Permutations and shifts

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Abstract. The entropy of a symbolic dynamical system is usually defined in terms of the growth rate of the number of distinct allowed factors of length n. Bandt, Keller and Pompe showed that, for piecewise monotone interval maps, the entropy is also given by the number of permutations defined by consecutive elements in the trajectory of a point. This result is the starting point of several works of Elizalde where he investigates permutations in shift systems, notably in full shifts and in beta-shifts. The goal of this talk is to survey Elizalde's results. I will end by mentioning the case of negative beta-shifts, which has been simultaneously studied by Elizalde and Moore on the one hand, and by Steiner and myself on the other hand.

Keywords: Dynamical systems, permutation entropy, beta-shifts.

1 Introduction

The following result motivates the subject.

Theorem 1 (Bandt-Keller-Pompe [BKP02]). *For piecewise monotonic maps, the topological entropy coincides with the permutation entropy.*

Let us introduce the permutation entropy of a totally ordered dynamical system. This notion was first introduced in [BP02] and then, studied in [BKP02], [Kel12], [KUU12], [Ami12] (and other papers). Let us also mention the book [Ami10].

From now on, we suppose that X is a totally ordered set and $T\colon X \to X$. For an integer $n \geq 1$ and a point $x \in X$ such that $x, T(x), \ldots, T^{n-1}(x)$ are pairwise distinct, $\mathrm{Pat}(T, n, x)$ denotes the permutation $\pi \in \mathcal{S}_n$ defined by

$$T^{\pi^{-1}(1)-1}(x) < T^{\pi^{-1}(2)-1}(x) < \cdots < T^{\pi^{-1}(n)-1}(x).$$

Otherwise stated, the relative order of $x, T(x), \ldots, T^{n-1}(x)$ corresponds to the permutation π.

Example 2. Suppose $T^3(x) < T(x) < x < T^2(x)$. Then $\mathrm{Pat}(T, 4, x) = 3241$.

A permutation π in $\mathcal{S}_n$ is *realized*, or *allowed*, in (X,T) if there exists $x \in X$ such that $\mathrm{Pat}(T,n,x)=\pi$. The set of allowed permutations of length n and the set of all allowed permutations are denoted by

$$\mathcal{A}(T,n)=\{\pi\in\mathcal{S}_n\colon \exists x\in X\ \ \mathrm{Pat}(T,n,x)=\pi\} \quad\text{and}\quad \mathcal{A}(T)=\bigcup_{n\geq 1}\mathcal{A}(T,n)$$

respectively. Then the *permutation entropy* of (X,T) is defined as

$$\lim_{n\to\infty}\frac{1}{n}\log\#\mathcal{A}(T,n)$$

provided that this limit exists. Theorem 1 states that this limit exists for piecewise monotonic maps, and coincides with the topological entropy. In particular this result implies that not all permutations are realized in a given piecewise monotonic map system. In fact, most of them are not since the number of permutations of length n is super-exponential.

Example 3 (Tent map). Let $X=[0,1]$ and $T(x)=\begin{cases} 2x & \text{if } x\in[0,\frac{1}{2}] \\ -2x+2 & \text{if } x\in[\frac{1}{2},1] \end{cases}.$

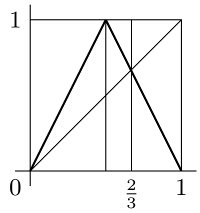

Fig. 1. The tent map

Clearly, any x close to 0 realizes the permutation 123 and any x close to 1 realizes the permutation 312. A simple case study shows that every $x\in]0,1/3[$ realizes the permutation $\pi=123$, every $x\in]1/3,2/5[$ realizes $\pi=132$, every $x\in]2/5,2/3[$ realizes $\pi=231$, every $x\in]2/3,4/5[$ realizes $\pi=213$, and finally, that every $x\in]0,1/3[$ realizes $\pi=312$. In particular, the permutation $\pi=321$ is not realizable.

The aim of this note is to provide a quick and understandable overview of the results of the following papers: [AEK08], [Eli09], [Eli11], [AE14], [EM] and [CS]. Of course, I do not claim to be exhaustive; thus many interesting results will not be mentioned. I will end by listing two open questions in this field.

