One step futher: an explicit solution to Robbins' problem when $n = 4$

Rémi Dendievel[*] Yvik Swan[†]

Abstract

Fix some $n \in \mathbb{N}$ and let $X_1, X_2, \dots, X_n$ be independent random variables drawn from the uniform distribution on $[0, 1]$. A decision maker is shown the variables sequentially and, after each observation, must decide whether or not to keep the current one, with payoff the overall rank of the selected observation. Decisions are final: no recall is allowed, no regret is tolerated. The objective is to act in such a way as to minimize the expected payoff. In this note we give the explicit solution to this problem, known as Robbins' problem of optimal stopping, when $n = 4$.

1 Introduction

Robbins' problem (of optimal stopping) consists in studying the mathematical properties of the optimal strategy in the following sequential selection problem.

> *Fix some $n \in \mathbb{N}$ and let $X_1, X_2, \dots, X_n$ be independent random variables drawn from the uniform distribution on $[0, 1]$. A decision maker is shown the variables sequentially and, after each observation, must decide whether or not to keep the current one. The payoff is R_k, the overall rank of the selected observation, with the convention*
>
> $$R_k = \sum_{i=1}^{n} \mathbb{I}(X_i \le X_k)$$
>
> *(and $\mathbb{I}(A)$ the indicator function of A). Decisions are final: no recall is allowed, no regret is tolerated. The total number of observations is known to the decision maker. The objective is to*

[*]Mathematics Department, Université libre de Bruxelles, Belgium
[†]Mathematics Department, Université de Liège, Belgium

act in such a way as to minimize the expected overal rank of the selected observation.

In the sequel we use the shorthand $RP(n)$ to refer to the above problem with n arrivals. Solving Robbins' problem consists in describing $\tau_n^\star$, the optimal stopping rule, computing $v(n)$, the optimal expected rank obtainable with n observations, understanding the main traits of $\tau_n^\star$ as n grows large and obtaining the limiting value $\lim_{n\to\infty} v(n) = v$. Coaxed by Prof. Herbert Robbins in the early 1990's (see Bruss 2005), several independent teams devoted a significant amount of effort on this seemingly innocuous problem. All have come to the conclusion that the problem is *"very hard"*. So much so that a complete solution to Robbins' problem still eludes us to this date.

Robbins and coauthors (see Chow *et al.* 1964) solve a *no-information* version of the problem, in which the decision maker is not given the *values* of the observations but only their *relative ranks.* Denoting $W(n)$ the corresponding expected rank, Chow *et al.* (1964) provide the optimal strategy and manage an analytic *tour de force* to prove that $W(n) \to W \approx 3.8695$, as $n \to \infty$. Clearly $W(n) \geq v(n)$ for all $n \geq 1$, and hence we deduce that

$$v \leq 3.8695.$$

Of course the full-information $RP(n)$ is much more favorable to the decision maker and we thus expect $v(n)$ and v to be, in fact, much smaller than $W(n)$ and W, respectively.

Taking advantage of the knowledge of the values of the arrivals it is natural to consider the class of stopping rules of the form

$$\tau^{(n)} = \inf\left\{k \geq 1 \,|\, X_k \leq c_k^{(n)}\right\}, \tag{1}$$

which we will call *memoryless threshold rules.* Bruss and Ferguson (1996) prove that there exists a unique optimal sequence (that is, optimal among memoryless threshold rules) which is stepwise increasing in n. Also it is shown in Assaf and Samuel-Cahn (1996) and in Bruss and Ferguson (1993) that if $\tau^{(n)}$ is given by a sequence of increasing thresholds $0 < a_1 \leq a_2 \leq \ldots \leq a_n = 1$, then

$$E\left(R_{\tau^{(n)}}\right) = 1 + \frac{1}{2}\sum_{k=1}^{n-1}(n-k)a_k^2\prod_{j=1}^{k-1}(1-a_j) + \frac{1}{2}\sum_{k=1}^{n}\prod_{j=1}^{k-1}(1-a_j)\sum_{j=1}^{k-1}\frac{(a_k-a_j)^2}{1-a_j}$$

