Semantics of Collinearity Among Regions

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Abstract. Collinearity is a basic arrangement of regions in the plane. We investigate the semantics of collinearity in various possible meanings for three regions and we combine these concepts to obtain definitions for four and more regions. The aim of the paper is to support the formalization of projective properties for modelling geographic information and qualitative spatial reasoning. Exploring the semantics of collinearity will enable us to shed light on elementary projective properties from which all the others can be inferred. Collinearity is also used to find a qualitative classification of the arrangement of many regions in the plane.

1. Introduction

Enriching the semantics of geographic information is a challenge that cannot avoid coping with the meaning of basic geometric concepts. Referring to the subdivision of geometric properties into topological, projective, and metric [3], we undertake the study of a basic projective property, which is collinearity among spatial objects. Collinearity is an elementary geometric notion among points and it is a very interesting issue how to extend this concept to two-dimensional regions.

The importance of semantics of collinearity relies on the fact that modelling all projective properties of spatial data can be done as a direct extension of such a property [1]. Notably, a model for representing the projective relations among spatial objects, the 5-intersection model, has been derived from the concept of collinearity among regions [2]. Most work on projective relations deals with point abstractions of spatial features and limited work has been devoted to extended objects [6, 7, 10]. In [4], the authors use spheres surrounding the objects to take into account the shape of objects in relative orientation. Early work on projective relations such as "between" was developed by [5].

Some interesting studies from theories of perception give support to our claim that collinearity is a basic property from which other properties may be derived [8]. "Emergent features" are visual properties possessed by configurations that are not present in the component elements of those same configurations. Examples of emergent features made up of two objects are proximity and orientation, emergent features made up of three objects are collinearity and symmetry, emergent features of four objects are surroundedness. Adding more objects, there are no further emergent features [9].

In Section 2, we describe the semantics of collinearity among three regions as an extension of the collinearity among points and we discuss the relevant formal properties. In section 3, we extend the concept of collinearity to four and more regions by taking two alternative ways, called step-wise collinearity and n-ary collinearity. In Section 4, we show that the collinearity of three regions is a concept that can be used to build a qualitative description about the arrangement of various regions in the plane. In Section 5, we draw short conclusions.

2. Semantics of collinearity among three regions

Collinearity among points is an elementary concept of projective geometry. At least three points are needed to define collinearity, and therefore it is intrinsically a ternary relation. Three points *x*,*y*,*z* are said to be *collinear* if they lie on the same line; we write *coll*(*x*,*y*,*z*) in the rest of the paper.

Two groups of properties of the *collinear* relation that are important to discuss for extending the concept to regions are symmetry and transitivity. By symmetry, we mean that we can exchange the order of arguments in the relation. Symmetry can be expressed by the following equations:

$\forall xyz \in R^2$, $coll(x, y, z) \Rightarrow coll(x, z, y)$, $coll(x, y, z) \Rightarrow coll(z, x, y)$.

By transitivity, we mean that given four points and two *collinear* relations holding among them, we can infer collinearity for any triplet of points out of that set of four points; we write:

$\forall xyzt \in R^2$, $coll(x, y, z) \wedge coll(y, z, t) \Rightarrow coll(x, z, t)$.

The extension of the collinearity relation among points to regions is rather complex. Indeed, its generalisation leads to different definitions of collinearity among regions. Basically, the collinearity relation is applied to points belonging to the three regions, but differences occur when one considers all the points of a region or just some of them. By different combinations of universal and existential quantifiers, we obtain eight different definitions that we call *collinear_1*, *collinear_2*, etc.; given three simple regions $A,B,C \in R^2$:

