Chapter 5

Option pricing models

- Objective and hypotheses
- Fundamental theorem of risk-neutral valuation
- Black & Scholes : a martingale approach
- Black & Scholes : an arbitrage approach

Objective and hypotheses

Objective

Give a rigorous proof of Black & Scholes formula for an European option on equity

Perfect market

- No investor is dominant (no market maker)
- Investors are rational (prefer more to less)
- Assets infinitely divisible
- No transaction costs
- No tax
- Short sales allowed

Risk-free asset

Existence of a constant, continuous risk-free rate r, the same for borrowing and deposit

Arbitrage-free market

= “no free lunch”

Underlying asset

- The underlying asset is an equity, paying no dividend in the duration of the contract
- The evolution of the underlying asset is driven by a GBM

$$dS_t = \delta S_t \cdot dt + \sigma S_t \cdot dW_t$$

$$S_t = S_0\, e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Fundamental theorem of risk-neutral valuation

- Self-financing strategy and contingent claim
  - Trading strategy
  - Self-financing strategy
  - Link with a contingent claim
- Fundamental theorem of risk-neutral valuation
  - Discounted value for the equity
  - Discounted value for the portfolio
  - Fundamental theorem

Delbaen, F. and Schachermaeyer, W. (1994) A general version of the fundamental theorem of asset pricing, *Math. Ann.*, 300, 463-520

Self-financing strategy and contingent claim

Trading strategy

Let consider two assets

a) An equity whose value is driven by a GBM

$$dS_t = \delta S_t \cdot dt + \sigma S_t \cdot dW_t$$

b) A risk-free asset β_t for which $\beta_t = \beta_0 e^{rt}$ or, more generally,

$$d\beta_t = r\beta_t \cdot dt$$

A trading strategy is a couple (a_t, b_t) for constructing a portfolio with

- a_t units of the equity
- b_t units of the risk-free asset

$(a_t, b_t \in \mathbb{R})$

The value of the portfolio is $V_t = a_t S_t + b_t \beta_t$

Self-financing strategy

A self-financing strategy is a trading strategy for which the variations of the portfolio value comes only from changes of the prices of S_t and β_t

So, the portfolio value shows

- no decrements by consumption
- no increments by paying dividends

For a self-financing portfolio, we have

$$dV_t = a_t \, dS_t + b_t \, d\beta_t$$

Link with a contingent claim

We are searching for the value of a contingent claim at time t, knowing the pay-off of this contingent claim at time T $(> t)$: $h(S_T)$

[For an European call option, $h(S_T) = (S_T - K)^+$]

A contingent claim is a "game" at time t with reward equal to the pay-off $h(S_T)$ at time T

The rational fee (at a financial point of view) for playing this game is the price (premium) of the contingent claim

Black, Scholes and Merton reasoning :

- after investing this rational amount of money, we can manage the portfolio (equity; risk-free) according to a self-financing strategy, for obtaining the same pay-off $h(S_T)$ as if the contingent claim has been purchased
- if the contingent claim were offered at any price other than this rational value, there would exist an arbitrage opportunity

So, the value of the contingent claim is

$$\begin{cases} V_t = a_t S_t + b_t \beta_t & \text{at time } t \\ V_T = h(S_T) & \text{at time } T \end{cases}$$

Fundamental theorem of risk-neutral valuation

Discounted value for the equity

$$S_t^* = e^{-rt} S_t$$

Apply Itô's lemma to $f(t,x) = e^{-rt}x$

$$\begin{aligned} dS_t^* &= (-re^{-rt}S_t + \delta S_t e^{-rt})dt + \sigma S_t e^{-rt}\, dW_t \\ &= S_t^*[(-r+\delta)dt + \sigma\, dW_t] \\ &= \sigma S_t^*\, d\left[\left(\frac{-r+\delta}{\sigma}\right)t + W_t\right] \\ &= \sigma S_t^*\, d\widetilde{W}_t \end{aligned}$$

where $(\widetilde{W}_t)$ is a SBM w.r.t. the equivalent martingale measure Q

So, the solution of this SDE is a GBM with 0 drift :