with $R_{\tau^{(n)}}$ the rank of the observation selected by applying the stopping rule $\tau^{(n)}$. Clearly $v(n) \leq E\left(R_{\tau^{(n)}}\right)$ for all n. It is straightforward to optimize this expression over all possible thresholds (at least numerically) to obtain the values for $V(n) = \inf_{\tau^{(n)}} E\left(R_{\tau^{(n)}}\right)$ reported in Table 1. See Bruss and Ferguson

n	1	2	3	4	5	20	50
V	1	1.25	1.4009	1.5065	1.5861	1.9890	2.1482

Table 1: Values of the memoryless optimal expected rank

(1996, Table 1b) (up to a minor correction of a typo for their $V(4)$) or Bruss and Ferguson (1993) where the computations are pushed as far as the case $n = 800$. Assaf and Samuel-Cahn (1996) further explore rules based on sub-optimal thresholds of the form $a_k^{(n)} = \sum_{j=0}^{m} c_j k^j/(n-k+c) \wedge 1$ and mention numerical computations showing that for $m = 2$ the optimal coefficients are $c_0 = 1.77$, $c_1 = 0.54$ and $c_2 = -0.27$ yielding $V = \lim_{n\to\infty} V(n) \leq 2.3268\cdots$ (our conclusion is slightly different to their value 2.3267; this is perhaps due to rounding errors in their computation) and therefore

$$v \leq 2.3268$$

(which is already an important improvement on the optimal no-information value). Although we still do not know the exact value of V, Bruss and Ferguson (1993) extrapolate $V = 2.32659$ and Assaf and Samuel-Cahn (1996) prove that $V \geq 2.29558$, hence not much improvement on v can be hoped for by further exploring memoryless threshold rules of the form (1).

Intriguingly we know that there must exist rules which provide strict improvement on those of the form (1) because Bruss and Ferguson (1993) prove that $v(n) < V(n)$ for all $n \geq 1$, i.e. even the optimal memoryless rule is strictly sub-optimal at every n for $RP(n)$. Meier and Sögner (2014) study variations on the memoryless threshold rules wherein relative ranks are taken into account and manage to lower the upper bound to obtain an expected rank of 2.31301. This improvement is, however, not significant enough even to answer whether or not v is strictly smaller than V or not.

Several authors (e.g. Gnedin 2007, Bruss and Swan 2009 and Gnedin and Iksanov 2011) have considered an alternative approach to Robbins's problem by embedding it in a Poisson process. Gnedin (2007) proves that the memoryless stopping rules remain sub-optimal even in a Poisson limiting model, i.e. there must exist stopping rules which take the history of the arrival process into account and which provide a strict improvement (even in a Poissonian limit) on the optimal memoryless threshold rule. As can be seen from Bruss and Swan (2009), embedding the problem in a Poisson arrival process yields several advantages and opens several new veins of research on this fascinating problem (see also Gnedin and Iksanov 2011) but still does not provide satisfactory solutions to the original problem.

Backward induction guarantees the existence of an optimal strategy $\tau_\star^{(n)}$ and provides, in principle, a way to compute it. Hence for each $n \geq 1$ there must exist threshold functions $h_k^{(n)} : [0,1]^{k-1} \to [0,1], k = 1, \ldots, n-1$ such that the optimal stopping rule is

$$\tau_\star^{(n)} = \inf \left\{ k \,|\, X_k \leq h_k^{(n)}(X_1, \ldots, X_{k-1}) \right\}.$$

Bruss and Ferguson (1993, 1996) prove that the threshold functions are pointwise increasing but depend in a non-monotone way on all the values of the previous arrivals and any loss of information results in the loss of optimality. This last point is referred to as *full history dependence* of the optimal policy. A consequence is that any direct computations related to this optimal strategy are fiendishly complicated and even computer simulations with modern-day technology cannot bring any intuition even for moderate values of n (double exponential complexity). We refer the reader to Bruss (2005) for further information on the problem and its history.

To this date the optimal policy was only explicitly known in the case $n = 2$ (basically trivial) and $n = 3$ (provided by Assaf and Samuel-Cahn 1996), with values $v(2) = 1.25$ and $v(3) = 1.3915\cdots$, respectively. The purpose of this note is to provide a modest complement to the literature by solving the case $n = 4$. We will derive the optimal threshold functions $h_1^{(4)}$, $h_2^{(4)}(x_1)$ and $h_3^{(4)}(x_1, x_2)$, whose behaviour is a complicated function of the past data, see Section 3 for details) and compute the value $v(4) = 1.4932\cdots$ which is remarkably close to the optimal memoryless value $V(4) = 1.5065$ from Table 1. For the sake of completeness we also provide a proof for the optimal strategies and values in the cases $n = 2$ and $n = 3$. As far as we can see there is no easy way to generalize our result to higher values of n.