2 Permutations and full shifts

Let $\mathbb{A}_k$ denote the k-letter alphabet $\{\mathtt{0},\mathtt{1},\ldots,\mathtt{k-1}\}$ and consider the map $\sigma_k\colon\mathbb{A}_k^{\mathbb{N}}\to\mathbb{A}_k^{\mathbb{N}},(a_m)\mapsto(a_{m+1})$. This map is continuous with respect to the prefix metric on $\mathbb{A}_k^{\mathbb{N}}$: for two distinct infinite words over $\mathbb{A}_k$, the longer is their common prefix, the closer they are. As the set $\mathbb{A}_k^{\mathbb{N}}$ is compact with respect to this metric, $(\mathbb{A}_k^{\mathbb{N}}, \sigma_k)$ is a topological dynamical system. The domain $\mathbb{A}_k^{\mathbb{N}}$ is usually called the *full shift* (over k symbols).

We use the notation $\overline{a_1 a_2 \cdots a_i}$ for the periodic sequence with period $a_1 a_2 \cdots a_i$, and $a_{[i,\infty)} = a_i a_{i+1} \cdots$ and $a_{[i,j)} = a_i a_{i+1} \cdots a_{j-1}$. Moreover, for $(a_m)_{m\geq 1} \in \mathbb{A}_k^{\mathbb{N}}$, we let

$$\widetilde{a} = \sup_{m\geq 1} a_{[m,\infty)}. \tag{1}$$

In this section, we suppose that $\mathbb{A}_k^{\mathbb{N}}$ is ordered by the lexicographic order. We have

$$\mathrm{Pat}(\sigma_k, n, (a_m)_{m\geq 1}) = \pi \iff a_{[\pi^{-1}(1),\infty)} <_{\mathrm{lex}} a_{[\pi^{-1}(2),\infty)} <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} a_{[\pi^{-1}(n),\infty)}.$$

Permutations in full shifts were first studied in [AEK08]. In this paper, the authors show that the smallest permutations that are not allowed (such permutations are also said to be *forbidden*) in $(\mathbb{A}_k^{\mathbb{N}}, \sigma_k)$ have length $k+2$. For example, for a binary alphabet, every permutation of length smaller than or equal to 3 is allowed, whereas it is easily checked that the permutation $\pi = 1423$ is not.

In [Eli09], Elizalde is interested in computing the quantity $N_+(\pi)$, which is the smallest k such that π is realized in $(\mathbb{A}_k^{\mathbb{N}}, \sigma_k)$:

$$N_+(\pi) = \min\{k \geq 1 \colon \pi \in \mathcal{A}(\sigma_k)\}.$$

In Section 4, we will use the analogous notation $N_-(\pi)$ in the case of negative β-shifts. This is the reason why we write $N_+(\pi)$ instead of following Elizalde's notation $N(\pi)$.

Example 4. Consider the permutation $\pi = 4217536 \in \mathcal{S}_7$. Then any infinite sequence $(a_m)_{m\geq 1}$ starting with $\mathtt{210221220}$ realizes π since

$$\begin{array}{ccccccc|c} 2 & 1 & 0 & 2 & 2 & 1 & 2 & 2\,0\cdots \\ 4 & 2 & 1 & 7 & 5 & 3 & 6 & \end{array}$$

where, for each m, $1 \leq m \leq 7$, we wrote $\pi(m)$ below a_m if $\pi(m) = i$. For instance, $a_{[1,\infty)} = \mathtt{210}\cdots <_{\mathrm{lex}} a_{[5,\infty)} = \mathtt{212}\cdots$, so $\pi(1) = 4 < \pi(5) = 5$. Note that we do not have uniqueness as $\mathrm{Pat}(\sigma_3, 7, \mathtt{210221220}\cdots) = \mathrm{Pat}(\sigma_3, 7, \mathtt{210221221}\cdots) = \pi$.