1. *coll_1*(*A*,*B*,*C*) $\equiv_{def}$ $\exists x \in A$ [$\exists y \in B$ [$\exists z \in C$ [*coll*(*x*,*y*,*z*)]]];
2. *coll_2*(*A*,*B*,*C*) $\equiv_{def}$ $\forall x \in A$ [$\exists y \in B$ [$\exists z \in C$ [*coll*(*x*,*y*,*z*)]]];
3. *coll_3*(*A*,*B*,*C*) $\equiv_{def}$ $\exists x \in A$ [$\forall y \in B$ [$\exists z \in C$ [*coll*(*x*,*y*,*z*)]]];
4. *coll_4*(*A*,*B*,*C*) $\equiv_{def}$ $\exists x \in A$ [$\exists y \in B$ [$\forall z \in C$ [*coll*(*x*,*y*,*z*)]]];
5. *coll_5*(*A*,*B*,*C*) $\equiv_{def}$ $\forall x \in A$ [$\forall y \in B$ [$\exists z \in C$ [*coll*(*x*,*y*,*z*)]]];
6. *coll_6*(*A*,*B*,*C*) $\equiv_{def}$ $\forall x \in A$ [$\exists y \in B$ [$\forall z \in C$ [*coll*(*x*,*y*,*z*)]]];
7. *coll_7*(*A*,*B*,*C*) $\equiv_{def}$ $\exists x \in A$ [$\forall y \in B$ [$\forall z \in C$ [*coll*(*x*,*y*,*z*)]]];
8. *coll_8*(*A*,*B*,*C*) $\equiv_{def}$ $\forall x \in A$ [$\forall y \in B$ [$\forall z \in C$ [*coll*(*x*,*y*,*z*)]]].

In the above definitions, the ordering of the quantifiers is the same as the ordering of variables in *coll*(*x*,*y*,*z*): we could consider other variants of these relations by exchanging the ordering of variables in *coll*(*x*,*y*,*z*), but they would not be significant since the collinear relation among points is symmetric. All the collinearity relations among regions can be hierarchically structured (Fig. 1); *Collinear_1* is at the top of the structure, all of the other cases are specialisations of it. This relation allows for weaker constraints on the arrangement of three regions. At the opposite, *collinear_8* represent the strongest notion of collinearity. Between them, we have the other relations ruled by different levels of dependency, such as:

$coll\_2(A, B, C) \Rightarrow coll\_1(A, B, C)$

$coll\_5(A, B, C) \Rightarrow coll\_2(A, B, C) \wedge coll\_3(A, B, C)$

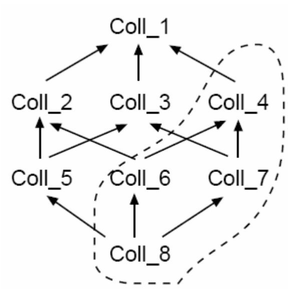


Fig. 1. Collinear relations' structure

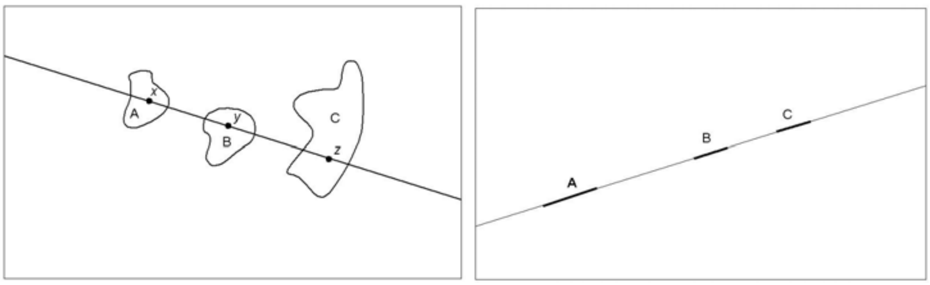
Relations *collinear_4*, *collinear_6*, *collinear_7*, and *collinear_8* can be only applied to degenerate cases where at least one region is a segment or a point. Only *collinear_8* is transitive, and only *collinear_1* and *collinear_ 8* are symmetric.

Collinear_1 relation

This relation is the easiest one to understand and surely the most useful. In short, three regions *A*, *B* and *C* are collinear if it exists at least one common line intersecting them (Fig. 2.a). This relation is *symmetric*, which means that the relation remains true under any permutation of the arguments. This can be expressed by the following relationships:

$coll\_1(A, B, C) \Rightarrow coll\_1(A, C, B)$;

$coll\_1(A, B, C) \Rightarrow coll\_1(B, A, C)$.



a. *collinear_1* b. *collinear_8*

Fig. 2. Symmetric collinear relations

Collinear_8 relation

The *collinear_8* relation is an extreme case which works only for degenerate regions (points or lines). It implies that each 3-tuple of points of *A*, *B* and *C* are collinear. It is only possible if the regions are collinear points or collinear lines (or a combination of both) (figure 2.b).

The remaining 6 relations are not symmetric, it means that a relation would not be necessarily maintained after permutation of the order of the regions in the relation. It is therefore necessary to consider a primary object, region *A*, for whom the relation stands and two reference objects, regions *B* and *C*.