$$S_t^* = S_0\, e^{-\frac{\sigma^2}{2}t + \sigma \widetilde{W}_t}$$

Discounted value for the portfolio

For the self-financing portfolio,

$$\begin{aligned} dV_t &= a_t\, dS_t + b_t\, d\beta_t \\ &= a_t(\delta S_t\, dt + \sigma S_t\, dW_t) + b_t r\beta_t\, dt \\ &= (a_t\delta S_t + b_t r\beta_t)\cdot dt + a_t\sigma S_t \cdot dW_t \end{aligned}$$

$$V_t^* = e^{-rt}V_t$$

Apply Itô's lemma to $f(t,x) = e^{-rt}x$

$$\begin{aligned} dV_t^* = [-re^{-rt}V_t &+ (a_t\delta S_t + b_t r\beta_t)e^{-rt}]\, dt \\ &+ a_t\sigma S_t e^{-rt}\, dW_t \\ &= e^{-rt}[(-ra_tS_t + a_t\delta S_t)dt + a_t\sigma S_t\, dW_t] \\ &= a_t S_t^*[(-r+\delta)dt + \sigma\, dW_t] \\ &= a_t S_t^* \sigma\, d\widetilde{W}_t \\ &= a_t\, dS_t^* \end{aligned}$$

The solution of this SDE is given by

$$V_t^* = V_0 + \int_0^t a_u\, dS_u^* = V_0 + \sigma \int_0^t a_u S_u^*\, d\widetilde{W}_u$$

Fundamental theorem

Under the martingale equivalent measure Q defined in Girsanov's theorem, $\widetilde{W}_t$ is a SBM adapted to the natural filtration of (W_t)

Then, the integral in the solution for V_t^* is an Itô stochastic integral, and so, it is a martingale

$$E_Q(V_T^*|\mathcal{F}_t) = V_t^*$$

But

$$V_T^* = e^{-rT}V_T = e^{-rT}h(S_T)$$

$$e^{-rt}V_t = E_Q(e^{-rT}h(S_T)|\mathcal{F}_t)$$

And, by introducing $\tau = T - t$,

$$V_t = e^{-r\tau}E_Q(h(S_T)|\mathcal{F}_t)$$

The price of a contingent claim is equal to the discounted value of the (conditional) expectation of its final value w.r.t. the risk-neutral measure

Black & Scholes : a martingale approach

- Pricing of a general contingent claim with underlying GBM
  - For the underlying equity
  - For the contingent claim
- Black & Scholes model for an European call option

Harrisson, M.J. and Pliska, S.R. (1981) Martingales and stochastic integrals in the theory of continuous trading, *Stoch. Proc. Appl.*, **11**, 215-260

Pricing of a general contingent claim with underlying GBM

For the underlying equity

Under the risk-neutral measure, the underlying equity behavior is, in mean, the same as the risk-free rate [Chapter 1 : $E_q(S_1) = S_0(1+R_F)$]

The (conditional) expectation is taken w.r.t. Q. We have then to use the $\left(\widetilde{W}_t\right)$ SBM :

$$dS_t = rS_t \cdot dt + \sigma S_t \cdot d\widetilde{W}_t$$

$$S_t = S_0\, e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma \widetilde{W}_t}$$

This last formula (at time t and at time T) leads to

$$S_T = S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma (\widetilde{W}_T-\widetilde{W}_t)}$$

For the contingent claim

$$\begin{aligned} V_t &= e^{-r\tau}E_Q(h(S_T)|\mathcal{F}_t) \\ &= e^{-r\tau}E_Q\left(h\left(S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma(\widetilde{W}_T-\widetilde{W}_t)}\right)\middle|\mathcal{F}_t\right) \end{aligned}$$

- S_t is $\mathcal{F}_t$-measurable (and is then considered as a constant in the conditional expectation)
- $\left(\widetilde{W}_T-\widetilde{W}_t\right)$ is independent of $\mathcal{F}_t$ (and for the exponential, the conditional expectation is an ordinary expectation)