2 Solution for the cases $n = 2$ and $n = 3$

The case $n = 2$ is nearly trivial. Indeed the threshold value at step 2 must be taken as 1, and only h_1 needs to be computed (here and throughout we drop the superscript (n) for the thresholds). Define $G(h)$ as the expected rank of the selected value by using a strategy with threshold $h_1 = h$. This expression is minimal for $h_1 = 1/2$ and we immediately conclude $v(2) = 5/4$ (which is obviously the same value as $V(2)$ in Table 1).

We now tackle the case $n = 3$. We know that $h_3 = 1$ and must determine the thresholds h_1 and $h_2(x_1)$. Define, in the same fashion as above, $G_{x_1}(h)$ as "the expected rank of the selected variable given $X_1 = x_1$ if we start to play at step 2 by using a threshold value set to h". Direct computations yield

$$G_{x_1}(h) = \frac{3}{2} + h^2 - h + (1 - x_1)(1 - h) + (h - x_1)_+, \tag{2}$$

where $y_+ = \max(y, 0)$.

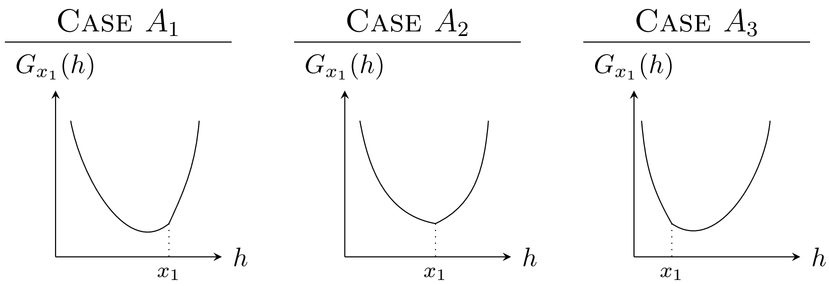

Figure 1: The three generic situations we must study in order to find the expression of the minimizer of G_{x_1}.

Minimizing in h this expected rank we find that we must distinguish three cases (see Figure 1) to get

$$\operatorname*{argmin}_{h\in[0,1]} G_{x_1}(h) = \begin{cases} \frac{1-x_1}{2} & \text{if } 0 \le x_1 < \frac{1}{3} \text{ (case } A_1) \\ x_1 & \text{if } \frac{1}{3} \le x_1 < \frac{2}{3} \text{ (case } A_2) \\ 1 - \frac{x_1}{2} & \text{if } \frac{2}{3} \le x_1 \le 1 \text{ (case } A_3) \end{cases}, \tag{3}$$

from which we deduce $h_2(x_1)$, the optimal threshold at step 2.

By the optimality principle, the value of the threshold h_1 must be a solution to the indifference equation

$$1 + 2h_1 = G_{h_1}(h_2(h_1)) \tag{4}$$

(i.e. the expected rank for choosing an arrival with value h_1 is the same as for continuing and acting optimally thereafter). Solutions of (4) are outside of $[0, 1]$ both when $h_1 < 1/3$ (case A_1) and $2/3 \le h_1 \le 1$ (case A_3). In situation A_2 the equation becomes

$$1 + 2h_1 = \frac{3}{2} + h_1^2 - h + (1 - h_1)^2,$$

with solution $h_1 = (5 - \sqrt{13})/4$. This leads to the same conclusion as Assaf and Samuel-Cahn (1996), namely that the optimal thresholds for $RP(3)$ are

$$h_1 = \frac{5 - \sqrt{13}}{4}, \quad h_2(x_1) = \begin{cases} x_1 & \text{if } h_1 \le x_1 \le \frac{2}{3} \\ 1 - x_1/2 & \text{if } \frac{2}{3} \le x_1 \le 1 \end{cases}$$

(and $h_3 = 1$) providing us with the value

$$v(3) = \frac{341}{144} - \frac{13}{48}\sqrt{13} = 1.39155\cdots$$

which is remarkably close to the corresponding memoryless value $V(3)$ in Table 1.

