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On Generalized Hölder-Zygmund Spaces

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Abstract

To study the regularity of functions, many functional spaces have been introduced during the 20th century. Among them, let us mention the Hölder-Zygmund spaces $\Lambda^\alpha(\mathbb{R}^d)$ and the Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ where $\alpha > 0$ somehow indicates the regularity of their elements ($p, q \in]0, +\infty]$). The Hölder-Zygmund spaces are particular cases of Besov spaces in the sense that $\Lambda^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$.

A generalization of Besov spaces has been introduced in the middle of the seventies and is still studied nowadays. This new type of space allows a deepest study of the regularity of functions (see e.g. [50]). In this thesis, we start from this generalization in order to introduce a generalization of Hölder-Zygmund spaces.

The first aim of this thesis is to show that most classical properties of Hölder-Zygmund spaces can be transposed to their generalized version. Among others, a complete characterization of these spaces in terms of wavelet coefficients is proved, which opens their use in the context of the signal analysis.

The second aim of this thesis is to introduce a generalized version of the pointwise Hölder spaces similarly to their global version. We then show that most properties of the global spaces can be transposed to their generalized pointwise version.

Finally, we study the regularity of some financial stochastic processes.

Résumé

Afin d'étudier la régularité des fonctions, divers espaces fonctionnels ont été introduits au XX$^{\text{ème}}$ siècle. Parmis eux, citons les espaces de Hölder-Zygmund $\Lambda^\alpha(\mathbb{R}^d)$ et les espaces de Besov $B^\alpha_{p,q}(\mathbb{R}^d)$, où $\alpha > 0$ indique d'une certaine manière la régularité de leurs éléments ($p, q \in]0, +\infty]$). Les espaces de Hölder-Zygmund sont des cas particuliers d'espaces de Besov, en ceci que $\Lambda^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$.

Une généralisation des espaces de Besov a été introduite durant les années 1970 et est encore étudiée aujourd'hui. Ce nouveau type d'espace permet une étude plus approfondie de la régularité (voir par ex. [50]). Dans cette thèse, nous partons de cette généralisation pour introduire une généralisation des espaces de Hölder-Zygmund.

Le premier objectif de cette thèse est de démontrer que la plupart des propriétés classiques des espaces de Hölder-Zygmund peuvent être transposées à leur version généralisée. Entre autre, une caractérisation complète de ces espaces en terme de coefficients en ondelettes est démontrée, ce qui ouvre leur utilisation dans le contexte de l'analyse du signal.

Le second objectif de cette thèse est d'introduire une version généralisée des espaces de Hölder ponctuels, de manière similaire à leur version globale. Nous démontrons ensuite que la plupart des propriétés et caractérisations des espaces globaux peuvent être transposées à leur version ponctuelle généralisée.

Diverses applications d'étude de régularité de processus stochastiques financiers seront entre autres illustrées.

Contents

Some basic notations

- $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ denotes the set of natural numbers.
- $\mathbb{N}^* = \{1, 2, 3, ...\}$.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^d = \{(x_1, ..., x_d) : x_i \in \mathbb{R} \quad \forall i \in \{1, ..., d\}\}$ denotes the d-dimensional euclidean space ($d \in \mathbb{N}^*$).
- $\mathbb{N}_0^d = \{(\alpha_1, ..., \alpha_d) : \alpha_i \in \mathbb{N}_0 \quad \forall i \in \{1, ..., d\}\}$ denotes the set of *multi-indices* ($d \in \mathbb{N}^*$).
- $B(x, R)$ denotes the open ball with center $x \in \mathbb{R}^d$ and radius $R > 0$.
- $B(x, \leq R)$ denotes the closed ball with center $x \in \mathbb{R}^d$ and radius $R > 0$.
- $\lfloor x \rfloor = \sup\{m \in \mathbb{Z} : m \leq x\}$ denotes the floor of $x \in \mathbb{R}$.
- $\lceil x \rceil = \inf\{m \in \mathbb{Z} : m \geq x\}$ denotes the ceiling of $x \in \mathbb{R}$.
- $(x)_+ = \max\{x, 0\}$ denotes the positive part of a real number $x \in \mathbb{R}$.
- $\alpha!$ denotes the value $\alpha! = \alpha_1!...\alpha_d!$ if $\alpha \in \mathbb{N}_0^d$ is a multi-index.
- $|\alpha|$ denotes the value $\alpha_1 + ... + \alpha_d$ if $\alpha \in \mathbb{N}_0^d$ is a multi-index.
- $\binom{m}{j} = \frac{m!}{(m-j)!j!}$ where m, $j \in \mathbb{N}_0$ and $m \geq j$.
- $C(A)$ denotes the space of continuous functions defined on $A \subseteq \mathbb{R}^d$.
- $C^p(\Omega)$ ($p \in \mathbb{N}_0 \cup \{\infty\}$) denotes the space of functions which are p-times continuously differentiable on Ω (where Ω is an open set of $\mathbb{R}^d$).
- $D(\Omega)$ denotes the subspace of $C^\infty(\Omega)$ made from compactly supported functions on $\Omega \subseteq \mathbb{R}^d$.
- $L^p(A)$ denotes the space of measurable functions on A satisfying $\|f\|_{L^p(A)} = \left(\int_A |f(x)|^p \, dx\right)^{1/p} < \infty$ (where $p \in]0, \infty[$ and A is a measurable set of $\mathbb{R}^d$).
- $L^\infty(A)$ denotes the space of measurable functions on A satisfying $\|f\|_{L^\infty(A)} = \sup_{ppA} |f| < \infty$ (where A is a measurable set of $\mathbb{R}^d$).
- $\|f\|_E = \sup_{x \in E} |f(x)|$ where f is a function defined on $E \subseteq \mathbb{R}^d$.

- $L^p = L^p(\mathbb{R}^d)$ if $p \in]0, \infty]$.
- $l^p = l^p(\mathbb{N}_0)$ denotes the space of sequences $(a_n)_{n\in\mathbb{N}_0}$ such that $\|(a_n)_{n\in\mathbb{N}_0}\|_{l^p} = (\sum_{n=0}^{\infty} |a_n|^p)^{1/p} < \infty$ ($p \in [0, \infty[$).
- $l^\infty = l^\infty(\mathbb{N}_0)$ denotes the space of sequences $(a_n)_{n\in\mathbb{N}_0}$ such that $\|(a_n)_{n\in\mathbb{N}_0}\|_{l^\infty} = \sup_{n\in\mathbb{N}_0} |a_n| < \infty$.
- $D'(\mathbb{R}^d)$ denotes the space of distributions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}$ denotes the Schwartz space, composed of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}'(\mathbb{R}^d) = \mathcal{S}'$ denotes the topological dual of the space $\mathcal{S}$, i.e. the space of all tempered distributions on $\mathbb{R}^d$.
- $\mathcal{F}f$ denotes the Fourier transform of the distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. If the function f belongs to $L^1(\mathbb{R}^d)$, this expression is equal to $\mathcal{F}f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx$.
- $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of the distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. If the function f belongs to $L^1(\mathbb{R}^d)$, this expression is equal to $\mathcal{F}^{-1}f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{+ix\xi} f(x)\, dx$.

Introduction

The Hölder condition has been first introduced by Otto Ludwig Hölder, and is more than 130 years old. It is formally defined as follows: a real or complex-valued function f satisfies a Hölder condition if there exist $C > 0$ and $\alpha \in]0,1[$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

for all x, y belonging to the domain of f. If such an inequality is satisfied, one says that f belongs to the Hölder space of index α. It is easy to define these spaces for $\alpha \geq 1$. They are used in many areas such as the theory of partial differential equation, harmonic analysis, stochastic differential equations and function spaces (see e.g. [4, 29, 48, 56, 118]). More recently, it gave birth to the multifractal analysis theory (see e.g. [13, 64, 67]).

The concept of Hölder-Zygmund space $\Lambda^\alpha(\mathbb{R}^d)$ ($\alpha > 0$) is directly linked with the Hölder condition and can be used to measure the global regularity through the parameter α. Let W_α be the Weierstraß function given by

$$W_\alpha(x) = \sum_{j=0}^{+\infty} 2^{-j\alpha} \cos(2^j \pi x),$$

where $\alpha \in]0,1[$. This function and the Brownian motion share a particular property: these functions are continuous but nowhere differentiable. A natural question that arises is how to determine which one of them is more regular. Using Hölder-Zygmund spaces, one can answer the question depending on the value of the parameter α. One can prove that the Weierstraß function is more regular than the Brownian motion if and only if $\alpha \geq 1/2$.

Hölder-Zygmund spaces are particular cases of Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ where $0 < p, q \leq +\infty$ and $\alpha > 0$ ([118]). Besov spaces were defined in 1959 by O. V. Besov in [18, 19]. A large number of references are now dedicated to them, see [20, 21, 117, 118, 119, 120, 121, 122]. These spaces were generalized in the middle of the seventies by several authors, with different starting points and in different contexts. They were first considered by the Russian school, and are still studied today in connection with embeddings, limiting embeddings, entropy numbers, probability theory and theory of stochastic processes (see e.g. [25, 46, 47, 85, 99]).

Meanwhile, S. Jaffard and Y. Meyer proposed a generalization of Hölder-Zygmund spaces in [65] for several purposes, such as Sobolev embeddings results, the study of regularity and multifractal purpose. This new type of Hölder-Zygmund spaces relies on the concept of modulus of continuity. Our approach is to propose a unified view. It consists in defining generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ as a particular case of generalized Besov spaces through the use of admissible sequences. This generalizes S. Jaffard's approach and helps to stand back from the different theories to obtain a better understanding of the underlying mechanisms behind the spaces. The generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ allow a more precise study of the regularity of a function, as illustrated by the following example. The Brownian motion is α-regular with $\alpha < \frac{1}{2}$, in the sense that it belongs to the space $\Lambda^{\alpha}(\mathbb{R}^d) = \Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ with $\sigma_j = 2^{-j\alpha}$. Using a result of A. Khintchine ([72]), this sequence can be replaced by $2^{-j/2}\sqrt{\log(j)}$, giving a better characterization of the regularity of the Brownian motion. The concepts of Besov spaces and Hölder-Zygmund spaces, as well as their generalized versions, are recalled in chapter 1.

As generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ measure the regularity of functions, it is natural to wonder what links these spaces and classical concepts of regularity, like $C^N(\mathbb{R}^d)$ spaces. For this purpose, the second chapter is dedicated to the study of the characterizations of the spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$. It is shown that the properties of classical Hölder-Zygmund spaces can be transposed to their generalized version. In particular, elements in the generalized Hölder-Zygmund spaces can be characterized through approximation with polynomials, smooth functions obtained from a convolution product, a Taylor decomposition formula, wavelet coefficients and Littlewood-Paley decomposition. We also show that those spaces can be obtained through a (generalized) real interpolation of Sobolev spaces. The results of this chapter have been published in [76, 77] (except theorem 115).

An important concept associated to Hölder spaces is the Hölder exponent of some function f, which measures the global regularity of f. It is defined as follows: if $f \in L^{\infty}(\mathbb{R}^d)$, the Hölder exponent of f is given by

$$H_f = \sup\{\alpha > 0 : f \in \Lambda^{\alpha}(\mathbb{R}^d)\}.$$

In chapter 3, we explain how to transpose this concept to the generalized Hölder spaces, so we can define a concept of generalized Hölder exponents. As opposed to Hölder spaces which allow to measure the regularity of functions through the Hölder exponent, the uniform irregular spaces somehow measure the irregularity of a function ([31, 32, 33]). We show in the same chapter that those spaces can be expressed in terms of generalized Hölder-Zygmund spaces. Finally, we discuss some considerations related to financial models. Only results of section 3.1 have been published in [76, 77].

Up to now, we have considered the study of global regularity through the generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$. There also exists a pointwise version of the Hölder-Zygmund spaces, which gives some indication about the regularity at a given point x_0. These spaces are denoted by $\Lambda^{\alpha}(x_0)$ and we can define a pointwise version of the Hölder exponent. It is formally defined as follows: if $f \in L^{\infty}_{loc}(\mathbb{R}^d)$, the Hölder exponent of f at x_0 is given by

$$h_f(x_0) = \sup\{\alpha > 0 : f \in \Lambda^{\alpha}(x_0)\}.$$

As Hölder-Zygmund spaces have been generalized through the use of admissible sequences, it is therefore natural to do the same for pointwise Hölder spaces. We introduce the generalized pointwise Hölder spaces in chapter 4 and we show that main properties of classical pointwise spaces are kept. In particular, we obtain a wavelet characterization of those spaces as well as a Taylor decomposition formula for their elements. Then, we study the pointwise and the global regularity of the Takagi function. We also show how to define a pointwise generalized Hölder exponent using these new spaces. Results of this chapter have been published in [78].

Several necessary theories and results are recalled in the appendix.

Chapter 1

Origins of the study of generalized Hölder-Zygmund spaces – State of the Art

1.1 Background: reminder of basic concepts

The Besov spaces are the central point of the study of this thesis. These spaces allow one to measure the smoothness of functions, and are a useful tool to solve partial differential equations. They were first introduced in the 1960's and are defined here below.

Let us introduce some notations linking spaces l^q and $L^p(\mathbb{R}^d)$. If $0 < p, q \leq \infty$ and $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of Lebesgue-measurable functions on $\mathbb{R}^d$, we set

$$\|(f_j)_{j\in\mathbb{N}_0}|l^q(L^p(\mathbb{R}^d))\| = \left(\sum_{j=0}^{\infty} \|f_j\|^q_{L^p(\mathbb{R}^d)}\right)^{1/q} \quad \text{if } q \neq \infty$$

and

$$\|(f_j)_{j\in\mathbb{N}_0}|l^\infty(L^p(\mathbb{R}^d))\| = \sup_{j\in\mathbb{N}_0} \|f_j\|_{L^p(\mathbb{R}^d)} \quad \text{else.}$$

To define Besov spaces, we need to introduce particular sequences of infinitely differentiable compactly supported functions which constitute a smooth resolution of unity.

Definition 1. Let $\Phi(\mathbb{R}^d)$ denote the set of sequences $(\varphi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying the following properties:

1. $\mathrm{supp}\varphi_0 \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq 2\}$;
2. $\mathrm{supp}\varphi_j \subseteq \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \qquad j \in \mathbb{N}^*$;
3. $\sup_{\xi\in\mathbb{R}^d} |D^\alpha \varphi_j(\xi)| \leq c_\alpha 2^{-j|\alpha|}, \qquad j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d$;
4. $\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^d$.

H. Triebel introduced the definition of Besov spaces in [118] as:

Definition 2 (Besov spaces)**.** Let $(\varphi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$ and $0 < p, q \leq \infty$, $s \in \mathbb{R}$. We define the *Besov space* $B^s_{p,q}(\mathbb{R}^d)$ by

$$B^s_{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,q}(\mathbb{R}^d)} := \|(2^{js}\mathcal{F}^{-1}(\varphi_j \mathcal{F} f))_{j\in\mathbb{N}_0} | l^q(L^p(\mathbb{R}^d))\| < \infty\}.$$

Remark 3.

1. This definition does not depend on the chosen sequence $(\varphi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$ (see e.g. [117, 118]).
2. The distributions $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ $(j \in \mathbb{N}_0)$ are associated to analytic functions defined on $\mathbb{R}^d$ by the Paley-Wiener theorem. So, they belong to $C^\infty(\mathbb{R}^d)$.
3. The spaces $B^s_{p,q}(\mathbb{R}^d)$ are quasi-Banach spaces and Banach spaces if $p, q \geq 1$ ([119]).

These spaces have many interesting properties, but they will not be discussed in further detail here. We refer to [43, 118, 119, 121] for a rather complete study.

Let us introduce Hölder spaces. We need to introduce the concept of finite difference. This concept is developed further in section 1.7 because it is one of the most important aspects of this thesis.

Definition 4. Let x, $h \in \mathbb{R}^d$ and a function f defined on $\mathbb{R}^d$. We define the *(forward) finite difference of order $m \in \mathbb{N}^*$ of f* by

$$\begin{aligned} \Delta^1_h f(x) &:= f(x+h) - f(x) \\ \Delta^m_h f(x) &:= \Delta^1_h \Delta^{m-1}_h f(x) \qquad (m \in \mathbb{N}^*). \end{aligned}$$

If the function f is continuously differentiable in a neighbourhood of $x \in \mathbb{R}^d$, we have (by definition)

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\Delta^1_{he_i} f(x)}{h} = D_{x_i} f(x).$$

More generally, we have the following result ([73], lemma 4.4). This result can be easily proved using the definition of a derivative and permuting and reducing all the limits to a single one, thanks to the regularity assumption.

Lemma 5. *If f is a n-times continuously differentiable function on a neighbourhood of $x \in \mathbb{R}^d$, then we have*

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}_0}} \frac{\Delta^n_{he_i} f(x)}{h^n} = D^n_{x_i} f(x)$$

The Hölder-Zygmund spaces can be interpreted as a continuum between the classical spaces $C^n(\mathbb{R}^d)$ of n-times continuously differentiable functions on $\mathbb{R}^d$. In this way, they allow a more precise measurement of the global regularity of functions.

Definition 6 (Hölder-Zygmund spaces)**.** Let $\alpha > 0$. We define the *Hölder-Zygmund space* $\Lambda^\alpha(\mathbb{R}^d)$ by

$$\Lambda^\alpha(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) : \sup_{j\in\mathbb{N}_0} 2^{j\alpha} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} < \infty\}. \tag{1.1}$$

The spaces $\Lambda^n(\mathbb{R}^d)$ do not coincide with the spaces $C^n(\mathbb{R}^d)$, but each function belonging to $\Lambda^{n+\varepsilon}(\mathbb{R}^d)$ with $\varepsilon > 0$ is n-times continuously differentiable ([73]). These spaces have many interesting properties. For example, these spaces can be characterized by Taylor decomposition of their elements and can even be characterized by wavelet coefficients[1] ([93]).

Remark 7.
1. Some authors ask the continuity of the elements of the Hölder spaces in the definition (see e.g. [73]). This is not necessary: indeed, we will see that functions of Hölder spaces can be modified on a set of null measure so that they become continuous (proposition 80).

2. These spaces are Banach spaces with the norm defined by

$$\|f\|_{\Lambda^\alpha(\mathbb{R}^d)} = \|f\|_{L^\infty} + \sup_{j\in\mathbb{N}_0} \sup_{0\neq|h|\leq 2^{-j}} \frac{\|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty}}{|h|^\alpha}$$

but they are not separable. They are indeed isomorphic to l^∞ ([118], p.87, Remark 1).

The Hölder spaces are closely linked to Besov spaces. Indeed, they are particular cases of Besov spaces as stated by the following result [118].

Theorem 8. *Let $\alpha > 0$. We have*

$$B^\alpha_{\infty,\infty}(\mathbb{R}^d) = \Lambda^\alpha(\mathbb{R}^d)$$

with equivalent norms.

To conclude this introduction to Besov and Hölder spaces, let us note that Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ in general can be expressed in terms of finite differences. This fact is expressed by the next result ([117]).

Theorem 9. *Let $0 < p, q \leq +\infty$, $\alpha > d(\frac{1}{\inf\{1,p\}} - 1)$ and $M \in \mathbb{N}_0$ such that $M > \alpha$. We have*

$$\begin{aligned} B^\alpha_{p,q}(\mathbb{R}^d) &= \left\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|^{(1),M}_{B^\alpha_{p,q}} < +\infty\right\} \\ &= \left\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|^{(2),M}_{B^\alpha_{p,q}} < +\infty\right\} \end{aligned}$$

[1] The concept of wavelet is developed further in section 5.4.

where

$$\|f\|^{(1),M}_{B^{\alpha}_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} |h|^{-\alpha q} \sup_{|y|\leq |h|} \|\Delta^M_y f\|^q_p \frac{dh}{|h|^d} \right)^{1/q}$$

and

$$\|f\|^{(2),M}_{B^{\alpha}_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} |h|^{-\alpha q} \|\Delta^M_h f\|^q_p \frac{dh}{|h|^d} \right)^{1/q}$$

are equivalent quasi-norms to $\|f\|_{B^{\alpha}_{p,q}}$ *(with the usual modification if* $q = \infty$*).*

1.2 An overview of admissible sequences

The core of generalized Besov and Hölder spaces relies on the notion of admissible sequence. We propose to study this concept separately before entering the definition of the generalized spaces.

Definition 10. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ of real positive numbers is called an *admissible sequence* if there exists two positive constants d_0 and d_1 such that

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \qquad j \in \mathbb{N}_0\,. \tag{1.2}$$

In the following, we will only consider admissible sequences which are not identically zero. This implies in particular that no element can be equal to 0.

For an admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$, let

$$\underline{\sigma}_j := \inf_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \qquad j \in \mathbb{N}_0\,.$$

The *lower and upper Boyd indices* are respectively defined by

$$\underline{s}(\sigma) := \lim_{j\to+\infty} \frac{\log_2(\underline{\sigma}_j)}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to+\infty} \frac{\log_2(\overline{\sigma}_j)}{j}.$$

Let us show that the two previous limits exist and are finite. For $\overline{s}(\sigma)$, this results from the fact that the sequence $(\log_2(\overline{\sigma}_j))_{j\in\mathbb{N}_0}$ is subadditive and from the following lemma (see lemma 189 in the appendix for a proof of this lemma).

Lemma 11 (Fekete (1923))**.** *For every subadditive sequence* $(a_n)_{n\in\mathbb{N}^*}$*, the limit* $\lim_{n\to+\infty} \frac{a_n}{n}$ *exists and is equal to* $\inf_{n\in\mathbb{N}^*} \frac{a_n}{n}$ *(the limit can be equal to* $-\infty$*).*

Indeed, if j, $l \in \mathbb{N}_0$ then

$$\frac{\sigma_{j+k+l}}{\sigma_k} = \frac{\sigma_{j+k+l}}{\sigma_{k+l}} \frac{\sigma_{k+l}}{\sigma_k} \qquad \forall k \in \mathbb{N}_0$$

which implies

$$\sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k+l}}{\sigma_k} \leq \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k+l}}{\sigma_{k+l}} \sup_{k\in\mathbb{N}_0} \frac{\sigma_{k+l}}{\sigma_k} \leq \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \sup_{k\in\mathbb{N}_0} \frac{\sigma_{k+l}}{\sigma_k}$$

hence the conclusion. The same proof can be used to show that $\underline{s}(\sigma)$ is well defined (we use the analogous result of lemma 11 for superadditive sequences).

The Boyd index $\overline{s}(\sigma)$ of an admissible sequence σ describes the asymptotic behaviour of $\overline{\sigma}_j$; similarly, the index $\underline{s}(\sigma)$ describes the asymptotic behaviour of $\underline{\sigma}_j$. We notice that for $\varepsilon > 0$, there exist two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1 2^{(\underline{s}(\sigma)-\varepsilon)j} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq c_2 2^{(\overline{s}(\sigma)+\varepsilon)j}, \qquad j,k \in \mathbb{N}_0\,. \tag{1.3}$$

Conversely, $\underline{s}(\sigma)$ and $\overline{s}(\sigma)$ are respectively the biggest and the lowest real numbers satisfying inequalities (1.3) for every $\varepsilon > 0$.

Remark 12. Let us remark that the supremum of the constants $d_0 > 0$ satisfying (1.2) also satisfies this inequality. It corresponds to $\underline{\sigma}_1$. Similarly, the infinum of constants $d_1 > 0$ satisfying (1.2) corresponds to $\overline{\sigma}_1$. If there exists a constant $d_0 > 1$ satisfying (1.2), then this implies that $\sigma_j \to +\infty$ (if the sequence is not identically null). Similarly, if there exists a constant $d_1 < 1$ satisfying (1.2), then $\sigma_j \to 0$. For Boyd indices: if $\overline{s}(\sigma) < 0$, then $\sigma_j \to 0$; if $\underline{s}(\sigma) > 0$, then $\sigma_j \to +\infty$.

The following results are immediate.

Lemma 13. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ be two admissible sequences.*

1. *The sequence $\sigma + \gamma = (\sigma_j + \gamma_j)_{j\in\mathbb{N}_0}$ is admissible;*
2. *the sequence $r\sigma = (r\sigma_j)_{j\in\mathbb{N}_0}$ is admissible for every positive real number r;*
3. *the sequence $\sigma\gamma = (\sigma_j\gamma_j)_{j\in\mathbb{N}_0}$ is admissible;*
4. *the sequence $\sigma^{-1} = (\sigma_j^{-1})_{j\in\mathbb{N}_0}$ is admissible.*

Lemma 14. *Let $\alpha \in \mathbb{R}$ and σ be an admissible sequence. The new sequence σ^α is admissible and*

1. *if $\alpha > 0$, then we have*
$$\underline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma) \quad \text{and} \quad \overline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma);$$
2. *if $\alpha < 0$, then we have*
$$\underline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma) \quad \text{and} \quad \overline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma).$$

Proof. The proof is immediate: we start from inequalities (1.3) for the sequences σ and σ^α, and we raise them to the power α and $1/\alpha$ respectively. □

Example 15. The following example will be fundamental for the basis of this thesis. It is closely linked to classical Hölder spaces. Let $\alpha \in \mathbb{R}$ and $\sigma_j := 2^{j\alpha}$ for every $j \in \mathbb{N}^*$. Because the sequence σ satisfies

$$2^\alpha \sigma_j = \sigma_{j+1},$$

it is an admissible sequence and we have

$$\underline{\sigma}_j = 2^{j\alpha} = \overline{\sigma}_j$$

so that

$$\underline{s}(\sigma) = \overline{s}(\sigma) = \alpha.$$

Example 16. The sequence σ defined by $\sigma_j = 2^{2^j}$ $(j \in \mathbb{N}^*)$ is not admissible. To show that, we proceed by reductio ad absurdum. As

$$\sigma_{j+1} = \sigma_j^2,$$

the inequalities (1.2) should imply $\sigma_j \leq C$ for some constant. This example gives an idea about how to interpret the definition of admissible sequence: inequalities (1.2) imply some limits on the speed of convergence and divergence.

The following examples come from [6] and [49].

Example 17. A function $\Phi :]0,1] \to]0,+\infty[$ is a *slowly varying function* if it is (Lebesgue-) measurable and satisfies

$$\lim_{x\to 0} \frac{\Phi(\lambda x)}{\Phi(x)} = 1, \qquad \forall \lambda > 0.$$

For all $s \in \mathbb{R}$, the sequence $\sigma = (2^{sj}\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ is admissible and satisfies $\underline{s}(\sigma) = \overline{s}(\sigma) = s$. An example of slowly varying function is given by $\Phi(x) = |\log(x)|$ (defined on some interval $]0, a[$ with $a < 1$ and extended outside of this interval by a constant).