If $a_i a_{i+1} \cdots <_{\mathrm{lex}} a_j a_{j+1} \cdots$ and $a_i = a_j$ then $a_{i+1} a_{i+2} \cdots <_{\mathrm{lex}} a_{j+1} a_{j+2} \cdots$. If $a_1 a_2 \cdots$ realizes the permutation π, this means that $\pi(i) < \pi(j)$, $a_i = a_j$ and $1 \leq i, j < n \implies \pi(i+1) < \pi(j+1)$. Thus, for a permutation $\pi \in \mathcal{S}_n$, it is natural to consider the circular permutation

$$\hat{\pi} = (\pi(1)\pi(2)\cdots\pi(n)). \tag{2}$$

Roughly, $N_+(\pi)$ is approximately equal to the number of descents in $\hat{\pi}$, i.e., the number of indices $k < n$ such that $\hat{\pi}(k) > \hat{\pi}(k+1)$. Indeed, if $1 \leq i, j < n$, $\pi(i) < \pi(j)$, and $\pi(i+1) = \hat{\pi}(\pi(i)) > \hat{\pi}(\pi(j)) = \pi(j+1)$, then $a_i < a_j$. So, for each descent in $\hat{\pi}$ where $\pi(1)$ is ignored we need one more symbol in order to realize π.

Example 5. We continue Example 4. One has $\hat{\pi} = (\underline{4}217536) = 7162\underline{3}\underline{4}5$ and $\hat{\pi}$ where $\pi(1) = 4$ is ignored, which is the sequence 716235, has 2 descents. By using the previous argument, we need at least 3 symbols to realize π: $\mathtt{0}, \mathtt{1}, \mathtt{2}$. More precisely, the permutation $\hat{\pi}$ also tells us the number of occurrences of those symbols in the prefix of length $n-1$ of any infinite sequence (a_m) realizing the permutation π:

$$\begin{array}{rccccccc} \hat{\pi} = & 7 & 1 & 6 & 2 & 3 & \underline{4} & 5 \\ & \mathtt{0} & \mathtt{1} & \mathtt{1} & \mathtt{2} & \mathtt{2} & & \mathtt{2} \end{array}$$

Then, the exact order of those $n-1$ digits in the prefix of any such (a_m) is given by π itself:

$$\begin{array}{rccccccc} \pi = & \underline{4} & 2 & 1 & 7 & 5 & 3 & 6 \\ & & \mathtt{2} & \mathtt{1} & \mathtt{0} & \mathtt{2} & \mathtt{2} & \mathtt{1} \end{array}$$

The previous discussion ignores specific situations, where more symbols are needed. The main result of [Eli09] is as follows:

Theorem 6 ([Eli09]). *Let $n \geq 2$. For any $\pi \in \mathcal{S}_n$,*

$$N_+(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon_+(\pi)$$

where $\operatorname{des}(\hat{\pi})$ *is the number of descents in $\hat{\pi}$ with $\pi(1)$ removed and*

$$\epsilon_+(\pi) = \begin{cases} 1 & \text{if } \pi \text{ ends with } 21 \text{ or with } (n-1)n, \\ 0 & \text{otherwise.} \end{cases}$$

Pursuing the previous discussion, in the case $\epsilon_+(\pi) = 0$ the prefix $z_1 z_2 \cdots z_{n-1}$ of any infinite sequence realizing the permutation π is given by

$$z_j = \#\{1 \leq i < \pi(j) \colon \text{either } i \notin \{\pi(n)-1, \pi(n)\} \text{ and } \hat{\pi}(i) > \hat{\pi}(i+1), \\ \text{or } i = \pi(n)-1 \text{ and } \hat{\pi}(i) > \hat{\pi}(i+2)\} \tag{3}$$

where it should be understood that z_j is really the digit corresponding to this number.

Example 7. We continue Example 5. We have $\epsilon_+(\pi) = 0$. By (3) we find $z_1 z_2 \cdots z_{n-1} = \mathtt{210221}$, as desired.

Example 8. Let $\pi = 346752189$. Then $\hat{\pi} = (\underline{3}46752189) = 81462759\underline{3}$ and $\epsilon_+(\pi) = 1$. In order to realize π, an infinite word $a_1 a_2 \cdots$ starting with $z_1 z_2 \cdots z_{n-1} = \mathtt{11232103}$ needs one more symbol. Indeed

$$a_n a_{n+1} \cdots >_{\text{lex}} z_{n-1} a_n \cdots = \mathtt{3} a_n \cdots \implies a_n > \mathtt{3}$$

and any infinite sequence starting with $\mathtt{112321034}$ realizes π.