3 Solution for the case $n = 4$

As anticipated, in this section we prove the main contribution of this note, namely

$$v(4) = 1.4932\cdots . \tag{5}$$

The dynamic programming approach requires to find the optimal behaviour at some specific step k given a length $k-1$ history, by letting k go backwards from n to 1. Our plan is thus simple : we start considering the best action at time $k = 4$, then we proceed backwards and end with the case $k = 1$. For each k, we fix a history $X_1 = x_1$, $X_2 = x_2$,..., $X_{k-1} = x_{k-1}$. We know from Bruss and Ferguson (1993) that the optimal action is defined by a threshold $h_k(x_1, \ldots, x_{k-1})$: keep X_k if less than $h_k(x_1, \ldots, x_{k-1})$, otherwise discard it. Our purpose is to determine the exact expressions for $h_k(x_1, \ldots, x_{k-1}), k = 1, 2, 3, 4$.

Step 4. Suppose that $(X_1, X_2, X_3) = (x_1, x_2, x_3)$ has been observed and we only enter the game at step 4 before learning the value of X_4. Since this is the last step, we must accept it whatever its value may be. This is the optimal behaviour, and $h_4(x_1, x_2, x_3) = 1$, for all $(x_1, x_2, x_3) \in [0, 1]^3$.

Step 3. Suppose that $(X_1, X_2) = (x_1, x_2)$ has been observed and we enter the game at step 3 before learning the value of X_3. Define $R_{x_1,x_2}(h)$ as the rank of a value chosen using threshold h at step 3 given the history (x_1, x_2). Its expected value is

$$G_{x_1,x_2}(h) := E(R_{x_1,x_2}(h)), \tag{6}$$

which can be computed directly to get

$$G_{x_1,x_2}(h) = \frac{3}{2} + h^2 - h + (2 - x_1 - x_2)(1 - h) + \sum_{i=1}^{2}(h - x_i)_+ \tag{7}$$

where $y_+ = \max(y, 0)$, for all $y \in \mathbb{R}$. Then the optimal threshold $h_3(x_1, x_2)$ must be given by

$$h_3(x_1, x_2) = \operatorname*{argmin}_{h \in [0,1]} G_{x_1,x_2}(h). \tag{8}$$

For each history (x_1, x_2), the graph of $G_{x_1,x_2}(\cdot)$ is composed of the reunion of three parabolae, as illustrated in Figure 2. In this Figure we read also that the behaviour of the minimum (mainly on which of the the three parabolae it is to be found) depends on the region of the square $[0,1]^2$ the pair (x_1, x_2) lies in, as illustrated in Figure 3. We do not go into detail.

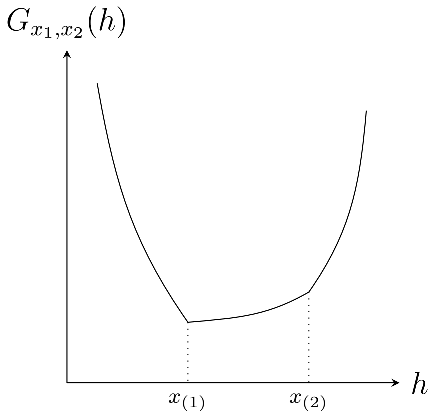

Figure 2: Graph of $G_{x_1,x_2}(\cdot)$ for one particular history. As in the case $n = 3$, the minimum will be given by the minimizer of one of the parabolae or by one of the past observations. In our case ($n = 4$), this leads to 5 cases.

Similarly as in the previous section for $RP(3)$ we need to distinguish 5 cases, and obtain

$$h_3(x_1, x_2) = \begin{cases} x_{(1)} & \text{for } (x_1, x_2) \in A_1 \\ x_{(2)} & \text{for } (x_1, x_2) \in A_2 \\ \tilde{x}_1 = \frac{3-(x_1+x_2)}{2} & \text{for } (x_1, x_2) \in B_1 \\ \tilde{x}_2 = \frac{2-(x_1+x_2)}{2} & \text{for } (x_1, x_2) \in B_2 \\ \tilde{x}_3 = \frac{1-(x_1+x_2)}{2} & \text{for } (x_1, x_2) \in B_3 \end{cases} . \tag{9}$$

where the A_i's and B_i's are shown on Figure 3, and where $x_{(1)}$ and $x_{(2)}$ are respectively $\min(x_1, x_2)$ and $\max(x_1, x_2)$.