Example 18. Let $(j_n)_{n\in\mathbb{N}_0}$ the increasing sequence defined by

$$j_0 = 0, \quad j_1 = 1, \quad j_{2n} = 2j_{2n-1} - j_{2n-2}, \quad j_{2n+1} = 2^{j_{2n}}, \quad n \in \mathbb{N}^*\,.$$

We set a sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ by

$$\sigma_j = \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ 2^{j_{2n}} 4^{(j-j_{2n+1})} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

The sequence σ is admissible : by proceeding by case-to-case, we remark that

$$\sigma_j \leq \sigma_{j+1} \leq 4\sigma_j \qquad \forall j \in \mathbb{N}_0\,.$$

The sequence σ oscillates infinitely between $(j)_{j\in\mathbb{N}_0}$ and $(2^j)_{j\in\mathbb{N}_0}$, i.e. we have

$$j \leq \sigma_j \leq 2^j \quad \forall j \in \mathbb{N}_0$$

and there exist infinitely many indices $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ such that $\sigma_n = n$ and $\sigma_m = 2^m$ respectively.

Example 19. Let $s \in \mathbb{R}$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be the sequence defined in the previous example. Then the new sequence

$$\gamma_j = 2^{js}\sigma_j$$

is admissible (lemma 13). Moreover, the sequence γ oscillates between $(j2^{js})_{j\in\mathbb{N}_0}$ and $(2^{j(s+1)})_{j\in\mathbb{N}_0}$, i.e.

$$j2^{js} \leq \gamma_j \leq 2^{j(s+1)} \quad \forall j \in \mathbb{N}_0$$

where the left inequality is attained as well as the right one for infinitely many indices j.

The reader should note that in section 2.7 we provide a method to construct an admissible sequence such that the Boyd indices can take any given values (this method consists in adapting the previous example).

Before concluding this section, let us consider some classical terminologies linked to admissible sequences.

Definition 20. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}^*}$ of positive real numbers is called

1. *almost increasing* if there is a positive constant d_0 such that

$$d_0\sigma_j \leq \sigma_k \quad \text{for all} \quad j,k \quad \text{such that} \quad 0 \leq j \leq k.$$

2. *strongly increasing* if it is almost increasing and, in addition, there is a natural number k_0 such that

$$2\sigma_j \leq \sigma_k \quad \text{for all} \quad j,k \quad \text{such that} \quad j+k_0 \leq k.$$

3. *of bounded growth* if there is a positive constant d_1 such that

$$\sigma_{j+1} \leq d_1\sigma_j \quad \text{for all} \quad j \in \mathbb{N}_0\,.$$

Example 21. The following examples illustrate these notions.

1. The sequence defined by $\sigma_j = 2^{j\alpha}(1+j)^\beta$ (where $\alpha > 0$, $\beta \in \mathbb{R}$) is strongly increasing and of bounded growth.
2. The sequence defined by $\sigma_j = j!$ is strongly increasing, but not of bounded growth.
3. The sequence defined by $\sigma_j = j$ is not strongly increasing but of bounded growth.

Let us note that each almost increasing and of bounded growth sequence is admissible, but the converse is false as it is shown by the admissible sequence $(2^{-j\alpha})_{j\in\mathbb{N}_0}$ for $\alpha > 0$.

1.3 Generalized Besov Spaces

In this section, we will present some of the generalizations of Besov spaces that have been proposed in the literature for more than 30 years. One of the most general definitions until now is the one given by the definition 31, presented at the end of this section.

Definition 22. The set $\mathcal{B}$ is defined by

$$\mathcal{B} = \left\{ \phi \in C(]0,\infty[) : \phi > 0, \quad \phi(1) = 1 \quad \text{and} \quad \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty \quad \forall t > 0 \right\}.$$

Definition 23. Let E be a quasi-normed space, $\phi \in \mathcal{B}$ and $0 < q \leq \infty$. We define

$$l^{q,\phi}(E) = \{(a_j)_{j\in\mathbb{N}_0} \in E^{\mathbb{N}_0} : \|(\phi(2^j)a_j)_j\|_{l^q(E)} < \infty\}.$$

This is a quasi-normed space.

Definition 24. Let $(\varphi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$ be a sequence of functions. For a chosen function $\phi \in \mathcal{B}$, $0 < p \leq \infty$ and $0 < q \leq \infty$, we define

$$B^{\phi}_{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f|B^{\phi}_{p,q}(\mathbb{R}^d)\| := \|(\mathcal{F}^{-1}(\varphi_j\mathcal{F}f))_{j\in\mathbb{N}_0}|l^{q,\phi}(L^p(\mathbb{R}^d))\| < \infty\}.$$

Remark 25.

1. This definition does not depend on the chosen sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ (see e.g. [6]).
2. Moreover, the distributions $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ $(j \in \mathbb{N}_0)$ are associated with analytic functions defined on $\mathbb{R}^d$ by the Paley-Wiener theorem. So, they belong to $C^\infty(\mathbb{R}^d)$.

Another (identical) definition is expressed in terms of admissible sequences. The idea is to replace the terms $\varphi(2^j)$ by σ_j.

Definition 26 (Generalized Besov spaces)**.** Let $(\varphi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $0 < p, q \leq \infty$. The *generalized Besov space* $B^{\sigma}_{p,q}$ is defined by

$$B^{\sigma}_{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f|B^{\sigma}_{p,q}(\mathbb{R}^d)\| := \|(\sigma_j\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)))_{j\in\mathbb{N}_0}|l^q(L^p(\mathbb{R}^d))\| < +\infty\}.$$

Remark 27. Let $\alpha \in \mathbb{R}$. By considering $\phi(2^j) = 2^{\alpha j} = \sigma_j$ $(j \in \mathbb{N}^*)$, we find the classical Besov spaces ([7]).

We note that the definition of the spaces $B^{\sigma}_{p,q}$ is equivalent to the one of $B^{\phi}_{p,q}$. Indeed, the following result has been demonstrated in [6].

Proposition 28. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $0 < p, q \leq \infty$. There exists a function $\phi_\sigma \in \mathcal{B}$ such that*

$$B^{\phi_\sigma}_{p,q} = B^{\sigma}_{p,q}$$

and so that $\phi_\sigma(2^j) = \sigma_j$ for all $j \in \mathbb{N}^*$.

Proof. Let σ be an admissible sequence. By considering the new sequence σ' defined by $\sigma'_0 = 1$, $\sigma'_n = \sigma_n$ $\forall n \geq 1$ (which does not change anything on the associated space $B^{\sigma}_{p,q}$), we can suppose that the first term of the sequence is 1. Then we construct a function $\phi_\sigma \in \mathcal{B}$ in the following way:

$$\phi_\sigma(t) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{2^j}(t-2^j) + \sigma_j, & t \in [2^j, 2^{j+1}[, \quad j \in \mathbb{N}_0 \\ 1, & t \in]0,1[. \end{cases}$$

So, $\phi_\sigma(2^j) = \sigma_j$ for all $j \in \mathbb{N}^*$, which is sufficient to conclude. □

The next result gives the converse part.

Lemma 29. *Let $\phi \in \mathcal{B}$ et $0 < p, q \leq \infty$. The sequence σ_ϕ defined by $\sigma_{\phi,j} = \phi(2^j)$ ($j \in \mathbb{N}_0$) is admissible and such that*

$$B^{\phi}_{p,q} = B^{\sigma_\phi}_{p,q}.$$

Proof: Indeed, we have for all $j \in \mathbb{N}_0$

$$\frac{\phi(22^j)}{\phi(2^j)} \leq \sup_{s>0} \frac{\phi(2s)}{\phi(s)} < +\infty$$

and

$$\frac{\phi(2^{-1}2^j)}{\phi(2^j)} \leq \sup_{s>0} \frac{\phi(2^{-1}s)}{\phi(s)} < +\infty.$$

□

The definition given in [50, 97] generalizes those definitions. They allow a change of scale in the support of the functions φ_j, where the scale is associated with a strongly increasing sequence. Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence, J a natural number different from zero and k_0 the natural number associated to the strongly increasing sequence N. We define the associate covering $\Omega^{N,J} = (\Omega^{N,J}_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^d$ by

$$\Omega^{N,J}_j = \left\{\xi \in \mathbb{R}^d : |\xi| \leq N_{j+Jk_0}\right\} \quad \text{if} \quad j = 0, 1, ..., Jk_0 - 1,$$

and

$$\Omega^{N,J}_j = \left\{\xi \in \mathbb{R}^d : N_{j-Jk_0} \leq |\xi| \leq N_{j+Jk_0}\right\} \quad \text{if} \quad j = Jk_0, Jk_0+1, ...$$

As for classical Besov spaces, we define a smooth resolution of unity associated to the covering $(\Omega^{N,J}_j)_{j\in\mathbb{N}_0}$.

Definition 30. We let $\Phi^{N,J}(\mathbb{R}^d)$ denotes the set of sequences $(\varphi^{N,J}_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying the following properties:

1. $\varphi^{N,J}_j(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d, j \in \mathbb{N}_0$;
2. $\text{supp}\varphi^{N,J}_j \subseteq \Omega^{N,J}_j$ for all $j \in \mathbb{N}_0$;

3. for all $\alpha \in \mathbb{N}_0^d$, there exists a constant $c_\alpha > 0$ such that for all $j \in \mathbb{N}_0$ we have
$$|D^\alpha \varphi_j^{N,J}(\xi)| \leq c_\alpha (1+|\xi|^2)^{-|\alpha|/2} \qquad \text{for all} \quad \xi \in \mathbb{R}^d;$$

4. there exists a constant $c_\varphi > 0$ such that
$$0 < \sum_{j=0}^{\infty} \varphi_j^{N,J}(\xi) = c_\varphi < \infty \qquad \text{for all} \quad \xi \in \mathbb{R}^d.$$

The construction (and so the existence) of such a sequence of functions is tricky. We refer to [50] for such constructions.

Definition 31. Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence and of bounded growth. Let $J \in \mathbb{N}^*$, $(\varphi_j^{N,J})_{j\in\mathbb{N}_0} \in \Phi^{N,J}(\mathbb{R}^d)$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. Let $0 < p, q \leq \infty$. The *generalized Besov space* $B^{\sigma,N}_{p,q}$ is defined by
$$B^{\sigma,N}_{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f|B^{\sigma,N}_{p,q}(\mathbb{R}^d)\| := \|(\sigma_j \mathcal{F}^{-1}(\varphi_j^{N,J}\mathcal{F}f)))_{j\in\mathbb{N}_0}|l^q(L^p(\mathbb{R}^d))\| < +\infty\}.$$

As for classical Besov spaces, the distributions $\mathcal{F}^{-1}(\varphi_j^{N,J}\mathcal{F}f)$ are associated with functions belonging to $C^\infty(\mathbb{R}^d)$. The assumption on N implies in particular that N is an admissible sequence. One can easily adapt the usual proofs for the classical properties of Besov spaces in [118] ([50]), to prove the independence of the definition from the choice of the sequence $(\varphi_j^{N,J})_{j\in\mathbb{N}_0}$, the inclusion of the Schwartz space into those spaces and so on.

Remark 32. If we consider the sequence $N_j = 2^j$ $(j \in \mathbb{N}_0)$, definition 31 is equivalent to definition 26 so that we have $B^{\sigma,(2^j)_j}_{p,q} = B^\sigma_{p,q}$.

The generalized Besov spaces have been studied since the mid of 1970's by many authors using many different approaches. Several references about these ones can be found in [50] with many historical explanations about them.

1.4 A central result

In this section, we introduce the generalized Hölder-Zygmund spaces that are studied in this thesis. We explain their origins and some first reasons why these have interesting properties.

The following result is the core of our work. It has been proved in [97].

Theorem 33. *Let $0 < p, q \leq \infty$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma^{-1})\overline{s}(N)^{-1} > n(1/p-1)_+$. Let $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$. We have*
$$B^{\sigma^{-1},N}_{p,q} =$$
$$\left\{ f \in L^{\max(1,p)} : \|f|B^{\sigma^{-1},N}_{p,q}\|^{(M)} := \|f\|_{L^p} + \left(\sum_{j=0}^{\infty} \sigma_j^{-q} \left(\sup_{|h|\leq N_j^{-1}} \|\Delta_h^M f\|_{L^p}\right)^q\right)^{1/q} < \infty \right\}$$
(with an obvious modification if $q = \infty$), in the sense of equivalent quasi-norms.

If we consider the particular case $N_j = 2^j$ and $p = q = +\infty$, then $\underline{N}_1 = 2$ and $\overline{s}(N) = \underline{s}(N) = 1$, so that the previous result can be restated in this case as follows.

Corollary 34. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$ and let $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})$. We have*

$$B^{\sigma^{-1}}_{\infty,\infty} = \left\{ f \in L^\infty : \|f|B^{\sigma}_{\infty,\infty}\|^{(M)} := \|f\|_{L^\infty} + \sup_{j\in\mathbb{N}_0} \sigma_j^{-1} \sup_{|h|\leq 2^{-j}} \|\Delta_h^M f\|_{L^\infty} < \infty \right\}$$

in the sense of equivalent quasi-norms.

Let us remark the immediate analogy with the classical Hölder-Zygmund spaces: it consists of replacing the sequence $2^{-j\alpha}$ of definition 6 by an arbitrary admissible sequence σ which satisfies the conditions of corollary 34. This means replacing the following control of the finite difference

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^M f\|_{L^\infty} \leq C 2^{-j\alpha}$$

by the more general control

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^M f\|_{L^\infty} \leq C\sigma_j.$$

Theorem 33 shows that an even more generalized control is possible: it consists of replacing the ball of dyadic radius on which we consider the finite difference ($h \in B(0, 2^{-j})$) by a ball of a more general radius linked to an arbitrary admissible sequence ($h \in B(0, N_j^{-1})$). The control on the finite difference under that last generalization is then

$$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^M f\|_{L^\infty} \leq C\sigma_j.$$

The next step is to study the conditions needed for the sequence σ to be able to apply the corollary 34. We provide some results that can help the reader to interpret these conditions and that are often used in the sequel.

Lemma 35. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. If $\underline{s}(\sigma^{-1}) > 0$, then there exists $C > 0$ such that*

$$\sum_{j=J}^{+\infty} \sigma_j \leq C\sigma_J \quad \forall J \in \mathbb{N}_0 .$$

Proof. Let $\varepsilon > 0$ such that $\underline{s}(\sigma^{-1}) - \varepsilon > 0$. Using inequalities (1.3), we obtain

$$\sum_{j=J}^{+\infty} \sigma_j = \sum_{j=0}^{+\infty} \sigma_{J+j} \leq C \sum_{j=0}^{+\infty} \sigma_J 2^{-(\underline{s}(\sigma^{-1})-\varepsilon)j} \leq C\sigma_J$$

for all $J \in \mathbb{N}_0$. □

So the condition $\underline{s}(\sigma^{-1}) > 0$ does not only ask for the sequence σ to converge to 0, but also asks the convergence to be "as quick" as some arbitrary power of 2. This is confirmed by inequalities (1.3) which implies

$$\sigma_j \leq C 2^{-(\underline{s}(\sigma^{-1})-\varepsilon)j}$$

(where the constant C depends on ε) where ε can be taken as arbitrarily small. Moreover, the Boyd index gives the power of 2 that should be used. One can easily prove the following result.

Lemma 36. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. It satisfies $\underline{s}(\sigma) > 0$ if and only if there exist $\varepsilon > 0$ and $C > 0$ such that $C2^{\varepsilon j} \leq \underline{\sigma}_j$ for all $j \in \mathbb{N}^*$.*

The next result allows for the interpretation of the condition $M > \overline{s}(\sigma^{-1})$ in the corollary 34.

Lemma 37. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. If $M \in \mathbb{N}_0$ satisfies $M > \overline{s}(\sigma^{-1})$, then there exists $C > 0$ such that*

$$\sum_{j=0}^{J} 2^{jM}\sigma_j \leq C 2^{JM}\sigma_J \quad \forall J \in \mathbb{N}_0 .$$

Proof. Let $\varepsilon > 0$ such that $\overline{s}(\sigma^{-1}) + \varepsilon < M - \varepsilon$. Using inequalities (1.3), we obtain

$$\frac{\sigma_k}{\sigma_{j+k}} \leq C 2^{jM} 2^{-\varepsilon j} \quad \forall j, k \in \mathbb{N}_0$$

which implies for $J \in \mathbb{N}_0$

$$2^{-JM}\sigma_J^{-1} \sum_{j=0}^{J} 2^{jM}\sigma_j \leq C \sum_{j=0}^{J} 2^{-\varepsilon(J-j)} \leq C$$

hence the conclusion. □

So the condition $M > \overline{s}(\sigma^{-1})$ means that the growth of the sequence $2^{jM}\sigma_j$ must be fast enough so that the term $2^{JM}\sigma_J$ is still "comparable" to the sum of the previous elements of the sequence $(\{2^M\sigma_1, 2^{2M}\sigma_2 ..., 2^{JM}\sigma_J\})$. The natural M we have to choose to apply corollary 34 must be large enough so that $2^{jM}\sigma_j$ grows sufficiently fast. We can also note that inequalities (1.3) imply in particular that $C \leq 2^{jM}\sigma_j$ for all $j \in \mathbb{N}_0$.

Another important characterization of spaces $B^{\sigma,N}_{p,q}$ is proved in [97] (Theorem 3.1). This characterization states that any element of $B^{\sigma,N}_{p,q}$ can be approximated by functions with compactly supported Fourier transforms. Let $0 < p \leq \infty$ and N be an admissible sequence such that $\underline{N}_1 > 1$. We easily show that N is a strongly increasing sequence, and so there exists a natural number $k_0 \in \mathbb{N}_0$ such that

$$N_k \geq 2N_j \quad \forall k, j \quad \text{satisfying} \quad k \geq j + k_0.$$

We set

$$\mathcal{U}^N_p = \{a = (a_j)_{j\in\mathbb{N}_0} : a_j \in \mathcal{S}' \cap L^p, \mathrm{supp}\mathcal{F}a_j \subset \{y : |y| \leq N_{j+k_0}\}, j \in \mathbb{N}_0\} .$$

Theorem 38. *Let $0 < p, q \leq \infty$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma)\overline{s}(N)^{-1} > n(1/p-1)_+$. We have*

$$B^{\sigma,N}_{p,q} = \Big\{ f \in \mathcal{S}' : \exists a = (a_j)_j \in \mathcal{U}^N_p \text{ such that } f = \lim_{k\to+\infty} a_k \text{ in } \mathcal{S}' \text{ and } \|f|B^{\sigma,N}_{p,q}\|^a := \|a_0\|_{L^p} + \|(\sigma_k(f-a_k))_{k\in\mathbb{N}^*}|l^q(L^p)\| < \infty \Big\}.$$

Moreover,

$$\|f|B^{\sigma,N}_{p,q}\|^X := \inf \|f|B^{\sigma,N}_{p,q}\|^a$$

(where the infimum is taken over all admissible systems $a \in \mathcal{U}^N_p$), is an equivalent quasi-norm in $B^{\sigma,N}_{p,q}$.

If we consider $N_j = 2^j$ and $p = q = +\infty$, this result can be restated in this case as follows:

Corollary 39. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma) > 0$. Then*

$$B^{\sigma}_{\infty,\infty} = \Big\{ f \in \mathcal{S}' : \exists a = (a_j)_j \in \mathcal{U}_\infty \text{ such that } f = \lim_{k\to+\infty} a_k \text{ in } \mathcal{S}' \text{ and } \|f|B^{\sigma}_{\infty,\infty}\|^a := \|a_0\|_{L^\infty} + \sup_{k\in\mathbb{N}^*} \sigma_k \|f-a_k\|_{L^\infty} < \infty \Big\}$$

where

$$\mathcal{U}_\infty = \left\{ a = (a_j)_{j\in\mathbb{N}_0} : a_j \in \mathcal{S}' \cap L^\infty, supp\mathcal{F}a_j \subset \{y : |y| \leq 2^{j+1}\}, j \in \mathbb{N}_0 \right\}.$$

1.5 Definition of generalized Hölder-Zygmund spaces

In this section, we introduce the definition of generalized Hölder-Zygmund spaces studied in this thesis. We show also that these spaces generalize a particular type of space studied from the wavelet[2] point of view some years ago by Stéphane Jaffard, Yves Meyer and Marianne Clausel ([65, 30]). Indeed, these authors already noticed that the useful wavelet characterization known for classical Hölder spaces could be generalized to a more broad scope which can be used in practice. It is remarkable that the path they follow to generalize Hölder spaces and the completely different path followed through the generalization of Besov point of view (by Hans-Gerd Leopold, Walter Farkas, Susana Moura, Alexandre Almeida, ...) merge into one unique notion several years after their work, when restricted to the Hölder case.

Let us give a definition of the generalized Hölder-Zygmund spaces. This definition relies on the principles exposed in the previous section.

[2]Wavelets are a mathematical tool used to study signals in mathematical analysis, in engineering and physical domains (see [89] for a good overview on the subject). This concept is introduced in section 5.4.

Definition 40. Let $\alpha > 0$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences. The *generalized Hölder-Zygmund space* $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ is defined by

$$\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) : \sup_{j\in\mathbb{N}_0} \sigma_j^{-1} \sup_{|h|\leq N_j^{-1}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} < \infty\}.$$

If $N_j = 2^j$ $(j \in \mathbb{N}_0)$, then we note $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ instead of $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ to simplify notations.

Remark 41. The application

$$f \mapsto |f|_{\Lambda^{\alpha}_{\sigma,N}} = \sup_{j\in\mathbb{N}_0} \sigma_j^{-1} \sup_{|h|\leq N_j^{-1}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty}$$

is a semi-norm on the space $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$. The application

$$f \mapsto \|f\|_{\Lambda^{\alpha}_{\sigma,N}} = \|f\|_{L^\infty} + |f|_{\Lambda^{\alpha}_{\sigma,N}}$$

is trivially a norm on this space. Moreover, the space $(\Lambda^{\alpha}_{\sigma,N}, \|.\|_{\Lambda^{\alpha}_{\sigma,N}})$ is complete. Let $(f_j)_{j\in\mathbb{N}_0}$ be a Cauchy sequence on this space. Because the space L^∞ is complete, the sequence f_j converges in L^∞ to a function f. We can choose a subsequence $(f_{k(j)})_{j\in\mathbb{N}_0}$ of the Cauchy sequence such that

$$|f_{k(j+1)} - f_{k(j)}|_{\Lambda^{\alpha}_{\sigma,N}} < 2^{-j} \qquad \forall j \in \mathbb{N}_0\,.$$

We set $g_1 = f_{k(1)}$ and $g_{j+1} = f_{k(j+1)} - f_{k(j)}$ $(j \in \mathbb{N}^*)$, so that we obtain $f = \sum_{j=1}^{+\infty} g_j$. We find

$$|f - f_{k(j)}|_{\Lambda^{\alpha}_{\sigma,N}} \leq \sum_{l=j+1}^{+\infty} |g_l|_{\Lambda^{\alpha}_{\sigma,N}} \leq \sum_{l=j+1}^{+\infty} 2^{-l+1}.$$

As a consequence, $\|f - f_{k(j)}\|_{\Lambda^{\alpha}_{\sigma,N}} \to 0$ if $j \to +\infty$. Proposition 4.4.4 of [106] leads to the following result:

Proposition 42. *Let $\alpha > 0$, σ and N be two admissible sequences. The space $(\Lambda^{\alpha}_{\sigma,N}, \|.\|_{\Lambda^{\alpha}_{\sigma,N}})$ is a Banach space.*

Remark 43. The choice of the order of the finite difference should take into account the speed of convergence of the sequence σ, so that the spaces are not reduced to the space of constant functions. Indeed, we prove in section 2.3 that if $m+1 < \underline{s}(\sigma^{-1})$ (where $m \in \mathbb{N}_0$), then the space $\Lambda^{\sigma,m}$ is composed of constant functions.

As already mentioned, another generalized version of the classical Hölder-Zygmund spaces has been introduced in [65] et [30], which relies on the concept of moduli of continuity. Let us introduce that concept.

Definition 44 (Modulus of continuity)**.** A non-decreasing function non identically null ω defined on $[0, +\infty[$ is a *modulus of continuity* if it satisfies the two following conditions:

1. $\omega(0) = 0$
2. there exists a constant $C_\omega > 0$ such that

$$\omega(2t) \le C_\omega \omega(t), \quad \forall t \in [0, +\infty[. \tag{1.4}$$

Obviously, such a function cannot be equal to 0 at any point distinct of 0. The generalized version of Hölder-Zygmund space linked to this concept is the following.

Definition 45. Let $\alpha > 0$ and ω be a modulus of continuity. We say that a function $f \in L^\infty(\mathbb{R}^d)$ belongs to the *generalized Hölder-Zygmund space* $\Lambda^{\omega,\alpha}(\mathbb{R}^d)$ if there exists a constant $C > 0$ such that

$$\sup_{|h|\le 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \le C\omega(2^{-j}) \qquad \forall j \in \mathbb{N}_0 .$$

Since only the behavior of ω near the origin really matters, we could define a modulus of continuity using germ functions. The purpose of the remaining part of this section is to study the link between these spaces and the generalized Hölder-Zygmund spaces based on admissible sequences.

Let ω be a modulus of continuity. We define $\sigma_j := \omega(2^{-j})$ for all $j \in \mathbb{N}_0$. The sequence σ so defined is admissible and shows that the generalized Hölder-Zygmund spaces linked to moduli of continuity are a particular case of the ones concerned by the definition 40. The reader might wonder what the exact link is between moduli of continuity and admissible sequences. The two following immediate results give an answer to that question.

Lemma 46. *Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. There exists a modulus of continuity ω such that $\sigma_j = \omega(2^{-j})$ $\forall j \in \mathbb{N}_0$ if and only if $(\sigma_j)_{j\in\mathbb{N}_0}$ is a non-increasing sequence.*

Lemma 47. *Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. There exists a modulus of continuity ω continuous at 0 and such that $\sigma_j = \omega(2^{-j})$ $\forall j \in \mathbb{N}_0$ if and only if $(\sigma_j)_{j\in\mathbb{N}_0}$ is a non-increasing sequence that converges to 0.*

Proof: We define the modulus of continuity by $\omega(r) := \sigma_j$ if $r \in [2^{-j}, 2^{-(j-1)}[$ $(j \in \mathbb{N}_0)$, and we extend it by a constant over the interval $[1, +\infty[$. □

Our considerations of the section 1.4 can now be restated as follow.

Proposition 48. *1. Let σ and N be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma^{-1})\overline{s}(N)^{-1} > 0$. We have*

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$$

for all $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$.

2. Let σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. We have

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \Lambda^{\sigma,M-1}(\mathbb{R}^d)$$

for all $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})$.