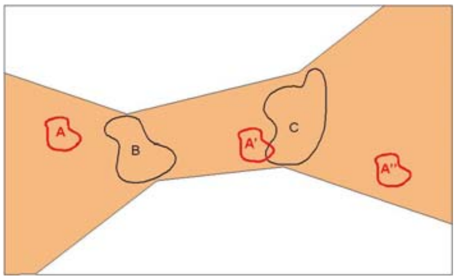


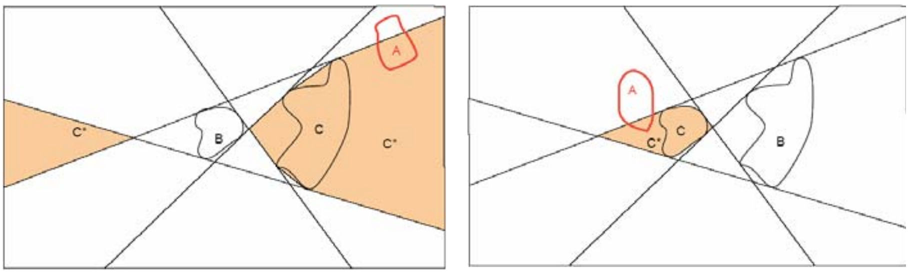
Fig. 3. *Collinear_2*

Collinear_2 relation

This relation means that every points of the primary object *A* have to be collinear with at least two points of *B* and *C*. It is said to be *partially symmetric*, which means that reference objects *B* and *C* can be exchanged:

$coll\_2(A,B,C) \Leftrightarrow coll\_2(A,C,B)$.

This relation has already been developed in previous work, and is the basis of the 5-intersection model [2]. At this stage, one can introduce the concept of *collinearity zone*: a *collinearity_2 zone* is the part of the plane where the relation *coll_2(A,B,C)* is true for any region *A* entirely contained in it. The *collinearity_2 zone* can be built using external and internal tangents of the regions *B* and *C* [2]. The concept is illustrated in Fig. 3.



a. b.

Fig. 4. *Collinear_3*. The zone *C** can be unbounded and separated in two parts (a) or bounded (b)

Collinear_ 3 relation

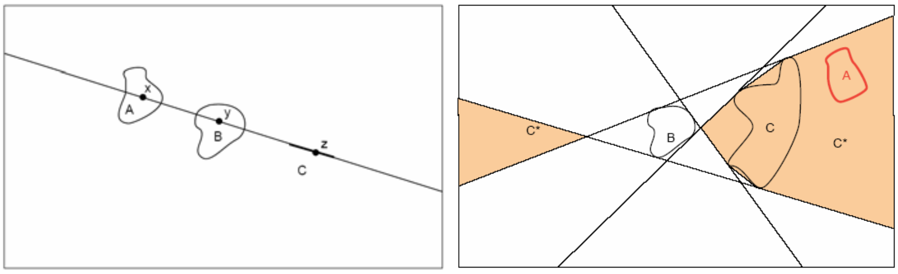
Let us call *C** a zone including *C* and bounded by the external and the internal tangents like illustrated in Fig. 4. The relation *coll_3* (*A*,*B*,*C*) is true if region *A* is at least partially contained in *C**. Depending on the relative size and shape of *B* and *C*, the zone *C** can be bounded or unbounded.

Collinear_ 4 relation

The *collinear_4* relation is illustrated in Fig. 5.a. This relation is verified only if the object *C* is a segment contained in a line that intersects both *A* and *B*.

Collinear_5 relation

Considering the zone *C**, as defined for *collinear_3*, the relation *coll_5(A,B,C)* is true if region *A* is entirely contained in *C** (Fig. 5.b).