Step 2. Suppose that $X_1 = x_1$. The optimal threshold $h_2(x_1)$ must be such that, if $X_2 = h_2(x_1)$, then the same payoff is obtained by selecting X_2 or rejecting it and acting optimally thereafter. In other words, $h_2(x_1)$ is the indifference value for X_2. Consequently the threshold $h_2(x_1)$ must be solution to

$$1 + 2h_2 + \mathbb{1}(h_2 > x_1) = g(x_1, x_2), \tag{10}$$

with $g(x_1, x_2) := G_{x_1,x_2}(h_3(x_1, x_2))$. The decomposition of h_3 given in (9) allows us to obtain the explicit expression of $g(x_1, x_2)$, on each of the regions

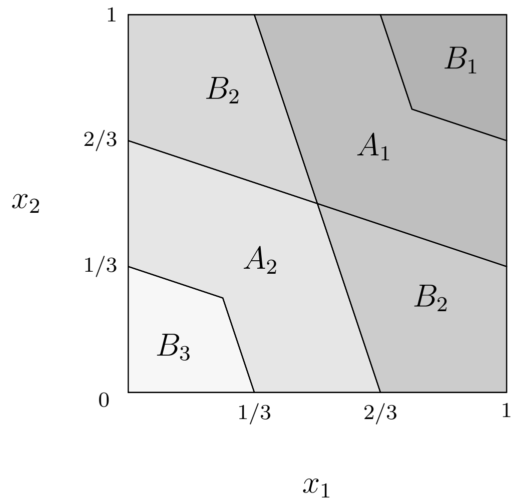

Figure 3: The regions A_1, A_2, B_1, B_2, B_3 are circumscribed by the borders of $[0,1]^2$ and the lines $x_2 = (3 - x_1)/3$, $x_2 = (2 - x_1)/3$, $x_2 = (1 - x_1)/3$, $x_2 = 3 - 3x_1$, $x_2 = 2 - 3x_1$, $x_2 = 1 - 3x_1$.

A_1, A_2, B_1, B_2, and B_3. After some work one notices that the optimal threshold $h_2(x_1)$ can be obtained explicitly by discussing separately over 6 different intervals for x_1.

When the history is $X_1 = 0$, we are faced with a RP(3) on $\{X_2, X_3, X_4\}$. Therefore the value of $h_2(0)$ is equal to the value of h_1 in a $RP(3)$, and (see Section 2)

$$h_2(0) = \frac{5 - \sqrt{13}}{4} =: a. \tag{11}$$

Similarly, if $X_1 = 1$, then we find again a $RP(3)$, hence

$$h_2(1) = a. \tag{12}$$

The endcases are therefore covered.

We now study $h_2(x_1)$ for small values of x_1. We know that $h_2(x_1)$ is a continuous functions of x_1 (see Bruss and Ferguson 1993). The graph of h_2 starts at $(0, a)$ which lies in A_2 (because $a > 1/3$) and ends at $(1, a)$ which lies in A_1 (for the same reason). We can therefore determine h_2 on the interval $[0, \beta_1]$ where β_1 is the first coordinate of the intersection of the graph of h_2 with one of the boundaries of the regions B_2 or B_3. For this reason we use the expression $G_{x_1,x_2}(x_{(2)})$ in (10) and the fact that $h_2 > x_1$ when we are close to $x_1 = 0$. Note that it is possible that the graph of h_2 intersects the line $x_2 = x_1$ before it reaches the border of B_2 or B_3. We find that the graph of h_2 intersects first the border between A_2 and B_3 at the point with x-coordinate equal to $\beta_1 = \frac{3}{2}\sqrt{2} - 2$. Therefore,

$$h_2(x_1) = \frac{1}{4}\left(5 - x_1 - \sqrt{x_1^2 + 6x_1 + 13}\right) =: h_{21}(x_1), \tag{13}$$

on $[0, \beta_1]$.