3. Let ω be a modulus of continuity. We set $\sigma = (\omega(2^{-j}))_{j\in\mathbb{N}^*}$. If $\underline{s}(\sigma^{-1}) > 0$, then

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\omega,\overline{s}(\sigma^{-1})}(\mathbb{R}^d).$$

A modulus of continuity satisfies the conditions of this result if, for example, there exist $C > 1$ and $J \in \mathbb{N}^*$ such that

$$\omega(2^{-j}) \geq C\omega(2^{-(j+1)}) \qquad \forall j \geq J.$$

This condition is often realized for interesting moduli of continuity or admissible sequences (i.e. the ones that give interesting properties to the associated spaces). For example, this is the case for the sequence $\sigma_j = \omega(2^{-j}) := 2^{-j\alpha}$ ($\alpha > 0$) which corresponds to the classical Hölder-Zygmund spaces. To end this section, we show some examples of admissible sequences which are moduli of continuity for which the proposition 48 can or cannot be applied.

Example 49. Let $0 < a < 1$. We define a modulus of continuity by

$$\omega(r) = \begin{cases} |\log_2(r)|^{-1} & \text{si } r \in]0, a[\\ |\log_2(a)|^{-1} & \text{si } r \geq a. \end{cases}$$

The sequence $\sigma = (\omega(2^{-j})^{-1})_{j\in\mathbb{N}_0}$ is such that $\underline{s}(\sigma) = 0$.

Example 50. We look back at the example 18. The sequence σ is non-decreasing so σ^{-1} defines a modulus of continuity by lemma 46. For all $j \in \mathbb{N}_0$, we have

$$\underline{\sigma}_j = \inf_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k} \geq 1$$

and by considering $n \in \mathbb{N}_0$ sufficiently large so that $j_{2n} + j \leq j_{2n+1}$, we obtain $\sigma_{j+j_{2n}} = \sigma_{j_{2n}} = 2^{j_{2n}}$. So

$$\underline{\sigma}_j = 1$$

which implies $\underline{s}(\sigma) = 0$.

Example 51. We look back at the example 19 with $s > 0$. The sequence γ is non-decreasing and γ^{-1} defines a modulus of continuity. Moreover, by lemma 52 which is proved below, we have $\underline{s}(\gamma) \geq s > 0$ and corollary 48 can be applied.

Lemma 52. *Let σ and γ be two admissible sequences. We have*

$$\underline{s}(\sigma\gamma) \geq \underline{s}(\sigma) + \underline{s}(\gamma)$$

and

$$\overline{s}(\sigma\gamma) \leq \overline{s}(\sigma) + \overline{s}(\gamma)$$

Proof. For every $j \in \mathbb{N}_0$, we have

$$\underline{\sigma\gamma}_j = \inf_{k \geq 0} \frac{\sigma_{j+k}\gamma_{j+k}}{\sigma_k \gamma_k} \geq \underline{\sigma}_j \underline{\gamma}_j$$

which implies

$$\frac{\log_2(\underline{\sigma\gamma}_j)}{j} \geq \frac{\log_2(\underline{\sigma}_j)}{j} + \frac{\log_2(\underline{\gamma}_j)}{j}.$$

This proves the first inequality. The second one can be proved in the same way. □

Before ending this section, let us provide generalizations of lemmata 37 and 35 (these results do not appear in [76, 77]).

Lemma 53. *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ and $N = (N_j)_{j \in \mathbb{N}_0}$ be two admissible sequences such that $\underline{s}(N) > 0$. If $M \in \mathbb{N}_0$ is such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$, then there exists a constant $C > 0$ such that*

$$\sum_{j=0}^{J} N_j^M \sigma_j \leq C N_J^M \sigma_J \quad \forall J \in \mathbb{N}_0 \,.$$

Proof. Let $\varepsilon > 0$ such that $\overline{s}(\sigma^{-1}) + \varepsilon < M(\underline{s}(N) - \varepsilon) - \varepsilon$. Using inequalities (1.3) we obtain

$$\frac{\sigma_k}{\sigma_{j+k}} \leq C \left(\frac{N_{j+k}}{N_k} \right)^M 2^{-\varepsilon j} \quad \forall j, k \in \mathbb{N}_0$$

which implies, for $J \in \mathbb{N}_0$,

$$N_J^{-M} \sigma_J^{-1} \sum_{j=0}^{J} N_j^M \sigma_j \leq C \sum_{j=0}^{J} 2^{-\varepsilon(J-j)} \leq C$$

which allows to conclude. □

One can proceed similarly to prove the following result.

Lemma 54. *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ and $N = (N_j)_{j \in \mathbb{N}_0}$ be two admissible sequences such that $\overline{s}(N) > 0$. If $L \in \mathbb{N}_0$ is such that $L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$, then there exists a constant $C > 0$ such that*

$$\sum_{j=J}^{+\infty} N_j^L \sigma_j \leq C N_J^L \sigma_J \quad \forall J \in \mathbb{N}_0 \,.$$

1.6 Some last remarks about generalized Besov and Hölder-Zygmund spaces

The purpose of this section is to discuss some last points about generalized Besov spaces. After this section, we will only be interested in generalized Hölder-Zygmund spaces. The first remark states that some results and references connect classical and generalized Besov spaces with wavelets. These comments are interesting as we will consider characterizations of generalized Hölder spaces in terms of wavelet coefficients in section 2.9. Secondly, as stated in the proposition 48, generalized Besov spaces coincide with generalized Hölder spaces in particular cases. It is thus natural to ask whether it is the case in general. The second remark gives a negative answer to that question. Finally, the last remark somehow connects the generalized Besov spaces with the classical Hölder-Zygmund spaces, and consequently with the usual notion of continuity.

Remark 55. A characterization of classical Besov spaces $B^s_{p,q}$ ($s \in \mathbb{R}$, $0 < p, q \leq +\infty$) in terms of wavelet decomposition has been proved in [120] (see also [35, 40, 41]). This result has been generalized to Weighted Besov spaces in [61], and recently to spaces $B^\sigma_{p,q}$ ($0 < p < +\infty$, $0 < q \leq +\infty$) in [6]. The existence of a topological isomorphism between the generalized Besov spaces and sequences spaces was also proved in the last reference. Other characterizations of these spaces were proved in [27]. In section 2.9, we show a simple characterization of space $B^\sigma_{\infty,\infty}$ in terms of wavelet coefficients under some assumptions on the admissible sequence σ.

Remark 56. The goal of this section is to show that if σ is an admissible sequence with $\underline{s}(\sigma) = 0$, then the spaces $B^\sigma_{\infty,\infty}$ can be different from every generalized Hölder-Zygmund space $\Lambda^{\psi,\alpha}$ (for all arbitrary admissible sequence ψ). By considering the sequence $\sigma_j^{-1} = \frac{1}{j}$ which corresponds to the admissible sequence given by example 49, we know that $\underline{s}(\sigma) = 0$ and proposition 3.13 of [28] implies that

$$B^\sigma_{\infty,\infty} \not\hookrightarrow L^\infty,$$

where the symbol "$\hookrightarrow$" means that the corresponding embedding is continuous. In particular, the space $B^\sigma_{\infty,\infty}$ can not be written as a generalized Hölder-Zygmund space.

This fact shows that the condition $\underline{s}(\sigma) > 0$ of theorem 33 is necessary and that the concept of generalized Besov space can not always be directly linked to some generalized Hölder-Zygmund spaces.

Remark 57. The following result was proved in [6] (proposition 4).

Proposition 58. *Let ϕ, $\psi \in \mathcal{B}$, $0 < p \leq \infty$, $0 < q_0, q_1 \leq \infty$. If $(\frac{\phi(2^j)}{\psi(2^j)})_{j\in\mathbb{N}_0} \in l^{\min\{q_1,1\}}$, then*

$$B^\psi_{p,q_0}(\mathbb{R}^d) \hookrightarrow B^\phi_{p,q_1}(\mathbb{R}^d).$$

We obtain the following result from the previous one.

Proposition 59. *Let $0 < p, q_0 \leq \infty$, $1 \leq q_1 \leq \infty$ and $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma) > 0$. For every $0 < \epsilon < \underline{s}(\sigma)$, we have*

$$B^{\sigma}_{p,q_0}(\mathbb{R}^d) \hookrightarrow B^{\underline{s}(\sigma)-\varepsilon}_{p,q_1}(\mathbb{R}^d).$$

Proof. Let $0 < \varepsilon < \underline{s}(\sigma)$. By inequalities (1.3), there exists $C > 0$ such that

$$c2^{(\underline{s}(\sigma)-\varepsilon/2)j}\sigma_0 \leq \sigma_j \qquad \forall j \in \mathbb{N}_0$$

which implies

$$\sum_{j=0}^{+\infty} 2^{j(\underline{s}(\sigma)-\varepsilon)}\sigma_j^{-1} < \infty.$$

Proposition 58 allows to conclude. □

Corollary 60. *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma) > 0$. There exists $s > 0$ such that*

$$B^{\sigma}_{\infty,\infty}(\mathbb{R})^d \hookrightarrow B^{s}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^s(\mathbb{R}^d).$$

In particular, the elements of $B^{\sigma}_{\infty,\infty}(\mathbb{R}^d)$ coincide almost everywhere with a continuous function defined on $\mathbb{R}^d$.

1.7 Some reminders on finite differences

The concept of finite differences is at the core of the future developments done throughout this thesis. In this regard, we need to recall the two different types of finite differences as well as their main properties that are used at a later stage.

The concept of forward finite difference has already been introduced in the section 1.1.

Example 61. We find the following expressions

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$

$$\Delta_h^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

for the (forward) finite difference of order 2 and 3 respectively.

Remark 62. We can easily check recursively that we have the following expression for the finite differences:

$$\Delta_h^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jh).$$

In the literature, there exists another definition of finite difference which is called "*central*" ([68]).

Definition 63. The *central finite difference of order* $m \in \mathbb{N}^*$ *associated to* f is defined recursively by

$$\delta_h^1 f(x) := f(x + h/2) - f(x - h/2),$$
$$\delta_h^m f(x) := \delta_h^1 \delta_h^{m-1} f(x),$$

where x, $h \in \mathbb{R}^d$.

Example 64. We find the expression

$$\delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$$

and

$$\delta_h^3 f(x) = f(x + 3\frac{h}{2}) - 3f(x + \frac{h}{2}) + 3f(x - \frac{h}{2}) - f(x - 3\frac{h}{2})$$

for the central finite difference of order 2 and 3 respectively.

Remark 65. As $\delta_h^1 f(x) = \Delta_h^1 f(x - \frac{h}{2})$, we can link the two concepts of finite difference together by the following formulae :

$$\delta_h^m f(x) = \Delta_h^m f(x - m\frac{h}{2}) \qquad \text{et} \qquad \delta_h^m f(x + m\frac{h}{2}) = \Delta_h^m f(x).$$

In particular we deduce the following formula for the central finite difference of f:

$$\delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + (j - \frac{m}{2})h).$$

Remark 66. In the following, we mostly consider the norm in the $L^p(\mathbb{R}^d)$ space (for $p \in [1, +\infty]$) of the finite difference of f. As

$$\|\Delta_h^m f\|_{L^p} = \|\delta_h^m f\|_{L^p}$$

for $f \in L^p(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, we can use either central or forward finite differences and move from one concept to another depending on our needs without modifying the results of this thesis.

For the results of this thesis, we need to connect and control finite differences of different orders together. First, let us remark that it is easy to control finite differences of order $m+1$ by finite differences of order m. Indeed, we have

$$|\Delta_h^{m+1} f(x)| = |\Delta_h^m f(x+h) - \Delta_h^m f(x)| \leq 2 \sup_{y \in \{x, x+h\}} |\Delta_h^m f(y)|.$$

However, the converse is less trivial: it is more complicated to control finite differences of order m by finite differences of order $m+1$. To obtain such a result, we need two lemmata.

Lemma 67. *Let $m \in \mathbb{N}^*$ and f be a function defined on $\mathbb{R}^d$. We have*

$$\Delta_{2h}^m f(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta_h^m f(x + jh)$$

for all x,$h \in \mathbb{R}^d$.

Proof. We proceed by induction on m. For $m = 1$, this results immediately from

$$\Delta_{2h}^1 f(x) = f(x+2h) - f(x+h) + f(x+h) - f(x) = \Delta_h^1 f(x+h) + \Delta_h^1 f(x).$$

We suppose that the result is true for m and we prove that it is still true for $m+1$. We obtain successively

$$\begin{aligned}
\Delta_{2h}^{m+1} f(x) &= \Delta_{2h}^m (\Delta_{2h}^1) f(x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta_h^m (\Delta_{2h}^1 f)(x + jh) \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta_h^m (\Delta_h^1 f(x + (j+1)h) + \Delta_h^1 f(x + jh)) \\
&= \sum_{j=0}^{m} \left(\binom{m+1}{j+1} - \binom{m}{j+1} \right) \Delta_h^{m+1} f(x + (j+1)h) + \sum_{j=0}^{m} \binom{m}{j} \Delta_h^{m+1} f(x + jh) \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta_h^{m+1} f(x + jh) + \sum_{j=1}^{m} \left(\binom{m+1}{j} - \binom{m}{j} \right) \Delta_h^{m+1} f(x + jh) + \\
&\qquad \binom{m}{m} \Delta_h^{m+1} f(x + (m+1)h) \\
&= \binom{m+1}{0} \Delta_h^{m+1} f(x) + \binom{m+1}{m+1} \Delta_h^{m+1} f(x + (m+1)h) + \sum_{j=1}^{m} \binom{m+1}{j} \Delta_h^{m+1} f(x + jh) \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} \Delta_h^{m+1} f(x + jh).
\end{aligned}$$

□

Lemma 68. *Let k, $m \in \mathbb{N}^*$ and f be a function defined on $\mathbb{R}^d$. We have*

$$\Delta_{kh}^m f(x) = \sum_{i_1=0}^{k-1} \dots \sum_{i_m=0}^{k-1} \Delta_h^m f(x + i_1 h + \dots + i_m h)$$

for all x, $h \in \mathbb{R}^d$.

Proof. We proceed by induction on m. For $m = 1$, there is nothing to prove. We suppose the result is true for m and we show that it is still true for $m+1$. We obtain successively

$$\begin{aligned}\Delta_{kh}^{m+1} f(x) &= \Delta_{kh}^{1}\Delta_{kh}^{m} f(x)\\ &= \Delta_{kh}^{m} f(x+kh) - \Delta_{kh}^{m} f(x)\\ &= \sum_{i_1=0}^{k-1} \dots \sum_{i_m=0}^{k-1} (\Delta_h^m f(x+kh+i_1h+\dots+i_mh) - \Delta_h^m f(x+i_1h+\dots+i_mh))\\ &= \sum_{i_1=0}^{k-1} \dots \sum_{i_m=0}^{k-1} \Delta_{kh}^{1}\Delta_h^m f(x+i_1h+\dots+i_mh),\end{aligned}$$

so the conclusion by the case $m = 1$. □

The next proposition is an important result that is used in section 3.1 to explicitly control finite differences of order m by finite differences of order $m+1$.

Proposition 69. *Let $m \in \mathbb{N}^*$ and f be a function defined on $\mathbb{R}^d$. We have*

$$|\Delta_h^m f(x+mh)| \le \frac{m}{2} \sup_{j\in\mathbb{N}_0} |\Delta_h^{m+1} f(x+jh)| + \frac{1}{2^m}|\Delta_{2h}^m f(x+(m-1)h)|$$

for all x, $h \in \mathbb{R}^d$.

Proof. We proceed by induction on m. For $m = 1$, we easily check that we have

$$\begin{aligned}|\Delta_h^1 f(x+h)| &= |f(x+2h) - f(x+h)|\\ &\le \frac{1}{2}|f(x+2h) - 2f(x+h) + f(x)| + \frac{1}{2}|f(x+2h) - f(x)|\\ &\le \frac{1}{2}|\Delta_h^2 f(x)| + \frac{1}{2}|\Delta_{2h}^1 f(x)|\end{aligned}$$

for all x, $h \in \mathbb{R}^d$. We suppose that the result is true for m and we prove that it is still true for $m+1$. Let x, $h \in \mathbb{R}^d$. By the induction hypothesis, we have

$$\begin{aligned}|\Delta_h^{m+1} f(x+(m+1)h)| &= |\Delta_h^m(\Delta_h^1 f)((x+h)+mh)|\\ &\le \frac{m}{2}\sup_{j\in\mathbb{N}_0} |\Delta_h^{m+1}(\Delta_h^1 f)((x+h)+jh)| +\\ &\qquad \frac{1}{2^m}|\Delta_{2h}^m(\Delta_h^1 f)((x+h)+(m-1)h)|.\end{aligned}$$

Let us remark that

$$\Delta_{2h}^m(\Delta_h^1 f) = \frac{1}{2}(\Delta_{2h}^m\Delta_{2h}^1 f - \Delta_{2h}^m\Delta_h^2 f).$$

By lemma 67, we have for all $y \in \mathbb{R}^d$

$$\begin{aligned}|\Delta_{2h}^m \Delta_h^2 f(y)| &\le \sum_{j=0}^{m} \binom{m}{j} |\Delta_h^{m+2} f(y+jh)| \\ &\le 2^m \sup_{j \in \{0,\ldots,m\}} |\Delta_h^{m+2} f(y+jh)|.\end{aligned}$$

So we obtain

$$|\Delta_{2h}^m (\Delta_h^1 f)(y)| \le \frac{1}{2} \left(|\Delta_{2h}^{m+1} f(y)| + \sup_{j \in \{0,\ldots,m\}} |\Delta_h^{m+2} f(y+jh)| 2^m \right)$$

for all $y \in \mathbb{R}^d$. As a result, we find

$$|\Delta_h^{m+1} f(x+(m+1)h)| \le \binom{m+1}{2} \sup_{j \in \mathbb{N}_0} |\Delta_h^{m+2} f(x+jh)| + \frac{1}{2^{m+1}} |\Delta_{2h}^{m+1} f(x+mh)|,$$

which allows to conclude. □

Chapter 2

Characterizations and basic properties of generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$

In this chapter we show that the main properties of the usual Hölder-Zygmund spaces are still satisfied in the generalized setting previously introduced. Consequently, such a generalization implies more flexibility for approximation purposes.

2.1 A reminder of the characterizations of Hölder-Zygmund spaces $\Lambda^{\alpha}(\mathbb{R}^d)$

Since the aim of the forthcoming subsections is to generalize properties related to the characterizations of the Hölder-Zygmund spaces $\Lambda^{\alpha}(\mathbb{R}^d)$, we first recall some of these properties. Proofs can be found in [42, 73, 84, 93] for example.

Let us first give a characterization in terms of convolution with smooth functions

Theorem 70. *Let $\alpha > 0$. Then $f \in \Lambda^{\alpha}(\mathbb{R}^d)$ if and only if there exists $C > 0$ and $\Phi \in D(\mathbb{R}^d)$ so that, with $\Phi_\delta(x) := \delta^{-d}\Phi(x/\delta)$,*

$$\|f - f \star \Phi_\delta\|_{L^\infty} \leq C\delta^{\alpha} \quad \forall \delta > 0.$$

Notation 71. Let $\mathbb{P}_m$ denote the set of polynomials of degree less or equal to $m \in \mathbb{N}_0$. The following result shows a close link between Hölder spaces and polynomial approximations.

Theorem 72. *Let $\alpha > 0$ and $f \in L^\infty(\mathbb{R}^d)$. The following assertions are equivalent:*

1. *$f \in \Lambda^{\alpha}(\mathbb{R}^d)$;*

2. *there exist a constant $C > 0$ and $J \in \mathbb{N}_0$ such that*

$$\inf_{P \in \mathbb{P}_{\lfloor \alpha \rfloor}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \leq C2^{-j\alpha}, \quad \forall x \in \mathbb{R}^d, j \geq J.$$

The next result establishes a strong connection between the somehow classical concept of regularity given by C^n spaces and the regularity expressed through Hölder spaces. Some authors (e.g. [73, 93]) even introduce these spaces in terms of such a characterization.

Proposition 73. *Let $\alpha > 0$. A function $f \in \Lambda^\alpha(\mathbb{R}^d)$ satisfies*[1]

1. $f \in C^{\lceil \alpha \rceil - 1}(\mathbb{R}^d)$,
2. $D^\nu f \in L^\infty(\mathbb{R}^d)$ $\forall |\nu| \le \lceil \alpha \rceil - 1$,
3. $\sup_{|h|\le 2^{-j}} \|\Delta_h^{\lfloor \alpha \rfloor + 1 - |\nu|} D^\nu f\|_{L^\infty} \le C 2^{-j(\alpha - |\nu|)} \quad \forall j \in \mathbb{N}_0, |\nu| \le \lceil \alpha \rceil - 1.$

Conversely, if $f \in C^{\lceil \alpha \rceil - 1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{|h|\le 2^{-j}} \|\Delta_h^{\lfloor \alpha \rfloor + 1 - |\nu|} D^\nu f\|_{L^\infty} \le C 2^{-j(\alpha - |\nu|)} \qquad \forall j \in \mathbb{N}_0, |\nu| = \lceil \alpha \rceil - 1,$$

then $f \in \Lambda^\alpha(\mathbb{R}^d)$.

It is also well known that the Hölder-Zygmund spaces are closely related to the Taylor approximation.

Theorem 74. *Let $N \in \mathbb{N}_0$, $\alpha > 0$ such that $N < \alpha < N + 1$. If $f \in \Lambda^\alpha(\mathbb{R}^d)$, then, for all $x \in \mathbb{R}^d$, we have*

$$f(x+h) = \sum_{|\nu|\le N} D^\nu f(x) \frac{h^\nu}{|\nu|!} + R_N(x,h), \quad \forall h \in \mathbb{R}^d$$

where $|R_N(x,h)| \le C|h|^\alpha$.

Conversely, if $f \in L^\infty(\mathbb{R}^d) \cap C^N(\mathbb{R}^d)$ satisfies

$$f(x+h) = \sum_{|\nu|\le N} D^\nu f(x) \frac{h^\nu}{|\nu|!} + R_N(x,h) \qquad \forall x, h \in \mathbb{R}^d,$$

where $|R_N(x,h)| \le C|h|^\alpha$, then $f \in \Lambda^\alpha(\mathbb{R}^d)$.

The next result is a characterization of Hölder-Zygmund spaces in terms of wavelet coefficients. The concept of wavelet and multiresolution analysis is recalled in section 5.4. A proof of this result can be found in [64, 93].

Theorem 75. *Let $\alpha > 0$ such that $\alpha \notin \mathbb{N}_0$. Given a multiresolution analysis of regularity $r > \alpha$ with $r \in \mathbb{N}_0$, the following assertions are equivalent:*

1. $f \in \Lambda^\alpha(\mathbb{R}^d)$;

[1] The function f in the next expressions should be understood by the reader as a function which is equal almost everywhere to the original function f and which is $(\lceil \alpha \rceil - 1)$-times continuously differentiable. The existence of such a function is demonstrated in proposition 80.

2. $\exists C > 0 : \begin{cases} |C_k| \leq C & \forall k \in \mathbb{Z}^d \\ |c^i_{j,k}| \leq C 2^{-j\alpha} & \forall j \in \mathbb{N}_0, \forall i \in \{1, \ldots, 2^d - 1\}, \forall k \in \mathbb{Z}^d, \end{cases}$

where C_k and $c^i_{j,k}$ denote wavelet coefficients of f associated respectively to the "father" and the "mother" wavelets (see section 5.4).

The following result shows that Hölder spaces can be written as a real interpolation of Sobolev spaces. Reminders about Sobolev spaces and the real interpolation theory are made in section 5.5. A proof of the following result can be found in [84].

Theorem 76. *Let N, M be two natural numbers and $\alpha > 0$ such that $N < \alpha < M$ and $\alpha = (1-\theta)N + \theta M$. Then*

$$\Lambda^\alpha(\mathbb{R}^d) = [W^\infty_N, W^\infty_M]_{\theta,\infty,J} = [W^\infty_N, W^\infty_M]_{\theta,\infty,K}.$$

2.2 Some preliminary results

The aim of this section is to present some basic results that are often used in the sequel.

Let $\rho \in D(\mathbb{R}^d)$ be a function whose support is included in the closed ball $B(0, \leq 1)$ and satisfies the following conditions:

1. $0 \leq \rho \leq 1$;
2. $\int_{\mathbb{R}^d} \rho(x) dx = 1$;
3. ρ is a radial function, i.e. we have $|x| = |y| \Rightarrow \rho(x) = \rho(y)$.

We set $\rho_\delta(x) := \delta^{-d} \rho(x/\delta)$ $\forall \delta > 0$. A classical example of such a function is given by

$$\rho(x) = \begin{cases} e^{-1/(1-|x|^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

up to a normalisation factor.

Proposition 77. *Let $m \in \mathbb{N}$, $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence and $f \in L^1_{loc}(\mathbb{R})$ be a function such that $\sup_{|h| \leq 2^{-j}} \|\Delta^m_h f\|_{L^\infty} \leq C\sigma_j$ $\forall j \in \mathbb{N}_0$. There exists $\Phi \in D(\mathbb{R})$ such that*

$$\sup_{\delta \leq 2^{-j}} \|f \star \Phi_\delta - f\|_{L^\infty} \leq C\sigma_j, \ \ \forall j \in \mathbb{N}_0\,.$$

Proof. We can increase m by 1, 2 or 3 units if necessary so that we can assume that m is equal to $2n$ where n is an odd integer. We set

$$\tilde{\Phi}(t) := \sum_{j=0}^{m/2-1} (-1)^j \binom{m}{j} \frac{1}{2j-m} \rho\left(\frac{t}{2j-m}\right),$$

where $\binom{m}{j} = \frac{m!}{(m-j)!j!}$ ($m, j \in \mathbb{N}_0$ and $m \geq j$). Let $c_m := \int \tilde{\Phi}(t)dt = \frac{1}{2}\binom{m}{m/2}$, $\Phi(t) := \tilde{\Phi}(t)/c_m$ and $\Phi_\delta(t) := \delta^{-1}\Phi(t/\delta)$. We have

$$\begin{aligned}
f \star \Phi_\delta(x) - f(x) &= \int f(x-t)\Phi_\delta(t)dt - f(x) \\
&= \int f(x-\delta t)\Phi(t)dt - f(x) \\
&= \frac{1}{c_m}\sum_{j=0}^{m/2-1}(-1)^j\binom{m}{j}\frac{1}{2j-m}\int f(x-\delta t)\rho\left(\frac{t}{2j-m}\right)dt - f(x) \\
&= \frac{1}{c_m}\sum_{j=0}^{m/2-1}(-1)^j\binom{m}{j}\int f(x-\delta(2j-m)t)\rho(t)dt - f(x) \\
&= \frac{1}{2c_m}\left(\sum_{\substack{j=0\\ j\neq m/2}}^{m}(-1)^j\binom{m}{j}\int f(x-\delta(2j-m)t)\rho(t)dt - 2c_m f(x)\right) \\
&= \frac{1}{2c_m}\int \Delta_{2\delta t}^m f(x)\rho(t)dt.
\end{aligned}$$

We conclude that

$$\sup_{\delta\leq 2^{-j}}\|f\star\Phi_\delta - f\|_{L^\infty} \leq \frac{2^m}{2c_m}C\sigma_j.$$

□

Lemma 78. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. If $f \in L^1_{loc}(\mathbb{R})$ is a function satisfying*

$$\|f\star\rho_{2^{-j}} - f\|_{L^\infty(\mathbb{R})} \leq C\sigma_j \quad \forall j\in\mathbb{N}_0,$$

then, for all $k\in\mathbb{N}_0$, we have

$$\|D^k\left(f\star\rho_{2^{-j}} - f\star\rho_{2^{-j+1}}\right)\|_{L^\infty(\mathbb{R})} \leq C'2^{jk}\sigma_j \qquad \forall j\in\mathbb{N}^*.$$

Proof. For $\delta > 0$, let us write

$$\begin{aligned}
f\star\rho_\delta - f\star\rho_{2\delta} =& \rho_\delta\star(f\star\rho_\delta - f\star\rho_{2\delta}) \\
&+ \rho_\delta\star(f - f\star\rho_\delta) \\
&- \rho_{2\delta}\star(f - f\star\rho_\delta)
\end{aligned}$$

One gets

$$\begin{aligned}
|D^k\rho_\delta\star(f\star\rho_\delta - f\star\rho_{2\delta})| &\leq ||D^k\rho_\delta||_{L^1}||f\star\rho_\delta - f\star\rho_{2\delta}||_{L^\infty} \\
&\leq C_1\delta^{-k}\left(||f\star\rho_\delta - f||_{L^\infty} + ||f - f\star\rho_{2\delta}||_{L^\infty}\right).
\end{aligned}$$

By considering $\delta = 2^{-j}$, the previous inequality leads to

$$|D^k \rho_{2^{-j}} \star (f \star \rho_{2^{-j}} - f \star \rho_{2^{-j+1}})| \leq C 2^{jk} C(\sigma_j + \sigma_{j-1}) \leq C 2^{jk} \sigma_j,$$

for all $j \in \mathbb{N}^*$. The two other terms in the decomposition of $f \star \rho_\delta(x) - f \star \rho_{2\delta}(x)$ can be handled in the same way. □

Remark 79. All the results from this section can easily be adapted to $\mathbb{R}^d$.

