Part II
FINANCIAL APPLICATIONS

4. Stochastic calculus
5. Option pricing models
6. Interest rate models

Chapter 4

Stochastic calculus

- Brownian motion
- Stochastic integral
- Stochastic differential
- Change of probability measure

Brownian motion

- Definition
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- Hitting time for a SBM
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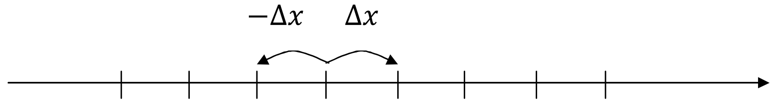
Definition

Argument

Let us consider a (discrete time) symmetrical random walk (X_t)



$$X_t = \sum_{k=1}^{n} Z_k \qquad Z_k \sim \begin{pmatrix} -\Delta x & \Delta x \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

with

- $t = n \cdot \Delta t$
- independent moves
- $X_0 = 0$

We know that

$$E(X_t) = 0$$
$$var(X_t) = \frac{(\Delta x)^2}{\Delta t} \cdot t$$

We want to define

- a continuous time stochastic process
- with positive constant instantaneous variance
  - if $var(X_t) \to \infty$, too "explosive" : the fluctuations will grow to infinity
  - if $var(X_t) \to 0$, no more random

So, we have to

- let Δt tend to 0
- in such a manner that $\frac{(\Delta x)^2}{\Delta t} \cdot t \to C \cdot t$

We can choose $C = 1$: if we want another constant σ, we will consider (σX_t)

Thanks to the CLT, we have

$$X_t = \sum_{k=1}^{n} Z_k \quad \to \quad \mathcal{N}(0; t)$$

Furthermore, a random walk has independent and stationary increments …

Definition

A continuous time stochastic process (W_t) is a standard brownian motion (SBM) if

- $W_0 = 0$
- (W_t) has independent increments
- (W_t) has stationary increments
- $W_t \sim \mathcal{N}(0; t)$

The notation "W" is for Wiener

Strictly speaking, a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{Pr}, \mathbf{F})$ is a SBM adapted to the filtration $\mathbf{F}$

Properties

Elementary properties

a) A SBM is a Gaussian process

b) If $s < t$, $(W_t - W_s) \triangleq W_{t-s} \sim \mathcal{N}(0; t-s)$

c) We have $E(W_t) = 0$, $var(W_t) = t$ and

$$cov(W_s, W_t) = \min(s, t)$$

Proof : if $s < t$,

$$\begin{aligned} cov(W_s, W_t) &= cov(W_s, W_t - W_s + W_s) \\ &= cov(W_s, W_t - W_s) + cov(W_s, W_s) \\ &= 0 + s \end{aligned}$$

Quadratic variation of a SBM

Let us consider a partition $\mathcal{P}_n$ of the time interval $[0; t]$ $(0 = t_0 < t_1 < \cdots < t_n = t)$ such that

$$\delta_n = \max\{t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1}\}$$

tends to 0 when $n \to \infty$

We define the quadratic variation of the SBM W_t, associated with the partition $\mathcal{P}_n$, by

$$Q_n(t) = \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)^2$$

Property : when $n \to \infty$, we have $Q_n(t) \xrightarrow{q.m.} t$

Lemma : if $X \sim \mathcal{N}(0; \sigma^2)$, then $var(X^2) = 2\sigma^4$

Since $\mu_4 = 3\sigma^4$, we have

$$var(X^2) = E(X^4) - E^2(X^2) = 3\sigma^4 - (\sigma^2)^2$$

Proof

- $$\begin{aligned} E\big(Q_n(t)\big) &= \textstyle\sum_{i=1}^{n} E\left(\left(W_{t_i} - W_{t_{i-1}}\right)^2\right) \\ &= \textstyle\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= t \end{aligned}$$

- $$\begin{aligned} var\big(Q_n(t)\big) &= \textstyle\sum_{i=1}^{n} var\left(\left(W_{t_i} - W_{t_{i-1}}\right)^2\right) \\ &= 2\textstyle\sum_{i=1}^{n}(t_i - t_{i-1})^2 \\ &\le 2\delta_n \textstyle\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= 2t\delta_n \\ &\to 0 \end{aligned}$$

so that $E((Q_n(t) - t)^2) \to 0$

Regularity properties

a) The paths of a SBM are continuous

We have to prove that $\lim_{\Delta t \to 0} W_{t+\Delta t} = W_t$

We give a proof for limit in probability. Let us choose an arbitrary $\varepsilon > 0$. We will prove that

$$\lim_{\Delta t \to 0} \Pr[|W_{t+\Delta t} - W_t| > \varepsilon] = 0$$

Since (Chebyshev's inequality)

$$\Pr\left[|W_{t+\Delta t} - W_t - 0| > h\sqrt{\Delta t}\right] \le \frac{1}{h^2}$$

we have

$$\Pr[|W_{t+\Delta t} - W_t| > \varepsilon] \le \frac{\Delta t}{\varepsilon^2} \to 0$$

b) The paths of a SBM are nowhere derivable

$$W_{t+\Delta t} - W_t \sim \mathcal{N}(0;\ \Delta t) \triangleq \sqrt{\Delta t} \cdot Z$$

with $Z \sim \mathcal{N}(0;1)$

$$\frac{W_{t+\Delta t} - W_t}{\Delta t} \triangleq \frac{Z}{\sqrt{\Delta t}}$$

that tends to $\pm\infty$, depending on the sign of Z

Interpretation of this property : a SBM is unpredictable over short time intervals

c) A SBM has unbounded variations. More precisely (with the same notations as for quadratic variation),

$$V_t = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} \left| W_{t_i} - W_{t_{i-1}} \right| = +\infty \qquad a.s.$$