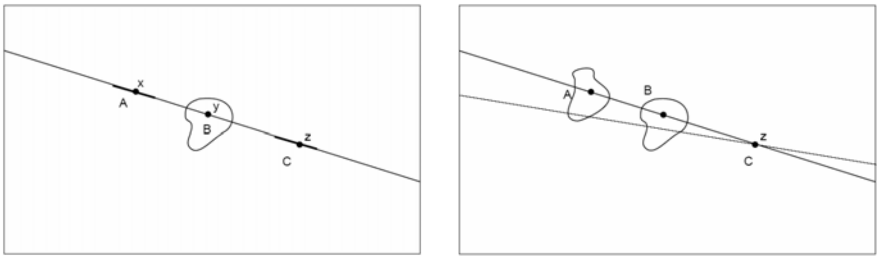


a. *collinear_4* b. *collinear_5*

Fig. 5. *Collinear_4* and *collinear_5*

Collinear_ 6 relation

The *collinear_6* relation is illustrated in Fig. 6. This relation is verified only in two cases: if objects *A* and *C* are segments belonging to a common line that is also intersecting *B* (Fig. 6.a); if *C* is a point and all lines passing through *A* and *C* intersect *B* (Fig. 6.b).

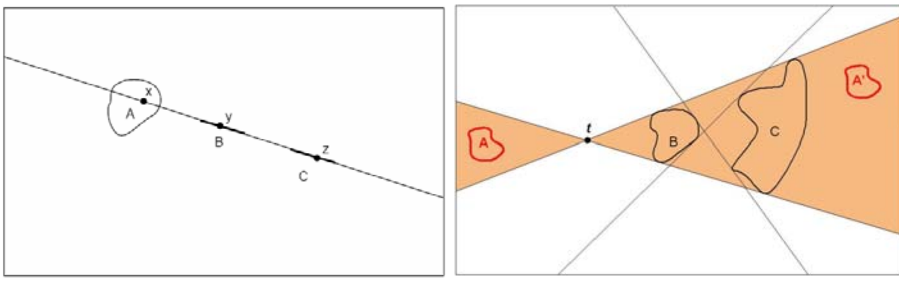


a. b.

Fig. 6. *Collinear_6*

Collinear 7 relation

Collinear_7 relation is partially symmetric. This relation is true only if *B* and *C* are two segments contained in a common line that intersects *A* (Fig. 7.a).



a. *collinear_7* b. another collinear relation

Fig. 7. *Collinear_7* and another collinear relation

One may argue that other definitions of collinearity could be proposed. For example, one could define a *collinearity zone* bounded by the external tangents of *B* and *C* as it is illustrated in Fig. 7.b. However, such a relation can be expressed using previous definitions; in this particular case, it is a combination of *collinear_5* and the convex hull of regions *B* and *C*:

$$coll\_9(A,B,C) \Leftarrow coll\_5(A,B,C) \vee coll\_5(A,C,B) \vee (A \subset CH(B \cup C)) .$$

Collinear_1 and *collinear_2* seem to be the most suitable definitions for expressing collinearity among three regions; *collinear_1* is the only usable symmetric case, and *collinear_2* is the most intuitive among the non-symmetric cases. Furthermore, when considering reference objects for *collinear_1*, one can see that the two relations share some common geometries; *coll_1 (A,B,C)* is true for any region *A* at least partially contained in the *collinearity_2 zone*.

As a twin concept of collinearity, one can define the fact of "being aside" by the negation of being collinear. We adopt the following definition:

$$aside(A,B,C) \Leftarrow \neg coll\_1(A,B,C) .$$

The part of the plane where a region *A* satisfies the *aside* relation corresponds to the complement of the *collinearity_2 zone* and may be called the *aside zone*.

3. Semantics of collinearity of four and more regions

We have two ways of extending the concept of collinearity to four and more regions. In the first way, given n regions *A*, *B*, *C*, *D*, *E*, …, we apply the already discussed definition of collinearity to groups of three regions taken in a given sequence: we call it *step-wise collinearity*. In the second way, we redefine collinearity directly from the concept of collinearity among n points: we call it *n-ary collinearity*.

Note that in the case of points, the two ways are equivalent since the transitive property holds for the relation collinear among three points; *n* points are collinear if there exists a single line that contains them all. If n points *x, y, z, t,* …, are collinear, we write *coll (x,y,z,t,…)*.