Example 9. Let $\pi = 24153$. Then $\hat{\pi} = (\underline{2}4153) = 54\underline{2}13$ and $\epsilon_+(\pi) = 0$. Then $z_1 z_2 \cdots z_{n-1} = \mathtt{1202}$ (the prefix defined by (3)). Any sequence starting with $\mathtt{1202121}$ or $\mathtt{1202201}$ realizes π. This illustrates that, unlike the prefix of length $n-1$, the nth letter is not fixed by the permutation. This choice comes specifically from the descent 41 in $\hat{\pi}$ where $\pi(1) = 2$ is removed.

As a corollary of Theorem 6, Elizalde obtains that for $n \geq 3$ and $\pi \in \mathcal{S}_n$, one has $N_+(\pi) \leq n-1$. In addition, he proves that for all $n \geq 3$, there are exactly 6 permutations $\pi \in \mathcal{S}_n$ such that $N_+(\pi) = n-1$. These 6 permutations are:

$$\begin{array}{cc} 1n2(n-1)3(n-2)\ldots, & \ldots(n-2)3(n-1)2n1, \\ n1(n-1)2(n-2)3\ldots, & \ldots 3(n-2)2(n-1)1n, \\ \ldots 4(n-1)3n21, & \ldots(n-3)2(n-2)1(n-1)n. \end{array}$$

In doing so, he answers a conjecture from [AEK08]. In fact, Elizalde shows much more by proving a closed formula for the number $a_{n,N}$ of permutations π of length n for which $N_+(\pi) = N$, for any n and N. In particular, for each fixed N, one has $a_{n,N} \sim nN^{n-1}$ as n tends to infinity, whence for each k, $\lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{A}(\sigma_k, n) = \lim_{n\to\infty} \frac{1}{n} \log(\sum_{N=1}^{k} a_{n,N}) = \log k$, in accordance with Theorem 1.

To end this section, let me also mention the work [AE14] where the authors consider other orderings of the elements of the full shift in the case of periodic orbits.

3 Permutations and positive β-shifts

Let $\beta > 1$. The β-transformation is the map $T_\beta \colon [0,1) \to [0,1), x \mapsto \{\beta x\}$ where $\{\cdot\}$ designates the fractional part of a real number. Instead of numbers $x \in [0,1)$, we will rather consider their *β-expansions* [Rén57]:

$$x = \sum_{k=1}^{\infty} \frac{d_{\beta,k}(x)}{\beta^k} \text{ with } d_{\beta,k}(x) = \left\lfloor \beta\, T_\beta^{k-1}(x) \right\rfloor.$$

Set $d_\beta(x) = d_{\beta,1}(x) d_{\beta,2}(x) \cdots$. The *$\beta$-shift* is the topological closure of the set $\{d_\beta(x) \colon x \in [0,1)\}$ of all β-expansions from $[0,1)$; it is denoted by Ω_β. Then σ_β denotes the shift map $\sigma_\beta \colon \Omega_\beta \to \Omega_\beta,\ (a_m) \mapsto (a_{m+1})$. This map is continuous and the β-shift is a compact metric space, hence $(\Omega_\beta, \sigma_\beta)$ is a topological dynamical system. For all $x, y \in [0,1)$, we have $\sigma_\beta(d_\beta(x)) = d_\beta(T_\beta(x))$ and $x < y \iff d_\beta(x) <_{\text{lex}} d_\beta(y)$. Thus, for all $x \in [0,1)$ and all $n \geq 1$, we have

$$\text{Pat}(T_\beta, n, x) = \text{Pat}(\sigma_\beta, n, d_\beta(x)),$$

with the lexicographical order on Ω_β. We note that if $a_1 a_2 \cdots = \lim_{i\to\infty} d_\beta(x_i)$ with (x_i) a sequence of $[0,1)$, then for all sufficiently large i and all $n \geq 1$, we have $\text{Pat}(\sigma_\beta, n, a_1 a_2 \cdots) = \text{Pat}(\sigma_\beta, n, d_\beta(x_i))$. Therefore $\mathcal{A}(T_\beta) = \mathcal{A}(\sigma_\beta)$. Moreover, if $1<\beta<\beta'$, then $d_\beta(1) <_{\text{lex}} d_{\beta'}(1)$, whence $\Omega_\beta \subseteq \Omega_{\beta'}$ and $\mathcal{A}(T_\beta) \subseteq \mathcal{A}(T_{\beta'})$ (this follows from Parry's theorem, which characterizes the β-shift [Par60]).