If V_t were finite $(= C$, say), then, for any partition $\mathcal{P}_n$,

$$\begin{aligned} Q_n(t) &= \textstyle\sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)^2 \\ &\leq \textstyle\sum_{i=1}^{n} \left|W_{t_i} - W_{t_{i-1}}\right| \cdot \max\limits_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right| \\ &\leq C \cdot \max\limits_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right| \end{aligned}$$

and the 2nd factor tends to 0 by continuity of the paths of the SBM. This is incompatible with the property of quadratic variation : $Q_n(t) \to t$

d) Self-similarity of a SBM

(= scaling effect = "fractals" property)

By definition, a stochastic process is H-self-similar if, for any $n \geq 1,\ t_1, \dots, t_n \in T$ and $\lambda > 0$,

$$\left(X_{\lambda t_1}, \dots, X_{\lambda t_n}\right) \triangleq \left(\lambda^H X_{t_1}, \dots, \lambda^H X_{t_n}\right)$$

H is the Hurst index of the stochastic process

Property : a SBM is $\frac{1}{2}$-self-similar :

$$\left(W_{\lambda t_1}, \dots, W_{\lambda t_n}\right) \triangleq \left(\sqrt{\lambda} W_{t_1}, \dots, \sqrt{\lambda} W_{t_1}\right)$$

Proof (for $n = 1$) :

$$W_{\lambda t} \sim \mathcal{N}(0;\ \lambda t) \equiv \sqrt{\lambda} \cdot \mathcal{N}(0;\ t) \sim \sqrt{\lambda} \cdot W_t$$

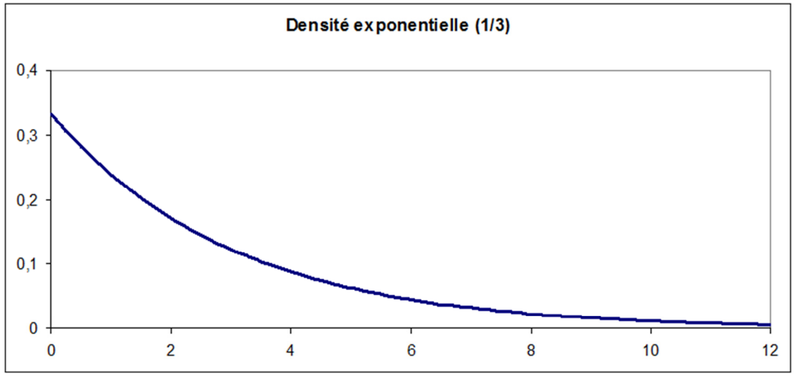
Interpretation : the pattern of any path of a SBM has a similar shape, independently of the length of the time interval

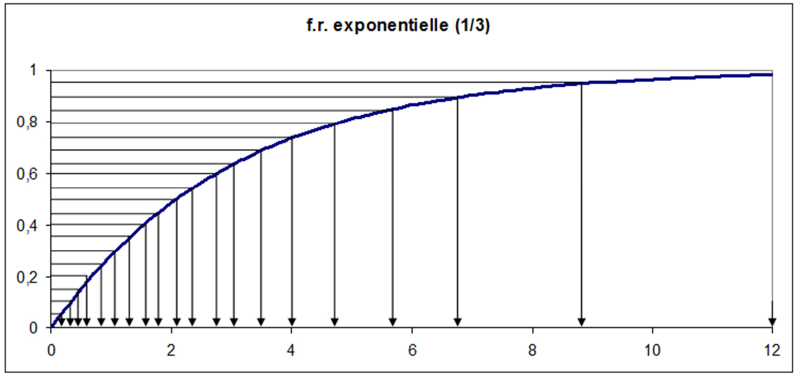
Simulation of a SBM

It is easy to obtain pseudo-random values for the law of X from $\mathcal{U}(0; 1)$ pseudo-random values :

$$F_X^{-1}(U) \triangleq X$$

$$\Pr[F_X^{-1}(U) \leq t] = \Pr[U \leq F_X(t)] = F_X(t)$$





For simulating a path of a SBM, we discretize the time variable : let the time interval $[0;t]$ be partitioned in n sub-intervals of length Δt :
$t = n \cdot \Delta t$

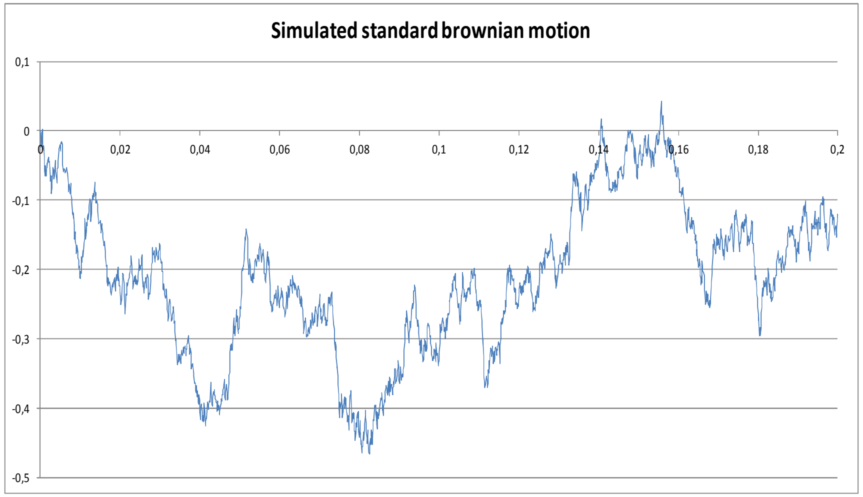
We know that

$$W_{\Delta t}, (W_{2\Delta t} - W_{\Delta t}), \ldots, (W_{n\Delta t} - W_{(n-1)\Delta t})$$

are i.i.d. r.v. $\sim \mathcal{N}(0;\Delta t)$

Algorithm :