Moreover, $\left(\widetilde{W}_T-\widetilde{W}_t\right) \sim \mathcal{N}(0;\tau)$, so that

$$\frac{\widetilde{W}_T-\widetilde{W}_t}{\sqrt{\tau}} \sim \mathcal{N}(0;1)$$

$$V_t = e^{-r\tau}\int_{-\infty}^{+\infty} h\left(S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}z}\right)\varphi(z)dz$$

<u>**Black & Scholes model for an European call option**</u>

Here, we have

$$h(S) = (S - K)^+ = \max(S - K, 0)$$

So,

$$h\left(S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}z}\right) = \left(S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}z} - K\right)^+$$

$$\begin{aligned}
&S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}z} \geq K \\
&\Leftrightarrow \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}z \geq -\ln\frac{S_t}{K} \\
&\Leftrightarrow z \geq \frac{-\ln\frac{S_t}{K} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\
&\Leftrightarrow z \geq -d_2
\end{aligned}$$

with

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$\begin{aligned}
C(t, S_t)& \\
&= e^{-r\tau}\int_{-d_2}^{+\infty}\left(S_t\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}z} - K\right)\varphi(z)dz \\
&= S_t\int_{-d_2}^{+\infty} e^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}z}\varphi(z)dz \\
&\qquad - e^{-r\tau}K\int_{-d_2}^{+\infty}\varphi(z)dz \\
&= S_t \cdot I - e^{-r\tau}K(1 - \Phi(-d_2)) \\
&= S_t \cdot I - e^{-r\tau}K \cdot \Phi(d_2)
\end{aligned}$$

$$\begin{aligned}
I &= \int_{-d_2}^{+\infty} e^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}z}\varphi(z)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty} e^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}z-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty} e^{-\frac{1}{2}\left(z-\sigma\sqrt{\tau}\right)^2}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_2-\sigma\sqrt{\tau}}^{+\infty} e^{-\frac{1}{2}y^2}dy \\
&= 1 - \Phi(-d_2 - \sigma\sqrt{\tau}) \\
&= \Phi(d_2 + \sigma\sqrt{\tau}) \\
&= \Phi(d_1)
\end{aligned}$$

with

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}$$
$$= \frac{\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$C(t, S_t) = S_t \cdot \Phi(d_1) - e^{-r\tau}K \cdot \Phi(d_2)$$

Note 1 : the martingale approach is also called

- Risk-neutral approach
- Probabilistic approach
- “change of numeraire” approach

Note 2 : The formula for a put is easily obtained from the call-put parity relation

$$\begin{aligned} &P(t, S_t) \\ &= -S_t + C(t, S_t) + e^{-r\tau}K \\ &= -S_t + S_t \cdot \Phi(d_1) - e^{-r\tau}K \cdot \Phi(d_2) + e^{-r\tau}K \\ &= -S_t\big(1 - \Phi(d_1)\big) + e^{-r\tau}K\big(1 - \Phi(d_2)\big) \\ &= -S_t\Phi(-d_1) + e^{-r\tau}K\Phi(-d_2) \end{aligned}$$

Black & Scholes : an arbitrage approach

- From the SDE to the PDE
- From the PDE to the heat equation
  - Change of variables
  - Heat equation
  - Limit conditions
- Solving heat equation
  - Heat equation
  - Limit conditions
- Black & Scholes model
  - Development of the solution of heat equation
  - Reverse change of variables
- Final note

Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities, *J. Political Economy*, **81**, 635-654

From the SDE to the PDE

Starting with the GBM

$$dS_t = \delta S_t \cdot dt + \sigma S_t \cdot dW_t$$

and applying Itô's lemma to $C(t, S_t)$, we have

$$\begin{aligned} dC(t, S_t) \\ = \left(C_t' + \delta S_t C_S' + \frac{\sigma^2 S_t^2}{2} C_{SS}''\right) dt + \sigma S_t C_S' dW_t \end{aligned}$$