In [Eli11], Elizalde introduces the notion of the shift complexity of a permutation. We will take the liberty of calling it the *positive* shift complexity as we will need an analogous definition in the next section for negative β-shifts. The *positive shift complexity* of a permutation $\pi \in \mathcal{S}_n$ is the quantity

$$B_+(\pi) = \inf\{\beta > 1 \colon \pi \in \mathcal{A}(T_\beta)\}. \tag{4}$$

The main result of [Eli11] is a method to compute $B_+(\pi)$. For $\pi \in \mathcal{S}_n$, let $z_1 z_2 \cdots z_{n-1}$ as in (3). Moreover, let

$$m = \pi^{-1}(n) \quad \text{ and } \quad \ell = \pi^{-1}(\pi(n)-1) \text{ if } \pi(n) \neq 1. \tag{5}$$

For a sequence $a = a_1 a_2 \cdots$ of finitely many nonnegative digits such that $a = \widetilde{a}$ (see (1)), let $b_+(a)$ be the unique solution $\beta \geq 1$ of

$$\sum_{j=1}^{\infty} \frac{a_j}{\beta^j} = 1.$$

Note that when a is an eventually periodic sequence, $b_+(a)$ is the unique real root greater than or equal to 1 of a polynomial.

Theorem 10. *[Eli11] Let $\pi \in \mathcal{S}_n$. Then $\pi \in \mathcal{A}(T_\beta) \iff \beta > b_+(a)$ where*

$$a = \begin{cases} z_{[m,n)}\overline{z_{[\ell,n)}} & \text{if } \pi(n) \neq 1, \\ z_{[m,n)}\overline{0} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) \neq 2, \\ z'_{[m,n)}\overline{0} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) = 2. \end{cases}$$

where the digits z_j are defined as in (3) and for every $1 \leq j < n$, $z'_j = z_j + 1$. In particular, $B_+(\pi) = b_+(a)$ and $B_+(\pi)$ is 1 or a Parry number, i.e., a number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic.

It directly follows from this theorem that $N_+(\pi) = 1 + \lfloor B_+(\pi) \rfloor$.

4 Permutations and negative β-shifts

In this section, I report recent results obtained by Steiner and myself [CS]. Equivalent results were obtained simultaneously by Elizalde and Moore [EM].

Let $\beta > 1$. Here we are interested in the $(-\beta)$-transformation $T_{-\beta} \colon (0,1] \to (0,1], x \mapsto \lfloor \beta x \rfloor + 1 - \beta x$. This maps is a generalization of T_β in the following sense: $T_{-\beta}(x) = \{-\beta x\}$, except for the (finitely many) following values of x: $\frac{1}{\beta}, \frac{2}{\beta}, \ldots, \frac{\lfloor \beta \rfloor}{\beta}$.

Again, instead of numbers $x \in (0,1]$, we will rather consider their $(-\beta)$-*expansions* [IS09,Ste13]:

$$x = -\sum_{k=1}^{\infty} \frac{d_{-\beta,k}(x)+1}{(-\beta)^k} \text{ with } d_{-\beta,k}(x) = \lfloor \beta\, T_{-\beta}^{k-1}(x) \rfloor.$$

Set $d_{-\beta}(x) = d_{-\beta,1}(x) d_{-\beta,2}(x) \cdots$. For all $x, y \in (0,1]$, we have $\sigma_{-\beta}(d_{-\beta}(x)) = d_{-\beta}(T_{-\beta}(x))$ and $x < y$ if and only if $d_{-\beta}(x) <_{\mathrm{alt}} d_{-\beta}(y)$. Here we use the *alternating lexicographical order* for sequences:

$$a_1 a_2 \cdots <_{\mathrm{alt}} b_1 b_2 \cdots \iff \exists i \geq 1,\ a_1 \cdots a_{i-1} = b_1 \cdots b_{i-1} \text{ and } \begin{cases} a_i < b_i & \text{if } i \text{ is odd,} \\ a_i < b_i & \text{if } i \text{ is even.} \end{cases}$$