3.1. Step-wise collinearity

Step-wise collinearity can be defined for four and more regions for all the eight kinds of collinearity of Section 2. Given a sequence of regions *A*, *B*, *C*, *D*, *E*, … we have the following definitions for relations *collinear_1* and *collinear_2*:

$$sw\_coll\_1(A,B,C,D,E,...) \Leftarrow coll\_1(A,B,C) \wedge coll\_1(B,C,D) \wedge coll\_1(C,D,E) \wedge ...$$

$$sw\_coll\_2(A,B,C,D,E,...) \Leftarrow coll\_2(A,B,C) \wedge coll\_2(B,C,D) \wedge coll\_2(C,D,E) \wedge ...$$

The step-wise collinearity for the other kinds of collinearity is defined analogously, but it doesn't lead to interesting results. The relation *sw_collinear_1* as defined above leads to a too much weak form of collinearity among n regions. The relation *sw_collinear_2* is more interesting (see Fig. 8): it imposes a "local" collinearity for each triplet, but the global arrangement may be curvilinear and could end up also in a circular one by adding more regions to the sequence.

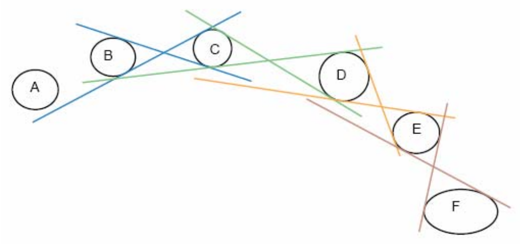


Fig. 8. *sw_collinear_2*

3.2. n-ary collinearity

A more sophisticated way of defining collinearity for four and more regions is to pick up different combinations of existential and universal quantifiers for points of every region. In this way, we could theoretically find quite a number of definitions, but we restrict ourselves to the following two:

$coll\_1(A,B,C,D,E,...) \Leftarrow \exists x \in A, \exists y \in B, \exists z \in C, \exists t \in D, \exists u \in E, ...$, such that $coll(x,y,z,t,u,...)$

$coll\_2(A,B,C,D,E,...) \Leftarrow \forall x \in A, \exists y \in B, \exists z \in C, \exists t \in D, \exists u \in E, ...$, such that $coll(x,y,z,t,u,...)$

The relation *collinear_1* among many regions is symmetric in the same sense of relation *collinear_1* for three regions: the order in which the regions are considered does not affect the relation. An illustration of *collinear_1* is given in Fig. 9. The relation *aside* defined as the negation of being *collinear_1* stands for more than three objects.

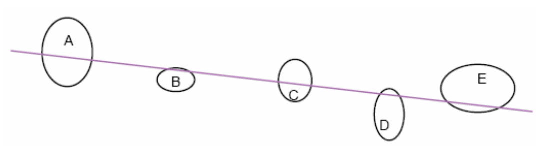


Fig. 9. n-ary *collinear_1*

The relation *collinear_2* among many regions is partially symmetric: we distinguish a primary object *A* and reference objects *B*, *C*, *D*, *E*, … but the reference objects only can be exchanged among themselves without affecting the relation *collinear_2*. It is possible to define a *collinearity zone* made up by the reference regions, which is the lieu of points a primary object can occupy to have the relation *collinear_2* satisfied. Geometrically, the *collinearity zone* can be obtained with the set intersection of all *collinearity zones* of all triplets of reference regions. This is illustrated in Fig. 10 for the *collinearity zone* formed by the regions *B*, *C*, and *D*, and in Fig. 11, where another region *E* is added to them. It is interesting to note that the *collinearity zone* tends to be narrower, when more reference regions are added. Therefore, the concept of collinearity we obtain put more constraints on the global arrangement of regions if they grow in number. The relation *collinear_2* implies the relation *collinear_1*, as it was for relations among three regions.

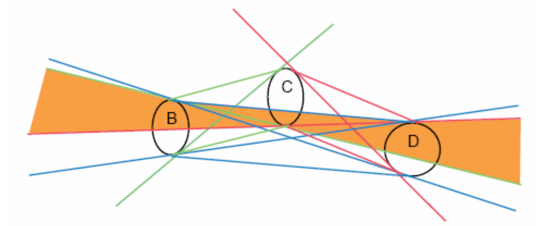


Fig. 10. n-ary *collinear_2* (with 3 reference objects)

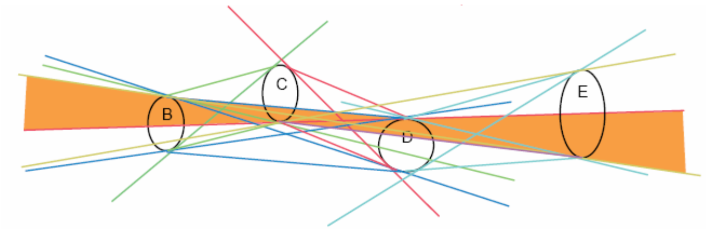


Fig. 11. n-ary *collinear_2* (with 4 reference objects)

4. Categorizing configurations using relation collinear_1

Collinearity is a high-level primitive concept that can be used to formulate a qualitative description of the configuration of many regions in the plane. Such a description highlights what the relative position of regions is alike and can give information on the global arrangement of regions.