Let us construct at time t a portfolio by

- buying X unit(s) of the equity
- selling 1 unit of the call option

The value of this portfolio is

$$V_t = XS_t - C(t, S_t)$$

We chose X such that this portfolio has no longer random component

By arbitrage free principle, the return of this portfolio is equal to the risk-free rate r :

if

$$\frac{dV_t}{V_t} = \alpha\, dt + \beta\, dW_t$$

then,

$$\beta = 0 \quad \Rightarrow \quad \alpha = r$$

We have

$$\begin{aligned} dV_t \\ &= X\, dS_t - dC(t, S_t) \\ &= X[\delta S_t\, dt + \sigma S_t\, dW_t] \\ &\quad - \left[\left(C_t' + \delta S_t C_S' + \frac{\sigma^2 S_t^2}{2} C_{SS}''\right) dt + \sigma S_t C_S' dW_t\right] \\ &= \left[X\delta S_t - \left(C_t' + \delta S_t C_S' + \frac{\sigma^2 S_t^2}{2} C_{SS}''\right)\right] dt \\ &\qquad + [X\sigma S_t - \sigma S_t C_S']\, dW_t \end{aligned}$$

And the return of the portfolio is equal to

$$\frac{dV_t}{V_t} = \frac{X\delta S_t - \left(C'_t + \delta S_t C'_S + \frac{\sigma^2 S_t^2}{2} C''_{SS}\right)}{XS_t - C(t,S_t)}\, dt + \frac{X\sigma S_t - \sigma S_t C'_S}{XS_t - C(t,S_t)}\, dW_t$$

$$= \alpha\, dt + \beta\, dW_t$$

From the system of equations

$$\begin{cases} \beta = 0 \\ \alpha = r \end{cases}$$

we eliminate X :

$$\beta = 0 \quad \Rightarrow \quad X = C'_S$$

$$\frac{C'_S \delta S_t - \left(C'_t + \delta S_t C'_S + \frac{\sigma^2 S_t^2}{2} C''_{SS}\right)}{C'_S S_t - C(t,S_t)} = r$$

$$-C'_t - \frac{\sigma^2 S_t^2}{2} C''_{SS} = rC'_S S_t - rC(t,S_t)$$

$$C'_t + rS_t C'_S + \frac{\sigma^2 S_t^2}{2} C''_{SS} - rC(t,S_t) = 0$$

is a PDE, with three limit conditions

- Terminal condition : $C(T,S_T) = (S_T - K)^+$
- Boundary condition 1 : $C(t,0) = 0$
- Boundary condition 2 : when $S_t \gg$,
  $$C(t,S_t) \sim S_t - e^{-r\tau}K$$

Note 1 : the parameter δ is no more present in this equation (just like the historical probability in the binomial model)

Note 2 : the proportion $X = C'_S$ for the portfolio with no random component can be interpreted as the "delta hedging" : the portfolio with

- a short position of 1 unit of call
- a long position with $C'_S = \Delta$ unit(s) of the underlying equity

is non risky (= is hedged)

From the PDE to the heat equation

Change of variables

The variables/unknown $(\tau, S\,; C)$ are replaced by new variables/unknown $(\theta, x\,; u)$

We also introduce the constant $m = r - \frac{\sigma^2}{2}$

new ← old	old ← new
$\theta = 2m^2\tau/\sigma^2$	$\tau = \theta\sigma^2/2m^2$
$x = \frac{2m}{\sigma^2}\left(\ln\left(\frac{S}{K}\right) + m\tau\right)$	$S = K \cdot e^{\frac{\sigma^2(x-\theta)}{2m}}$
$u = e^{r\tau} \cdot C$	$C = e^{-r\theta\sigma^2/2m^2} \cdot u$

Heat equation

The partial derivatives of C in the PDE are given by

$$C'_t = C'_\theta \cdot \theta'_t + C'_x \cdot x'_t$$

$$C'_S = C'_\theta \cdot \theta'_S + C'_x \cdot x'_S = C'_x \cdot x'_S$$

$$\begin{aligned} C''_{SS} &= (C'_x)'_S \cdot x'_S + C'_x \cdot (x'_S)'_S \\ &= (C''_{x\theta} \cdot \theta'_S + C''_{xx} \cdot x'_S) \cdot x'_S + C'_x \cdot x''_{SS} \\ &= C''_{xx} \cdot (x'_S)^2 + C'_x \cdot x''_{SS} \end{aligned}$$

