Multifractal analysis of the divergence of wavelet series

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Introduction

Let us denote $\mathbb{T} = \mathbb{R} / \mathbb{Z}$. We are interested in the pointwise convergence of the Fourier partial sums

$$S_n f : x \mapsto \sum_{k=-n}^{n} < f, e_k > e_k(x) \quad \text{where} \quad e_k : x \mapsto e^{2i\pi kx}$$

- Du Bois Reymond (1873) : There is $f \in \mathcal{C}(\mathbb{T})$ such that $S_n f(x)$ diverges at 0
- Kolmogorov (1926) : There is $f \in L^1(\mathbb{T})$ such that $S_n f(x)$ diverges at every x
- Kahane et Katznelson (1966) : If $A \subset \mathbb{T}$ is a F_σ of Lebesgue measure zero, there is $f \in \mathcal{C}(\mathbb{T})$ such that $S_n f(x)$ diverges at every $x \in A$
- Carleson et Hunt (1967) : If $f \in L^p(\mathbb{T})$ $(1 < p < +\infty)$, $S_n f$ converges almost everywhere.

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- Carleson et Hunt (1967) : If $f \in L^p(\mathbb{T})$ ($1 < p < +\infty$), $S_n f$ converges almost everywhere.

Question.
Let x be a divergent point of the Fourier series of $f \in L^p(\mathbb{T})$. Characterization of the divergence rate ? What about the size of the set of the points with a given divergence rate ?

Hausdorff dimension

Let $B \subset \mathbb{R}^n$ and $s > 0$. We set

$$\mathcal{H}^s_\delta(B) = \inf \left\{ \sum_{j \in \mathbb{N}} \operatorname{diam}(B_j)^s \ : \ (B_j)_{j \in \mathbb{N}} \ \delta-\text{covering of } B \right\}.$$

and we define the s-dimensional Hausdorff outer measure $\mathcal{H}^s$ by

$$\mathcal{H}^s(B) = \sup_{\delta > 0} \mathcal{H}^s_\delta(B) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(B)$$

There is a critical value of s for which the graph of $s \mapsto \mathcal{H}^s(B)$ “jumps” from $+\infty$ to 0. This critical value is called the Hausdorff dimension $\dim_{\mathcal{H}}(B)$ of B :

$$\boxed{\dim_{\mathcal{H}}(B) = \sup\{s \geq 0 : \mathcal{H}^s(B) = +\infty\}}$$

Divergence of Fourier series

Nikolsky's inequality. If $f \in L^p(\mathbb{T})$,

$$\|S_n f\|_\infty \leq C_p n^{1/p} \|f\|_p$$

Question.
Let $\beta \in [0, 1/p]$. What can we say about the size of the set $\left\{x : |S_n f(x)| \approx n^\beta\right\}$?

Aubry (2006)

If $p > 1$ and $f \in L^p(\mathbb{T})$, then

$$\dim_{\mathcal{H}} \left\{x : \limsup_{n\to\infty} n^{-\beta} |S_n f(x)| > 0\right\} \leq 1 - \beta p, \quad \forall \beta \in [0, 1/p].$$

Moreover, if $\beta \in [0, 1/p]$ is fixed, this result is optimal : Given a set E such that $\dim_{\mathcal{H}} E < 1 - \beta p$, there is $f \in L^p(\mathbb{T})$ such that

$$\limsup_{n\to\infty} n^{-\beta} |S_n f(x)| = +\infty \quad \forall x \in E.$$

- Divergence index at x :

$$\beta_f(x) := \limsup_{n\to+\infty} \frac{\log |S_n f(x)|}{\log n}$$

- Level set : $E(\beta, f) := \{x : \beta_f(x) = \beta\}$
- Multifractal spectrum of the divergence : $\beta \mapsto \dim_{\mathcal{H}} E(\beta, f)$

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Bayart, Heurteaux (2011)

Quasi-all (in the sense of Baire category theorem) function $f \in L^p(\mathbb{T})$ satisfies

$$\dim_{\mathcal{H}} E(\beta, f) = 1 - \beta p, \quad \forall \beta \in [0, 1/p].$$

For these functions, one has in particular

$$\dim_{\mathcal{H}} \left\{x : \limsup_{n\to\infty} n^{-\beta}|S_n f(x)| > 0\right\} = 1 - \beta p, \quad \forall \beta \in [0, 1/p].$$

Wavelet basis

The Fourier series of a continuous function may diverge at some points. Is this property inherent to any orthogonal decomposition ?

