P_H} L_f(x, y),$$

and

$$\max_{(x,y) \in P_{H(s)}} L_f(x, y) \tag{9}$$

have the same sets of optimal solutions.

Proof. Since $P_H \subseteq P_{H(s)}$, it suffices to show that every optimal solution of Problem (9) belongs to P_H. So let (x^*, y^*) be such an optimal solution and consider any edge $e \in E$.

Assume first that (x^*, y^*) violates (5) for some $j \in e$, so that $y^*_e > x^*_j$. By definition of $P_{H(s)}$, this is possible only if $s_e = -1$, we thus have $a_e < 0$. We can now decrease y^*_e to x^*_j without leaving the polyhedron $P_{H(s)}$, since the resulting vector satisfies $y^*_e - \sum_{i \in e} x^*_i = -\sum_{i \in e \setminus \{j\}} x^*_i \geq -(|e| - 1)$. As $a_e < 0$, this contradicts the assumption that (x^*, y^*) is an optimal solution of (9).

Now assume that (x^*, y^*) violates (6). Then $y^*_e < \sum_{i \in e} x^*_i - |e| + 1$, $s_e = +1$, and $a_e > 0$. We can now increase y^*_e to $\sum_{i \in e} x^*_i - |e| + 1$ without leaving $P_{H(s)}$, as $\sum_{i \in e} x^*_i - |e| + 1 \leq x^*_j$ for all $j \in e$. We again obtain a contradiction to the optimality of (x^*, y^*). □

We next observe that (b) implies (d) in Theorem 1. For this, consider the *generalized set covering polytope* $Q(A)$ corresponding to a matrix A with entries in $\{-1, 0, +1\}$, defined as

$$Q(A) := \{x \colon Ax \geq 1 - n(A), 0 \leq x \leq 1\},$$

where $n(A)$ denotes the column vector whose i^{th} component $n_i(A)$ is the number of negative entries in row i of A. We recall a fundamental result on balanced matrices:

Theorem 2 (Conforti and Cornuéjols [8, 10])**.** *Let M be a matrix with entries in $\{-1, 0, +1\}$. Then M is balanced if and only if for each submatrix A of M, the generalized set covering polytope $Q(A)$ is integral.*

Since $P_{H(s)}$ is exactly the generalized set covering polytope $Q(M_{H(s)})$, we obtain

Proposition 2. *Given $s \in \{-1, 1\}^E$, if $M_{H(s)}$ is balanced, then $P_{H(s)}$ is integral.*

We next show that (a) implies (c) in Theorem 1. This is equivalent to showing

Proposition 3. *Let $s \in \{-1, 1\}^E$ and assume that $H(s)$ has a negative special cycle C. Then there exists $f \in \mathcal{P}(H)$ with sign pattern s such that some optimal vertex of P_H with respect to L_f is not integer.*

Proof. Let $C = (i_1, e_1, i_2, e_2, \dots, i_p, e_p, i_{p+1})$ be a negative cycle in $H(s) = (V, E)$, with $i_{p+1} = i_1$, and consider the sets $E^+_C := E_C \cap E^+$ and $E^-_C := E_C \cap E^-$. Moreover, consider the partition of the set V_C given by the following subsets:

$$\begin{aligned}
V^{++} &:= \{i_k \in V_C \colon e_{i_k - 1} \in E^+_C \text{ and } e_{i_k} \in E^+_C\},\\
V^{--} &:= \{i_k \in V_C \colon e_{i_k - 1} \in E^-_C \text{ and } e_{i_k} \in E^-_C\},\\
V^{+-} &:= \{i_k \in V_C \colon e_{i_k - 1} \in E^+_C \text{ and } e_{i_k} \in E^-_C\},\\
V^{-+} &:= \{i_k \in V_C \colon e_{i_k - 1} \in E^-_C \text{ and } e_{i_k} \in E^+_C\}.
\end{aligned}$$

Note that the following identity holds for all $x \in \mathbb{R}^V$:

$$\sum_{e_k \in E^-_C} x_{i_k} - \sum_{e_k \in E^+_C} x_{i_{k+1}} = \sum_{j \in V^{--}} x_j + \sum_{j \in V^{+-}} x_j - \sum_{j \in V^{+-}} x_j - \sum_{j \in V^{++}} x_j = \sum_{j \in V^{--}} x_j - \sum_{j \in V^{++}} x_j. \tag{10}$$

We will define an objective function $h = L_f$ corresponding to some $f \in \mathcal{P}(H)$ with sign pattern s, as well as a fractional point $(\hat{x}, \hat{y}) \in P_H$ such that $h(\hat{x}, \hat{y}) > h(x, y)$ for all integer points $(x, y) \in P_H$. This will imply the result.