Next, on some interval $[\beta_1, \beta_2]$ with β_2 to be determined, we consider (10) with $g(x_1, x_2) = G_{x_1,x_2}(\tilde{x}_3)$ because the graph entered the region B_3. The value of β_2 is either the x-coordinate of the point at which the graph of h_2 enters a new region, or the point at which the solution h_2 of (10) stops being strictly larger than x_1. Therefore, on $[\beta_1, \beta_2]$, we have

$$h_2(x_1) = \sqrt{8x_1 + 54} - x_1 - 7 =: h_{22}(x_1), \tag{14}$$

and we can also check that $h_{21}(\beta_1) = h_{22}(\beta_1)$. We find that the graph of h_2 crosses the line $x_2 = x_1$ before it reaches another region. Therefore β_2 is the solution of $h_{22}(x_1) = x_1$, thus $\beta_2 = \frac{\sqrt{30}-5}{2}$.

By symmetry, these arguments also apply for large values of x_1 (i.e. close to 1). One finds easily that

$$h_2(x_1) = \begin{cases} \dfrac{3}{2} - \dfrac{1}{4}(x_1 + \sqrt{x_1^2 - 4x_1 + 16}) & \text{for } x_1 \in [\beta_5, 1] \\ \sqrt{12x_1 + 42} - 6 - x_1 & \text{for } x_1 \in [\beta_4, \beta_5] \\ -\dfrac{(4x_1^2 - 6x_1 + 5)}{2(x_1 - 4)} & \text{for } x_1 \in [\beta_3, \beta_4] \end{cases} \tag{15}$$

where

$$\beta_3 = \frac{7 - \sqrt{19}}{6}, \quad \beta_4 = \frac{1}{2}(11 - 3\sqrt{11}), \quad \beta_5 = \frac{1}{2}(7 - 3\sqrt{3}). \tag{16}$$

The left-hand-side of (10) was equal to $1 + 2h_2$ as we started at $x_1 = 1$ and moved to the left. At β_3, we have $h_2(x_1) = x_1$. At this point, $h_2(x_1)$ is not strictly lower than x_1 anymore.

Finally we need to obtain h_2 for intermediate values of $x_1 \in [\beta_2, \beta_3]$; to this end we need to consider separately the cases $x_1 \in [\beta_2, 1/4)$ and $x_1 \in [1/4, \beta_3]$. We get the dichotomy (i) $h_2 < x_1$ then the LHS of (10) is strictly smaller than its RHS, (ii) $h_2 > x_1$ then the LHS of (10) is strictly larger than its RHS. This can be interpreted in a probabilistic way: if h_2 is taken smaller than x_1, the expected payoff is better if we could stop on this value (LHS<RHS), while it is a bad choice to stop on $X_2 = h_2$ if $h_2 > x_1$ since the expected payoff is then worse than what expected if one continues the game (LHS>RHS). From these two observations, we conclude that $h_2 = x_1$.

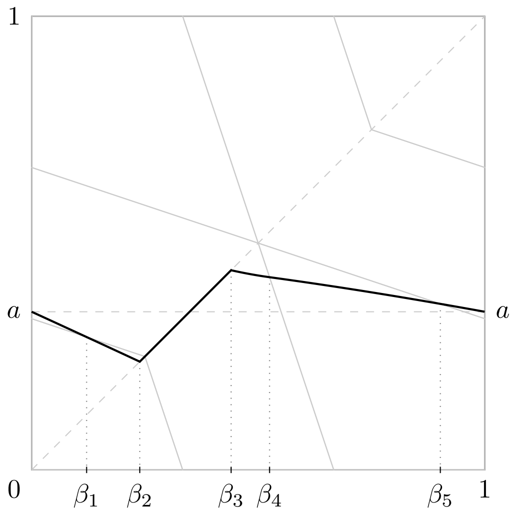

Figure 4: Plot of $h_2(x_1)$ for $x_1 \in [0,1] = [0,\beta_1] \cup [\beta_1,\beta_2] \cup [\beta_2,\beta_3] \cup [\beta_3,\beta_4] \cup [\beta_4,\beta_5] \cup [\beta_5,1]$. Although there are 6 different expressions, it can be checked that $h_2(\cdot)$ is differentiable at β_i for $i \in \{1,4,5\}$.