Haar basis (1910). In this orthonormal basis of $L^2(\mathbb{R})$, the expansion of any continuous function converges uniformly on any compact.
This basis is given by

$$\begin{cases} \varphi_k : x \mapsto \varphi(x-k), & k \in \mathbb{Z} \\ \psi_{j,k} : x \mapsto 2^{j/2}\psi(2^j x - k), & j \in \mathbb{N}, k \in \mathbb{Z} \end{cases}$$

where $\varphi = \mathbf{1}_{[0,1[}$ and $\psi = \mathbf{1}_{[0,\frac{1}{2}[} - \mathbf{1}_{[\frac{1}{2},1[}$

$\longrightarrow$ Prototype of wavelet basis. Uniform convergence of the expansion of any continuous function on any compact set (Walter 1995)

Divergence of wavelet series

Wavelet basis of $L^2(\mathbb{R})$. Orthonormal basis of the form

$$\begin{cases} \varphi_k : x \mapsto \varphi(x-k), & k \in \mathbb{Z} \\ \psi_{j,k} : x \mapsto 2^{j/2}\psi(2^j x - k), & j \in \mathbb{N}, k \in \mathbb{Z} \end{cases}$$

Classical assumptions. The wavelet ψ is

- well localized : ψ is rapidly decreasing, i.e. for all $N \in \mathbb{N}$, there is $C_N > 0$ such that
$$|\psi(x)| \leq \frac{C_N}{(1+|x|)^N}, \quad \forall x \in \mathbb{R}$$
- oscillating : there is $M \in \mathbb{N}$ such that
$$\int_{\mathbb{R}} x^m \psi(x) dx = 0 \quad \forall m \in \{0, \dots, M-1\}.$$
- regular : ψ is at least piecewise continuous

We are interested in the pointwise convergence of the wavelet expansion of f

$$\sum_{k\in\mathbb{Z}} < f,\varphi_k > \varphi_k(x) + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} < f,\psi_{j,k} > \psi_{j,k}(x)$$

Remark. Unlike the Fourier series, there is no “natural order” for the wavelets. Consequently, the notion of pointwise convergence has no natural definition.

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- We study the pointwise convergence of

$$\sum_{k\in\mathbb{Z}} | < f, \varphi_k > \varphi_k(x)| + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} | < f, \psi_{j,k} > \psi_{j,k}(x)|$$

  which does not depend on the chosen order (similar behavior, unlike Fourier series)

- We consider the periodic case : An orthonormal wavelet basis of $L^2(\mathbb{T})$ is given by the constant function equal to 1 and the periodized wavelets

$$\Psi_{j,k} : x \mapsto \sum_{l\in\mathbb{Z}} \psi_{j,k}(x-l), \quad j \in \mathbb{N},\, k \in \{0, \ldots, 2^j - 1\}$$

Case of L^p spaces

Hölder's inqualities If $f \in L^p(\mathbb{T})$,

$$|< f, \Psi_{j,k} >| \leq C_\Psi 2^{j(\frac{1}{p}-\frac{1}{2})} \|f\|_p \implies \| < f, \Psi_{j,k} > \Psi_{j,k}\|_\infty \leq C 2^{\frac{j}{p}}$$

since $\|\Psi_{j,k}\|_\infty \leq 2^{-\frac{j}{2}}$.

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Aubry (2006)

If $p > 1$ and $f \in L^p(\mathbb{T})$, then for all $\beta \in [0, 1/p]$,

$$\dim_{\mathcal{H}} \left\{ x : \limsup_{J\to\infty} 2^{-\beta J} \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} | < f, \Psi_{j,k} > \Psi_{j,k}(x)| > 0 \right\} \leq 1 - \beta p$$

Conversely, if ψ is the Haar wavelet and if $\beta \in [0, 1/p]$ is fixed, given a set E such that $\dim_{\mathcal{H}} E < 1 - \beta p$, there exists $f \in L^p(\mathbb{T})$ such that

$$\limsup_{J\to\infty} 2^{-\beta J} \left| \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} < f, \Psi_{j,k} > \Psi_{j,k}(x) \right| = +\infty \quad \forall x \in E.$$

Multifractal analysis of the divergence

- Divergence rate at x :

$$\gamma_f(x) = \sup \left\{ \gamma : \exists C > 0, \exists (j_n, k_n), \, | < f, \psi_{j_n,k_n} > \psi_{j_n,k_n}(x)| \geq C 2^{\gamma j_n} \right\}$$

- Multifractal spectrum of the divergence :

$$D_f : \gamma \mapsto \dim_{\mathcal{H}} \left\{ x : \gamma_f(x) = \gamma \right\}$$

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Remarks.

- Since the wavelet is rapidly decreasing, we have

$$\gamma_f(x) = \limsup_{J \to \infty} \frac{\log\left(\sum_{j=0}^{J} \sum_{k \in \mathbb{Z}} | < f, \psi_{j,k} > \psi_{j,k}(x)|\right)}{\log 2^J}$$

- A divergence rate gives a divergence of the wavelet series only if it is positive !