The closure of the set of all $(-\beta)$-expansions $\{d_{-\beta}(x) : x \in (0,1]\}$ forms the $(-\beta)$-*shift*, which is denoted by $\Omega_{-\beta}$. The shift map $\sigma_{-\beta} \colon \Omega_{-\beta} \to \Omega_{-\beta},\ (a_m) \mapsto (a_{m+1})$ is continuous. For all $x \in (0,1]$, one has

$$\mathrm{Pat}(x, T_{-\beta}, n) = \mathrm{Pat}(d_{-\beta}(x), \sigma_{-\beta}, n),$$

with the alternating lexicographical order on the $(-\beta)$-shift. Therefore $\mathcal{A}(T_{-\beta}) = \mathcal{A}(\sigma_{-\beta})$. From [Ste13], we know that if $1 < \beta < \beta'$ then $d_{-\beta}(1) <_{\mathrm{alt}} d_{-\beta'}(1)$ and $\Omega_{-\beta} \subseteq \Omega_{-\beta'}$, whence $\mathcal{A}(T_{-\beta}) \subseteq \mathcal{A}(T_{-\beta'})$.

Similarly to (4), the *negative shift complexity* of a permutation $\pi \in \mathcal{S}_n$ is the quantity

$$B_-(\pi) = \inf\{\beta > 1 \colon \pi \in \mathcal{A}(T_{-\beta})\}.$$

Let φ be the substitution defined by $\varphi(\mathtt{0}) = \mathtt{1}$, $\varphi(\mathtt{1}) = \mathtt{100}$, with the unique fixed point $u = \varphi(u)$, i.e.,

$$u = \mathtt{10011100100100111001 1} \cdots .$$

If $\widetilde{a} = a$ and $a \leq u$, we set $b_-(a) = 1$. If $\widetilde{a} = a$ and $a >_{\mathrm{alt}} u$, then let $b_-(a)$ be the largest positive root of $1 + \sum_{j=1}^{\infty} (a_j + 1)(-x)^{-j}$ [EM]. If a is eventually periodic with preperiod of length q and period of length p, then $b_-(a)$ is the largest positive solution of

$$(-x)^{p+q} + \sum_{k=1}^{p+q} (a_k + 1)\,(-x)^{p+q-k} = (-x)^q + \sum_{k=1}^{q} (a_k + 1)\,(-x)^{q-k}.$$

Since we are dealing with an order different from the lexicographic order, the discussion from Section 2 about the first $n-1$ digits of any sequence realizing a given permutation has to be adapted (see the examples at the end of this section). We define $n-1$ digits $z_1 z_2 \cdots z_{n-1}$ by

$$\begin{aligned} z_j = \#\{1 \leq i < \pi(j) \colon & \text{ either } i \notin \{\pi(n)-1, \pi(n)\} \text{ and } \hat{\pi}(i) < \hat{\pi}(i+1), \\ & \text{ or } i = \pi(n)-1 \text{ and } \hat{\pi}(i) < \hat{\pi}(i+2)\} \end{aligned}$$

where it should be understood that z_j is really the digit corresponding to this number. So, roughly, we now have one new digit for each ascent in $\hat{\pi}$ where $\pi(1)$ is removed (see Theorem 13 below). Let m, ℓ as in (5) and

$$r = \pi^{-1}(\pi(n) + 1) \quad \text{if } \pi(n) \neq n.$$

When

$$z_{[\ell,n)} = z_{[r,n)} z_{[r,n)} \quad \text{or} \quad z_{[r,n)} = z_{[\ell,n)} z_{[\ell,n)}, \quad \text{if } \pi(n) \notin \{1, n\}, \tag{6}$$

we also use the following digits: for $0 \le i < |r - \ell|$, $1 \le j < n$,

$$z_j^{(i)} = z_j + \begin{cases} 1 & \text{if } \pi(j) \ge \pi(r+i) \text{ and } i \text{ is even, or } \pi(j) \ge \pi(\ell+i) \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise} \end{cases}$$

where, again, $z_j^{(i)}$ really is the digit corresponding to this number.