2.3 Generalized Hölder spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ and $C^k(\mathbb{R}^d)$ spaces

The next result binds the regularity of the elements of $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ to the classical notion of differentiability.

Proposition 80. *Let $m \in \mathbb{N}^*$, $k \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence such that*

$$\sum_{j=1}^{+\infty} 2^{jk} \sigma_j < \infty.$$

If $f \in L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^m f\|_{L^\infty} \leq C \sigma_j \quad \forall j \in \mathbb{N}_0,$$

then f is k-times continuously differentiable (in the sense that f coincides almost everywhere on $\mathbb{R}^d$ with a k-times continuously differentiable function).

Proof. Let Φ be the function defined by proposition 77 and $f_1 := f \star \Phi_{2^{-1}}$, $f_j := f \star (\Phi_{2^{-j}} - \Phi_{2^{-j+1}})$ $(j > 1)$. By proposition 77, we have $||f_j||_{L^\infty} \leq C\sigma_j$ for all $j \in \mathbb{N}^*$, where the constant C does not depend on j. We thus get

$$\sum_{j=1}^{k} ||f_j||_{L^\infty} \leq C \sum_{j=1}^{k} \sigma_j,$$

for all $k \in \mathbb{N}^*$. So, the series $\sum_{j=1}^{+\infty} f_j$ converges uniformly on $\mathbb{R}^d$ to a function which coincides almost everywhere with f. Moreover, we have

$$\|D^\alpha f_j\|_{L^\infty} \leq C 2^{jk} \sigma_j \quad \forall j \in \mathbb{N}^*, |\alpha| \leq k.$$

Therefore, the series $\sum_{j=1}^{\infty} D^\alpha f_j$ converges uniformly, which leads to the conclusion. □

Remark 81. This result does not depend on the order m of the finite difference.

Under the assumptions of theorem 33, this result can be rewritten in the following way.

Corollary 82. *Let $k \in \mathbb{N}_0$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$ and*

$$\sum_{j=1}^{+\infty} 2^{jk}\sigma_j < \infty.$$

We have

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\bar{s}(\sigma^{-1})}(\mathbb{R}^d) \subseteq C^k(\mathbb{R}^d).$$

If $\underline{s}(\sigma^{-1}) > 0$, the last result is always satisfied for $k = 0$. In this case, the elements of $\Lambda^{\sigma,\bar{s}(\sigma^{-1})}(\mathbb{R}^d)$ are continuous.

Remark 83. If we apply corollary 82 to the admissible sequence $\sigma_j = 2^{-j\alpha}$ ($\alpha > 0$), we recover the differentiability properties of classical Hölder-Zygmund spaces. Corollary 82 gives additional information about differentiability properties of those spaces. Indeed, it is well known that the elements of $\Lambda^m(\mathbb{R}^d)$ are at least $(m-1)$-times continuously differentiable ($m \in \mathbb{N}^*$) and that the elements of $\Lambda^{m+\epsilon}(\mathbb{R}^d)$ are m-times continuously differentiable for any $\epsilon > 0$. A question that naturally arises is the following: can we sensibly modify the admissible sequences of the usual Hölder spaces $\Lambda^m(\mathbb{R}^d)$ so that its elements become m-times continuously differentiable? The answer is also given by corollary 82, which states that if we replace the sequence $(2^{-jm})_j$ with $\sigma_j^{(m)} = 2^{-jm}j^{-1}\log(j)^{-(1+\epsilon)}$ ($j \in \mathbb{N}_0$, $j > 1$, $\epsilon > 0$), then we have $\Lambda^{\sigma^{(m)},m}(\mathbb{R}^d) \subset C^m(\mathbb{R}^d)$. This is a direct consequence of the study of Bertrand series, which is recalled in the next result:

Lemma 84. *Let α, $\beta \in \mathbb{R}$. The series*

$$\sum_{j\geq 2} \frac{1}{j^\alpha(\log(j))^\beta}$$

converges if and only if $\alpha > 1$ or ($\alpha = 1$ and $\beta > 1$)

Remark 85. Let $m \in \mathbb{N}_0$. If σ is an admissible sequence satisfying

$$\sum_{j=1}^{+\infty} 2^{j(m+1)}\sigma_j < \infty, \tag{2.1}$$

then the generalized Hölder-Zygmund space $\Lambda^{\sigma,m}(\mathbb{R}^d)$ is composed of constant functions.

Let us prove this assertion. If $f \in \Lambda^{\sigma,m}(\mathbb{R}^d)$, then the function f belongs to $C^{m+1}(\mathbb{R}^d)$ by proposition 80. By lemma 5, we find

$$\frac{|\Delta^{m+1}_{2^{-j}e_i} f(x)|}{2^{-j(m+1)}} \to |D^{m+1}_{x_i} f(x)|$$

and

$$\frac{|\Delta^{m+1}_{2^{-j}e_i} f(x)|}{2^{-j(m+1)}} \leq C 2^{j(m+1)}\sigma_j \to 0 \text{ if } j \to +\infty,$$

so that $D^{m+1}_{x_i} f = 0$ for all $i \in \{1, \dots, d\}$. For all $j \in \{1, \dots, d\}$, the function f can be written as

$$f(x_1, ..., x_j, ..., x_d) = \sum_{i=0}^{m} a^{(i)}_{x_1,...,x_{j-1},x_{j+1},...,x_d} x_j^i.$$

As $f \in L^\infty(\mathbb{R}^d)$, we have

$$f(x_1, ..., x_j, ..., x_d) = a^{(0)}_{x_1,...,x_{j-1},x_{j+1},...,x_d}$$

so that $D_{x_j} f = 0$. As a conclusion, the function f is constant.

The condition (2.1) implies that the sequence $(\sigma_j)_{j\in\mathbb{N}^*}$ converges faster to 0 than $(2^{-j(m+1)})_{j\in\mathbb{N}^*}$. In particular, if $m + 1 < \underline{s}(\sigma^{-1})$ $(m \in \mathbb{N}_0)$, then the spaces $\Lambda^{\sigma,m}$ are composed of constant functions.

2.4 A characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ in terms of convolution product

The aim of this section is to generalize theorem 70 to spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$. The following result gives the converse of proposition 77.

Proposition 86. *Let $m \in \mathbb{N}^*$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence satisfying*

$$\forall J \in \mathbb{N}_0, \quad \sum_{j=0}^{J} 2^{jm}\sigma_j \leq C 2^{Jm}\sigma_J \tag{2.2}$$

and

$$\forall J \in \mathbb{N}_0, \quad \sum_{j=J}^{+\infty} \sigma_j \leq C\sigma_J. \tag{2.3}$$

If $f \in L^\infty(\mathbb{R}^d)$ is a function for which there exists $\Phi \in D(\mathbb{R}^d)$ satisfying

$$\|f \star \Phi_{2^{-j}} - f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0,$$

then we have

$$\sup_{|h|\leq 2^{-j}} \|\Delta^m_h f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

Proof. We keep the same notations as the ones introduced in the proof of proposition 80. We know that $\Delta^m_h f = \sum_{j=1}^{+\infty} \Delta^m_h f_j$ with uniform convergence on $\mathbb{R}^d$. For all $N \in \mathbb{N}^*$, we have

$$\begin{aligned}\|\Delta_h^m f\|_{L^\infty} &\leq \sum_{j=1}^{N} \|\Delta_h^m f_j\|_{L^\infty} + \sum_{j=N+1}^{+\infty} \|\Delta_h^m f_j\|_{L^\infty} \\ &\leq C \sum_{j=1}^{N} |h|^m \sup_{|\alpha|=m} \|D^\alpha f_j\|_{L^\infty} + \sum_{j=N+1}^{+\infty} 2^m \|f_j\|_{L^\infty} \\ &\leq C|h|^m \sum_{j=1}^{N} 2^{jm}\sigma_j + C \sum_{j=N+1}^{+\infty} \sigma_j,\end{aligned}$$

so

$$\|\Delta_h^m f\|_{L^\infty} \leq C(1 + |h|^m 2^{Nm})\sigma_N.$$

We get

$$\sup_{|h|\leq 2^{-N}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_N \qquad \forall N \in \mathbb{N}_0,$$

which ends the proof. □

The following corollary is the main result of this section.

Corollary 87. (D.K., S. Nicolay) Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. We have

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \exists \Phi \in D(\mathbb{R}^d) \quad \sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \sup_{\delta\leq 2^{-j}} \|f \star \Phi_\delta - f\|_{L^\infty} \right) < \infty \right\}. \quad (2.4)$$

Moreover, the norm

$$\|f\|_{L^\infty} + \inf \left\{ \sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \sup_{\delta\leq 2^{-j}} \|f \star \Phi_\delta - f\|_{L^\infty} \right) \right\},$$

where the infimum is taken on the set of functions $\Phi \in D(\mathbb{R}^d)$ satisfying (2.4) and $\sup_{|\alpha|=\lfloor\overline{s}(\sigma^{-1})\rfloor+1} \|D^\alpha\Phi\|_{L^1(\mathbb{R}^d)} \leq 2^{\lfloor\overline{s}(\sigma^{-1})\rfloor+1}$, is equivalent to $\|f\|_{\Lambda^{\sigma,\overline{s}(\sigma^{-1})}}$.

Proof. This is a consequence of propositions 48, 77, 86 and from lemmata 35 and 53. □

2.5 A polynomial characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$

The following result shows that generalized Hölder-Zygmund spaces and polynomials approximations are closely linked. It is a generalization of theorem 72.

Theorem 88. *Let $m \in \mathbb{N}^*$, $f \in L^\infty(\mathbb{R}^d)$ be a continuous function on $\mathbb{R}^d$ and $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. The following assertions are equivalent:*

1. *there exists a constant* $C > 0$ *such that* $\sup_{|h|\le 2^{-j}} \|\Delta_h^m f\|_{L^\infty} \le C\sigma_j \ \forall j \in \mathbb{N}_0$,
2. *there exist a constant* $C > 0$ *and a natural number* J *such that*

$$\inf_{P\in\mathbb{P}_{m-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \le C\sigma_j,$$

$\forall x \in \mathbb{R}^d, j \ge J$.

Notation 89. We define the following notation:

$$B_h(x_0, r) := \{x \in B(x_0, r) : [x, x + mh] \subset B(x_0, r)\},$$

where $[a, b]$ refers to the segment in $\mathbb{R}^d$ joining the points a and b.

We need the following classical result ([24]).

Theorem 90. *Let* $m \in \mathbb{N}^*$ *and* $f \in L^\infty(\mathbb{R}^d)$*. There exists a constant* $C > 0$ *(which depends only on* m *and* d*) such that for all* $x_0 \in \mathbb{R}^d$ *and* $r > 0$*, we have*

$$\inf_{P\in\mathbb{P}_{m-1}} \|f - P\|_{L^\infty(B(x_0,r))} \le C \sup_{|h|\le r} \|\Delta_h^m f\|_{L^\infty(B_h(x_0,r))}.$$

Proof of theorem 88. Implication $1 \Rightarrow 2$ is a consequence of theorem 90. Let us prove that $2 \Rightarrow 1$. We remark that for all $x \in \mathbb{R}^d$ and $j \ge J$, there exists a polynomial $P \in \mathbb{P}_{m-1}$ such that

$$\sup_{y\in B(x,2^{-j})} |f(y) - P(y)| \le C\sigma_j.$$

For any polynomial $P \in \mathbb{P}_{m-1}$, we have

$$\begin{aligned} |\Delta_h^m f(x)| &= |\Delta_h^m (f - P)(x)| \\ &\le 2^m \sup_{y\in\{x,\ldots,x+mh\}} |f(y) - P(y)|. \end{aligned}$$

Then, for $|h| \le 2^{-(j+m)}$, we have

$$\begin{aligned} |\Delta_h^m f(x)| &\le 2^m C\sigma_j \\ &\le 2^m C d_0^{-m} \sigma_{j+m}, \end{aligned}$$

which ends the proof. □

One gets the following corollary.

Corollary 91. (D.K., S. Nicolay) Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. If $M \in \mathbb{N}_0$ is such that $M > \overline{s}(\sigma^{-1})$, then

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left(\sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \inf_{P\in\mathbb{P}_{M-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \right) \right) < \infty \right\}.$$

Moreover, the semi-norm $\sup_{x\in\mathbb{R}^d} \left(\sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \inf_{P\in\mathbb{P}_{M-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \right) \right)$ is equivalent to $|f|_{\Lambda^{\sigma,\overline{s}(\sigma^{-1})}}$.

2.6 A characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ in terms of derivatives

The goal of this section is to present a characterization of the generalized Hölder spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ in terms of derivatives of their elements. The following result generalizes proposition 73.

Proposition 92. (D.K., S. Nicolay) Let σ be an admissible sequence and N, M be two natural numbers such that $N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M$. If f belongs to $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$, then we have

1. $f \in C^N(\mathbb{R}^d)$,
2. $D^\nu f \in L^\infty(\mathbb{R}^d)$ $\forall |\nu| \leq N$,
3. $\sup_{|h|\leq 2^{-j}} \|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} \leq C\sigma_j 2^{j|\nu|} \quad \forall j \in \mathbb{N}_0, |\nu| \leq N$.

Conversely, if $f \in C^N(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} \leq C\sigma_j 2^{j|\nu|} \qquad \forall j \in \mathbb{N}_0, |\nu| = N,$$

then $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Proof. Let $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$. Using the same notations as in proposition 80, we have

$$\sum_{j=1}^{+\infty} D^\nu f_j = D^\nu f \text{ (uniformly)} \quad \forall |\nu| \leq N.$$

By lemma 78, we get

$$\sum_{j=1}^{+\infty} \|D^\nu f_j\|_{L^\infty} \leq C \sum_{j=1}^{+\infty} 2^{jN}\sigma_j < +\infty,$$

which proves the two first assertions. Let $\nu \in \mathbb{N}_0^d$ be a multi-index such that $|\nu| \leq N$, $h \in \mathbb{R}^d$ and $J \in \mathbb{N}_0$ such that $|h| \leq 2^{-J}$. By the mean value theorem, we have

$$\begin{aligned}
\|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} &\leq \sum_{j=1}^{J} \|\Delta_h^{M-|\nu|} D^\nu f_j\|_{L^\infty} + \sum_{j=J+1}^{+\infty} \|\Delta_h^{M-|\nu|} D^\nu f_j\|_{L^\infty} \\
&\leq \sum_{j=1}^{J} |h|^{M-|\nu|} \sup_{|\alpha|=M-|\nu|} \|D^{\alpha+\nu} f_j\|_{L^\infty} + C \sum_{j=J+1}^{+\infty} |h|^{N-|\nu|} \sup_{|\alpha|=N-|\nu|} \|D^{\alpha+\nu} f_j\|_{L^\infty} \\
&\leq C \sum_{j=1}^{J} |h|^{M-|\nu|} 2^{Mj}\sigma_j + C \sum_{j=J+1}^{+\infty} |h|^{N-|\nu|}\sigma_j 2^{Nj} \\
&\leq C 2^{J|\nu|}\sigma_J.
\end{aligned}$$

Let us prove the converse. Let $|h| \leq 2^{-j}$; by the mean value theorem, we have

$$\begin{aligned}\|\Delta_h^M f\|_{L^\infty} &\leq C|h|^N \sup_{|\nu|=N} \|\Delta_h^{M-N} D^\nu f\|_{L^\infty}\\ &\leq C 2^{-jN} 2^{jN} \sigma_j = C\sigma_j.\end{aligned}$$

Theorem 33 leads to the conclusion. □

We can explain the last result in the following way: the value of $\underline{s}(\sigma^{-1})$ characterizes the number of times a function of $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}$ is differentiable. On the other hand, the value of $\overline{s}(\sigma^{-1})$ is linked to the order of the finite difference.

The same proof as the one given in proposition 92 leads to the following result, which ends our study of the generalized Hölder spaces in terms of derivatives.

Lemma 93. *Let $N \in \mathbb{N}_0$, σ be an admissible sequence and $f \in C^N(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If there exists a natural number $M > N$ such that*

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{M-N} D^\nu f\|_{L^\infty} \leq C\sigma_j 2^{jN} \quad \forall |\nu| = N$$

then $f \in \Lambda^{\sigma,M-1}(\mathbb{R}^d)$.

The next result is a characterization of the generalized Hölder-Zygmund spaces in terms of derivatives.

Corollary 94. (D.K., S. Nicolay) Let σ be an admissible sequence and N, M be two natural numbers such that $N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M$. Then

$$\begin{aligned}B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) \cap C^N(\mathbb{R}^d) :&\\ \sup_{|h|\leq 2^{-j}} \|\Delta_h^{M-N} D^\nu f\|_{L^\infty} \leq C\sigma_j 2^{jN} \quad \forall j \in \mathbb{N}_0, |\nu| = N\}.& \quad (2.5)\end{aligned}$$

2.7 A Characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ in terms of Taylor decomposition

The first goal of this section is to present a particular case of admissible sequences called strong admissible sequences, which rely on strong assumptions about the convergence to 0. This concept has already been introduced by S. Jaffard and Y. Meyer ([65]) and by M. Clausel ([30]) in the particular case of moduli of continuity (in section 1.5, we have seen that a modulus of continuity leads to an admissible sequence, whereas the converse is false). Examples of such sequences are then provided to bring a better understanding of the underlying conditions.

Secondly, a characterization of spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ in terms of a Taylor decomposition is presented, under the assumption that the admissible sequence σ is strong. This characterization refines the polynomial approximation of those spaces obtained in the section 2.5:

the characterization given by corollary 91 gives an approximation of functions in terms of polynomials, but the choice of those polynomials depends on the scale j. The characterization presented here not only allows to break away from this dependence on the scale j, but also gives the expression of the polynomial that approximates f. This polynomial is expressed only in terms of derivatives of the function f.

First, we define the concept of strong admissible sequence.

Definition 95. An admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ is *strong of order* $N \in \mathbb{N}^*$ if it satisfies

$$\sum_{j=0}^{J} 2^{Nj}\sigma_j \leq C2^{NJ}\sigma_J, \tag{2.6}$$

$$\sum_{j=J}^{+\infty} 2^{(N-1)j}\sigma_j \leq C2^{(N-1)J}\sigma_J \tag{2.7}$$

for all $J \in \mathbb{N}_0$.

Before providing some examples of strong admissible sequences in the sequel, let us study the implications of being a strong admissible sequences on the Boyd indices. Those preliminary results lead to a better understanding of the underlying conditions. They also lead to an easier method for constructing such sequences.

The following result shows that the concept of strong admissible sequence implies strong conditions on the values of the Boyd indices $\underline{s}(\sigma^{-1})$ and $\overline{s}(\sigma^{-1})$.

Lemma 96. *If $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ is a strong admissible sequence of order $N \in \mathbb{N}^*$, then*

$$N-1 \leq \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) \leq N.$$

Proof. We have

$$2^{Nk}\sigma_k \leq \sum_{l=0}^{j+k} 2^{Nl}\sigma_l \leq C2^{N(j+k)}\sigma_{j+k},$$

so that

$$\frac{\sigma_k}{\sigma_{j+k}} \leq C2^{Nj},$$

which implies

$$\lim_{j\to+\infty} \frac{\log_2(\sup_k \frac{\sigma_k}{\sigma_{j+k}})}{j} \leq N$$

and $\overline{s}(\sigma^{-1}) \leq N$. Similarly, we have

$$2^{(N-1)(j+k)}\sigma_{j+k} \leq \sum_{l=k}^{+\infty} 2^{(N-1)l}\sigma_l \leq C2^{(N-1)k}\sigma_k$$

and

$$\frac{2^{(N-1)j}}{C} \leq \frac{\sigma_k}{\sigma_{j+k}},$$

so $N-1 \leq \underline{s}(\sigma^{-1})$. □

The following lemma gives some partial converse to the previous result.

Lemma 97. *Let $N \in \mathbb{N}^*$. If $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ is an admissible sequence satisfying*

$$N - 1 < \underline{s}(\sigma^{-1}) \quad \text{and} \quad \overline{s}(\sigma^{-1}) < N$$

then σ is a strong admissible sequence of order N.

Proof. Inequalities (2.6) and (2.7) follow from lemmata 53 and 54. □

The following result clarifies the concept of strong admissible sequence.

Lemma 98. *If $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ is a strong admissible sequence of order $N \in \mathbb{N}^*$, then there exist two constants C_1, $C_2 > 0$ such that*

$$C_1 2^{-jN} \leq \sigma_j \leq C_2 2^{-j(N-1)} \qquad \forall j \in \mathbb{N}_0\,.$$

As a consequence, we have

$$\Lambda^N(\mathbb{R}^d) \hookrightarrow \Lambda^{\sigma,\overline{s}(\sigma^{-1})} \hookrightarrow \Lambda^{N-1}(\mathbb{R}^d).$$

Proof. The second inequality follows from the definition. Let us prove the first one. We have

$$2^0\sigma_0 \leq \sum_{j=0}^{J} 2^{Nj}\sigma_j \leq C2^{NJ}\sigma_J \qquad \forall J \in \mathbb{N}_0\,.$$

The conclusion can be deduced from lemma 96. □

As a result, generalized Hölder spaces associated to strong admissible sequences "lie between" classical Hölder spaces superscripted with two consecutive natural numbers.

Example 99. Let us give two examples of strong admissible sequences.

1. The admissible sequence $\sigma_j = 2^{-j\alpha}$ where $\alpha \in]0,+\infty[\setminus \mathbb{N}_0$ is strong of order $\lfloor\alpha\rfloor + 1$.
2. If $\alpha \in]0,+\infty[\setminus \mathbb{N}_0$ and $\beta \in \mathbb{R}$, the admissible sequence $\sigma_j = 2^{-j\alpha}j^\beta$ is strong of order $\lfloor\alpha\rfloor + 1$.

Example 100. Let us now give two examples of admissible sequences that are not strong.

1. If $\alpha \in \mathbb{N}^*$, then the sequence $\sigma_j = 2^{-j\alpha}$ is not strong of order $\alpha + 1$. It is not even strong for any order $N \in \mathbb{N}^*$. Indeed, in the case where $\alpha + 1 \leq N$, we have

$$\sum_{j=J}^{+\infty} 2^{(N-1)j}\sigma_j = +\infty$$

and condition (2.7) cannot be satisfied. If $\alpha > N$, then

$$2^{NJ}\sigma_J \to 0 \quad \text{if } J \to +\infty$$

and inequality (2.6) should imply $\sum_{j=0}^{J} 2^{Nj}\sigma_j \to 0$ if $J \to \infty$, which is absurd. Finally, if $\alpha = N$, we obtain

$$\sum_{j=0}^{J} 2^{Nj}\sigma_j = J$$

so that inequality (2.6) can not be satisfied.

2. Similarly, one can easily show that the sequence $\sigma_j = 2^{-j\alpha}j^{\beta}$ with $\alpha \in \mathbb{N}^*$ and $\beta \in \mathbb{R}$ is not strong for any natural number N.

Example 101. Let us consider the examples described in section 1.2 in order to determine if they are strong or not. If the Boyd indices are known, then lemmata 96, 97 and 98 can be used to give an answer to the question.

1. Let us consider example 18. The sequence $\psi = \sigma^{-1}$ converges to 0. Let us prove that ψ is not strong. If $j \in \mathbb{N}^*$, as ψ is constant for all the indices in the intervals of type $[j_{2n}, j_{2n+1}] \cap \mathbb{N}_0$, we have

$$\inf_{k\geq 0}\left(\frac{\psi_k}{\psi_{j+k}}\right) = 1$$

so that $\underline{s}(\psi^{-1}) = 0$. We can easily check that we have

$$\sup_{k\geq 0}\left(\frac{\psi_k}{\psi_{j+k}}\right) = 2^{2j}$$

so that $\overline{s}(\psi^{-1}) = 2$. Hence the conclusion follows from lemma 96.

2. Let us consider example 19. The sequence $\eta_j = 2^{-js}\psi_j$ satisfies

$$\inf_{k\geq 0}\left(\frac{\eta_k}{\eta_{j+k}}\right) = 2^{js}\inf_{k\geq 0}\left(\frac{\psi_k}{\psi_{j+k}}\right) = 2^{js}$$

and

$$\sup_{k\geq 0}\left(\frac{\eta_k}{\eta_{j+k}}\right) = 2^{js}2^{2j}.$$

This implies $\underline{s}(\eta^{-1}) = s$ and $\overline{s}(\eta^{-1}) = s + 2$, which leads to the conclusion.