Sobolev and Besov spaces

Sobolev spaces. If $p \geq 1$ and $s \in \mathbb{R}$,

$$\begin{aligned} L^{p,s} &:= \left\{ f \in L^p(\mathbb{R}) : \mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}f\big) \in L^p(\mathbb{R}) \right\} \\ &= \left\{ f \in L^p(\mathbb{R}) : D^k f \in L^p(\mathbb{R})\ \forall k \leq s \right\} \quad \text{if } s \in \mathbb{N} \end{aligned}$$

Besov spaces. If $p, q > 0$ and $s \in \mathbb{R}$,

$$B_p^{s,q} := \left\{ f \in L^p(\mathbb{R}) : \left(2^{sj} \|\mathcal{F}^{-1}\left(\phi_j \mathcal{F} f\right)\|_{L^p(\mathbb{R})}\right)_{j\in\mathbb{N}} \in l^q \right\}$$

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We will work in the Besov spaces. Thanks to the inclusions

$$B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty}$$

for all $p \geq 1$, $s \in \mathbb{R}$, we will get similar results in Sobolev spaces.

Besov spaces and wavelets

We use a L^∞ normalization of the wavelets, i.e. we set

$$\varphi_k(x) := \varphi(x-k) \quad \text{and} \quad \psi_{j,k}(x) := \psi(2^j x - k) \quad j \in \mathbb{N},\ k \in \mathbb{Z}$$

The wavelet coefficients of f are denoted

$$C_k := \int_{\mathbb{R}} f(x)\varphi_k(x)dx \quad \text{and} \quad c_{j,k} := 2^j \int_{\mathbb{R}} f(x)\psi_{j,k}(x)dx$$

Caracterization of Besov spaces. Let $s \in \mathbb{R}$ and $p, q > 0$. Then

$$f \in B_p^{s,q} \Longleftrightarrow \begin{cases} \left(\sum_{k\in\mathbb{Z}} \left| c_{j,k} 2^{(s-\frac{1}{p})j} \right|^p \right)^{1/p} = \varepsilon_j \quad \text{with} \quad \varepsilon_j \in l^q \\ \left(\sum_{k\in\mathbb{Z}} |C_k|^p \right)^{1/p} < +\infty \end{cases}$$

$$f \in B_p^{s,q} \Longrightarrow \left(\sum_{k\in\mathbb{Z}} \left| c_{j,k} 2^{(s-\frac{1}{p})j} \right|^p \right)^{1/p} = \varepsilon_j \quad \text{with} \quad \varepsilon_j \in l^q$$

In particular,

$$\exists C > 0 \ : \ \forall j \quad \sum_{k\in\mathbb{Z}} |c_{j,k} 2^{(s-\frac{1}{p})j}|^p \leq C \quad \Longrightarrow \boxed{|c_{j,k}| \leq C^{\frac{1}{p}} 2^{(\frac{1}{p}-s)j}}$$

$$f \in B_p^{s,q} \Longrightarrow \left(\sum_{k \in \mathbb{Z}} \left| c_{j,k} 2^{(s-\frac{1}{p})j} \right|^p \right)^{1/p} = \varepsilon_j \quad \text{with} \quad \varepsilon_j \in l^q$$

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Consequence. If $f \in B_p^{s,q}$, then

$$\boxed{\gamma_f(x) \leq \frac{1}{p} - s, \quad \forall x \in \mathbb{R}}$$

Remark. The interesting case is $s < \frac{1}{p}$.

Proposition

If $f \in B_p^{s,q}$, then for all $\gamma \in [-s, \frac{1}{p} - s]$, we have

$$\dim_{\mathcal{H}} \left\{x : \ \gamma_f(x) \geq \gamma\right\} \leq 1 - sp - \gamma p.$$

In particular, the divergence spectrum of f satisfies

$$D_f(\gamma) \leq 1 - sp - \gamma p.$$

Idea. We set

$$E_\gamma^\varepsilon := \limsup_{j \to +\infty} \bigcup_{k : |c_{j,k}| \geq 2^{\gamma j}} \left] k2^{-j} - 2^{(\varepsilon-1)j}, k2^{-j} + 2^{(\varepsilon-1)j} \right[$$

Since $\sum_k |c_{j,k} 2^{(s-\frac{1}{p})j}|^p \leq C$, one has $\#\left\{k : |c_{j,k}| \geq 2^{\gamma j}\right\} \leq C\ 2^{(1-sp-\gamma p)j}$ so that

$$\dim_{\mathcal{H}}(E_\gamma^\varepsilon) \leq \frac{1 - sp - \gamma p}{1 - \varepsilon}.$$