The function h with sign pattern s is defined by

$$h(x,y) := \mu \left(\sum_{j \in V \setminus V_C} x_j \right) + \sum_{j \in V^{--}} x_j - \sum_{e \in E_C^-} y_e - \sum_{j \in V^{++}} x_j + \sum_{e \in E_C^+} y_e + \varepsilon \,(\sum_{e \in E^+} y_e - \sum_{e \in E^-} y_e),$$

where μ is chosen large enough so that all x_j with $j \in V \setminus V_C$ take value one when maximizing h over P_H, and where ε is small and positive. Define $(\hat{x}, \hat{y})$ as follows:

- $\hat{x}_i = \frac{1}{2}$, for $x_i \in V_C$,
- $\hat{x}_i = 1$, for $x_i \in V \setminus V_C$,
- $\hat{y}_e = \frac{1}{2}$, for $e \in E_C^+$,
- $\hat{y}_e = 0$, for $e \in E_C^-$,
- $\hat{y}_e = \frac{1}{2}$ for $e \in E \setminus E_C$ containing at least one vertex in V_C, and
- $\hat{y}_e = 1$ for $e \in E \setminus E_C$ containing no vertex in V_C.

It can be verified that $(\hat{x}, \hat{y}) \in P_H$ and that

$$h(\hat{x}, \hat{y}) = \mu |V \setminus V_C| + \tfrac{|V^{--}|}{2} - \tfrac{|V^{++}|}{2} + \tfrac{|E_C^+|}{2} + \varepsilon\, t(\hat{y}) = \mu |V \setminus V_C| + \tfrac{|E_C^-|}{2} + \varepsilon\, t(\hat{y}),$$

where $t(\hat{y})$ stands for the multiplier of ε, and where the second equality follows from identity (10).

It remains to prove that $h(\hat{x}, \hat{y}) > h(x, y)$ for all integer points $(x, y) \in P_H$. By the choice of μ, we may assume $x_j = 1$ for all $j \in V \setminus V_C$. Moreover, by integrality of $(x, y) \in P_H$, we have $y_e = \prod_{i \in e} x_i$ for all $e \in E$. Now, since C is a special cycle, and using (10) again, we can express $h(x, y)$ as follows:

$$\begin{aligned} h(x,y) &= \mu |V \setminus V_C| + \sum_{j \in V^{--}} x_j - \sum_{e_k \in E_C^-} x_{i_k} x_{i_{k+1}} - \sum_{j \in V^{++}} x_j + \sum_{e_k \in E_C^+} x_{i_k} x_{i_{k+1}} + \varepsilon\, t(y) \\ &= \mu |V \setminus V_C| + \sum_{e_k \in E_C^-} [x_{i_k}(1 - x_{i_{k+1}})] - \sum_{e_k \in E_C^+} [(1 - x_{i_k}) x_{i_{k+1}}] + \varepsilon\, t(y). \end{aligned}$$

Since there is an odd number of negative edges in C, we derive that

$$h(x,y) \le \mu |V \setminus V_C| + \tfrac{|E_C^-| - 1}{2} + \varepsilon\, t(y) < h(\hat{x}, \hat{y})$$

for every integer point $(x, y) \in P_H$ if ε is small enough. □

The basic idea of the previous proof is to reduce the construction of a fractional vertex of P_H to the quadratic case. For this, all variables not corresponding to a node in V_C are set to one, letting them disappear from the monomial expressions corresponding to the edges in E_C. This construction only works for special cycles.

In order to conclude the proof of Theorem 1, it remains to show that (c) implies (b), that is:

Proposition 4. *Given $s \in \{-1, 1\}^E$, if $H(s)$ has no negative special cycle, then the matrix $M_{H(s)}$ is balanced.*

Proof. Assume that $M_{H(s)}$ is not balanced and let B be a smallest submatrix of $M_{H(s)}$ with two non-zero entries per row and two non-zero entries per column, such that the sum of its entries is congruent with 2 modulo 4.

Let $V_B \subseteq V$ be the set of vertices with their associated column in B, let $E_B \subseteq E$ be the set of edges associated with at least one row of B, let E_B^+ be the set of positive edges in E_B, and E_B^- be the set of negative edges in E_B.

Since each column of B has exactly two non-zero entries, each vertex in V_B is contained in exactly two edges in E_B. Moreover, by definition of $M_{H(s)}$, every edge $e \in E_B$ must contain exactly two vertices of V_B (if $e \in E^-$, then the entries corresponding to both vertices appear in the same row of B; if $e \in E^+$, then the entries corresponding to the two vertices appear in two rows associated with y_e). Consequently, in view of the minimality of B, the vertices in V_B and the edges in E_B define a special cycle.

Since rows corresponding to edges $e \in E_B^+$ are associated with constraints of type (5) and contain two non-zero entries by definition, the sum of these entries must be zero. This means that the sum of entries of rows corresponding to edges $e \in E_B^-$ has a value congruent with 2 modulo 4. Notice that each edge $e \in E_B^-$ has exactly one row of type (6) associated with it, and both entries of this row take value -1. This implies that there must be an odd number of negative edges in the special cycle defined by B. □

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