3. We can modify the previous example in order to construct an admissible sequence which satisfies the conditions of lemma 97. This example has been published in [80]. Let $s_0 \geq 0$ and $s_1 > 0$. Let $(j_n)_{n\in\mathbb{N}_0}$ be defined as in example 18. We define σ by induction in the following way: we put $\sigma_0 = 1$ and

$$\sigma_{j+1} = \begin{cases} \sigma_j 2^{s_0} & \text{if } j_{2n} \leq j < j_{2n+1}, \\ \sigma_j 2^{s_0+s_1} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

Let us prove that $\underline{s}(\sigma) = s_0$ and $\overline{s}(\sigma) = s_0 + s_1$. Let us first remark that the sequence can be rewritten as

$$\sigma_j = 2^{s_0 j}\psi_j$$

where ψ is an admissible sequence satisfying

$$\psi_{j+1} = \begin{cases} \psi_j & \text{if } j_{2n} \leq j < j_{2n+1}, \\ \psi_j 2^{s_1} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

The following lemma allows to determine the Boyd indices of σ from the ones of ψ.

Lemma 102. *Let $\alpha \in \mathbb{R}$ and σ be an admissible sequence. We have*

$$\underline{s}(2^{\alpha j}\sigma_j) = \alpha + \underline{s}(\sigma) \quad \text{and} \quad \overline{s}(2^{\alpha j}\sigma_j) = \alpha + \overline{s}(\sigma).$$

Proof. This is an immediate consequence of the definition of the Boyd indices. □

Let us now determine the Boyd indices of the sequence ψ. Let $j \in \mathbb{N}_0$; for some sufficiently large value of $n \in \mathbb{N}_0$, we have $j_{2n} \leq j_{2n} + j < j_{2n+1}$, which implies

$$\underline{\psi}_j = \inf_{k \geq 0} \frac{\psi_{j+k}}{\psi_k} = \frac{\psi_{j_{2n}+j}}{\psi_{j_{2n}}} = 1.$$

So, we have $\underline{s}(\psi) = 0$. Similarly, for a fixed value $j \in \mathbb{N}_0$, there exists a natural number n sufficiently large so that $j_{2n+1} \leq j_{2n+1} + j < j_{2n+2}$, which implies

$$\overline{\psi}_j = \sup_{k \geq 0} \frac{\psi_{j+k}}{\psi_k} = \frac{\psi_{j_{2n+1}+j}}{\psi_{j_{2n+1}}} = 2^{s_1 j}.$$

We thus have $\overline{s}(\psi) = s_1$. The Boyd indices of the sequence σ are therefore $\underline{s}(\sigma) = s_0$ and $\overline{s}(\sigma) = s_0 + s_1$.

An interesting consequence can be deduced from this example: let $N \in \mathbb{N}^*$; for any α, β satisfying $N - 1 < \alpha \leq \beta < N$, we can construct a strong admissible sequence σ for which $\underline{s}(\sigma^{-1}) = \alpha$ and $\overline{s}(\sigma^{-1}) = \beta$.

Remark 103. The same proof as in proposition 92 applied to strong admissible sequences of order $L \in \mathbb{N}^*$ shows that proposition 92 is still true for $L - 1 \leq \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) \leq L$ (where the limit values are allowed).

Let us generalize theorem 74 to generalized Hölder-Zygmund spaces. We need a lemma to prove this result.

Lemma 104. *Let $f \in C^k(\mathbb{R}^d)$ where $k \in \mathbb{N}_0$. We have*

$$f(x+h) = \sum_{|\alpha| \leq k} D^\alpha f(x) \frac{h^\alpha}{|\alpha|!} + R_k(x,h) \frac{|h|^k}{k!} \qquad \text{for all } x,\, h \in \mathbb{R}^d$$

where

$$|R_k(x,h)| \leq \sum_{|\alpha|=k} \sup_{|l| \leq |h|} \|\Delta_l^1 D^\alpha f\|_{L^\infty}.$$

Proof. By considering the function g defined on $\mathbb{R}$ by $g : t \mapsto f(x + th)$, we can assume that $d = 1$. Using integration by parts, we find

$$\begin{aligned}
f(x+h) - f(x) &= \int_0^h Df(x+t)dt \\
&= Df(x+t)(t-h)|_0^h - \int_0^h D^2 f(x+t)(t-h)dt \\
&= Df(x)h - \int_0^h D^2 f(x+t)(t-h)dt \\
&= \cdots \\
&= \sum_{j=1}^{k-1} D^j(x)\frac{h^j}{j!} + (-1)^{k-1}\int_0^h D^k f(x+t)\frac{(t-h)^{k-1}}{(k-1)!}dt \\
&= \sum_{j=1}^{k-1} D^j(x)\frac{h^j}{j!} + (-1)^{k-1}\int_0^h D^k f(x)\frac{(t-h)^{k-1}}{(k-1)!}dt + \\
&\qquad (-1)^{k-1}\int_0^h (D^k f(x+t) - D^k f(x))\frac{(t-h)^{k-1}}{(k-1)!}dt \\
&= \sum_{j=1}^{k} D^j(x)\frac{h^j}{j!} + (-1)^{k-1}\int_0^h (D^k f(x+t) - D^k f(x))\frac{(t-h)^{k-1}}{(k-1)!}dt,
\end{aligned}$$

which leads to the conclusion if we set

$$R_k(x,h) = \frac{(-1)^{k-1}\int_0^h (D^k f(x+t) - D^k f(x))\frac{(t-h)^{k-1}}{(k-1)!}dt}{\int_0^h \frac{|t-h|^{k-1}}{(k-1)!}dt}.$$

□

The following result gives the Taylor approximation of elements in $\Lambda^{\sigma,\bar{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Corollary 105. (D.K., S. Nicolay) Let σ be a strong admissible sequence of order $N \in \mathbb{N}^*$. If $f \in \Lambda^{\sigma,\bar{s}(\sigma^{-1})}(\mathbb{R}^d)$, then, for all $x \in \mathbb{R}^d$, we have

$$f(x+h) = \sum_{|\nu| \le N-1} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_{N-1}(x,h)\frac{|h|^{N-1}}{(N-1)!}, \quad \forall h \in \mathbb{R}^d$$

where $|R_{N-1}(x,h)| \le C\sigma_j 2^{j(N-1)}$, $\forall |h| \le 2^{-j}$.

Conversely, if $f \in L^\infty(\mathbb{R}^d) \cap C^{N-1}(\mathbb{R}^d)$ satisfies

$$f(x+h) = \sum_{|\nu| \le N-1} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_{N-1}(x,h)\frac{|h|^{N-1}}{(N-1)!} \qquad \forall x, h \in \mathbb{R}^d \tag{2.8}$$

with $\sup_{x,|h|\le 2^{-j}} |R_{N-1}(x,h)| \le C\sigma_j 2^{j(N-1)}$ $\forall j \in \mathbb{N}^*$, then $f \in \Lambda^{\sigma,\bar{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Proof. Let $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$. We know that $f \in C^{N-1}(\mathbb{R}^d)$. By lemma 104, we get

$$f(x+h) = \sum_{|\nu|\leq N-1} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_{N-1}(x,h)\frac{|h|^{N-1}}{(N-1)!},$$

where $|R_{N-1}(x,h)| \leq C \sup_{\substack{|l|\leq|h| \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty}$. Proposition 92 leads to the conclusion.

The converse result is a consequence of theorem 88. □

2.8 A characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ in terms of Littlewood-Paley decomposition

The goal of this section is to prove a characterization of the generalized Hölder spaces in terms of the Littlewood-Paley decomposition. The results obtained in this section are a particular case of theorem 33. However, since they can easily be obtained from the previous results, they are presented here in order to have a consistent theory without using results from [97]. Moreover, they allow a better understanding of the hidden mechanisms behind the theory.

Let us recall the definition of Littlewood-Paley decomposition. Let $\hat{\varphi}$ be a function of the Schwartz space such that

$$\hat{\varphi}(\xi) = 1 \quad \text{if} \quad |\xi| \leq \frac{1}{2}$$

$$\hat{\varphi}(\xi) = 0 \quad \text{if} \quad |\xi| \geq 1$$

and

$$\hat{\psi}(\xi) := \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi).$$

We set $S_j(f) = \mathcal{F}^{-1}(\hat{\varphi}(2^{-j}\xi)\mathcal{F}f)$ and $\Delta_j(f) := S_{j+1}(f) - S_j(f) = \mathcal{F}^{-1}(\hat{\psi}(2^{-j}\xi)\mathcal{F}f)$ for all $f \in \mathcal{S}'(\mathbb{R}^d)$. Such a definition is motivated by the Bernstein inequalities (see appendix). It gives the following Littlewood-Paley decomposition[2]: we have

$$Id = S_0 + \Delta_0 + \Delta_1 + \ldots,$$

with convergence in $\mathcal{S}'(\mathbb{R}^d)$.

Since $\psi(x) = 2^d\varphi(2x) - \varphi(x)$ $(x \in \mathbb{R}^d)$, the function ψ has a vanishing integral. If the function f belongs to the space L^p for some $p \in [1,\infty]$, the functions S_jf and Δ_jf can be interpreted as a convolution product between f and a regular function that belongs to the Schwartz space. Indeed, one can prove[3] that functions $S_j(f)$ and $\Delta_j(f)$ belong to the space $L^p(\mathbb{R}^d)$ and that

$$S_j(f) = 2^{jd}\varphi(2^j\cdot) \star f \quad \text{and} \quad \Delta_j(f) = 2^{jd}\psi(2^j\cdot) \star f$$

for all $j \in \mathbb{Z}$.

[2] A proof of this result is given by proposition 191 in the appendix.

[3] See e.g. lemma 192 for a proof.

Remark 106. The support of the Fourier transform of $S_j(f)$ is included in the ball $B(0, \leq 2^j)$ and the support of the Fourier transform of $\Delta_j f$ is included in the annulus $B(0, \leq 2^{j+1}) \backslash B(0, < 2^{j-1})$. As a consequence, we have the following result: if $f \in L^p(\mathbb{R}^d)$ ($p \in [1, +\infty]$) then the functions $S_j(f)$ and $\Delta_j(f)$ belong to $C^\infty(\mathbb{R}^d)$ and satisfy

$$\|D^\alpha S_j(f)\|_{L^p} \leq 2^{j|\alpha|} \|S_j(f)\|_{L^p} \quad \text{and} \quad \|D^\alpha \Delta_j(f)\|_{L^p} \leq 2^{(j+1)|\alpha|} \|\Delta_j f\|_{L^p} \quad \forall \alpha \in \mathbb{N}_0^d .$$

This is a consequence of classical Bernstein's inequalities (see e.g. [93], p.32).

Remark 107. The function φ satisfies $\int_{\mathbb{R}^d} \varphi(x)dx > 0$ and $\int_{\mathbb{R}^d} x^\alpha \varphi(x)dx = 0$ for all $\alpha \in \mathbb{N}_0^d \backslash \{0\}$. Indeed, as $D^\alpha \mathcal{F}\varphi(0) = 0$ for $\alpha \neq 0$, we have

$$0 = \mathcal{F}(x^\alpha \varphi(x))(0) = \int_{\mathbb{R}^d} x^\alpha \varphi(x)dx.$$

As $\psi(x) = 2^d \varphi(2x) - \varphi(x)$, the moments of ψ are vanishing, i.e.

$$\int_{\mathbb{R}^d} x^\alpha \psi(x)dx = 0 \quad \forall \alpha \in \mathbb{N}_0^d .$$

For more information about the Littlewood-Paley decomposition, the reader can refer to [84].

The proof of the following result is essentially inspired from proposition 1.1.[4] in [65].

Proposition 108. *Let* $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ *be a strong admissible sequence of order* $N \in \mathbb{N}^*$ *and* $f \in L^\infty(\mathbb{R}^d)$. *If*

$$\|\Delta_j f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$$

then $f \in \Lambda^{\sigma, \overline{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Proof. By the Bernstein inequalities, we have $\Delta_j f \in C^\infty(\mathbb{R}^d)$ and

$$\|D^\alpha \Delta_j f\|_{L^\infty} \leq C 2^{|\alpha| j} \sigma_j, \quad \forall |\alpha| \leq N, \forall j \in \mathbb{N}_0 .$$

So, $S_0 f + \sum_{j=0}^{+\infty} \Delta_j f$ converges uniformly on $\mathbb{R}^d$. Let $x_0 \in \mathbb{R}^d$; we set

$$P_j(x - x_0) = \sum_{|\alpha| \leq N-1} \frac{(x - x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j f(x_0) \quad \forall j \in \mathbb{N}_0 \cup \{-1\} \tag{2.9}$$

(where $\Delta_{-1} = S_0$) and

$$P(x - x_0) = \sum_{j \geq -1} P_j(x - x_0). \tag{2.10}$$

Obviously, the assumption on σ implies that those polynomials are well-defined and of a degree less or equal to $N - 1$. Let $x \in \mathbb{R}^d$ and $j_0 \in \mathbb{N}^*$ such that

$$2^{-j_0} \leq |x - x_0| < 2^{-j_0+1}.$$

[4]The present proof also corrects some small mistakes found in [65], such as the introduction of polynomials in the integrals.

We have

$$|f(x) - P(x-x_0)| \leq \sum_{j \leq j_0} |\Delta_j f(x) - P_j(x-x_0)| + \sum_{j > j_0} |\Delta_j f(x) - P_j(x-x_0)|.$$

The Taylor formula gives a bound for the first term:

$$\begin{aligned} &\sum_{j \leq j_0} |x-x_0|^N \sup_{|\alpha|=N} \|D^\alpha \Delta_j f\|_{L^\infty} \\ &\leq C 2^{-Nj_0} \sum_{j \leq j_0} 2^{Nj} \sigma_j \\ &\leq C \sigma_{j_0}. \end{aligned}$$

The second term is bounded by

$$\begin{aligned} &\sum_{j > j_0} \left(\|\Delta_j f\|_{L^\infty} + \sum_{|\alpha| \leq N-1} |x-x_0|^{|\alpha|} \|D^\alpha \Delta_j f\|_{L^\infty} \right) \\ &\leq \sum_{j > j_0} C \left(\sigma_j + \sum_{|\alpha| \leq N-1} |x-x_0|^{|\alpha|} 2^{|\alpha| j} \sigma_j \right) \\ &\leq C 2^{-(N-1)j_0} \sum_{j > j_0} 2^{(N-1)j} \sigma_j \\ &\leq C \sigma_{j_0}. \end{aligned}$$

□

The previous result can be improved significantly: the next result shows that we can get rid of the assumption about the strong admissible sequence.

Proposition 109. *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$ and $f \in L^\infty(\mathbb{R}^d)$. If*

$$\|\Delta_j f\|_{L^\infty} \leq C \sigma_j \quad \forall j \in \mathbb{N}_0$$

then $f \in \Lambda^{\sigma, \overline{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Proof. Let $M > \overline{s}(\sigma^{-1})$ be a natural number. We know that $\underline{s}(\sigma^{-1}) > 0$, so that

$$f = S_0(f) + \sum_{j \in \mathbb{N}_0} \Delta_j f \quad \text{uniformly on } \mathbb{R}^d.$$

Let $x_0 \in \mathbb{R}^d$ and $J \in \mathbb{N}_0$; let us set

$$P_j(x-x_0) = \sum_{|\alpha| \leq M-1} \frac{(x-x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j f(x_0) \quad \forall j \in \mathbb{N}_0 \cup \{-1\} \tag{2.11}$$

with $\Delta_{-1} = S_0$ and

$$P_{x_0,J}(x-x_0) = \sum_{j=-1}^{J} P_j(x-x_0). \tag{2.12}$$

It is a polynomial of degree less or equal to $M-1$. Let $x \in \mathbb{R}^d$ be such that $|x-x_0| \leq 2^{-J}$. We have

$$|f(x)-P_{x_0,J}(x-x_0)| \leq \left|\sum_{j=-1}^{J}\left(\Delta_j f(x) - \sum_{|\alpha|\leq M-1}\frac{(x-x_0)^\alpha}{|\alpha|!}D^\alpha\Delta_j f(x_0)\right)\right| + \left|\sum_{j=J+1}^{+\infty}\Delta_j f(x)\right|.$$

Since $\underline{s}(\sigma^{-1}) > 0$, the second term is bounded by $C\sigma_J$. By the Taylor formula, the first term is bounded by

$$\begin{aligned}\sum_{j=-1}^{J} |x-x_0|^M \sup_{|\alpha|=M} \|D^\alpha \Delta_j f\|_{L^\infty} &\\ &\leq C2^{-JM}\sum_{j=-1}^{J} 2^{jM}\sigma_j \\ &\leq C\sigma_J,\end{aligned}$$

thanks to lemma 53, where the constant C is independent of x and J. Theorem 88 allows to conclude. □

Remark 110. The previous proof gives interesting information about the spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$. Indeed, equalities (2.11) and (2.12) give the polynomials that can be used in corollary 91. They are strongly linked to the Littlewood-Paley decomposition. Moreover, when the sequence σ is strong of order $N \in \mathbb{N}^*$, the polynomial given by (2.9) and (2.10) is independent of the scale $j \in \mathbb{N}_0$.

The following result essentially uses ideas exposed in [65].

Proposition 111. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a strong admissible sequence of order $N \in \mathbb{N}^*$ and $f \in L^\infty(\mathbb{R}^d)$. If $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ then*

$$\|\Delta_j f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

Proof. By corollary 105 and remark 107, we have

$$
\begin{aligned}
|\Delta_j f(x)| &= \left| \int_{\mathbb{R}^d} f(x+t) 2^{jd} \psi(2^j t) dt \right| \\
&\leq \int_{\mathbb{R}^d} |R_{N-1}(x,t)| \frac{|t|^{N-1}}{(N-1)!} 2^{jd} |\psi(2^j t)| dt \\
&\leq C \int_{\mathbb{R}^d} \sup_{\substack{x,|l|\leq|t| \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} \frac{|t|^{N-1}}{(N-1)!} 2^{jd} |\psi(2^j t)| dt \\
&\leq C \int_{\mathbb{R}^d} \sup_{\substack{x,|l|\leq|t|/2^j \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} 2^{-j(N-1)} |t|^{N-1} |\psi(t)| dt.
\end{aligned}
$$

One has

$$
\int_{|t|\leq 1} \sup_{\substack{x,|l|\leq|t|/2^j \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} 2^{-j(N-1)} |t|^{N-1} |\psi(t)| dt \leq C\sigma_j
$$

and, for $m \in \mathbb{N}_0$,

$$
\begin{aligned}
&\int_{2^m\leq|t|\leq 2^{m+1}} \sup_{\substack{x,|l|\leq|t|/2^j \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} 2^{-j(N-1)} |t|^{N-1} |\psi(t)| dt \\
&\leq C_M \int_{2^m\leq|t|\leq 2^{m+1}} \sup_{\substack{x,|l|\leq|t|/2^j \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} 2^{-j(N-1)} |t|^{N-1} \frac{1}{(1+|t|)^M} dt
\end{aligned}
$$

(where $M \in \mathbb{N}^*$ can be chosen arbitrarily large)

$$
\begin{aligned}
&\leq C_M \int_{\frac{1}{2}\leq|t|\leq 1} \sup_{\substack{|l|\leq 2^{m+1-j} \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} 2^{m(N+d-1)} 2^{-j(N-1)} 2^{-mM} dt \\
&\leq C_M 2^{m(N+d)} 2^{-j(N-1)} 2^{-mM} \int_{\frac{1}{2}\leq|t|\leq 1} \sup_{\substack{|l|\leq 2^{-j} \\ |\nu|=N-1}} \|\Delta_l^1 D^\nu f\|_{L^\infty} dt \\
&\leq C_M 2^{m(N+d-M)} \sigma_j
\end{aligned}
$$

(where a classical property of the finite differences has been applied in the last but one line, see lemma 67). Putting these inequalities together gives the desired result. □

The previous proof relies on the Taylor expansion of elements of generalized Hölder spaces. We now give another proof slightly less intuitive and more technical. It leads to a more general result.

Proposition 112. *Let* $N \in \mathbb{N}_0$ *and* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence such that*

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < N + 2.$$

If $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ *then*

$$\|\Delta_j f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0 .$$

Proof. We can suppose that the function φ defining the Littlewood-Paley decomposition is an even function. Indeed, one can easily show that this result does not depend on the considered function φ (one can use the same proof as for the independence of the choice of the unit partition for the classical Besov spaces, see [118] or [50]). As a consequence, the function ψ is also even. In proposition 92, we have $f \in C^N(\mathbb{R}^d)$ and

$$D^\alpha f \in L^\infty(\mathbb{R}^d) \quad \text{and} \quad \sup_{|h|\leq 2^{-j}} \|\Delta_h^2 D^\alpha f\|_{L^\infty} \leq C\sigma_j 2^{jN} \quad \forall |\alpha| \leq N.$$

We note that

$$\begin{aligned}\Delta_j D_{y_k} f(x) &= \mathcal{F}^{-1}(iy_k\hat{\psi}(2^{-j}\xi)\mathcal{F}f)(x)\\ &= D_{y_k}\Delta_j f(x)\end{aligned}$$

for all $k \in \{1, ..., d\}$. By induction, we find

$$\Delta_j D^\alpha f(x) = D^\alpha \Delta_j f(x) \quad \forall |\alpha| \leq N.$$

Then, we have

$$\begin{aligned}\Delta_j D^\alpha f(x) &= 2^{jd}\int_{\mathbb{R}^d} D^\alpha f(x-y)\psi(2^j y)dy\\ &= \frac{2^{jd}}{2}\int_{\mathbb{R}^d}(D^\alpha f(x+y) - 2D^\alpha f(x) + D^\alpha f(x-y))\psi(2^j y)dy\end{aligned}$$

because ψ is even with vanishing integrals. One of the Bernstein's inequalities states that

$$\|\Delta_j f\|_{L^\infty} \leq C2^{-jN} \sup_{|\alpha|=N} \|D^\alpha \Delta_j f\|_{L^\infty}$$

(see e.g. [84], prop.3.2 p.24), so that

$$\begin{aligned}\|\Delta_j f\|_{L^\infty} &\leq C2^{-jN}2^{jd}\int_{\mathbb{R}^d} \sup_{|\alpha|=N} \|\Delta_y^2 D^\alpha f\|_{L^\infty}|\psi(2^j y)|dy\\ &\leq C2^{-jN}\int_{\mathbb{R}^d} \sup_{|\alpha|=N} \|\Delta_{2^{-j}y}^2 D^\alpha f\|_{L^\infty}|\psi(y)|dy.\end{aligned}$$

On one hand, we have

$$\int_{|y|\leq 1} \sup_{|\alpha|=N} \|\Delta^2_{2^{-j}y} D^\alpha f\|_{L^\infty} |\psi(y)| dy \leq C\sigma_j 2^{jN}$$

and, on the other hand, we get

$$\begin{aligned}
&\int_{2^m\leq|y|\leq 2^{m+1}} \sup_{|\alpha|=N} \|\Delta^2_{2^{-j}y} D^\alpha f\|_{L^\infty} |\psi(y)| dy \\
&\qquad \leq \int_{2^m\leq|y|\leq 2^{m+1}} \sup_{\substack{|\alpha|=N \\ |h|\leq 2^{-j}}} \|\Delta^2_{2^{m+1}h} D^\alpha f\|_{L^\infty} |\psi(y)| dy \\
&\qquad \leq C 2^{2(m+1)} \int_{2^m\leq|y|\leq 2^{m+1}} \sup_{\substack{|\alpha|=N \\ |h|\leq 2^{-j}}} \|\Delta^2_h D^\alpha f\|_{L^\infty} |\psi(y)| dy \\
&\qquad \text{(lemma 67)} \\
&\qquad \leq C_M 2^{2m} 2^{Nj} \sigma_j \int_{2^m\leq|y|\leq 2^{m+1}} \frac{1}{(1+|y|)^M} dy \\
&\qquad \leq C_M 2^{2m} 2^{Nj} \sigma_j 2^{md} 2^{-mM}
\end{aligned}$$

for $M \in \mathbb{N}_0$ sufficiently large. Hence the conclusion follows by taking e.g. $M = d + 3$. □

In particular, we have obtained the following results:

Corollary 113. *Let* $N \in \mathbb{N}_0$ *and let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence such that*

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < N + 2.$$

Let $f \in L^\infty(\mathbb{R}^d)$*; the following assertions are equivalent:*

1. $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$*;*
2. $\exists C > 0 : \|\Delta_j f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$.

Corollary 114. *Let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be a strong admissible sequence of order* $N \in \mathbb{N}^*$ *and* $f \in L^\infty(\mathbb{R}^d)$*. The following assertions are equivalent:*

1. $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$*;*
2. $\exists C > 0 : \|\Delta_j f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$.

2.9 A characterization of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ in terms of wavelet coefficients

The goal of this section is to generalize theorem 75. Reminders about wavelets are done in section 5.4.

If we keep the same notations as the ones introduced in section 2.8, we note that $\Delta_j f(x) = 2^{jd} \int_{\mathbb{R}^d} f(t)\psi(2^j(x-t))dt$. So,

$$\Delta_j f(k2^{-j}) = 2^{jd} \int_{\mathbb{R}^d} f(t)\psi(k - 2^j t)dt$$

for all $k \in \mathbb{Z}^d$ and $j \in \mathbb{N}_0$. One can therefore conclude that the Littlewood-Paley decomposition seems to be very similar to a wavelet decomposition. Indeed, the wavelet coefficients of f somehow looks like a discretization of the functions $\Delta_j(f)$ at points $k2^{-j}$. Since Hölder spaces can be characterized in terms of the Littlewood-Paley decomposition, it is natural to ask whether those spaces admit a wavelet characterization as well.

The aim of this section is to obtain a wavelet characterization of the generalized Hölder-Zygmund spaces. For that purpose, we follow some ideas expressed in [64, 93]. We consider the Lemarié-Meyer wavelets (where the functions ϕ and ψ^i belong to the Schwartz space, [83]) and the Daubechies wavelets (where the functions ϕ and ψ^i are compactly supported and can be taken arbitrarily regular[5], [37]).

Theorem 115. (D.K., S. Nicolay) Let σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$.

1. Let us consider the Daubechies wavelets. If $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$, then there exists $C > 0$ such that
$$\begin{cases} |C_k| \leq C & \forall k \in \mathbb{Z}^d \\ |c_{j,k}^i| \leq C\sigma_j & \forall j \in \mathbb{N}_0, \forall i \in \{1,\dots,2^d-1\}, \forall k \in \mathbb{Z}^d. \end{cases} \tag{2.13}$$
If the assumption $\underline{s}(\sigma^{-1}) > 0$ is replaced by σ is a strong admissible sequence of order $N \in \mathbb{N}^*$, then this result holds for the Lemarié-Meyer wavelets.