But,

$$C'_\theta = e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot \left(-\frac{r\sigma^2}{2m^2} \cdot u + u'_\theta\right)$$

$$C'_x = e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot u'_x \qquad C''_{xx} = e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot u''_{xx}$$

$$\theta'_t = -\frac{2m^2}{\sigma^2}$$

$$x'_t = -\frac{2m^2}{\sigma^2} \qquad x'_S = \frac{2m}{\sigma^2 S} \qquad x''_{SS} = -\frac{2m}{\sigma^2 S^2}$$

Then,

$$C_t' = \frac{2m^2}{\sigma^2} e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot \left(\frac{r\sigma^2}{2m^2} \cdot u - u_\theta' - u_x'\right)$$

$$C_S' = \frac{2m}{\sigma^2 S} e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot u_x'$$

$$C_{SS}'' = \frac{2m}{\sigma^2 S^2} e^{-\frac{r\theta\sigma^2}{2m^2}} \cdot \left(\frac{2m}{\sigma^2} u_{xx}'' - u_x'\right)$$

And the PDE becomes

$$-\frac{2m^2}{\sigma^2} \cdot u_\theta' + \left(-\frac{2m^2}{\sigma^2} + \frac{2mr}{\sigma^2} - m\right) \cdot u_x' + \frac{2m^2}{\sigma^2} \cdot u_{xx}'' = 0$$

The coefficient of u_x' being equal to 0, the PDE becomes

$$u_\theta' = u_{xx}''$$

= 1D heat flow equation with length (x) and time (θ) variables

Limit conditions

a) Terminal condition : $C(T, S_T) = (S_T - K)^+$

When $t = T$ (or $\tau = 0$), then

$$\theta = 0 \quad \text{and} \quad S_T = K \cdot e^{\frac{\sigma^2 x}{2m}}$$

$$u(0, x) = C(T, S_T) = K \cdot \left(e^{\frac{\sigma^2 x}{2m}} - 1\right)^+$$

When $x < 0$, we have $u(0, x) = 0$

When $x \geq 0$, we have $u(0, x) = K \cdot \left(e^{\frac{\sigma^2 x}{2m}} - 1\right)$

So,

$$u(0, x) = K \cdot \left(e^{\frac{\sigma^2 x}{2m}} - 1\right) \cdot \mathbf{1}_{\mathbb{R}^+}(x) = v(x)$$

= initial condition for $u(\theta, x)$

b) Boundary condition 1 : $C(t,0) = 0$

When $S \to 0$, then $x \to -\infty$, and

$$\lim_{x\to-\infty} u(\theta,x) = 0$$

c) Boundary condition 2 :

when $S \gg$, $C(t,S) \sim S - e^{-r\tau}K$

When $S \to +\infty$, then

$$x \to +\infty \text{ and } C(t,S) = S - e^{-r\tau}K$$

$$\begin{aligned} u(\theta,x) &= e^{r\tau}(S - e^{-r\tau}K) \\ &= e^{r\tau}S - K \\ &= e^{\frac{r\theta\sigma^2}{2m^2}} \cdot K \cdot e^{\frac{\sigma^2(x-\theta)}{2m}} - K \\ &= K\left(e^{\frac{\sigma^2}{2m}\left(x-\theta+\frac{r\theta}{m}\right)} - 1\right) \\ &\sim Ke^{\frac{\sigma^2 x}{2m}} \end{aligned}$$

Solving heat equation

The question is to solve the problem

$$u'_\theta = u''_{xx}$$

with IC : $u(0,x) = v(x)$

BC1 : $\lim_{x\to-\infty} u(\theta,x) = 0$

BC2 : if $x \gg$, then $u(\theta,x) \sim Ke^{\frac{\sigma^2 x}{2m}}$

Heat equation

$$u(\theta,x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} v(y)\frac{e^{-\frac{(x-y)^2}{4\theta}}}{\sqrt{\theta}}dy$$