Theorem 11. *[CS,EM] Let $\pi \in \mathcal{S}_n$ and $\beta > 1$. Then $\pi \in \mathcal{A}(T_{-\beta}) \iff \beta > b_-(a)$ where*

$$a = \begin{cases} z_{[m,n)}\, \overline{z_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \neq 1, \text{ and (6) does not hold,} \\ \min_{0 \le i < |r-\ell|} z^{(i)}_{[m,n)}\, \overline{z^{(i)}_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \neq 1, \text{ and (6) holds,} \\ \overline{z_{[m,n)} 0} & \text{if } n-m \text{ is even and } \pi(n) = 1, \\ z_{[m,n)}\, \overline{z_{[r,n)}} & \text{if } n-m \text{ is odd and (6) does not hold,} \\ \min_{0 \le i < |r-\ell|} z^{(i)}_{[m,n)}\, \overline{z^{(i)}_{[r,n)}} & \text{if } n-m \text{ is odd and (6) holds.} \end{cases} \tag{7}$$

In particular $B_-(\pi) = b_-(a)$ and if $a >_{\text{alt}} u$, then $B_-(\pi)$ is a Perron number, i.e., an algebraic integer all of whose Galois conjugates α satisfying $|\alpha| < b_-(a)$.

Theorem 12. *[CS] Let $\pi \in \mathcal{S}_n$ and a as in (7). We have $B_-(\pi) = 1$ if and only if $a = \overline{\varphi^k(0)}$ for some $k \ge 0$.*

Theorem 13. *[CS,EM] Let $\pi \in \mathcal{S}_n$ and a as in (7). Then the minimal number of distinct symbols of a sequence w satisfying $\mathrm{Pat}(w, \sigma_{-\beta}, n) = \pi$ is*

$$N_-(\pi) = 1 + \lfloor B_-(\pi) \rfloor = 1 + \mathrm{asc}(\hat{\pi}) + \epsilon_-(\pi),$$

where $\mathrm{asc}(\hat{\pi})$ denotes the number of ascents in $\hat{\pi}$ with $\hat{\pi}(\pi(n)) = \pi(1)$ removed and

$$\epsilon_-(\pi) = \begin{cases} 1 & \text{if (6) holds or } a = \overline{\mathrm{asc}(\hat{\pi})0}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $N_-(\pi) \le n-1$ for all $\pi \in \mathcal{S}_n$, $n \ge 3$, with equality for $n \ge 4$ if and only if

$$\pi \in \{12 \cdots n,\ 12 \cdots (n-2)n(n-1),\ n(n-1) \cdots 1,\ n(n-1) \cdots 312\}.$$

Example 14.

1. Let $\pi = 3421$. Then $n = 4$, $\hat{\pi} = \underline{3}142$, $z_{[1,4)} = 110$, $m = 2$, $\pi(n) = 1$, $r = 3$. We obtain that $a = \overline{z_{[2,4)}0} = \overline{100} = \overline{\varphi^2(0)}$, thus $B_-(\pi) = b_-(a) = 1$. Indeed, we have $\mathrm{Pat}(1100\,\overline{10011}, \sigma_{-\beta}, n) = \pi$.

2. Let $\pi = 892364157$. Then $n = 9$, $\hat{\pi} = 536174\underline{8}92$, $z_{[1,9)} = \mathtt{33012102}$, $m = 2$, $\ell = 5$, $r = 1$, thus $a = z_{[2,9)}\,\overline{z_{[1,9)}} = \overline{\mathtt{30121023}}$, and $b_-(a)$ is the unique root $x > 1$ of
$$x^8 - 4x^7 + x^6 - 2x^5 + 3x^4 - 2x^3 + x^2 - 3x + 4 = 1.$$
We get $B_-(\pi) \approx 3.831$, and we have $\mathrm{Pat}(\mathtt{330121023}\,\overline{\mathtt{301210220}}, \sigma_{-\beta}, n) = \pi$.
3. Let $\pi = 453261$. Then $n = 6$, $\hat{\pi} = \underline{4}62531$, $z_{[1,6)} = \mathtt{11001}$, $m = 5$, $\pi(n) = 1$, $r = 4$, thus $a = z_5\,\overline{z_4 z_5} = \overline{\mathtt{10}}$, and $b_-(a) = 2$. We have $\mathrm{Pat}(\mathtt{110010}\,\overline{\mathtt{2}}, \sigma_{-\beta}, n) = \pi$.
4. Let $\pi = 7325416$. Then $n = 7$, $\hat{\pi} = 6521\underline{4}\underline{7}3$, $z_{[1,7)} = \mathtt{100100}$, $m = r = 1$, $\ell = 4$. Hence (6) holds, and $z^{(0)}_{[1,7)} = \mathtt{200100}$, $z^{(1)}_{[1,7)} = \mathtt{200210}$, $z^{(2)}_{[1,7)} = \mathtt{211210}$. Since $n - m$ is even, we have
$$a = \min_{i\in\{0,1,2\}} z^{(i)}_{[1,7)}\,\overline{z^{(i)}_{[4,7)}} = \min\{\mathtt{200}\,\overline{\mathtt{100}}, \mathtt{200}\,\overline{\mathtt{210}}, \mathtt{211}\,\overline{\mathtt{210}}\} = \mathtt{211}\,\overline{\mathtt{210}}.$$
Therefore, $B_-(\pi) \approx 2.343$ is the largest positive root of
$$\begin{aligned} 0 &= (x^6 - 3x^5 + 2x^4 - 2x^3 + 3x^2 - 2x + 1) - (-x^3 + 3x^2 - 2x + 2) \\ &= x^6 - 3x^5 + 2x^4 - x^3 - 1. \end{aligned}$$
We have $\mathrm{Pat}(\mathtt{211}(\mathtt{210})^{2k+2}\,\overline{\mathtt{2}}, \sigma_{-\beta}, n) = \pi$ for $k \geq 0$.