2. Conversely, if $f \in L^\infty_{loc}(\mathbb{R}^d)$ and (2.13) holds, then $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Proof. Let $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ and let us prove that (2.13) holds. Let us consider the Daubechies wavelets. Let $M \in \mathbb{N}_0$ and $j_0 \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})$ and

$$\text{supp}\psi^i \subseteq B(0, \leq 2^{j_0}) \quad \forall i \in \{1,...,2^d-1\}.$$

We have

$$|C_k| = \left|\int_{\mathbb{R}^d} f(x)\phi(x-k)dx\right| \leq C\|f\|_{L^\infty}.$$

[5]Using the notation of the following, we consider that the multiresolution analysis is at least of regularity $r > \overline{s}(\sigma^{-1})$.

Using corollary 91, for $k \in \mathbb{Z}^d$ and $j \geq j_0$, let $P_{k/2^j,j-j_0}$ be a polynomial of degree less or equal to $M-1$ such that

$$\|f(\cdot) - P_{k/2^j,j-j_0}(\cdot - k/2^j)\|_{L^\infty(B(k/2^j,2^{-(j-j_0)}))} \leq C\sigma_{j-j_0}.$$

One gets

$$\begin{aligned}
|c^i_{j,k}| &= 2^{jd}\left|\int_{\mathbb{R}^d} f(x)\psi^i(2^j x - k)dx\right| \\
&= 2^{jd}\left|\int_{B\left(k/2^j,2^{-(j-j_0)}\right)} \left(f(x) - P_{k/2^j,j-j_0}(x - k/2^j)\right)\psi^i(2^j x - k)dx\right| \\
&\leq C\sigma_j \sup_{i\in\{1,\ldots,2^d-1\}} \|\psi^i\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Let us consider the Lemarié-Meyer wavelets and let σ be a strong admissible sequence of order $N \in \mathbb{N}^*$. We have $|C_k| \leq C\|f\|_{L^\infty}$ and

$$\begin{aligned}
|c^i_{j,k}| &= 2^{jd}\left|\int_{\mathbb{R}^d} f\left(\frac{k}{2^j} + (x - \frac{k}{2^j})\right)\psi^i(2^j x - k)dx\right| \\
&= 2^{jd}\int_{\mathbb{R}^d} |R_{N-1}\left(\frac{k}{2^j}, x - \frac{k}{2^j}\right)|\frac{|x-\frac{k}{2^j}|^{N-1}}{(N-1)!}|\psi^i(2^j x - k)|dx \\
&= 2^{jd}\int_{\mathbb{R}^d} |R_{N-1}\left(\frac{k}{2^j}, y\right)|\frac{|y|^{N-1}}{(N-1)!}|\psi^i(2^j y)|dy \\
&\leq C\int_{\mathbb{R}^d} \sup_{\substack{|h|\leq|y|/2^j \\ |\alpha|=N-1}} \|\Delta^1_h D^\alpha f\|_{L^\infty}|y|^{N-1}2^{-j(N-1)}|\psi^i(y)|dy
\end{aligned}$$

by using the same notations as the ones used in corollary 105. A similar proof as the one used in proposition 111 allows to conclude.

Now, suppose that we have

$$|C_k| \leq C \quad \text{and} \quad |c^i_{j,k}| \leq C\sigma_j \qquad \forall j \in \mathbb{N}_0, \forall i \in \{1,\ldots,2^d-1\}, \forall k \in \mathbb{Z}^d.$$

We need to check that conditions of proposition 92 are satisfied. For that purpose, we will use some of the ideas exposed in [64]. Let N, $M \in \mathbb{N}_0$ such that

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M \leq r.$$

Let us denote

$$f_{-1}(x) = \sum_{k\in\mathbb{Z}^d} C_k\phi(x-k)$$

and

$$f_j(x) = \sum_{i=1}^{2^d-1}\sum_{k\in\mathbb{Z}^d} c^i_{j,k}\psi^i(2^j x - k)$$

for all $j \in \mathbb{N}_0$. This last series converges uniformly on every compact set, because of the wavelet coefficient assumption and because of the decay of ϕ and ψ. Indeed, one has

$$\sum_{i=1}^{2^d-1} \sum_{|k|\leq L} |c_{j,k}^i \psi^i(2^j x - k)| \leq C\sigma_j \sum_{|k|\leq L} \frac{1}{(1+|2^j x - k|)^{2^{d+1}}} \leq C\sigma_j \quad (\forall L > 0),$$

where the constant C is uniform on x and j. The sequence of functions

$$\left(\sum_{i=1}^{2^d-1} \sum_{|k|\leq L} |c_{j,k}^i \psi^i(2^j x - k)| \right)_{L\in\mathbb{N}_0}$$

is a sequence of continuous functions on $\mathbb{R}^d$, increasing and converging to a continuous and bounded function on $\mathbb{R}^d$ (for the Daubechies wavelets, the continuity comes from the fact that the series can be reduced to a finite sum on each ball of $\mathbb{R}^d$; for the Meyer wavelets, we can check the continuity by using the convergence of the sequences and their rapidly decreasing properties). By applying a classical theorem of Dini, this series converges uniformly on every compact set of $\mathbb{R}^d$. So, the limit f_j is well-defined and has the same regularity as the wavelets. Let us write

$$g(x) = \sum_{j=-1}^{+\infty} f_j(x).$$

For all $j \geq -1$, we have

$$|f_j(x)| \leq C\sigma_j$$

and the series g converges uniformly on $\mathbb{R}^d$ to a function that belongs to $L^\infty(\mathbb{R}^d)$. So, one gets $f = g$. Similarly, by using the decay properties of $D^\beta\psi^i$ and $D^\beta\phi$, we obtain

$$|D^\beta f_j(x)| \leq C 2^{|\beta| j}\sigma_j, \quad \forall |\beta| \leq M.$$

So, we can differentiate the series $\sum_j f_j$ term by term up to order N. This proves that $f \in C^N(\mathbb{R}^d)$ and $|D^\beta f(x)| \leq C$ for all $|\beta| \leq N$. Let $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = N$, and let $h \in \mathbb{R}^d$ and $j_0 \in \mathbb{N}_0$ be such that $|h| < 2^{-j_0}$. We have

$$\begin{aligned}
&\|\Delta_h^{M-N} D^\alpha f\|_{L^\infty} \\
&\leq \sum_{j\leq j_0} \|\Delta_h^{M-N} D^\alpha f_j\|_{L^\infty} + \sum_{j=j_0+1}^{+\infty} 2^{M-N}\|D^\alpha f_j\|_{L^\infty} \\
&\leq C\sum_{j\leq j_0} |h|^{M-N} \sup_{|\beta|=M-N} \|D^{\beta+\alpha} f_j\|_{L^\infty} + C\sum_{j=j_0+1}^{+\infty} 2^{Nj}\sigma_j \\
&\leq C|h|^{M-N} 2^{Mj_0}\sigma_{j_0} + C 2^{N(j_0+1)}\sigma_{j_0+1} \\
&\leq C 2^{Nj_0}\sigma_{j_0}.
\end{aligned}$$

□

2.10 Generalized Hölder-Zygmund spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ and generalized (real) interpolation of Sobolev spaces

Several recent results express the generalized Besov spaces as an interpolation of other spaces. For example, generalized Besov spaces can be obtained through an interpolation of classical Besov spaces (see [7]). In our framework, this means that generalized Hölder spaces can be obtained as an interpolation of classical Hölder spaces.

On the other hand, it is well known that classical Besov spaces $B^s_{p,q}$ ($p, q \in [1, +\infty]$, $s \in \mathbb{R}$) can be obtained through an interpolation of Sobolev spaces $W^p_m(\mathbb{R}^d)$ ([3, 84]). A natural question to ask is whether this result is still true for generalized Hölder spaces. The goal of this section is to generalize theorem 76.

Classical concepts associated with interpolation spaces as well as the definition of Sobolev spaces are recalled in section 5.5. Let us now introduce some new concepts. In the sequel, we consider two Banach spaces A_0 and A_1, which are continuously embedded in a topological vector space V. So, spaces $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined Banach spaces. We recall that the operator J is defined for all $t > 0$ and $a \in A_0 \cap A_1$ by

$$J(t, a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

Let us give the definition of the generalized J-method of interpolation.

Definition 116. Let $\sigma = (\sigma_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two admissible sequences. We define the *generalized interpolation space* $[A_0, A_1]^\star_{\sigma,\psi,J}$ in the following way: we say that a belongs to $[A_0, A_1]^\star_{\sigma,\psi,J}$ if a can be written as $a = \sum_{j\in\mathbb{Z}} u_j$ with convergence in $A_0 + A_1$, where $u_j \in A_0 \cap A_1$ and $(\sigma_j J(\psi_j, u_j))_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

We recall that the operator K is defined for all $t > 0$ and $a \in A_0 + A_1$ by

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

Let us give the definition of the generalized K-method of interpolation.

Definition 117. Let $\sigma = (\sigma_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two admissible sequences. The *generalized interpolation space* $[A_0, A_1]^\star_{\sigma,\psi,K}$ is defined in the following way: we say that a belongs to $[A_0, A_1]^\star_{\sigma,\psi,K}$ if $a \in A_0 + A_1$ and $(\sigma_j K(\psi_j, a))_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

If $\sigma_j = 2^{-j\alpha}$ and $\psi_j = 2^j$, one recovers the classical real interpolation spaces $[A_0, A_1]_{\alpha,\infty,J}$ and $[A_0, A_1]_{\alpha,\infty,K}$.

The next result shows that the generalized J-method of interpolation and the generalized K-method of interpolation are equivalent under specific conditions.

Proposition 118. *Let N, $M \in \mathbb{N}_0$ and σ be an admissible sequence such that*

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M.$$

If A_1 is continuously embedded in A_0, one has

$$[A_0, A_1]^{\star}_{\theta,2^{j(M-N)},J} = [A_0, A_1]^{\star}_{\theta,2^{j(M-N)},K}$$

where θ is the admissible sequence defined by

$$\theta_j = \begin{cases} 2^{jN}\sigma_{-j}^{-1} & \forall j \in -\mathbb{N}_0 \\ (\theta_{-j})^{-1} & \forall j \in \mathbb{N}^* . \end{cases}$$

Proof. Let $f \in [A_0, A_1]^{\star}_{\theta,2^{j(M-N)},J}$. This function can be written as $f = \sum_{j\in\mathbb{Z}} f_j$ where the series converges in A_0 and where the functions f_j satisfy

$$\|f_j\|_{A_0} + 2^{j(M-N)}\|f_j\|_{A_1} \leq C\theta_j^{-1} \qquad \forall j \in \mathbb{Z}.$$

Let us set $b_j = \sum_{l=-\infty}^{j-1} f_l$ and $c_j = \sum_{l=j}^{+\infty} f_l$ for all $j \in \mathbb{Z}$. We have $b_j \in A_0$ and $c_j \in A_1$. Let us prove that the inequality

$$\theta_j(\|b_j\|_{A_0} + 2^{j(M-N)}\|c_j\|_{A_1}) \leq C \qquad \forall j \in \mathbb{Z}$$

holds.

1. If $j < 0$, then

$$\begin{aligned} \|b_j\|_{A_0} &\leq \sum_{l=-\infty}^{j-1} \|f_l\|_{A_0} \\ &\leq C \sum_{l=-j+1}^{+\infty} \theta_{-l}^{-1} = C \sum_{l=-j+1}^{+\infty} 2^{lN}\sigma_l \\ &\leq C2^{-jN}\sigma_{-j} = C\theta_j^{-1}, \end{aligned}$$

and

$$\begin{aligned} \|c_j\|_{A_1} &\leq \sum_{l=j}^{+\infty} \|f_l\|_{A_1} \\ &\leq C\sum_{l=j}^{+\infty} \theta_l^{-1} 2^{-l(M-N)} \\ &\leq C\sum_{l=1}^{-j} 2^{l(M-N)}\theta_{-l}^{-1} + C\sum_{l=0}^{+\infty} 2^{-l(M-N)}\theta_l^{-1} \\ &\leq C2^{-jM}\sigma_{-j} + C \leq C2^{-j(M-N)}\theta_j^{-1}. \end{aligned}$$

2. If $j \geq 0$, then

$$\begin{aligned}\|b_j\|_{A_0} &\leq \sum_{l=-\infty}^{0} \|f_l\|_{A_0} + C\sum_{l=1}^{j-1} \|f_l\|_{A_1} \\ &\leq C + C\sum_{l=1}^{j-1} 2^{-lM}\sigma_l^{-1} \\ &\leq C \leq C\theta_j^{-1},\end{aligned}$$

and

$$\begin{aligned}\|c_j\|_{A_1} &\leq \sum_{l=j}^{+\infty} \|f_l\|_{A_1} \leq C\sum_{l=j}^{+\infty} 2^{-lM}\sigma_l^{-1} \\ &\leq C2^{-jM}\sigma_j^{-1},\end{aligned}$$

where we have used the relation $\overline{s}(\sigma^{-1}) < M$ in the last inequality.

Let $f \in [A_0, A_1]^{\star}_{\theta,2^{j(M-N)},K}$; for all $j \in \mathbb{Z}$, there exist $b_j \in A_0$ and $c_j \in A_1$ such that $f = b_j + c_j$ and

$$\|b_j\|_{A_0} + 2^{j(M-N)}\|c_j\|_{A_1} \leq C\theta_j^{-1}.$$

Let us write $b_0 = \sum_{j=-\infty}^{-1}(b_{j+1} - b_j)$, with convergence in A_0. Similarly, let $c_0 = \sum_{j=0}^{+\infty}(c_j - c_{j+1})$, with convergence in A_1. Let us set

$$f_j = \begin{cases} b_{j+1} - b_j & \text{if } j \in -\mathbb{N}^*, \\ c_j - c_{j+1} & \text{if } j \in \mathbb{N}_0. \end{cases}$$

As $b_{j+1} - b_j = c_j - c_{j+1}$ (for all $j \in \mathbb{Z}$), we have $f = \sum_{j\in\mathbb{Z}} f_j$ in A_0, where $f_j \in A_1$ for all $j \in \mathbb{Z}$. Moreover, we have

$$\|f_j\|_{A_0} = \|b_{j+1} - b_j\|_{A_0} \leq C\theta_j^{-1}$$

and

$$\|f_j\|_{A_1} = \|c_{j+1} - c_j\|_{A_1} \leq C2^{-j(M-N)}\theta_j^{-1},$$

which leads to the conclusion. □

The following theorem is the main result of this section, and show that generalized Hölder spaces can be obtained through a generalized interpolation of Sobolev spaces. It is a consequence of the characterization of generalized Hölder spaces by derivatives and by the Littlewood-Paley decomposition.

Theorem 119. (D.K., S. Nicolay) Let N, $M \in \mathbb{N}_0$ and σ be an admissible sequence such that

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M.$$

We have

$$\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = [W^\infty_N, W^\infty_M]^\star_{\theta,2^{j(M-N)},J} = [W^\infty_N, W^\infty_M]^\star_{\theta,2^{j(M-N)},K}$$

where θ is the admissible sequence defined by

$$\theta_j = \begin{cases} 2^{jN}\sigma^{-1}_{-j} & \forall j \in -\mathbb{N}_0 \\ (\theta_{-j})^{-1} & \forall j \in \mathbb{N}^* . \end{cases}$$

Proof. Let us prove that we have $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = [W^\infty_N, W^\infty_M]^\star_{\theta,2^{j(M-N)},J}$.

Let $f \in \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$. We denote

$$u_j = \begin{cases} 0 & \text{if } j \in \mathbb{Z},\ j > 1, \\ S_0(f) & \text{if } j = 1, \\ \Delta_{-j}(f) & \text{if } j \in \mathbb{Z},\ j < 1. \end{cases}$$

By Bernstein's inequalities, the series $\sum_{j\in\mathbb{Z}} u_j$ converges in W^N_∞ and we have $u_j \in W^M_\infty$. Moreover, we have $\theta_j J(2^{j(M-N)}, u_j) \leq C$ by theorem 33.

Let $f \in [W^\infty_N, W^\infty_M]^\star_{\theta,2^{j(M-N)},J}$. Let us check that the conditions of proposition 92 are satisfied. Let $(f_j)_{j\in\mathbb{Z}}$ be a sequence of functions of $W^\infty_M(\mathbb{R}^d)$ such that $\sum_{j\in\mathbb{Z}} f_j = f$ with convergence in $W^\infty_N(\mathbb{R}^d)$ and such that

$$\theta_j J(2^{j(M-N)}, f_j) \in l^\infty(\mathbb{Z}).$$

By modifying the functions f_j on some negligible set, we can suppose that they belong to the space $C^{M-1}(\mathbb{R}^d)$ (see remark 197). Let $|\alpha| \leq N$. We have

$$\sum_{l=0}^{+\infty} \|D^\alpha f_l\|_{L^\infty} \leq C \sum_{l=0}^{+\infty} 2^{-l(M-N)}\theta_l^{-1} = C \sum_{l=0}^{+\infty} 2^{-lM}\sigma_l^{-1}$$

which is bounded because of the inequality $\overline{s}(\sigma^{-1}) < M$. Moreover,

$$\sum_{l=-\infty}^{-1} \|D^\alpha f_l\|_{L^\infty} \leq C \sum_{l=-\infty}^{-1} \theta_l^{-1} = C \sum_{l=1}^{+\infty} 2^{lN}\sigma_l.$$

Putting these inequalities together, we find that $f \in C^N(\mathbb{R}^d)$ and $D^\alpha f \in L^\infty$ $\forall |\alpha| \leq N$. Let $h \in \mathbb{R}^d$ be such that $|h| \leq 2^{-j}$ and $|\alpha| = N$. We note that

$$\Delta_h^{M-N} D^\alpha f = \sum_{l\in\mathbb{Z}} \Delta_h^{M-N} D^\alpha f_l \qquad \text{(uniformly)}.$$

By successive applications of the mean value theorem and by proposition 198, we have

$$\begin{aligned} \sum_{l=0}^{+\infty} \|\Delta_h^{M-N} D^\alpha f_l\|_{L^\infty} &\leq C|h|^{M-N} \sum_{l=0}^{+\infty} \|f_l\|_{W^\infty_M} \\ &\leq C 2^{-j(M-N)} \sum_{l=0}^{+\infty} 2^{-l(M-N)}\theta_l^{-1} \\ &\leq C 2^{-j(M-N)} \leq C 2^{Nj}\sigma_j, \end{aligned}$$

and

$$\begin{aligned}\sum_{l=-\infty}^{-1} \|\Delta_h^{M-N} D^\alpha f_l\|_{L^\infty} &= \sum_{l=-j}^{-1} \|\Delta_h^{M-N} D^\alpha f_l\|_{L^\infty} + \sum_{l=-\infty}^{-j-1} \|\Delta_h^{M-N} D^\alpha f_l\|_{L^\infty} \\ &\leq C|h|^{M-N} \sum_{l=-j}^{-1} \|f_l\|_{W_M^\infty} + C \sum_{l=-\infty}^{-j-1} \|f_l\|_{W_N^\infty} \\ &\leq C2^{-j(M-N)} \sum_{l=-j}^{-1} 2^{-l(M-N)}\theta_l^{-1} + C \sum_{l=-\infty}^{-j-1} \theta_l^{-1} \\ &\leq C2^{-j(M-N)} \sum_{l=1}^{j} 2^{lM}\sigma_l + C \sum_{l=j+1}^{+\infty} 2^{lN}\sigma_l \\ &\leq C2^{jN}\sigma_j.\end{aligned}$$

One can conclude, since we have $\sup_{|h|\leq 2^{-j}} \|\Delta_h^{M-N} D^\alpha f\|_{L^\infty} \leq C2^{Nj}\sigma_j$.

□

In particular, the previous result can be applied to strong admissible sequences. The result can be restated as follows:

Corollary 120. *Let σ be a strong admissible sequence of order $N \in \mathbb{N}^*$ such that $\overline{s}(\sigma^{-1}) < N$. We have*

$$\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = [W_{N-1}^\infty, W_N^\infty]^\star_{\theta,2^j,J} = [W_{N-1}^\infty, W_N^\infty]^\star_{\theta,2^j,K}$$

where θ is the admissible sequence defined by

$$\theta_j = \begin{cases} 2^{j(N-1)}\sigma_{-j}^{-1} & \forall j \in -\mathbb{N}_0, \\ (\theta_{-j})^{-1} & \forall j \in \mathbb{N}^*. \end{cases}$$

In this last result, the assumption $N - 1 < \underline{s}(\sigma^{-1})$ is not necessary: inequality (2.7) is sufficient.

Remark 121. In particular, theorem 119 can be applied to classical Hölder spaces to obtain theorem 76. For $\alpha > 0$, let N and M be two natural numbers satisfying $N < \alpha < M$. We have

$$\Lambda^\alpha(\mathbb{R}^d) = [W_N^\infty, W_M^\infty]^\star_{\theta,2^{j(M-N)},J} = [W_N^\infty, W_M^\infty]^\star_{\theta,2^{j(M-N)},K}$$

where

$$\theta_j = 2^{-\alpha j}2^{Nj} \qquad \forall j \in \mathbb{Z}.$$

The following result, which is easy to prove, is taken from [84] (Prop.2.1.(A)).

Lemma 122. *Let $\lambda \in]0,1[$. For all $\rho > 1$, we have*

$$[A_0, A_1]^\star_{2^{-j\lambda},2^j,J} = [A_0, A_1]^\star_{\rho^{-j\lambda},\rho^j,J}.$$

So, if λ satisfies $\alpha = (1-\lambda)N + \lambda M$, then

$$\Lambda^\alpha(\mathbb{R}^d) = [W_N^\infty, W_M^\infty]^\star_{2^{-j\lambda},2^j,J} = [W_N^\infty, W_M^\infty]^\star_{2^{-j\lambda},2^j,K}.$$

2.11 A weak result of Lions-Peetre type for spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$

The results of this section are based on the idea of expressing a given function belonging to some Hölder-Zygmund space as a weighted sum of a more regular and a less regular function. This idea is an important concept from the real variables Lions-Peetre method of interpolation theory. Proof of the next result can be found in [73].

Theorem 123. *Let* $0 < \alpha_1 < \alpha < \alpha_2 < +\infty$*. Then,* $f \in \Lambda^{\alpha}(\mathbb{R})$ *if and only if there exists* $C > 0$ *such that for each* $0 < \lambda < 1$*, there exist* $F_1^\lambda \in \Lambda^{\alpha_1}(\mathbb{R})$*,* $F_2^\lambda \in \Lambda^{\alpha_2}(\mathbb{R})$ *satisfying* $f = F_1^\lambda + F_2^\lambda$ *and*

$$\|F_1^\lambda\|_{\Lambda^{\alpha_1}(\mathbb{R})} \leq C\lambda^{\alpha-\alpha_1}$$
$$\|F_2^\lambda\|_{\Lambda^{\alpha_2}(\mathbb{R})} \leq C\lambda^{\alpha-\alpha_2}.$$

The aim of this section is to look how theorem 123 can be transposed in the general setting. Indeed, we have the following partial result:

Proposition 124. *Let* $m \in \mathbb{N}^* \setminus \{1\}$*,* $\alpha > 0$ *satisfying* $1 \leq \alpha \leq m$*,* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence and* $f \in L^\infty(\mathbb{R})$ *be a function such that*

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

If

$$\sum_{j=1}^{+\infty} 2^{j(m-\alpha)}\sigma_j < +\infty,$$

then, for all $\lambda \in]0,1[$*, there exist two functions* $F_1^\lambda \in \Lambda^{m-(\alpha-1)}(\mathbb{R})$*,* $F_2^\lambda \in \Lambda^{m-\alpha}(\mathbb{R})$ *such that* $f = F_1^\lambda + F_2^\lambda$ *and for* $K_\lambda = \lfloor 2log_2(1/\lambda)\rfloor + 1$*, we have*

$$\sup_{|h|\leq 2^{-l}} \|\Delta_h^m F_2^\lambda\|_{L^\infty} \leq C_1 2^{-l(m-\alpha)} \sum_{j=K_\lambda+1}^{+\infty} 2^{j(m-\alpha)}\sigma_j$$

and

$$\sup_{|h|\leq 2^{-l}} \|\Delta_h^m F_1^\lambda\|_{L^\infty} \leq C_2 2^{-l(m-\alpha+1)} \sum_{j=1}^{K_\lambda} 2^{j(m-\alpha+1)}\sigma_j$$

where C_1 *and* C_2 *are two constants independent of* λ*.*

Proof. Let Φ be the function defined by proposition 77 and $f_1 := f \star \Phi_{2^{-1}}$, $f_j := f \star (\Phi_{2^{-j}} - \Phi_{2^{-j+1}})$ $(j > 1)$. By proposition 77, we have $||f_j||_{L^\infty} \leq C\sigma_j$ for all $j \in \mathbb{N}^*$, where the constant C does not depend on j. We thus get

$$\sum_{j=1}^{k} ||f_j||_{L^\infty} \leq C \sum_{j=1}^{k} \sigma_j,$$

for all $k \in \mathbb{N}^*$, which implies $f = \sum_{j=1}^{+\infty} f_j$ (with uniform convergence). By the mean value theorem and lemma 78, one has

$$|\Delta_h^m f_j(x)| \leq C|h|^m \|D^m f_j\|_{L^\infty} \leq C|h|^m 2^{mj}\sigma_j$$

and

$$|\Delta_h^m f_j(x)| \leq 2^m \|f_j\|_{L^\infty} \leq 2^m C\sigma_j,$$

for all $j \in \mathbb{N}^*$. This implies

$$\begin{aligned}|\Delta_h^m f_j(x)| &= |\Delta_h^m f_j(x)|^{1-\alpha/m} |\Delta_h^m f_j(x)|^{\alpha/m} \\ &\leq C|h|^{m-\alpha} 2^{j(m-\alpha)}\sigma_j\end{aligned}$$

for all $0 \leq \alpha \leq m$.

Let $\lambda \in]0,1[$ and $M \in \mathbb{N}^*$ satisfying $M - 1 \leq 2\log_2(1/\lambda) < M$. We set $F_1^\lambda := \sum_{j=1}^{M} f_j$ and $F_2^\lambda := \sum_{j=M+1}^{+\infty} f_j$. From the previous results, we have

$$|\Delta_h^m F_2^\lambda(x)| \leq C|h|^{m-\alpha} \sum_{j=M+1}^{+\infty} 2^{j(m-\alpha)}\sigma_j$$

and

$$|\Delta_h^m F_1^\lambda(x)| \leq C|h|^{m-\alpha+1} \sum_{j=1}^{M} 2^{j(m-\alpha+1)}\sigma_j.$$

□

Remark 125. Proposition 124 can easily be adapted to $\mathbb{R}^d$.