The variables θ and x are present only in

$$w(\theta,x) = \frac{e^{-\frac{(x-y)^2}{4\theta}}}{\sqrt{\theta}}$$

$$w'_\theta = \frac{1}{\sqrt{\theta}} e^{-\frac{(x-y)^2}{4\theta}} \left(\frac{(x-y)^2}{4\theta^2} - \frac{1}{2\theta}\right)$$

$$w'_x = \frac{1}{\sqrt{\theta}} e^{-\frac{(x-y)^2}{4\theta}} \left(-\frac{2(x-y)}{4\theta}\right)$$

$$w''_{xx} = \frac{1}{\sqrt{\theta}} e^{-\frac{(x-y)^2}{4\theta}} \left(\left(-\frac{2(x-y)}{4\theta}\right)^2 - \frac{2}{4\theta}\right)$$

and we have $w'_\theta = w''_{xx}$

Note : by using the substitution $z = \frac{y-x}{\sqrt{2\theta}}$, the solution of the heat equation can be written

$$u(\theta, x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} v(y) \frac{e^{-\frac{(x-y)^2}{4\theta}}}{\sqrt{\theta}} dy$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v\left(z\sqrt{2\theta} + x\right) e^{-\frac{z^2}{2}} dz$$
$$= E\left(v\left(Z\sqrt{2\theta} + x\right)\right)$$

where $Z \sim \mathcal{N}(0; 1)$

Limit conditions

a) Initial condition : $u(0, x) = v(x)$

$$u(0, x) = E\left(v(x)\right) = v(x)$$

b) Boundary condition 1 : $\lim_{x \to -\infty} u(\theta, x) = 0$

$$\lim_{x \to -\infty} e^{-\frac{(x-y)^2}{4\theta}} = 0$$

c) Boundary condition 2 :

if $x \gg$, then $u(\theta, x) \sim K e^{\frac{\sigma^2 x}{2m}}$

$$u(\theta, x) = E\left(v\left(Z\sqrt{2\theta} + x\right)\right)$$
$$\sim E\left(v(x)\right)$$
$$= K \cdot \left(e^{\frac{\sigma^2 x}{2m}} - 1\right) \cdot \mathbf{1}_{\mathbb{R}^+}(x)$$
$$\sim K e^{\frac{\sigma^2 x}{2m}}$$

Black & Scholes model

Development of the solution of heat equation

With the specific expression for $v(y)$, we have

$$u(\theta,x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} v(y)\,\frac{e^{-\frac{(x-y)^2}{4\theta}}}{\sqrt{\theta}}\,dy$$
$$= \frac{K}{2\sqrt{\pi\theta}}\int_0^{+\infty}\left(e^{\frac{\sigma^2 y}{2m}} - 1\right)e^{-\frac{(x-y)^2}{4\theta}}\,dy$$
$$= \frac{K}{2\sqrt{\pi\theta}}\int_0^{+\infty} e^{\frac{\sigma^2 y}{2m}}\, e^{-\frac{(x-y)^2}{4\theta}}\,dy - \frac{K}{2\sqrt{\pi\theta}}\int_0^{+\infty} e^{-\frac{(x-y)^2}{4\theta}}\,dy$$
$$= I_1 - I_2$$

The exponent in I_1 is equal to

$$-\frac{1}{4\theta}\left[(y^2 - 2xy + x^2) - \frac{2\theta\sigma^2 y}{m}\right]$$
$$= -\frac{1}{4\theta}\left[\left(y - \left(x + \frac{\theta\sigma^2}{m}\right)\right)^2 - \left(\frac{\theta^2\sigma^4}{m^2} + \frac{2\theta\sigma^2 x}{m}\right)\right]$$

and we have

$$I_1 = \frac{K}{2\sqrt{\pi\theta}}\, e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} \cdot \int_0^{+\infty} \exp\left(-\frac{1}{4\theta}\left(y - \left(x + \frac{\theta\sigma^2}{m}\right)\right)^2\right) dy$$