- Generate n pseudo-random values $u_1, \ldots, u_n$ values of a $\mathcal{U}(0;1)$ r.v.
- Take the reciprocal of these values to obtain pseudo-random normal values
$$W_{j\Delta t} - W_{(j-1)\Delta t} = F_N^{-1}(u_j\,; 0, \Delta t)$$
- Cumulate these values
$$W_{k\Delta t} = \sum_{j=1}^{k} \left(W_{j\Delta t} - W_{(j-1)\Delta t}\right)$$
- Using continuity of the path, connect the points by line segments



Associated BM

Arithmetic BM

An ABM with drift α $(\in \mathbb{R})$ and volatility σ (> 0), associated to the SBM (W_t), is a stochastic process (X_t) defined by

$$X_t = \alpha t + \sigma W_t$$

Properties

- An ABM is a Gaussian process
- Moments :

$$\mu_X(t) = \alpha t$$
$$\sigma_X^2(t) = \sigma^2 t$$
$$c_X(s,t) = \sigma^2 \min(s,t)$$

This process can be generalized for beginning at a value x_0 instead of 0 :

$$X_t = x_0 + \alpha t + \sigma W_t$$

Brownian bridge

A Brownian bridge over the time interval $[0;1]$, associated to the SBM (W_t), is a stochastic process (X_t) defined by

$$X_t = W_t - tW_1$$

Properties

- A Brownian bridge is a Gaussian process
- $X_0 = X_1 = 0$
- Moments :

$$\mu_X(t) = 0$$
$$\sigma_X^2(t) = t(1-t)$$
$$c_X(s,t) = \min(s,t) - st$$

For the covariance function,

$$\begin{aligned} c_X(s,t) &= cov(W_s - sW_1, W_t - tW_1) \\ &= \min(s,t) - s\min(1,t) \\ &\qquad -t\min(s,1) + st\min(1,1) \\ &= \min(s,t) - st \end{aligned}$$

Brownian motion and martingales

Let us consider a probability space $(\Omega, \mathcal{F}, \Pr, \mathbf{F})$ where $\mathbf{F}$ is the natural filtration of a SBM (W_t)

(In this section, we will suppose $0 \leq s < t$)

Examples of martingales

a) (W_t) is a martingale

$$\begin{aligned} E(W_t|\mathcal{F}_s) &= E(W_t - W_s + W_s|\mathcal{F}_s) \\ &= E(W_t - W_s|\mathcal{F}_s) + E(W_s|\mathcal{F}_s) \\ &= E(W_t - W_s) + E(W_s|\mathcal{F}_s) \\ &= 0 + W_s \end{aligned}$$

b) $(W_t^2 - t)$ is a martingale

$$\begin{aligned} E(W_t^2 - t|\mathcal{F}_s) &= E(W_t^2 - W_s^2 + W_s^2|\mathcal{F}_s) - t \\ &= E(W_t^2 - W_s^2|\mathcal{F}_s) \\ &\quad + E(W_s^2|\mathcal{F}_s) - t \\ &= E(W_t^2 - W_s^2|\mathcal{F}_s) + W_s^2 - t \end{aligned}$$

But, $W_t^2 - W_s^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s)$

so that

$$\begin{aligned} &E(W_t^2 - W_s^2|\mathcal{F}_s) \\ &\quad = E((W_t - W_s)^2|\mathcal{F}_s) + 2E(W_s(W_t - W_s)|\mathcal{F}_s) \\ &\quad = (t - s) + 2W_s\, E(W_t - W_s|\mathcal{F}_s) \\ &\quad = (t - s) + 2W_s\, E(W_s - W_s) \\ &\quad = t - s \end{aligned}$$

and we have

$$\begin{aligned} E(W_t^2 - t|\mathcal{F}_s) &= (t - s) + W_s^2 - t \\ &= W_s^2 - s \end{aligned}$$

c) Counter-example : (W_t^3) is not a martingale

We know that

$$E((W_t - W_s)^3|\mathcal{F}_s) = E((W_t - W_s)^3) = 0$$

$$\begin{aligned} 0 &= E(W_t^3 - 3W_t^2W_s + 3W_tW_s^2 - W_s^3|\mathcal{F}_s) \\ &= E(W_t^3|\mathcal{F}_s) - 3W_sE(W_t^2|\mathcal{F}_s) \\ &\qquad +3W_s^2E(W_t|\mathcal{F}_s) - W_s^3 \\ &= E(W_t^3|\mathcal{F}_s) - 3W_sE((W_t^2 - t) + t|\mathcal{F}_s) \\ &\qquad +3W_s^2W_s - W_s^3 \\ &= E(W_t^3|\mathcal{F}_s) - 3W_s[(W_s^2 - s) + t] + 2W_s^3 \\ &= E(W_t^3|\mathcal{F}_s) - W_s^3 + 3W_s(s - t) \end{aligned}$$

so that

$$E(W_t^3|\mathcal{F}_s) = W_s^3 - 3W_s(s - t) \neq W_s^3$$

Reciprocal (without proof)

If a stochastic process (X_t) is such that (X_t) and $(X_t^2 - t)$ are martingales, then (X_t) is a SBM