Chapter 3

Some applications of the generalized Hölder-Zygmund spaces

3.1 Generalized Hölder exponents

The main objective of this section is to give some conditions on admissible sequences $\sigma^{(\alpha)}$ ($\alpha > 0$) that lead to embedded generalized Hölder spaces, i.e. spaces such that $\alpha < \beta$ implies $\Lambda^{\sigma^{(\beta)},\beta}(\mathbb{R}^d) \subseteq \Lambda^{\sigma^{(\alpha)},\alpha}(\mathbb{R}^d)$. For such spaces, we can define a generalized Hölder exponent of any function $f \in L^\infty(\mathbb{R}^d)$ by

$$H_f^{\sigma^{(\cdot)}} = \sup\{\alpha > 0 : f \in \Lambda^{\sigma^{(\alpha)},\alpha}(\mathbb{R}^d)\}.$$

This exponent gives information about the regularity of the function f.

3.1.1 Preliminary results

The two main results of this section are the following. The first one is expressed in terms of d_0 and the second one is expressed in terms of d_1 (where d_0 and d_1 come from (1.2)).

Proposition 126. *Let $p \in [1,\infty]$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $f \in L^p(\mathbb{R}^d)$ satisfying*

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^M f\|_{L^p} \leq C\sigma_j \qquad \forall j \in \mathbb{N}_0$$

with $M \in \mathbb{N}^* \setminus \{1\}$. We have three different cases:

1. *if $1 < 2^{M-1}d_0$, then we have*

$$\sup_{|h|\leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C\sigma_J + C2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^*, \tag{3.1}$$

2. *if $1 > 2^{M-1}d_0$, then we have*

$$\sup_{|h|\leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C(2^{M-1}d_0)^{-J}\sigma_J + C2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^*, \tag{3.2}$$

3. if $1 = 2^{M-1}d_0 \Leftrightarrow d_0 = 2^{-(M-1)}$*, then we have*

$$\sup_{|h|\le 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \le CJ\sigma_J + C2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^* . \tag{3.3}$$

Proposition 127. *Let* $p \in [1,\infty]$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence and* $f \in L^p(\mathbb{R}^d)$ *satisfying*

$$\sup_{|h|\le 2^{-j}} \|\Delta_h^{M} f\|_{L^p} \le C\sigma_j \qquad \forall j \in \mathbb{N}_0$$

with $M \in \mathbb{N}^* \setminus \{1\}$*. We have three different cases:*

1. if $1 < 2^{M-1}d_1$*, then we have*

$$\sup_{|h|\le 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \le Cd_1^J + C2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^*, \tag{3.4}$$

2. if $1 > 2^{M-1}d_1$*, then we have*

$$\sup_{|h|\le 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \le C2^{-(M-1)j} \qquad \forall J \in \mathbb{N}^*, \tag{3.5}$$

3. if $1 = 2^{M-1}d_1 \Leftrightarrow d_1 = 2^{-(M-1)}$*, then we have*

$$\sup_{|h|\le 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \le CJ2^{-(M-1)J} \qquad \forall J \in \mathbb{N}^* . \tag{3.6}$$

The proposition 128 links finite differences of different orders. The proof can be obtained through an easy adaptation of proposition 69.

Proposition 128. *Let* $m \in \mathbb{N}^*$, $p \in [1,+\infty]$ *and* $f : \mathbb{R}^d \to \mathbb{R}$*. We have*

$$\|\Delta_h^m f\|_{L^p} \le \frac{m}{2}\|\Delta_h^{m+1} f\|_{L^p} + \frac{1}{2^m}\|\Delta_{2h}^m f\|_{L^p}$$

for all $h \in \mathbb{R}^d$*.*

Proof of proposition 126. Let us consider the following sum:

$$\sum_{j=0}^{J} 2^{-j(M-1)} \left(2^{M-1} \sup_{|h|\le 2^{-J}} \|\Delta_{2^j h}^{M-1} f\|_{L^p} - \sup_{|h|\le 2^{-J}} \|\Delta_{2(2^j h)}^{M-1} f\|_{L^p} \right)$$
$$= 2^{M-1} \sup_{|h|\le 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} - 2^{-J(M-1)} \sup_{|h|\le 2^{-J}} \|\Delta_{2^{J+1}h}^{M-1} f\|_{L^p}, \quad J \in \mathbb{N}_0 .$$

For all $J \in \mathbb{N}_0$, we have

$$2^{M-1} \sup_{|h| \leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C \sum_{j=0}^{J} 2^{-j(M-1)} \sup_{|h| \leq 2^{-(J-j)}} \|\Delta_h^M f\|_{L^p} + 2^{-J(M-1)} \sup_{|h| \leq 2^{-J}} \|\Delta_{2^{J+1}h}^{M-1} f\|_{L^p}$$
$$\leq C \left(\sum_{j=0}^{J} (2^{-(M-1)} d_0^{-1})^j \right) \sigma_J + C 2^{-J(M-1)},$$

hence the conclusion. □

Proof of proposition 127. We proceed similarly as in the proof of proposition 126. We have

$$2^{M-1} \sup_{|h| \leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C \sum_{j=0}^{J} 2^{-j(M-1)} \sup_{|h| \leq 2^{-(J-j)}} \|\Delta_h^M f\|_{L^p} + 2^{-J(M-1)} \sup_{|h| \leq 2^{-J}} \|\Delta_{2^{J+1}h}^{M-1} f\|_{L^p}$$
$$\leq C \left(\sum_{j=0}^{J} (2^{-(M-1)} d_1^{-1})^j \right) d_1^J + C 2^{-J(M-1)},$$

which ends the proof. □

Let us note that each right member of inequalities (3.1)-(3.6) are new admissible sequences, by lemma 13.

Propositions 126 and 127 can be stated in terms of the Boyd index $\overline{s}(\sigma^{-1})$ instead of d_0 and d_1.

Corollary 129. *Let $p \in [1, \infty]$, $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence and $f \in L^p(\mathbb{R}^d)$ satisfying*

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^M f\|_{L^p} \leq C \sigma_j \qquad \forall j \in \mathbb{N}_0$$

with $M \in \mathbb{N}^* \setminus \{1\}$. We have the two following cases:

1. *if $\overline{s}(\sigma^{-1}) < M-1$, then*

$$\sup_{|h| \leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C \sigma_J + C 2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^*,$$

2. *if $\overline{s}(\sigma^{-1}) \geq M-1$, then*

$$\sup_{|h| \leq 2^{-J}} \|\Delta_h^{M-1} f\|_{L^p} \leq C_\varepsilon 2^{J(\overline{s}(\sigma^{-1}) + \varepsilon - (M-1))} \sigma_J + C 2^{-J(M-1)} \qquad \forall J \in \mathbb{N}^*,$$

   for all $\varepsilon > 0$.

Proof. Using (1.3), we have

$$\begin{aligned}2^{M-1}\sup_{|h|\le 2^{-J}}\|\Delta_h^{M-1}f\|_{L^p} &\le C\sum_{j=0}^{J}2^{-j(M-1)}\sup_{|h|\le 2^{-(J-j)}}\|\Delta_h^M f\|_{L^p}\\ &\qquad + 2^{-J(M-1)}\sup_{|h|\le 2^{-J}}\|\Delta_{2^{J+1}h}^{M-1}f\|_{L^p}\\ &\le C_\varepsilon\sigma_J\sum_{j=0}^{J}2^{j(\overline{s}(\sigma^{-1})+\varepsilon-(M-1))}+C2^{-J(M-1)}\end{aligned}$$

for all $\varepsilon>0$, hence the conclusion follows. □

Let us mention that corollary 129 expressed in terms of $\overline{s}(\sigma^{-1})$ is a weaker result than propositions 126 and 127 stated in terms of d_0 and d_1.

Remark 130. As a consequence, it is equivalent for finite differences in definition 40 to consider any bigger order than $\lfloor\overline{s}(\sigma^{-1})\rfloor+1$ (we can use (1.3) to prove it). This result can also be obtained from theorem 33 without assumption $\underline{s}(\sigma^{-1})>0$ being necessary.

3.1.2 Decreasing generalized Hölder spaces

The classical Hölder spaces $\Lambda^\alpha(\mathbb{R}^d)$ are decreasing in the following sense: if $\alpha<\beta$, then $\Lambda^\beta(\mathbb{R}^d)\subset\Lambda^\alpha(\mathbb{R}^d)$. If $\sigma^{(\alpha)}$ ($\alpha>0$) is a family of admissible sequences, let us set

$$\sigma^{(\cdot)}:\alpha>0\mapsto\sigma^{(\alpha)}.$$

The spaces $\Lambda^{\sigma^{(\alpha)},\alpha}$ are not necessary embedded as the classical Hölder spaces.

Definition 131. A family of admissible sequence $\sigma^{(\cdot)}$ is *decreasing* if $\alpha<\beta$ implies $\Lambda^{\sigma^{(\beta)},\beta}(\mathbb{R}^d)\subseteq\Lambda^{\sigma^{(\alpha)},\alpha}(\mathbb{R}^d)$.

A natural question arises: under which conditions can we obtain a family of decreasing admissible sequences? The following result gives an answer to that question:

Proposition 132. (D.K., S. Nicolay) A family of admissible sequences $\sigma^{(\cdot)}$ is decreasing if it satisfies the three following conditions:

1. for all $m\in\mathbb{N}_0$ and $\alpha,\ \beta>0$ for which $m\le\alpha<\beta<m+1$, there exist $C,\ J>0$ such that
$$\sigma_j^{(\beta)}\le C\sigma_j^{(\alpha)}\quad\forall j\ge J;$$
2. for all $m\in\mathbb{N}^*$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in]0,\varepsilon_0[$, there exist $C,\ J>0$ such that
$$2^{-jm}\le C\sigma_j^{(m-\varepsilon)}\quad\forall j\ge J;$$
3. for all $m\in\mathbb{N}^*$, at least one of the two following conditions is satisfied:

(a)
- if $1 < 2^m d_1^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$2^{-jm}(2^m d_1^{(m)})^j \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J;$$
- if $1 > 2^m d_1^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$2^{-jm} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J;$$
- if $1 = 2^m d_1^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$j2^{-jm} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J.$$

(b)
- if $1 < 2^m d_0^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$\sigma_j^{(m)} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J;$$
- if $1 > 2^m d_0^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$\sigma_j^{(m)}(2^m d_0^{(m)})^{-j} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J;$$
- if $1 = 2^m d_0^{(m)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$j\sigma_j^{(m)} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J.$$

(where $d_0^{(m)}$ and $d_1^{(m)}$ are some constants satisfying inequalities (1.2) for the admissible sequence $\sigma^{(m)}$).

Proof. It is an immediate consequence of propositions 126 and 127. □

Remark 133. In particular, conditions of proposition 132 imply that for all $0 < \alpha < \beta$, there exist $C, J > 0$ such that[1]

$$\sigma_j^{(\beta)} \leq C\sigma_j^{(\alpha)} \quad \forall j \geq J. \tag{3.7}$$

Moreover, for $0 < \alpha < m$, there exists $C > 0$ such that

$$2^{-jm} \leq C\sigma_j^{(\alpha)} \quad \forall j \in \mathbb{N}_0\,. \tag{3.8}$$

It is useless to check whether the conditions of the previous result are satisfied or not if the family of admissible sequences does not even satisfy these two simplified conditions.

[1]This result is a consequence of condition 3 of proposition 132. This is obvious from the inequalities expressed in terms of $d_0^{(m)}$. For the ones expressed in terms of $d_1^{(m)}$, we proceed case by case, by rewriting inequalities exclusively in terms of $d_1^{(m)}$, and using relation (1.2).

Remark 134. At first glance, inequality (3.8) can be seen as a strong restriction, but it is not. Indeed, condition (1.2) in the definition of admissible sequences already requires a similar restriction[2]. Moreover, the example 135 shows that an admissible sequence with a high speed of convergence to 0 can lead to useless spaces.

Example 135. Consider the admissible sequences $(2^{-j\delta\alpha})_j$ of usual Hölder spaces, and let us set $\sigma_j^{(\alpha)} := 2^{-j\delta\alpha}$ with $\delta > 0$ for all $j \in \mathbb{N}_0$. If $\delta = 1$, we recover the usual Hölder spaces. If $0 < \delta < 1$, it is easy to check that $\Lambda^{\sigma^{(\alpha)},\alpha} = \Lambda^{\delta\alpha}(\mathbb{R}^d)$ (this is a consequence of remark 130, or also of theorem 33). Let us now consider the case $\delta > 1$. The spaces $\Lambda^{\sigma^{(\alpha)},\alpha}$ do not satisfy the conditions of proposition 132. It is easy to check that if $1 < \delta \leq 2$, then the generated spaces are not embedded into one another. If $\delta > 2$, then there exists $0 < \epsilon < 1$ such that for all $\alpha > \epsilon$, the space $\Lambda^{\sigma^{(\alpha)},\alpha}$ is composed of constant functions[3].

In particular, this example shows that we can construct admissible sequences such that condition 2 of proposition 132 is not satisfied, and such that the associated spaces are composed of constant functions. Those spaces are embedded (even equal). So, this shows that proposition 132 only gives sufficient conditions to construct decreasing families of admissible sequences.

As announced previously, the concept of decreasing families of admissible sequences allows the definition of a notion which characterizes the global regularity of functions.

Definition 136. Let $\sigma^{(\cdot)}$ be a decreasing family of admissible sequences. The *generalized Hölder exponent associated with* $\sigma^{(\cdot)}$ of a function $f \in L^\infty(\mathbb{R}^d)$ is defined by

$$H_f^{\sigma^{(\cdot)}} = \sup\{\alpha > 0 : f \in \Lambda^{\sigma^{(\alpha)},\alpha}\}.$$

Example 137. Let $0 < a < 1$. Let us consider the admissible sequence σ defined by

$$\sigma_j = \frac{1}{j} \quad \text{for all } j \in \mathbb{N}^*.$$

It is easy to check that any constants $0 < d_0 < 1$ and $d_1 \geq 1$ satisfy inequalities (1.2) for j sufficiently large. By lemma 14, the family $(\sigma^\alpha)_{\alpha>0}$ is a family of admissible sequences. We easily check, by using proposition 132, that it is a decreasing family of admissible sequences. The spaces $\Lambda^{\sigma^\alpha,\alpha}$ are made of functions such that their finite differences $\Delta_h^{\lfloor\alpha\rfloor+1}$ have a slight decrease in the argument $|h|$. Moreover, it is easy to check that for all α', $\alpha > 0$, we have $\Lambda^\alpha(\mathbb{R}^d) \subset \Lambda^{\sigma^{\alpha'},\alpha'}$. Let us remark that elements of those spaces are continuous.

Example 138. Let us consider a function $g : \alpha > 0 \mapsto g(\alpha) \geq 0$. Let us define the family of admissible sequences $(\sigma^{(\alpha)})_{\alpha>0}$ by $\sigma_j^{(\alpha)} := 2^{-j\alpha} j^{g(\alpha)}$ for all $j \in \mathbb{N}^*$. Proposition 132 proves that it is a decreasing family of admissible sequences. Let us notice that the condition 3.(b) of this result is satisfied but not the 3.(a) if $g > 0$.

[2]It is important to remember that all convergences of exponential types, like $(2^{2^j})_j$, can not define an admissible sequence.

[3]This is proved in remark 85.

Example 139. Let us define the family of sequences $(\sigma^{(\alpha)})_{\alpha\geq 0}$ in the following way:

- if $\alpha \in \mathbb{N}^*$,

$$\sigma_j^{(\alpha)} := 2^{-j\alpha}\frac{1}{j} \quad \text{for all } j \in \mathbb{N}^*;$$

- if $\alpha \in]0,+\infty[\setminus \mathbb{N}^*$, $\sigma_j^{(\alpha)} := 2^{-j\alpha}\ \forall j \in \mathbb{N}^*$.

It is a family of admissible sequences. Condition 3.(b) of proposition 132 is not satisfied contrary to condition 3.(a). It is a decreasing family of admissible sequences.

Remark 140. The two previous examples show that conditions 3.(a) and 3.(b) of proposition 132 are complementary, i.e. one is not the consequence of the other.

Example 141. Proposition 132 gives an idea to construct recursively a decreasing family of admissible sequences. For all $m \in \mathbb{N}^*$, we would like to define

$$\sigma_j^{(m+1)} := \sigma_j^{(m)}\left(2^{m+1}d_0^{(m+1)}\right)^j.$$

Such a definition is not complete because it depends on itself through the factor $d_0^{(m+1)}$. We could then ask what are the values of $d_0^{(m+1)}$ which satisfy (1.2). It is easy to check that a sufficient condition is given by $(d_0^{(m)})^{-1} \leq 2^{m+1}$ $(m \in \mathbb{N}^*)$. If we set an initial condition $\sigma^{(1)}$ such that $(d_0^{(1)})^{-1} \leq 2^2$, we can construct a family of admissible sequences by setting

$$\begin{aligned}
\sigma_j^{(m)} &= \sigma_j^{(m-1)}\left(2^m d_0^{(m)}\right)^j \\
&= \sigma_j^{(m-2)}\left(2^{m-1}d_0^{(m-1)}\right)^j\left(2^m d_0^{(m)}\right)^j \\
&= \cdots \\
&= \sigma_j^{(1)} 2^{j\sum_{n=2}^m n}\left(\prod_{n=2}^m d_0^{(n)}\right)^j.
\end{aligned}$$

Let us study the conditions under which this construction leads to a decreasing family of admissible sequences. By construction, if we consider values of $d_0^{(m)}$ in the interval $[2^{-(m+1)}, 2^{-m}[$, condition 3 of proposition 132 is satisfied. The last condition to check is

$$2^{-j(m+1)} \leq C\sigma_j^{(m)} \quad \forall j \in \mathbb{N}_0.$$

To validate such a condition, we choose $\sigma^{(1)}$ such that $2^{-2j} \leq C\sigma_j^{(1)}$ for all j. Then, we find

$$\begin{aligned}
\sigma_j^{(m)} &= \sigma_j^{(1)} 2^{j\sum_{n=2}^m n} \left(\prod_{n=2}^m d_0^{(n)}\right)^j \\
&\geq \sigma_j^{(1)} \prod_{n=2}^m \left(d_0^{(n-1)}\right)^{-j} \left(\prod_{n=2}^m d_0^{(n)}\right)^j \\
&\geq C 2^{-2j} \left(\frac{d_0^{(m)}}{d_0^{(1)}}\right)^j .
\end{aligned}$$

So, a sufficient condition to apply proposition 132 is that for all $m \geq 2$, there exists $C > 0$ such that

$$2^{-j(m-1)} \leq C \left(\frac{d_0^{(m)}}{d_0^{(1)}}\right)^j \qquad \forall j \in \mathbb{N}_0 .$$

Let us consider the particular case where $d_0^{(m)} := 2^{m+1}$ for all $m \in \mathbb{N}^*$. We easily check that the family of sequences $(\sigma^{(m)})_{m\in\mathbb{N}^*}$ satisfies the previous sufficient conditions. We can thus apply proposition 132. It is a decreasing family of admissible sequences and we note that

$$\sigma_j^{(m)} = \sigma_j^{(1)} 2^{-(m-1)j} \geq C 2^{-(m+1)j} \quad \forall j \in \mathbb{N}_0 .$$

3.2 Another definition of Hölder-Zygmund spaces

In this section, we prove that definition 6 is equivalent to the following one: a function f defined on $\mathbb{R}^d$ belongs to the space $\Lambda^\alpha(\mathbb{R}^d)$ ($\alpha > 0$) if it is bounded almost everywhere and satisfies

$$\sup_{\substack{|h|\leq 2^{-j} \\ x\in\mathbb{R}^d}} |\Delta_h^{\lfloor\alpha\rfloor+1} f(x)| \leq C 2^{-j\alpha} \qquad \forall j \in \mathbb{N}_0 . \tag{3.9}$$

This means that the Lebesgue-measurability can be omitted in the definition of Hölder spaces if a slightly more restrictive inequality is imposed.

This is a consequence of the developments discussed in section 3.1. This result is not obvious at first glance as there exist non-measurable functions satisfying inequality (3.9).

Lemma 142. *Let $\alpha > 0$ and $m \in \mathbb{N}^*$. If the function $f : \mathbb{R}^d \to \mathbb{R}$ is bounded almost everywhere and satisfies $|\Delta_h^m f(x)| \leq C|h|^\alpha$ $\forall x, h \in \mathbb{R}^d$, then it is bounded.*

Proof. Let us note that

$$\begin{aligned}|f(x)| - |\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}f(x+mh-jh)| &\leq \left||f(x)| - |\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}f(x+mh-jh)|\right| \\ &\leq |\sum_{j=0}^{m}(-1)^j\binom{m}{j}f(x+mh-jh)| \\ &\leq |\Delta_h^m f(x)| \\ &\leq C|h|^\alpha,\end{aligned}$$

for all x, $h \in \mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be a set such that its complement is negligible and such that $|f(y)| \leq C'$ for all $y \in E$. Let also $x \in E^c$ and $h \in \mathbb{R}^d$ such that $|h| \leq 1$ and $|f(x+(m-j)h)| \leq C'$ for all $j \in \{0, 1, ..., m-1\}$ (such a real number h exists because the set E is the complementary of a negligible set). Putting these inequalities together, we find that $|f(x)| \leq C''$.

□

Let f be a function defined on $\mathbb{R}^d$ and satisfying (3.9). Let us prove the result for $\alpha \in \mathbb{N}^*$ and $\alpha > 1$ (if $\alpha < 1$, the function is continuous and so measurable). Using results of section 3.1 (which can easily be adapted to inequality (3.9) thanks to lemma 142), we have

$$\sup_{\substack{|h| \leq 2^{-j} \\ x \in \mathbb{R}^d}} |\Delta_h^\alpha f(x)| \leq Cj2^{-j\alpha} \qquad \forall j \in \mathbb{N}^* \,. \tag{3.10}$$

By induction, we can reduce the order of the finite difference to 1 to find that the function is continuous on $\mathbb{R}^d$, which ends the proof.

Remark 143. The equivalence of definitions proved in this section concerns the classical Hölder-Zygmund spaces. However, the same proof can also be applied to spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ to obtain similar results on generalized Hölder spaces, under some technical conditions on the sequence σ.

3.3 The uniform irregular Hölder spaces $I^\alpha(\mathbb{R}^d)$ expressed as a particular case of generalized Hölder spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$

The irregular Hölder spaces, introduced in [31, 32, 33], allow to study the irregularity of functions. Roughly speaking, they are the counterpart of Hölder spaces, and are obtained by reversing the inequality controlling the finite difference.

Definition 144. Let $\alpha > 0$ and $f \in L^\infty(\mathbb{R}^d)$. A function f belongs to the *irregular Hölder space* $I^\alpha(\mathbb{R}^d)$ if there exists a constant $C > 0$ such that

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \geq C2^{-j\alpha} \qquad \forall j \in \mathbb{N}_0 \,. \tag{3.11}$$

We can define an Hölder exponent linked to these spaces in the following way.

Definition 145. The upper global Hölder exponent (or uniform irregularity exponent) of a function $f \in L^\infty(\mathbb{R}^d)$ is defined by

$$\overline{H_f} = \inf\{\alpha > 0 : f \in I^\alpha(\mathbb{R}^d)\}.$$

The aim of this section is to show that these spaces can be expressed in terms of generalized Hölder spaces. This allows to apply our previous results.

Let us remark that the inequality (3.11) is not equivalent to $f \notin \Lambda^\alpha(\mathbb{R}^d)$. Indeed, if $f \notin I^\alpha(\mathbb{R}^d)$, then for every $C > 0$, there exists a strictly increasing sequence $(j_n)_{n\in\mathbb{N}^*}$ of integers such that

$$\sup_{|h|\leq 2^{-j_n}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq C 2^{-j_n\alpha} \qquad \forall n \in \mathbb{N}^* . \tag{3.12}$$

This inequality is a weaker condition than the one used for usual Hölder spaces. We are led to the following natural definition.

Definition 146. Let $\alpha > 0$ and $f \in L^\infty(\mathbb{R}^d)$. We say that f belongs to the *weak Hölder space* $C_w^\alpha(\mathbb{R}^d)$ if $f \notin I^\alpha(\mathbb{R}^d)$.

The sequence $(2^{-j_n\alpha})_{n\in\mathbb{N}^*}$ can be a non-admissible sequence (e.g. if j_n is of exponential type, see section 1.2). Moreover, $|h| \leq 2^{-j_n}$ seems closer to the spaces $\Lambda^\alpha_{\sigma,N}(\mathbb{R}^d)$ with $N_n = 2^{j_n}$ ($n \in \mathbb{N}^*$) than the spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$. Again, the sequence N_j can be non-admissible. Nevertheless, the following result shows that this can be reduced to a particular case of generalized Hölder spaces of type $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$, for some admissible sequence σ.