Let us show that

$$\boxed{x \notin E_\gamma^\varepsilon \Longrightarrow \gamma_f(x) \leq \gamma}$$

We wish to estimate $|c_{j,k}\psi_{j,k}(x)|$. Let us recall that

$$E_\gamma^\varepsilon = \limsup_{j\to+\infty} \bigcup_{k\in E_{j,\gamma}} \left] k2^{-j} - 2^{(\varepsilon-1)j}, k2^{-j} + 2^{(\varepsilon-1)j} \right[$$

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$$E^{\varepsilon}_{\gamma} = \limsup_{j\to+\infty} \bigcup_{k\in E_{j,\gamma}} \left]k2^{-j} - 2^{(\varepsilon-1)j}, k2^{-j} + 2^{(\varepsilon-1)j}\right[$$

1. If $|c_{j,k}| < 2^{\gamma j}$, then $|c_{j,k}\psi_{j,k}(x)| \leq 2^{\gamma j}$.
2. If $|c_{j,k}| \geq 2^{\gamma j}$, then from the fast decay of the wavelets,

$$\forall N,\ \exists C_N \text{ such that } \quad |\psi(2^j x - k)| \leq \frac{C_N}{(1+|2^j x - k|)^N}.$$

Since $x \notin E^{\varepsilon}_{\gamma}$ and $|c_{j,k}| \geq 2^{\gamma j}$, we have $|2^j x - k| \geq 2^{\varepsilon j}$ if $j \gg$ and therefore

$$|\psi(2^j x - k)| \leq C_N 2^{-\varepsilon N j}.$$

Let us recall that $|c_{j,k}| \leq C 2^{-(s-1/p)j}$, hence

$$|c_{j,k}\psi_{j,k}(x)| \leq C_N C 2^{-(s-1/p)j} 2^{-\varepsilon N j} \leq 2^{\gamma j} \text{ if } j \gg .$$

Optimality of the result

Theorem

Quasi-all fonction $f \in B^{s,q}_p$ satisfies

$$\gamma_f(x) \in \left[-s, \frac{1}{p} - s\right] \quad \forall x$$

and

$$D_f(\gamma) = \dim_{\mathcal{H}} \left\{x : \gamma_f(x) = \gamma\right\} = 1 - sp - \gamma p, \quad \forall \gamma \in \left[-s, \frac{1}{p} - s\right].$$

Steps of the proof.

1. Construction of a “saturation function” F_a
2. Construction of the dense G_δ set from F_a

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1. Construction of a "saturation function" F_a such that
   - $F_a \in B_p^{s,q}$ using the wavelet characterization of the Besov spaces
   - $\gamma_{F_a}(x) \in \left[-s, \frac{1}{p} - s\right]$ for all x : the wavelet does not vanish "too often"
   - $\mathcal{T}_\alpha \subseteq \left\{x : \gamma_{F_a}(x) \geq \frac{1}{p} - s - \frac{1}{\alpha p}\right\}$ where $\mathcal{T}_\alpha$ denotes the points α-aproximable by dyadic numbers with a condition of non-annulation of the wavelet
   - Using ubiquity techniques, $\dim_{\mathcal{H}} \mathcal{T}_\alpha = \frac{1}{\alpha}$
   - Deduce that $\dim_{\mathcal{H}} \left\{x : \gamma_{F_a}(x) = \frac{1}{p} - s - \frac{1}{\alpha p}\right\} = \frac{1}{\alpha}$, for all $\alpha \geq 1$

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2. Construction of the dense G_δ set :
   - The set $\{f_n : n \in \mathbb{N}\}$ of finite wavelet series with rational coefficients is dense in $B_p^{s,q}$
   - $g_n = f_n + \frac{1}{N_n} F_a$ has the same divergence properties than F_a and $\{g_n : n \in \mathbb{N}\}$ is still dense in $B_p^{s,q}$
   - We consider the dense G_δ set $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B(g_n, r_n)$, where $r_n = \frac{1}{2N_n^a} 2^{-\frac{N_n}{p}}$

Other results

- Similar results obtained with the notion of prevalence. The idea is to consider the coefficients

$$c_{j,k} = \frac{\xi_{j,k}}{j^a} 2^{(\frac{1}{p}-s)j} 2^{-\frac{1}{p}J}$$

  where $\xi_{j,k} \sim^{iid} \mathcal{N}(0,1)$

- Similar results obtained with the notion of lineability, considering the linear span of the functions F_a, $a > \frac{1}{p} + \frac{1}{q}$.

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Thank you for your attention !

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