Exponential Brownian motion

An EBM is a stochastic process (X_t) defined by

$$X_t = e^{\sigma W_t - \frac{\sigma^2 t}{2}}$$

with $\sigma > 0$

Property : an EBM is a martingale

$$\begin{aligned} E(e^{\sigma W_t}|\mathcal{F}_s) &= E\left(e^{\sigma(W_t - W_s)} \cdot e^{\sigma W_s}|\mathcal{F}_s\right) \\ &= e^{\sigma W_s} \cdot E\left(e^{\sigma(W_t - W_s)}|\mathcal{F}_s\right) \\ &= e^{\sigma W_s} \cdot E\left(e^{\sigma(W_t - W_s)}\right) \\ &= e^{\sigma W_s} \cdot e^{\frac{\sigma^2(t-s)}{2}} \end{aligned}$$

so that

$$\begin{aligned} E(X_t|\mathcal{F}_s) &= E\left(e^{\sigma W_t - \frac{\sigma^2 t}{2}}\middle|\mathcal{F}_s\right) \\ &= e^{\sigma W_s} \cdot e^{\frac{\sigma^2(t-s)}{2}} \cdot e^{-\frac{\sigma^2 t}{2}} \\ &= e^{\sigma W_s} \cdot e^{-\frac{\sigma^2 s}{2}} \\ &= X_s \end{aligned}$$

Particular case : if $s = 0$,

$$E\left(e^{\sigma W_t - \frac{\sigma^2 t}{2}}\right) = X_0 = 1$$

Using BM as a "noise"

Objective : express a stochastic process (X_t) as the "superposition" of

- a deterministic function f_t
- a non predictable "noise" (= martingale)

We can use

a) a SBM as an additive random noise :

$$X_t = f_t + \sigma W_t$$

b) An EBM as a multiplicative random noise :

$$X_t = f_t \cdot e^{\sigma W_t - \frac{\sigma^2 t}{2}}$$

In both case, $E(X_t) = f_t$

Hitting time for a SBM

Definition and property

For any fixed $a > 0$, we define the hitting time T_a as the first time the SBM W_t hits the value a :

$$\min\{t \in T : W_t = a\}$$

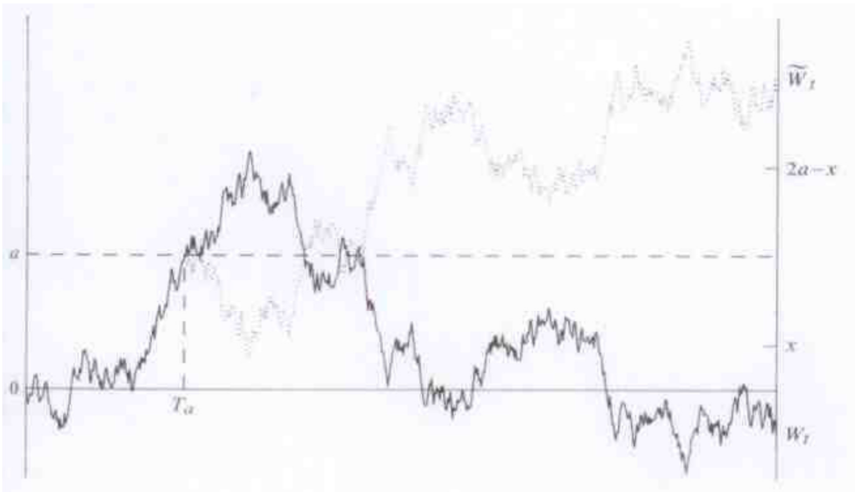
(and $+\infty$ if $W_t \neq a \ \forall t \in T$)

Property : the hitting time is a stopping time

Reflection principle

By symmetry, knowing that $T_a \leq t$, the events $[W_t > a]$ and $[W_t < a]$ have the same probability :

$$\Pr([W_t > a] | [T_a \leq t]) = \Pr([W_t < a] | [T_a \leq t]) = \frac{1}{2}$$



Distribution of hitting time and maximum

- By total probabilities formula,

$$\begin{aligned}\Pr[W_t > a] &= \Pr([W_t > a] | [T_a \leq t]) \cdot \Pr[T_a \leq t] \\ &\quad + \Pr([W_t > a] | [T_a > t]) \cdot \Pr[T_a > t] \\ &= \frac{1}{2} \Pr[T_a \leq t]\end{aligned}$$

So,

$$\begin{aligned}F_{T_a}(t) &= \Pr[T_a \leq t] \\ &= 2 \Pr[W_t > a] \\ &= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right) \\ &= 2\,\Phi\left(-\frac{a}{\sqrt{t}}\right)\end{aligned}$$

- If we define $M_t = \max\{W_s : 0 \leq s \leq t\}$,

$$\Pr[M_t \geq a] = \Pr[T_a \leq t] = 2\,\Phi\left(-\frac{a}{\sqrt{t}}\right)$$

Stochastic integral

- Definition
  - Motivation
  - Classical Riemann integral
  - Stieltjes-Riemann integral
  - Generalization ?
  - Choice of a definition
  - Definition
- Properties
  - Conditions of existence
  - Properties

Definition

Motivation

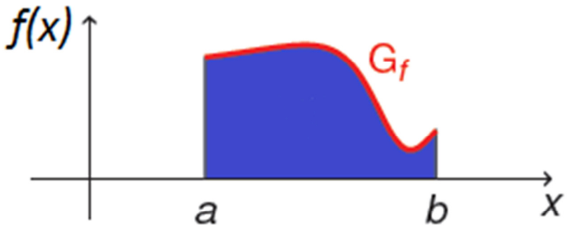
- The definition of the integral of a function $f(x)$ is concerned with small variations of the variable x
- The definition of the differential of a function f $(df(x) = f'(x) \cdot dx)$ is also concerned with small variations of the variable x