5 Comparing the positive and negative β-shifts

In Table 1, we give the values of the shift complexity $B(\pi)$ for all permutations of length up to 4, and we compare them with the values obtained by [Eli11] for the positive β-shift. Here $B(\pi)$ has to be understood as $B_-(\pi)$ or $B_+(\pi)$ accordingly. Note that much more permutations satisfy $B_-(\pi) = 1$ for the negative β-shift than $B_+(\pi) = 1$ for the positive one.

6 Open problems

Let me conclude with two open problems.

- Count all permutations with $B_-(\pi) \leq N$ or $B_-(\pi) < N$, in particular with $B_-(\pi) = 1$. From Theorem 1 we know that
$$\lim_{n\to\infty} \frac{1}{n} \log \#\{\pi \in \mathcal{S}_n : B_-(\pi) < \beta\} = \lim_{n\to\infty} \frac{1}{n} \log \#\{\pi \in \mathcal{S}_n : B_-(\pi) \leq \beta\} = \log \beta$$
What are the precise asymptotics of
$$c_n = \#\{\pi \in \mathcal{S}_n : B_-(\pi) = 1\}?$$
The first values are given by $(c_n)_{2\leq n\leq 9} = 2, 5, 12, 19, 34, 57, 82, 115$.
- Describe the permutations given by the transformations
$$T_{\beta,\alpha} : [0,1) \to [0,1),\ x \mapsto \beta x + \alpha - \lfloor \beta x + \alpha \rfloor.$$

$B(\pi)$	root of	π, negative β-shift	π, positive β-shift
1	$\beta - 1$	$12, 21$ $123, 132, 213, 231, 321$ $1324, 1342, 1432, 2134, 2143, 2314$ $2431, 3142, 3214, 3241, 3421, 4213$	$12, 21$ $123, 231, 312$ $1234, 2341, 3412, 4123$
1.465	$\beta^3 - \beta^2 - 1$		$1342, 2413, 3124, 4231$
1.618	$\beta^2 - \beta - 1$	312 $1423, 3412, 4231$	$132, 213, 321$ $1243, 1324, 2431, 3142, 4312$
1.755	$\beta^3 - 2\beta^2 + \beta - 1$	$2341, 2413, 3124, 4123$	
1.802	$\beta^3 - 2\beta^2 - 2\beta + 1$		4213
1.839	$\beta^3 - \beta^2 - \beta - 1$	4132	$1432, 2143, 3214, 4321$
2	$\beta - 2$	1234, 1243	$2134, 3241$
2.247	$\beta^3 - 2\beta^2 - \beta + 1$	4321	4132
2.414	$\beta^2 - 2\beta - 1$		$2314, 3421$
2.618	$\beta^2 - 3\beta + 1$		1423
2.732	$\beta^2 - 2\beta - 2$	4312	

Table 1. $B(\pi)$ for the $(-\beta)$-shift and the β-shift, for all permutations of length up to 4.

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