We therefore know the expression of h_2 for all values of x_1 on $[0,1]$; this is represented in Figure 4.

Step 1. The much sought-after threshold h_1 is solution to

$$1 + 3h_1 = g(h_1), \tag{17}$$

where $g(x_1)$ is the expected rank of the selected variable if one starts the game at step 2 with the history $X_1 = x_1$ and acts optimally thereafter.

Let us try to find a solution $h_1 \in [0,\epsilon]$. The right-hand-side of (17) is an integral where the integrating variable represents the value of X_2; when $X_2 = u \leq h_2(h_1)$, one must accept X_2, while one must reject $X_2 = u$ if $u > h_2(h_1)$. The behaviour when one moves on to step 3 depends on the region the history (h_1, u) lies in: A_2, B_2, or B_3. The expression of $G_{h_1,u}$ will depend on this.

For the sake of concision, we will only write out the complete expression of the integral for the smaller values of h_1. We thus have

$$\begin{aligned} g(h_1) = &\int_0^{h_1} (1+2u)\,du + \int_{h_1}^{h_2(h_1)} (2+2u)\,du \\ &+ \int_{h_2(h_1)}^{(2-h_1)/3} G_{h_1,u}(u)\,du \\ &+ \int_{(2-h_1)/3}^{1} G_{h_1,u}((1-(h_1+u))/2)\,du. \end{aligned}$$

The function $h_2(\cdot)$ is defined on 6 different intervals. Thus the need to write at least 6 integrals in order to keep explicit expressions around. Also look at the change in the path made vertically through the regions A_1, A_2, B_1, B_2, B_3. When the regions or the order of the regions in which we cross them changes, we must write a separate integral. Summing things up, we need 11 divisions of $[0, 1]$ on which the expression of the integral is each time different. The solution to (17) is found on $[\beta_2, \beta_3]$, with β_2 and β_3 defined above. The software Mathematica came in handy for this task, yielding

$$\begin{aligned} h_1 &= \left(\tfrac{6}{1849}\sqrt{123199} - \tfrac{87150}{79507}\right)^{1/3} - \tfrac{846}{1849}\left(\tfrac{6}{1849}\sqrt{123199} - \tfrac{87150}{79507}\right)^{-1/3} + \tfrac{53}{43} \\ &= 0.27502\cdots . \end{aligned}$$

Wrapping up we finally obtain (computations not included)

$$\begin{aligned} V(4) = &-\frac{5553791}{8640} + \frac{767}{80\sqrt{3}} + \frac{2609\sqrt{11}}{216} + \frac{3281\sqrt{19}}{216} - \frac{59(53-\alpha_1+\alpha_2)}{1548} \\ &+ \frac{85(53-\alpha_1+\alpha_2)^2}{44376} - \frac{53(53-\alpha_1+\alpha_2)^3}{2862252} + \frac{(53-\alpha_1+\alpha_2)^4}{11449008} \\ &+ \frac{1}{192}(842 - 532\sqrt{3} + 31\sqrt{13} + 216\,\mathrm{ArcCsch}(2\sqrt{3}) \\ &- 216\,\mathrm{ArcSinh}(\tfrac{3-\sqrt{3}}{4}) - \frac{2025\log(12)}{8} + \frac{2025\log(252)}{8} \\ &+ \frac{1}{288}(2586985 - 779844\sqrt{11} \\ &+ 72900\log(\tfrac{3}{7}(-1+\sqrt{11}))) - \frac{2025}{8}\log(17+\sqrt{19}) \\ = &\ 1.49329\cdots , \end{aligned}$$

with

$$\begin{aligned} \alpha_1 &= \left(\frac{5076}{14525 + 43\sqrt{123199}}\right)^{1/3}, \\ \alpha_2 &= \left(6(-14525 + 43\sqrt{123199})\right)^{1/3}. \end{aligned}$$

All Mathematica computations are available on Yvik Swan's webpage.[1]