Here, we will look at the time variations "through a SBM", which has

- unbounded variations
- non differentiable paths

The convergence being no more defined in the classical way, we have to give new definitions

Classical Riemann integral



Let $\mathcal{P}_n$ be a partition of $[a; b]$

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

with

$$t_i - t_{i-1} = \Delta_i$$

$$\delta_n = \max(\Delta_1, \Delta_2, \ldots, \Delta_n)$$

and let choose

$$u_i \in \,]t_{i-1};\ t_i[$$

The Riemann integral is defined by

$$\int_a^b f(u)\,du = \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} \sum_{i=1}^{n} f(u_i) \cdot \Delta_i$$

It can be prove that if f is sufficiently "regular" (continuous by parts e.g.), this integral

- exists
- is independent of $\mathcal{P}_n$
- is independent of the choice of u_i in $]t_{i-1};\ t_i[$

Stieltjes-Riemann integral

This is the same notion as ordinary Riemann integral, but the measure along horizontal axis is no more the length of segments, but the length through another function g

$$\int_a^b f(u)\,dg(u) = \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} f(u_i)\cdot\big(g(t_i)-g(t_{i-1})\big)$$

This integral has the same properties as the ordinary Riemann integral (with, furthermore, regularity conditions for g)

Example

$$\int_{-\infty}^{+\infty} u\,dF_X(u) = \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} u_i\cdot \Pr[t_{i-1}<X\le t_i] = E(X)$$

Note : from now on, the interval of integration becomes $[0;T]$ instead of $[a;b]$

Generalization ?

Let (X_t) be a stochastic process and (W_t) a SBM.
How can we define $\int_0^T X_u\, dW_u$?

Problems

a) Convergence "point by point" is the convergence a.s. (incompatible with the unbounded variation of the SBM)
   ➔ Solution : give a definition with another convergence mode (q.m.)

b) The definition is no more independent of the choice of u_i in $]t_{i-1};\ t_i[$
   ➔ Solution : make a choice for u_i

Let us examine the particular case of

$$"\int_0^T W_u\, dW_u" = \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} W_{u_i}\cdot\left(W_{t_i}-W_{t_{i-1}}\right)$$

We will need the following lemma

$$\begin{cases} a(b-a) = \frac{1}{2}[(b^2-a^2)-(b-a)^2] \\ b(b-a) = \frac{1}{2}[(b^2-a^2)+(b-a)^2] \end{cases}$$

• First choice : $u_i = t_{i-1}$

$$"\int_0^T W_u\, dW_u"$$
$$= \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} W_{t_{i-1}}\cdot\left(W_{t_i}-W_{t_{i-1}}\right)$$
$$= \frac{1}{2}\lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} \left\{\left(W_{t_i}^2-W_{t_{i-1}}^2\right)-\left(W_{t_i}-W_{t_{i-1}}\right)^2\right\}$$
$$= \frac{1}{2}\lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \left(W_T^2 - Q_n(T)\right)$$
$$= \frac{1}{2}(W_T^2 - T)$$

(this last convergence is in q.m.)

• Second choice : $u_i = t_i$

$$"\int_0^T W_u \, dW_u" = \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} \sum_{i=1}^{n} W_{t_i} \cdot \left(W_{t_i} - W_{t_{i-1}}\right)$$

$$= \frac{1}{2} \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} \sum_{i=1}^{n} \left\{\left(W_{t_i}^2 - W_{t_{i-1}}^2\right) + \left(W_{t_i} - W_{t_{i-1}}\right)^2\right\}$$

$$= \frac{1}{2} \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} \left(W_T + Q_n(T)\right)$$

$$= \frac{1}{2}(W_T^2 + T)$$

• Third choice : $u_i = \frac{t_{i-1} + t_i}{2}$

It can be shown that

$$"\int_0^T W_u \, dW_u" = \frac{1}{2} W_T^2$$

Note

- First choice : Itô integral
- Third choice : Stratonovich integral

Choice of a definition

• Stratonovich integral give the same result as in the deterministic case : if $f(0) = 0$, by integrating by parts,

$$\int_0^T f(u) \, df(u) = \frac{1}{2} f^2(T)$$

• Itô integral has two interesting properties

a) Non-anticipativity : for the i-th interval $]t_{i-1}; \ t_i[$, the integrand X_t is known at time t_{i-1}

b) We know that the stochastic process $(W_t^2 - t)$ is a martingale ; so is the Itô integral

➔ Itô integral is chosen for applications in finance

Definition

Let (X_t) be a stochastic process adapted to the natural filtration of the SBM (W_t). We define

$$I_T = \int_0^T X_u \, dW_u = \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} I_T^{(n)}$$

where

$$I_T^{(n)} = \sum_{i=1}^{n} X_{t_{i-1}} \cdot \left(W_{t_i} - W_{t_{i-1}}\right)$$

More precisely, it can be prove that there exists a r.v. I_T such that

$$\lim_{\substack{n \to +\infty \\ \delta_n \to 0}} E\left[\left(I_T^{(n)} - I_T\right)^2\right] = 0$$

so that $I_T^{(n)}$ converges in q.m. to I_T

Note : the hypothesis implies that $X_{t_{i-1}}$ is independent of $\left(W_{t_i} - W_{t_{i-1}}\right)$