And, with the substitution

$$z = \frac{y - \left(x + \frac{\theta\sigma^2}{m}\right)}{\sqrt{2\theta}}$$

$$I_1 = K \cdot e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} \cdot \int_{-\left(x + \frac{\theta\sigma^2}{m}\right)/\sqrt{2\theta}}^{+\infty} \varphi(z)\,dz$$
$$= K \cdot e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} \cdot \left(1 - \Phi\left(-\frac{x + \frac{\theta\sigma^2}{m}}{\sqrt{2\theta}}\right)\right)$$
$$= K \cdot e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} \cdot \Phi\left(\frac{x + \frac{\theta\sigma^2}{m}}{\sqrt{2\theta}}\right)$$

On the other hand, with the substitution

$$z = \frac{y-x}{\sqrt{2\theta}}$$

We have

$$\begin{aligned} I_2 &= \frac{K}{2\sqrt{\pi\theta}} \int_0^{+\infty} e^{-\frac{(x-y)^2}{4\theta}} dy \\ &= K \cdot \int_{-x/\sqrt{2\theta}}^{+\infty} \varphi(z) dz \\ &= K \cdot \left(1 - \Phi\left(-\frac{x}{\sqrt{2\theta}}\right)\right) \\ &= K \cdot \Phi\left(\frac{x}{\sqrt{2\theta}}\right) \end{aligned}$$

Finally,

$$u(\theta, x) = K \cdot e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} \cdot \Phi\left(\frac{x + \frac{\theta\sigma^2}{m}}{\sqrt{2\theta}}\right) - K \cdot \Phi\left(\frac{x}{\sqrt{2\theta}}\right)$$

Reverse change of variables

a)

$$\begin{aligned} &\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx) \\ &= \frac{\sigma^2}{4m^2}\left(2m^2\tau + \frac{4m^2}{\sigma^2}\left(\ln\left(\frac{S}{K}\right) + m\tau\right)\right) \\ &= \frac{\sigma^2\tau}{2} + \ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau \\ &= \ln\left(\frac{S}{K}\right) + r\tau \end{aligned}$$

so that

$$\begin{aligned} K \cdot e^{\frac{\sigma^2}{4m^2}(\theta\sigma^2 + 2mx)} &= K \cdot \exp\left[\ln\left(\frac{S}{K}\right) + r\tau\right] \\ &= K \cdot \left(\frac{S}{K} \cdot e^{r\tau}\right) \\ &= e^{r\tau} \cdot S \end{aligned}$$

b)

$$\begin{aligned}\frac{x+\frac{\theta\sigma^2}{m}}{\sqrt{2\theta}} &= \frac{\frac{2m}{\sigma^2}\left(\ln\left(\frac{S}{K}\right)+m\tau\right)+2m\tau}{\frac{2m}{\sigma}\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)+m\tau+\sigma^2\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma^2\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ &= d_1\end{aligned}$$

c)

$$\begin{aligned}\frac{x}{\sqrt{2\theta}} &= \frac{\frac{2m}{\sigma^2}\left(\ln\left(\frac{S}{K}\right)+m\tau\right)}{\frac{2m}{\sigma}\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)+m\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ &= d_2\end{aligned}$$

Finally,

$$u\big(\theta(\tau,S),x(\tau,S)\big) = e^{r\tau}S\cdot\Phi(d_1) - K\cdot\Phi(d_2)$$

And then,

$$\begin{aligned}C &= e^{-r\tau}\cdot u \\ &= e^{-r\tau}\cdot\left(e^{r\tau}S\cdot\Phi(d_1) - K\cdot\Phi(d_2)\right) \\ &= S\cdot\Phi(d_1) - e^{-r\tau}K\cdot\Phi(d_2)\end{aligned}$$

Final note

The general scheme used in the arbitrage approach

1) Evolution of the underlying equity : GBM
2) Portfolio $(+X \cdot S - 1 \cdot C)$:
   with X such that the portfolio has no risky component
3) Arbitrage free reasoning :
   return = risk-free rate $\longrightarrow$ PDE
4) Solving the PDE

is quite general for the pricing of different assets in stochastic finance