We consider the primitive relation *collinear_1* and its negation *aside*. For three regions, we can distinguish between two configurations: in one configuration, the three regions are *collinear_1* (Fig. 12.a), while in the other configuration the three regions are *aside* (Fig. 12.b). For four regions, the relation *collinear_1* can be checked on various combinations of three regions obtaining a range of five different cases. The two extremes of this range are made up by the cases where all possible triplets of regions are *aside* (Fig. 12.c) and where all of them are *collinear_1* (Fig. 12.g). Then, there are intermediate cases where three triplets are *aside* and one of them is *collinear_1* (Fig. 12.d), two triplets are *aside* and two of them are *collinear_1* (Fig. 12.e) and one triplet is *aside* and three of them are *collinear_1* (Fig. 12.f).

Extending the counting of configurations to n regions in general, we can say that the number of different triplets corresponds to $k=\binom{n}{3}$. The number of possible relations *collinear_1* or *aside* for these triplets is 2^k: by grouping them in such a way the number of triplets being *collinear_1* is the same, we can distinguish *k*+1 different configurations. Such a number can be used as a rough measure of the "amount of collinearity" of a certain configuration of regions, ranging from 0, where all regions are *aside*, to *k*, where all regions are *collinear_1*. An intermediate value would state an intermediate degree of collinearity inside the configuration.

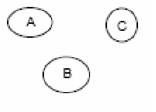
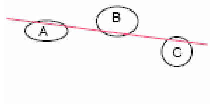
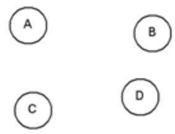
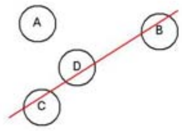
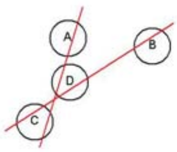
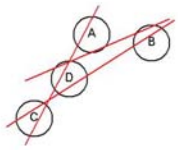
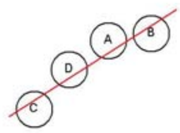
			
a. aside(A,B,C)	b. coll_1(A,B,C)	c. aside(A,B,C) ∧ aside(B,C,D) ∧ aside(C,D,A) ∧ aside(A,B,D)	d. aside(A,B,C) ∧ coll_1(B,C,D) ∧ aside(C,D,A) ∧ aside(A,B,D)
			
e. aside(A,B,C) ∧ coll_1(B,C,D) ∧ coll_1(C,D,A) ∧ aside(A,B,D)	f. aside(A,B,C) ∧ coll_1(B,C,D) ∧ coll_1(C,D,A) ∧ coll_1(A,B,D)	g. coll_1(A,B,C) ∧ coll_1(B,C,D) ∧ coll_1(C,D,A) ∧ coll_1(A,B,D)	

Fig. 12. Categorizing configurations using primitive *collinear_1*

5. Conclusions

While projective geometry provides a precise description of an arrangement of points in the plane, it is not evident how a similar qualitative description can be obtained for objects having a two-dimensional extension. A fundamental step in this process is exploring the meaning of three regions being collinear. Such a concept must be intrinsically an approximation since three regions cannot be collinear in a strict sense having different sizes and shapes. We found different ways of extending the concept from points to regions leading to a hierarchy of collinear relations, from the most permissive to the most stringent, and we commented their formal properties.

As a second step, we explored ways of combining the collinear relations among three regions to describe collinearity among many regions. This step is not obvious, since collinearity among regions loses the transitivity property, which in the case of points makes possible the extension from three to many points. We distinguished two categories of collinearity for many regions: the step-wise collinearity and the n-ary collinearity.

Finally, we used the concept of collinearity as a basic criterion to build a qualitative description of the arrangements of many regions in the plane: the number of collinear relations among triplets of regions expresses a measure of the collinearity of all the regions. In essence, a low number of collinear relations indicates an "encircling" arrangement of regions, while a high number of collinear relations indicates a tendency that the regions are located along the same line.

Acknowledgements

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