Properties

Condition of existence

If (X_t) is a stochastic process adapted to the natural filtration of the SBM (W_t), then

$$\int_0^T X_u \, dW_u$$

exists if

- paths of (X_t) are continuous
- $E\left(\int_0^T X_u \, du\right)$ is finite

Properties

a) $\int_0^T \left(\lambda_1 X_u^{(1)} + \lambda_2 X_u^{(2)}\right) dW_u$

$$= \lambda_1 \int_0^T X_u^{(1)}\, dW_u + \lambda_2 \int_0^T X_u^{(2)}\, dW_u$$

b) $E\left(\int_0^T X_u\, dW_u\right) = 0$

Proof :

$$E\left(X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right)\right)$$
$$= E\left(X_{t_{i-1}}\right) \cdot E\left(W_{t_i} - W_{t_{i-1}}\right)$$
$$= 0$$

c) $var\left(\int_0^T X_u\, dW_u\right) = \int_0^T E(X_u^2)\, du$

Proof :

$$var\left(\int_0^T X_u\, dW_u\right) = E\left[\left(\int_0^T X_u\, dW_u\right)^2\right]$$

$$= \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} E\left(X_{t_{i-1}}^2\left(W_{t_i} - W_{t_{i-1}}\right)^2\right)$$
$$+2 \lim_{\substack{n\to+\infty \\ \delta_n\to 0}} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} E\begin{pmatrix} X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right) \\ \cdot X_{t_{j-1}}\left(W_{t_j} - W_{t_{j-1}}\right) \end{pmatrix}$$

But

$$E\left(X_{t_{i-1}}^2\left(W_{t_i} - W_{t_{i-1}}\right)^2\right)$$
$$= E\left(X_{t_{i-1}}^2\right) \cdot E\left(\left(W_{t_i} - W_{t_{i-1}}\right)^2\right)$$
$$= E\left(X_{t_{i-1}}^2\right) \cdot (t_i - t_{i-1})$$

and the first term is equal to $\int_0^T E(X_u^2)\, du$

Furthermore, for $i < j$,

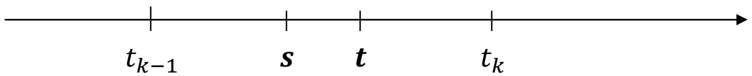
$$E\left(X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right) \cdot X_{t_{j-1}}\left(W_{t_j} - W_{t_{j-1}}\right)\right)$$
$$= E\left(X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right) X_{t_{j-1}}\right) \cdot E\left(W_{t_j} - W_{t_{j-1}}\right)$$
$$= 0$$

d) The stochastic process (I_t) for $t \in [0;T]$ is a martingale

For $s < t$,

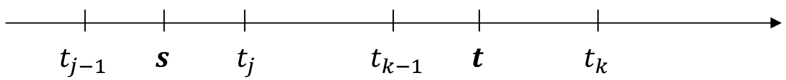
$$E(I_t|\mathcal{F}_s) = \lim_{\substack{n\to+\infty \\ \delta_n \to 0}} \sum_{i=1}^{n} E\left(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s\right)$$

• If $s, t \in]t_{k-1};\ t_k]$



$$\begin{aligned} E(I_t|\mathcal{F}_s) &= I_s + E\left(X_{t_{k-1}}(W_t - W_s)|\mathcal{F}_s\right) \\ &= I_s + X_{t_{k-1}} \cdot E\ (W_t - W_s|\mathcal{F}_s) \\ &= I_s + X_{t_{k-1}} \cdot E\ (W_t - W_s) \\ &= I_s \end{aligned}$$

• If $s \in]t_{j-1};\ t_j]$ and $t \in]t_{k-1};\ t_k]$ with $j < k$



$$\begin{aligned} E(I_t|\mathcal{F}_s) &= I_s + E\left(X_{t_{j-1}}\left(W_{t_j} - W_s\right)|\mathcal{F}_s\right) \\ &\quad + \sum_{i=j+1}^{k-1} E\left(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s\right) \\ &\quad + E\left(X_{t_{k-1}}(W_t - W_{t_{k-1}})|\mathcal{F}_s\right) \\ &= I_s + (a) + (b) + (c) \end{aligned}$$

$$\begin{aligned} (a) &= X_{t_{j-1}} \cdot E\left(W_{t_j} - W_s|\mathcal{F}_s\right) \\ &= X_{t_{j-1}} \cdot E\left(W_{t_j} - W_s\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} (b):\ E\left(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s\right) &= E\left(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\right) \\ &= E\left(X_{t_{i-1}}\right) \cdot E\left(W_{t_i} - W_{t_{i-1}}\right) \\ &= 0 \end{aligned}$$

$(c) = 0$: same reasoning as (b)

e) The stochastic process (I_t) has continuous paths (without proof)

Stochastic differential

- Definition
  - In the deterministic case
  - In the stochastic case
- Properties
  - Formal multiplication rules
  - Properties
- Examples
  - Simple examples
  - Arithmetic Brownian motion
  - Geometric Brownian motion
- Use of the stochastic differential
  - Evolution of financial variables
  - Classical stochastic differentials in finance

Definition

In the deterministic case

$$dX(t) = f(t) \cdot dt$$
$$\Leftrightarrow \quad X(t) = X(0) + \int_0^t f(u)du$$

Generalization ?