[1]https://sites.google.com/site/yvikswan/

Summarizing, we have obtained the following optimal thresholds :

$$h_1 = \left(\tfrac{6}{1849}\sqrt{123199} - \tfrac{87150}{79507}\right)^{1/3} - \tfrac{846}{1849}\left(\tfrac{6}{1849}\sqrt{123199} - \tfrac{87150}{79507}\right)^{-1/3} + \tfrac{53}{43},$$

$$h_2(x_1) = \begin{cases} \frac{1}{4}\left(5 - x_1 - \sqrt{x_1^2 + 6x_1 + 13}\right) & \text{if } x_1 \in [0, \frac{3}{2}\sqrt{2} - 2] \\ \sqrt{8x_1 + 54} - x_1 - 7 & \text{if } x_1 \in [\frac{3}{2}\sqrt{2} - 2, \frac{\sqrt{30}-5}{2}] \\ x_1 & \text{if } x_1 \in [\frac{\sqrt{30}-5}{2}, \frac{7-\sqrt{19}}{6}] \\ -\dfrac{(4x_1^2 - 6x_1 + 5)}{2(x_1 - 4)} & \text{if } x_1 \in [\frac{7-\sqrt{19}}{6}, \frac{1}{2}(11 - 3\sqrt{11})] \\ \sqrt{12x_1 + 42} - 6 - x_1 & \text{if } x_1 \in [\frac{1}{2}(11 - 3\sqrt{11}), \frac{1}{2}(7 - 3\sqrt{3})] \\ \frac{3}{2} - \frac{1}{4}(x_1 + \sqrt{x_1^2 - 4x_1 + 16}) & \text{if } x_1 \in [\frac{1}{2}(7 - 3\sqrt{3}), 1] \end{cases},$$

$$h_3(x_1, x_2) = \begin{cases} x_{(1)} & \text{if } (x_1, x_2) \in A_1 \\ x_{(2)} & \text{if } (x_1, x_2) \in A_2 \\ \tilde{x}_1 = \frac{3-(x_1+x_2)}{2} & \text{if } (x_1, x_2) \in B_1 \\ \tilde{x}_2 = \frac{2-(x_1+x_2)}{2} & \text{if } (x_1, x_2) \in B_2 \\ \tilde{x}_3 = \frac{1-(x_1+x_2)}{2} & \text{if } (x_1, x_2) \in B_3 \end{cases},$$

and, of course, $h_4 = 1$. Approximate values of the β_i's, rounded to the 5th decimal:

$$\beta_1 = 0.12132, \beta_2 = 0.23861, \beta_3 = 0.44018, \beta_4 = 0.52506, \beta_5 = 0.90192.$$

Acknowledgments

This note was written after a conference held in Brussels on September 10 and 11, 2015 in honor of Prof. F. T. Bruss who was supervisor of both the authors' PHDs. His enthusiasm for his craft is an inspiration for both of us and we thank him for his guidance during those important early moments of our career.

References

[1] Assaf, D. and Samuel-Cahn, E. (1996). The secretary problem; minimizing the expected rank with i.i.d. random variables, *Adv. Appl. Prob.*, Vol. 28, pp. 828-852.

[2] Bruss, F. T. (2005). What is known about Robbins' problem?, *J. Appl. Prob.*, Vol. 42, pp. 108-120.

[3] Bruss, F. T. and Ferguson, T. S. (1993). Minimizing the expected rank with full information, *J. Appl. Prob.*, Vol. 30, pp. 616 - 626.

[4] Bruss, F. T. and Ferguson, T. S. (1996). Half-Prophets and Robbins' problem of Minimizing the expected rank, *Springer Lecture Notes in Stat. 114*, Vol. 1 in honor of J.M. Gani, pp. 1-17.

[5] Bruss, F. T. and Swan, Y. (2009). A continuous-time approach to Robbins' problem of minimizing the expected rank, *J. Appl. Prob.*, Vol. 46, pp. 1-18.

[6] Chow, Y. S., Moriguti, S., Robbins, H. and Samuels, S. M. (1964). Optimal selection based on relative ranks, *Israel J. Math.*, Vol. 2, pp. 81-90.

[7] Gnedin, A. V. (2007). Optimal Stopping with Rank-Dependent Loss, *J. Appl. Prob.*, Vol. 44, pp. 996-1011.

[8] Gnedin, A. V. and Iksanov, A. (2011). Moments of random sums and Robbins' problem of optimal stopping, *J. Appl. Prob.*, Vol. 48, pp. 1197-1199.

[9] Meier, M. and Sögner, L. (2014). A New Upper Bound for Robbins' problem. Available at SSRN 2408149.