- One term with "dt" (trend)
- One term with "dW_t" (noise)

In the stochastic case

If the stochastic processes (a_t) and (b_t) are integrables and adapted to the natural filtration of the SBM (W_t), we define

$$dX_t = a_t \cdot dt + b_t \cdot dW_t$$

by

$$X_t = X_0 + \int_0^t a_u \, du + \int_0^t b_u \, dW_u$$

Properties

Formal multiplication rules

We will neglect terms smaller than dt $(= o(dt))$

- $(dt)^2 \approx 0$

- $dt \times dW_t \approx 0$

$$E(dt \cdot dW_t) = dt \cdot E(dW_t) = 0$$
$$var(dt \cdot dW_t) = (dt)^2 \cdot var(dW_t) = (dt)^3$$

- $(dW_t)^2 \approx dt$

$$E((dW_t)^2) = var(dW_t) = dt$$
$$var((dW_t)^2) = 2\big(var(dW_t)\big)^2 = 2(dt)^2$$

	1	dW_t	dt
1	1	dW_t	dt
dW_t	dW_t	dt	0
dt	dt	0	0

Properties

a) Linearity : if $(X_t^{(1)})$ and $(X_t^{(2)})$ are defined w.r.t. the same SBM (W_t),

$$d\left(\lambda_1 X_t^{(1)} + \lambda_2 X_t^{(2)}\right) = \lambda_1\, dX_t^{(1)} + \lambda_2\, dX_t^{(2)}$$

b) Product : if

$$dX_t^{(k)} = a_t^{(k)} \cdot dt + b_t^{(k)} \cdot dW_t \qquad (k = 1, 2)$$

then

$$d\left(X_t^{(1)} X_t^{(2)}\right)$$
$$= X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + b_t^{(1)} b_t^{(2)} dt$$

Proof

Taylor formula for n variables $x = (x_1, \dots, x_n)$

$$df(x) \approx \sum_{i=1}^{n} f'_{x_i}\, dx_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f''_{x_i x_j}\, dx_i\, dx_j$$

applied to $f(x_1, x_2) = x_1.x_2$ give

$$d\left(X_t^{(1)} X_t^{(2)}\right)$$
$$= X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + \tfrac{1}{2} \cdot 2\left(dX_t^{(1)} \cdot dX_t^{(2)}\right)$$

and

$$dX_t^{(1)} \cdot dX_t^{(2)}$$
$$= \left(a_t^{(1)} dt + b_t^{(1)} dW_t\right)\left(a_t^{(2)} dt + b_t^{(2)} dW_t\right)$$
$$= b_t^{(1)} b_t^{(2)} (dW_t)^2$$

c) Compound function (= Itô's lemma)

If $dX_t = a_t \cdot dt + b_t \cdot dW_t$ and if $f(t, x)$ is a deterministic function, derivable (one time w.r.t. t and twice w.r.t. x), then

$$df(t, X_t)$$
$$= \left(f'_t(t, X_t) + a_t f'_x(t, X_t) + \tfrac{b_t^2}{2} f''_{xx}(t, X_t)\right) \cdot dt$$
$$+ b_t f'_x(t, X_t) \cdot dW_t$$

Proof : by Taylor,

$$df(t, X_t)$$
$$= f'_t\, dt + f'_x\, dX_t$$
$$+ \tfrac{1}{2}[f''_{tt}\, (dt)^2 + 2 f''_{tx}\, (dt)(dX_t) + f''_{xx} (dX_t)^2]$$
$$= f'_t\, dt + f'_x\, dX_t + \tfrac{1}{2} f''_{xx} (dX_t)^2$$

and

$$(dX_t)^2 = (a_t \cdot dt + b_t \cdot dW_t)^2 = b_t^2\, dt$$

Examples

Simple examples

a) $f(t,x) = g(t)x$ and $X_t = W_t$

$$d(g(t)W_t) = g'(t)W_t\,dt + g(t)\,dW_t$$

$$\int_0^T d(g(t)W_t) = g(T)W_T$$
$$= \int_0^T g'(t)W_t\,dt + \int_0^T g(t)\,dW_t$$

$$\int_0^T g(t)\,dW_t = g(T)W_T - \int_0^T g'(t)W_t\,dt$$

(= integration by parts)

b) $f(t,x) = x^2$ and $X_t = W_t$

$$d(W_t^2) = \frac{1}{2}2\,dt + 2W_t\,dW_t$$

$$\int_0^T d(W_t^2) = W_T^2 = \int_0^T dt + 2\int_0^T W_t\,dW_t$$

$$\int_0^T W_t\,dW_t = \frac{1}{2}(W_T^2 - T)$$

c) $f(t,x) = e^x$ and $dX_t = a_t\,dt + b_t\,dW_t$

$$d(e^{X_t}) = \left(a_t\,e^{X_t} + \frac{b_t^2}{2}\,e^{X_t}\right)dt + b_t\,e^{X_t}\,dW_t$$
$$= e^{X_t}\left[\left(a_t + \frac{b_t^2}{2}\right)dt + b_t\,dW_t\right]$$
$$= e^{X_t}\left[dX_t + \frac{b_t^2}{2}dt\right]$$

Arithmetic Brownian motion

Definition : $X_t = X_0 + \alpha t + \sigma W_t$

$$dX_t = \alpha\, dt + \sigma\, dW_t$$

Geometric Brownian motion

Definition : $S_t = S_0\, e^{\mu t + \sigma W_t}$

$f(t,x) = S_0\, e^{\mu t + \sigma x}$ and $X_t = W_t$

$$\begin{aligned} dS_t &= \left(\mu S_t + \frac{\sigma^2}{2} S_t\right) dt + \sigma S_t\, dW_t \\ &= \delta S_t\, dt + \sigma S_t\, dW_t \end{aligned}$$

with $\delta = \mu + \frac{\sigma^2}{2}$

So, the GBM can be written

$$S_t = S_0\, e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Moments : $e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma W_t}$ being a log-normal r.v.,

$$E(S_t) = S_0\, e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2 t}{2}} = S_0\, e^{\delta t}$$

$$\begin{aligned} var(S_t) &= S_0^2\, e^{2\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right) \\ &= S_0^2\, e^{2\delta t}\left(e^{\sigma^2 t} - 1\right) \end{aligned}$$

Use of the stochastic differential

Evolution of a financial variable

$$dX_t = a_t\,dt + b_t\,dW_t$$

is an equation that describe the evolution of a financial variable

- For an equity, we have solved the equation : GBM
- For an option, we will solve it
- For a yield curve, the evolution of a state variable r_t will be describe by a stochastic differential and we will deduce $R_t(s)$

However, we will not study the techniques for solving a general SDE

Classical stochastic differentials in finance

For an Itô stochastic differential, the stochastic processes (a_t) and (b_t) are deterministic functions of t and X_t

Here, these functions do not depend explicitly on the time variable t

$$a_t = a(X_t) \qquad b_t = b(X_t)$$

- Arithmetic Brownian motion
$$dX_t = \alpha\,dt + \sigma\,dW_t$$
- Geometric Brownian motion
$$dX_t = \delta X_t\,dt + \sigma X_t\,dW_t$$
- Ornstein-Uhlenbeck process
$$dX_t = \delta(\theta - X_t)\,dt + \sigma\,dW_t$$
- Square-root process
$$dX_t = \delta(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t$$

Change of probability measure

- Radon-Nikodym theorem
  - Discrete case
  - General case
- Girsanov theorem
  - Girsanov theorem
  - Generalization

Radon-Nikodym theorem

Discrete case

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n, \dots\}$ be the set of possible outcomes in a random situation with probability measure $\Pr$:

$$\Pr(\{\omega_i\}) = p_i \qquad (\sum p_i = 1)$$

Let Q be another probability measure for this random situation :

$$Q(\{\omega_i\}) = q_i \qquad (\sum q_i = 1)$$

The r.v. L is defined by

$$L(\omega_i) = \frac{q_i}{p_i}$$

This r.v. has the following properties

- L positive
- $E_p(L) = \sum \frac{q_i}{p_i} p_i = 1$
- For any r.v. X,

$$E_q(X) = \sum X(\omega_i) q_i = \sum X(\omega_i) \frac{q_i}{p_i} p_i = E_p(L \cdot X)$$

and, in the particular case where $X = \mathbf{1}_A$,

$$Q(A) = E_p(L \cdot \mathbf{1}_A)$$

General case

Let $\Pr$ and Q be two probability measures on $(\Omega, \mathcal{F})$

We say that Q is absolutely continuous w.r.t. $\Pr$ ($Q \ll \Pr$) if

$$\forall A \in \mathcal{F}, \quad Q(A) = 0 \quad \Longrightarrow \quad \Pr(A) = 0$$

If $Q \ll \Pr$ and $\Pr \ll Q$, the two measures are said equivalent

Radon-Nikodym theorem

Q is absolutely continuous w.r.t. Pr
if and only if there exist a positive r.v. L such that

$$\forall A \in \mathcal{F}, \quad Q(A) = \int_A L(\omega)\, d\mathrm{Pr}(\omega)$$

or, equivalently,

$$Q(A) = E_Q(\mathbf{1}_A) = E_{\mathrm{Pr}}(L \cdot \mathbf{1}_A)$$

L is named Radon-Nikodym derivative and one writes

$$L = \frac{dQ}{d\mathrm{Pr}}$$

Property : by putting $A = \Omega$, we have

$$1 = Q(\Omega) = \int_\Omega L(\omega)\, d\mathrm{Pr}(\omega) = E_{\mathrm{Pr}}(L)$$

Girsanov theorem

Girsanov theorem

The definition of a SBM depends heavily on the probability measure : independent and stationary increments, normal distribution, …

Let us consider a SBM (W_t) on $(\Omega, \mathcal{F}, \mathrm{Pr})$ for the time interval $[0; T]$.

The stochastic process $(\widetilde{W}_t)$, defined by $\widetilde{W}_t = W_t + qt$, is an ABM, but no more a SBM :

$$E(\widetilde{W}_t) = qt \neq 0$$

The EBM $L_t = e^{-qW_t - \frac{q^2 t}{2}}$ is a positive stochastic process, martingale, with $E_p(L_t) = 1$. We will use it as a Radon-Nikodym derivative

Girsanov theorem

• The function

$$Q(A) = \int_A L_T(\omega)\, d\mathrm{Pr}(\omega) \qquad (A \in \mathcal{F})$$

is a probability measure

• The Q measure is equivalent to the Pr measure

• Under Q, $(\widetilde{W}_t)$ is a SBM, adapted to the natural filtration of (W_t)

The Q measure is the equivalent martingale measure

Generalization

Let (W_t) be a SBM on $(\Omega, \mathcal{F}, \mathrm{Pr})$ for the time interval $[0; T]$ and $(\widetilde{W}_t)$ the associated ABM with drift μ and volatility σ :

$$\widetilde{W}_t = \mu t + \sigma W_t$$

Then, $(\widetilde{W}_t)$ is an ABM with drift ν and volatility σ under the probability measure

$$Q(A) = \int_A L_T(\omega)\, d\mathrm{Pr}(\omega) \qquad (A \in \mathcal{F})$$

where

$$L_t = e^{\frac{\nu-\mu}{\sigma^2}\widetilde{W}_t - \left(\frac{\nu^2-\mu^2}{2\sigma^2}\right)t}$$