Proposition 147. *Let $\alpha > 0$, $M = \lfloor\alpha\rfloor + 1$ and $f \in L^\infty(\mathbb{R}^d)$. We have $f \in C_w^\alpha(\mathbb{R}^d)$ if and only if for every $C > 0$, there exists a strictly increasing sequence $(j_n)_{n\in\mathbb{N}^*}$ of integers (depending on C) such that $f \in \Lambda^{\sigma((j_n)_n),\alpha}(\mathbb{R}^d)$ where $\sigma((j_n)_n)$ is the admissible sequence defined by*

$$\sigma((j_n)_n)_j = \inf\{2^{-\alpha j_n}; 2^{(M-\alpha)j_{n+1}}2^{-jM}\} \quad \forall j \in \{j_n, ..., j_{n+1}-1\}, \quad \forall n \in \mathbb{N}^* .$$

Proof. 1. Let $C > 0$ and $(j_n)_n$ the sequence associated to C such that

$$\sup_{|h|\leq 2^{-j_n}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq C 2^{-j_n\alpha} \qquad \forall n \in \mathbb{N}^* .$$

Let us prove that we have

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq C \sigma((j_n)_n)_j \qquad \forall j \in \mathbb{N}^* .$$

Let $n \in \mathbb{N}^*$ and $j \in \{j_n, ..., j_{n+1}-1\}$. We have

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq \sup_{|h|\leq 2^{-j_n}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq C 2^{-\alpha j_n}.$$

We note that[4]

$$\sup_{|h|\leq 2r} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \leq 2^M \sup_{|h|\leq r} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \quad \forall r > 0.$$

Using this result, we find

$$\begin{aligned}\sup_{|h|\leq 2^{-j_{n+1}}2^{(j_{n+1}-j)}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} &\leq 2^{(j_{n+1}-j)M} \sup_{|h|\leq 2^{-j_{n+1}}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} \\ &\leq C2^{(M-\alpha)j_{n+1}}2^{-jM}.\end{aligned}$$

2. Let us prove that $(\sigma((j_n)_n)_j)_{j\in\mathbb{N}^*}$ is an admissible sequence with associated constants $d_0 = 2^{-M}$ and $d_1 = 1$ (these constants are optimal without any additional assumptions on the sequence $(j_n)_n$), i.e. we have

$$2^{-M}\sigma((j_n)_n)_j \leq \sigma((j_n)_n)_{j+1} \leq \sigma((j_n)_n)_j \quad \forall j \in \mathbb{N}^*.$$

Let $n \in \mathbb{N}^*$. The result is immediate for $j \in \{j_n, ..., j_{n+1}-2\}$. Let us prove the result for $j = j_{n+1}-1$. In order to simplify the notations, let us denote $\sigma((j_n)_n)_j$ by σ_j. We consider the following different cases:

(a) if $\sigma_{j_{n+1}-1} = 2^{-\alpha j_n}$ and $\sigma_{j_{n+1}} = 2^{-\alpha j_{n+1}}$, we have $\sigma_{j_{n+1}} < \sigma_{j_{n+1}-1}$ and

$$\sigma_{j_{n+1}-1} \leq 2^{(M-\alpha)j_{n+1}}2^{-(j_{n+1}-1)M} = 2^{-\alpha j_{n+1}}2^M = \sigma_{j_{n+1}}2^M,$$

(b) if $\sigma_{j_{n+1}-1} = 2^{-\alpha j_n}$ and $\sigma_{j_{n+1}} = 2^{(M-\alpha)j_{n+2}}2^{-j_{n+1}M}$, we have

$$\sigma_{j_{n+1}} \leq 2^{-\alpha j_{n+1}} \leq 2^{-\alpha j_n} = \sigma_{j_{n+1}-1}$$

and

$$\begin{aligned}\sigma_{j_{n+1}-1} &\leq 2^{-\alpha j_{n+1}}2^M \\ &= \sigma_{j_{n+1}}2^{(M-\alpha)(j_{n+1}-j_{n+2})}2^M \\ &\leq 2^M\sigma_{j_{n+1}},\end{aligned}$$

(c) if $\sigma_{j_{n+1}-1} = 2^{-\alpha j_{n+1}}2^M$ and $\sigma_{j_{n+1}} = 2^{-\alpha j_{n+1}}$, we have $\sigma_{j_{n+1}-1} = \sigma_{j_{n+1}}2^M$,

(d) if $\sigma_{j_{n+1}-1} = 2^{-\alpha j_{n+1}}2^M$ and $\sigma_{j_{n+1}} = 2^{(M-\alpha)j_{n+2}}2^{-j_{n+1}M}$, we have

$$\begin{aligned}\sigma_{j_{n+1}-1} &= 2^{(M-\alpha)j_{n+1}}2^{-j_{n+1}M}2^M \\ &\leq 2^{(M-\alpha)j_{n+2}}2^{-j_{n+1}M}2^M = \sigma_{j_{n+1}}2^M\end{aligned}$$

and

$$\sigma_{j_{n+1}} \leq 2^{-\alpha j_{n+1}} = \sigma_{j_{n+1}-1}2^{-M}.$$

This ends the proof.

□

Remark 148. We proved that $d_1 = 1$ in proposition 147. The sequence $\sigma((j_n)_n)$ is thus non-increasing.

[4]This is a consequence of lemma 67.

3.4 Application to financial models

Example 149. Let us set $\sigma_j := (2^{-j})^{\frac{1}{2}} |\log|\log(2^{-j})||^{\frac{1}{2}}$ for all $j \in \mathbb{N}^*$. This is a strong admissible sequence of order 1. A. Khintchine proved in [72] that the trajectories of Brownian motions belong almost surely to $\Lambda^{\sigma,\alpha}(\mathbb{R})$ (where $0 < \alpha < 1$). So, its wavelet coefficients satisfy

$$\begin{cases} |C_k| \leq C & \forall k \in \mathbb{Z}^d \\ |c^i_{j,k}| \leq C\sigma_j & \forall j \in \mathbb{N}_0, \forall i \in \{1, \ldots, 2^d - 1\}, \forall k \in \mathbb{Z}^d. \end{cases}$$

It is known that the trajectories of a Brownian motion do not belong to $\Lambda^{1/2}(\mathbb{R})$ ([98, 102]). This example is a typical case where the generalized Hölder-Zygmund spaces give more information than what is given by classical Hölder spaces. Let us note that, in the classical case, even if the Brownian motion does not belong to $\Lambda^{1/2}(\mathbb{R})$, the usual Hölder exponent is still 1/2, which demonstrates the lack of accuracy of the classical Hölder spaces in this particular case.

Example 150. Let us consider the *geometric Brownian motion.* This stochastic process is a basic example used in financial models of stock indices, and is used in the famous Black and Scholes model ([62]). It is given by the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where $(W(t))_{t\geq 0}$ is a Brownian motion, $\mu \in \mathbb{R}$ is a real number determining the mean value of the stock and σ represents the volatility of the stock index. A solution of this stochastic equation is given by

$$S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}, \qquad t \geq 0,$$

where $S(0) > 0$ is a constant. Let us remark that we can rewrite this process as

$$S(t) = f(W(t))$$

where f is an infinitely continuously differentiable function on $\mathbb{R}$. So, we have

$$|f(W(t)) - f(W(s))| \leq \max_{z\in K} |f'(z)||W(t) - W(s)|$$

by the mean value theorem for some compact set K. We can conclude from example 149 that the trajectories of $S(t)$ belong almost surely to $\Lambda^{\sigma,\alpha}(\mathbb{R})$ (where σ is the sequence defined in example 149). Moreover, the trajectories do not belong to $\Lambda^{1/2}(\mathbb{R})$ (otherwise we could apply the mean value theorem to the logarithm of $S(t)$ in order to prove that the trajectories of the Brownian motion also belong to $\Lambda^{1/2}(\mathbb{R})$).

Example 151. Let us consider the *Hull and White one-factor model.* This model was introduced in 1990 by John C. Hull and Alan White to model interest rates, and is popular

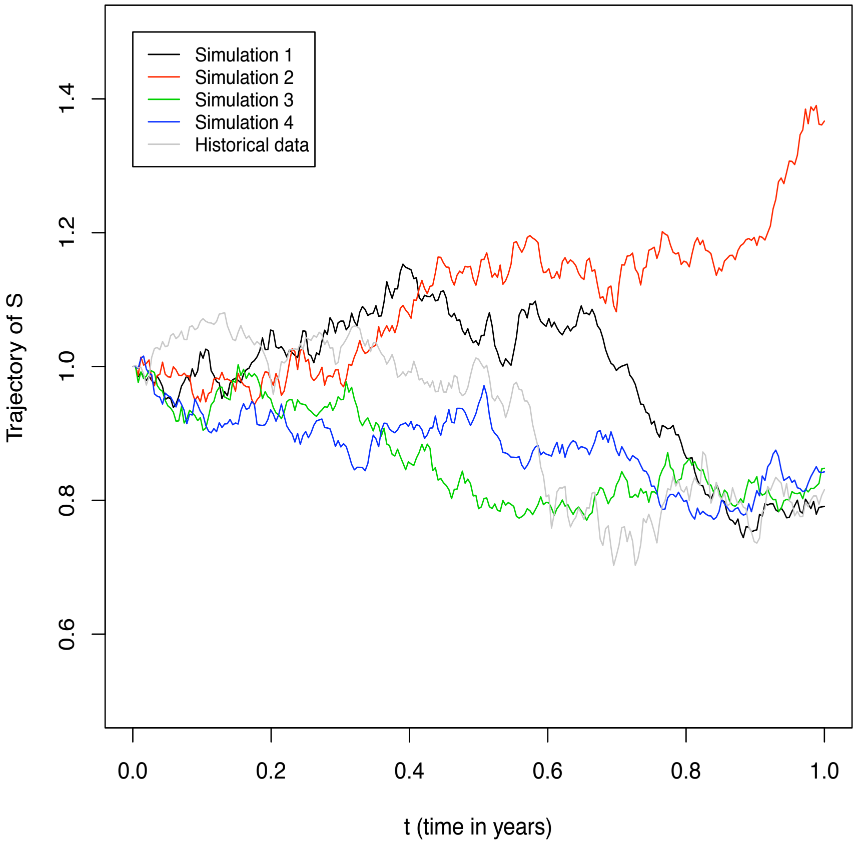

Figure 3.1: Simulations of trajectories of the stochastic process S, with $\mu = 0,49\%$ and $\sigma = 19,7\%$. The estimation of parameters μ and σ is based on observed values of Eurostoxx 50 during the year 2011 (these observed values are represented in grey in this graphic).

among financial market and actuarial fields ([23]). It is expressed by the following stochastic differential equation:

$$dr(t) = a(\theta(t) - r(t))dt + \sigma dW(t).$$

In this equation, the stochastic process $r(t)$ should be understood as a short rate model and $(W(t))_{t\geq 0}$ is a Brownian motion. All other factors are deterministic: the function $\theta(t)$ is the asymptotic interest rate (which can vary over time because of macroeconomical changes in the future), $a > 0$ represents the attractive force towards which the interest rate converges to $\theta(t)$, and σ represents the volatility of the interest rate. The solution of this stochastic equation is given by

$$r(t) = r(0)e^{-at} + e^{-at}a\int_0^t \theta(s)e^{as}ds + \sigma e^{-at}\int_0^t e^{as}dW(s),$$

where $r(0) > 0$ represents the initial observed value of the interest rate. When θ is a constant function, then the model is more commonly called the *Ornstein-Uhlenbeck stochastic process* or the *Vasicek* model. Under the Hull and White model, the price at time t of a zero-coupon bond that gives one unit of monetary at time T ($T > t$) can be written as

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

where $A(t,T)$ and $B(t,T)$ are two regular deterministic functions. This expression can be used to fit the model to the observed initial yield curve and to predict values of bonds. More information about those models can be found in [23]. In practice, it is common to use the Nelson-Siegel or Svensson model to describe the behaviour of θ. These models write the function θ as exponential polynomials, and are used to calibrate the model to market data (see e.g. [75]). So, the function θ can be assumed to be infinitely continuously differentiable. Let us set

$$g(t) := r(0)e^{-at} + e^{-at}a\int_0^t \theta(s)e^{as}ds \in C^\infty(\mathbb{R}).$$

By the mean value theorem, we find

$$|\Delta_h^1 g(t)| \leq |h| \sup_K |g'|$$

where t, $t+h \in K$ for some compact set K. Let us define the stochastic process Y by

$$Y(t) = \int_0^t e^{as}dW(s) \quad (t \geq 0).$$

Let us remark that

$$Y(t) = e^{at}W(t) - a\int_0^t e^{as}W(s)ds$$

by the integration by parts theorem for stochastic integrals (this classical result is a consequence of Ito's formula). Using the Leibniz formula for finite difference, we have

$$\Delta_h^1(e^{at}W(t)) = \Delta_h^1(e^{at})W(t+h) + e^{at}\Delta_h^1(W(t)).$$

By example 149, we have $t \mapsto e^{at}W(t) \in \Lambda^{\sigma,\alpha}(\mathbb{R})$ with $\sigma_j := (2^{-j})^{\frac{1}{2}}|\log|\log(2^{-j})||^{\frac{1}{2}}$ ($j \in \mathbb{N}^*$) and $0 < \alpha < 1$. Therefore, by proposition 92, we get $t \mapsto \int_0^t e^{as}W(s)ds \in \Lambda^{2^{-j}\sigma_j, 1+\alpha}$ (in the sense that its trajectories belong almost surely to this space). Through the Leibniz formula and linearity of finite difference, one finally gets that $r \in \Lambda^{\sigma,\alpha}$. The trajectories of r do not belong to $\Lambda^{1/2}$. Otherwise, we could prove that the trajectories of the Brownian motion

$$W(t) = e^{-at}(Y(t) + a\int_0^t e^{as}W(s)ds)$$

also belong to this space, which is in contradiction with the results stated in example 149. We thus get that the trajectories of $t \mapsto P(t,T)$ (with $T > t$) belong almost surely to $\Lambda^{\sigma,\alpha}$ and do not belong to $\Lambda^{1/2}$ (the same ideas as the ones exposed in example 150 can be applied).

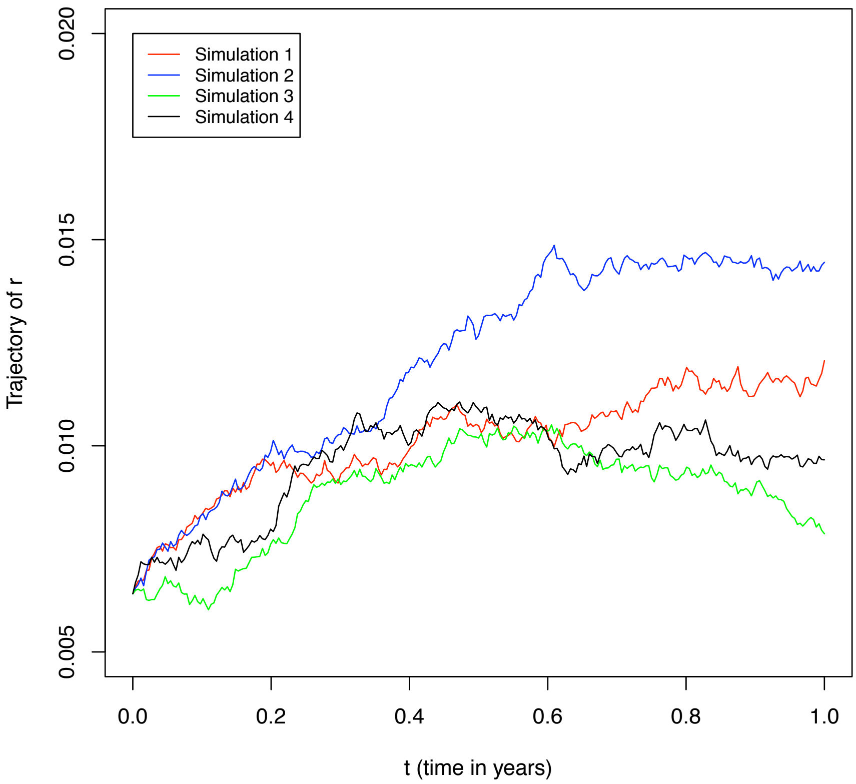

Figure 3.2: Simulations of trajectories of the stochastic process r, with $a = 9,46\%$ and $\sigma = 0,21\%$. The estimation of parameters a and σ is based on observed values of Euribor 1 Week of year 2011.

Example 152. Let us define the stochastic process $(X(t))_{t\geq 0}$ by

$$X(t) = \int_0^t W(s)ds.$$

This stochastic process is a Gaussian process and a martingale. It represents the area between a Brownian motion and the horizontal axis. Using the same arguments as the ones exposed in example 151, we easily check that the trajectories of X belong almost surely to $\Lambda^{\sigma,1+\alpha}$ with $\sigma_j := (2^{-j})^{\frac{3}{2}}|\log|\log(2^{-j})||^{\frac{1}{2}}$ $(j \in \mathbb{N}^*)$ and $0 < \alpha < 1$.

The reader should note that all ideas expressed in this section could be used to determine the generalized Hölder exponent of most stochastic processes which are derived from stochastic integration.

Chapter 4

Characterizations and properties of pointwise generalized Hölder-Zygmund spaces

The study of global regularity of a function can be made through the use of generalized Hölder-Zygmund spaces. For many functions used as models for signals, however, the regularity can change drastically from one point to another. It is therefore natural to define a notion of pointwise regularity. For example, the function $f(x) = x^{m+1}\sin(x^{-m})$ ($m \in \mathbb{N}_0$) is clearly irregular at $x = 0$ although it admits a continuous extension on $\mathbb{R}$ by $f(0) = 0$. This function is represented in figure 4.1 for $m = 2$. This function is clearly infinitely continuously differentiable on $\mathbb{R}_0$ (but is not continuously differentiable on $\mathbb{R}$). It could be interesting in this case to determine "how much" the function is irregular at $x = 0$. Another example is given by the Takagi function. We treat this case in section 4.6.

For those purposes, we introduce in this section pointwise Hölder spaces and their generalized versions. Later on, we also look at some characterizations of these spaces. Their properties are very similar to their global versions, although these spaces can not be manipulated in the same way.

In the sequel, the notation $\|f\|_E$ stands for $\|f\|_E = \sup_{x\in E}|f(x)|$.

4.1 Definition of pointwise generalized Hölder-Zygmund spaces $\Lambda^M_{\sigma,N}(x_0)$ and $\Lambda^{\sigma,M}(x_0)$

Let us recall the definition of classical pointwise Hölder spaces, which are denoted by $\Lambda^\alpha(x_0)$ ($\alpha > 0$, $x_0 \in \mathbb{R}^d$).

Definition 153. Let $\alpha > 0$ and $x_0 \in \mathbb{R}^d$. We say that a function $f \in L^\infty_{loc}(\mathbb{R}^d)$ belongs to

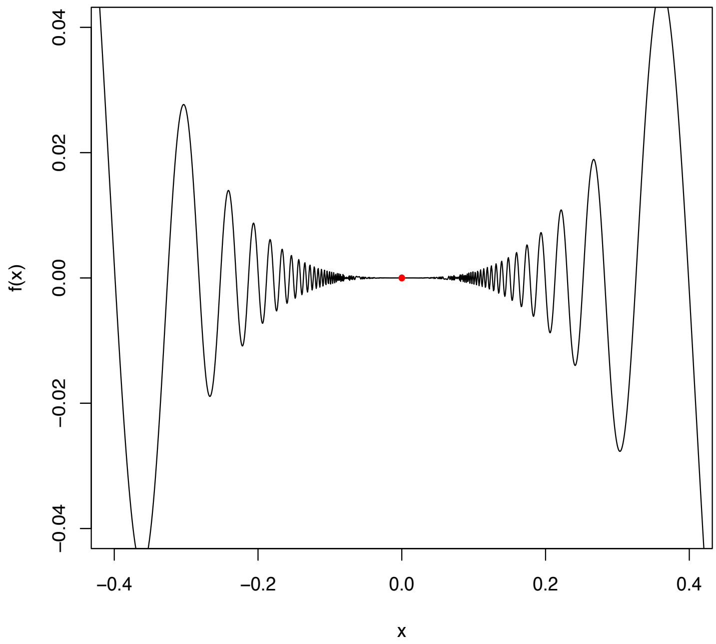

Figure 4.1: Graphic of the function $f(x) = x^3 \sin(x^{-2})$. The red dot represents the continuous extension at $x = 0$ of the function f.

the space $\Lambda^\alpha(x_0)$ if there exist two constants $C, J > 0$ such that[1]

$$\inf_{P \in \mathbb{P}_{\lfloor \alpha \rfloor}} \|f - P\|_{B(x_0, 2^{-j})} \leq C 2^{-j\alpha} \quad \forall j \geq J.$$

By analogy with the global Hölder-Zygmund spaces, we generalize this definition as follows.

Definition 154. Let $M \in \mathbb{N}_0$, σ and N be two admissible sequences, and $x_0 \in \mathbb{R}^d$. We say that a function $f \in L^\infty_{loc}(\mathbb{R}^d)$ belongs to the space $\Lambda^M_{\sigma,N}(x_0)$ if there exist two constants $C, J > 0$ such that

$$\inf_{P \in \mathbb{P}_M} \|f - P\|_{B(x_0, N_j^{-1})} \leq C \sigma_j \quad \forall j \geq J.$$

In the case $N_j = 2^j$ ($j \in \mathbb{N}_0$), we denote these spaces by $\Lambda^{\sigma,M}(x_0) = \Lambda^M_{\sigma,N}(x_0)$.

Remark 155. 1. The previous definition can be rewritten in the following way. A function $f \in L^\infty_{loc}(\mathbb{R}^d)$ belongs to the space $\Lambda^M_{\sigma,N}(x_0)$ if there exist two constants

[1] We remind the reader that the notation $\mathbb{P}_M$ refers to the set of polynomials defined on $\mathbb{R}^d$ of degree less or equal to M.

$C, J > 0$ such that for all $j \geq J$, there exists a polynomial $P_{x_0,j}$ of degree less or equal to M which satisfies

$$\sup_{|h|\leq N_j^{-1}} |f(x_0+h) - P_{x_0,j}(h)| \leq C\sigma_j. \tag{4.1}$$

2. Let σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. If a function f belongs to a generalized Hölder space $\Lambda^{\sigma,M}(\mathbb{R}^d)$, then it belongs in particular to the pointwise space $\Lambda^{\sigma,M}(x_0)$ for all $x_0 \in \mathbb{R}^d$. The converse is not true: if a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(x_0)$ for all $x_0 \in \mathbb{R}^d$, then it doesn't necessary belong[2] to $\Lambda^{\sigma,M}(\mathbb{R}^d)$ (the constant C depends on x_0).

Remark 156. We have introduced a generalization of global and pointwise Hölder-Zygmund spaces in terms of admissible sequences. Thus, it is natural to complete our study of Hölder-Zygmund spaces with the local point of view of these spaces. We propose a definition of local Hölder spaces that can be considered as a particular case of the global spaces. This means that all of our results which are true for the global case can be transposed to the local case.

Definition 157. Let $\alpha > 0$, σ be an admissible sequence and $x_0 \in \mathbb{R}^d$. A function $f \in L^\infty_{loc}(\mathbb{R}^d)$ belongs to the space $\Lambda^{\sigma,\alpha}_{loc}(x_0)$ if there exists a function $g \in \Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ such that $f = g$ on a neighbourhood of x_0, i.e. there exists $C > 0$ such that

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} g\|_{L^\infty(\mathbb{R}^d)} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

In particular, there exists a neighbourhood ν of x_0 such that

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty(\nu)} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

4.2 A characterization of pointwise generalized Hölder-Zygmund spaces in terms of finite differences

The definition of generalized pointwise Hölder spaces is expressed in terms of polynomials. Because of the analogy between pointwise Hölder spaces and global Hölder spaces, it is natural to wonder whether these spaces can be expressed in terms of finite differences. The result here below provides the answer.

Proposition 158. (D.K., S. Nicolay) Let $M \in \mathbb{N}_0$, σ and N be two admissible sequences such that $N_j \to +\infty$. If $f \in L^\infty_{loc}(\mathbb{R}^d)$ is a function which is continuous in a neighbourhood of $x_0 \in \mathbb{R}^d$, then the following conditions are equivalent:

[2] Such an example is given in remark 164.

1. $f \in \Lambda^M_{\sigma,N}(x_0)$,
2. there exist $C > 0$ and $J > 0$ such that

$$\sup_{|h|\le N_j^{-1}} \|\Delta_h^{M+1} f\|_{B_h(x_0,N_j^{-1})} \le C\sigma_j \quad \forall j \ge J.$$

Proof. Let us prove that 2. ⇒ 1. Let $r > 0$ such that f is continuous in $B(x_0, r)$ and $J' \ge J$ such that $N_j^{-1} \le r$ for all $j \ge J'$. Because f is bounded on $B(x_0, r)$, using Whitney theorem ([24], theorem 1'), we have

$$\inf_{P\in\mathbb{P}_M} \|f - P\|_{B(x_0,N_j^{-1})} \le C \sup_{|h|\le N_j^{-1}} \|\Delta_h^{M+1} f\|_{B_h(x_0,N_j^{-1})} \quad \forall j \ge J'.$$

Let us prove that 1. ⇒ 2. By assumption, there exists $J \in \mathbb{N}_0$ such that, for all $j \ge J$, there exists a polynomial $P_{x_0,j}$ of degree less or equal to M satisfying

$$\sup_{|h|\le N_j^{-1}} |f(x_0 + h) - P_{x_0,j}(h)| \le C\sigma_j.$$

Let $j \ge J$, x, $h \in \mathbb{R}^d$ such that $[x, x + (M + 1)h] \subseteq B(x_0, N_j^{-1})$ and let us set $Q_{x_0,j}(.) = P_{x_0,j}(. - x_0)$. We have

$$\begin{aligned} |\Delta_h^{M+1} f(x)| &= |\Delta_h^{M+1}(f - Q_{x_0,j})(x)| \\ &\le 2^{M+1} \sup_{y\in B(x_0,N_j^{-1})} |f(y) - Q_{x_0,j}(y)|. \end{aligned}$$

So, we find

$$\sup_{|h|\le N_j^{-1}} \|\Delta_h^{M+1} f\|_{B_h(x_0,N_j^{-1})} \le 2^{M+1} C\sigma_j.$$

□

Remark 159. Proposition 158 is the pointwise version of theorem 88.

4.3 A characterization of pointwise generalized Hölder-Zygmund spaces in terms of Taylor decomposition

In section 2.7, we have seen that, under some assumptions on σ, generalized Hölder-Zygmund spaces admit a Taylor decomposition of their elements (see corollary 105). The goal of this section is to prove a similar property for pointwise spaces.

Let us recall the following multidimensional Markov inequality: let $p \in]0, +\infty]$, $i \in \{1, ..., d\}$ and $S \subset \mathbb{R}^d$ be a bounded convex set with non-empty interior; we have

$$\|D_i P\|_{L^p(S)} \le Cn^2 \|P\|_{L^p(S)}$$

for every polynomials of degree less or equal to $n-1$, where the constant $C>0$ only depends on S and p, and does not depend on the polynomial P or n (see [44], theorem 4.1.). We deduce the following inequality: if $x_0 \in \mathbb{R}^d$, then we have

$$\|D_i P\|_{L^\infty(B(x_0,R))} \leq C R^{-1} n^2 \|P\|_{L^\infty(B(x_0,R))} \tag{4.2}$$

for all $R>0$ and for all polynomials of degree less or equal to $n-1$.

We need a lemma to prove the main result of this section.

Lemma 160. *Let $M \in \mathbb{N}_0$, σ and N be two admissible sequences such that $M < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$ and $\underline{N}_1 \geq 1$. If $f \in \Lambda^{M}_{\sigma,N}(x_0)$, then the sequence $(P_{x_0,j})_{j\in\mathbb{N}_0}$ composed of polynomials satisfying* (4.1) *verifies*

$$|D^\beta P_{x_0,l}(x_0) - D^\beta P_{x_0,j}(x_0)| \leq C N_j^{|\beta|}\sigma_j \quad \forall j \leq l, \forall |\beta| \leq M.$$

Proof. Using Markov's inequality (4.2), we have

$$\begin{aligned}
&\|D^\beta(P_{x_0,j} - P_{x_0,j+1})\|_{B(x_0,N_{j+1}^{-1})}\\
&\leq C N_j^{|\beta|}\|P_{x_0,j} - P_{x_0,j+1}\|_{B(x_0,N_{j+1}^{-1})}\\
&\leq C N_j^{|\beta|}\left(\|P_{x_0,j} - f\|_{B(x_0,N_j^{-1})} + \|P_{x_0,j+1} - f\|_{B(x_0,N_{j+1}^{-1})}\right)\\
&\leq C N_j^{|\beta|}(\sigma_j + \sigma_{j+1})\\
&\leq C N_j^{|\beta|}\sigma_j
\end{aligned}$$

for all $|\beta| \leq M$ and $j \in \mathbb{N}_0$. If $j \leq l$, we find

$$\begin{aligned}
&\|D^\beta(P_{x_0,j} - P_{x_0,l})\|_{B(x_0,N_l^{-1})}\\
&\leq \sum_{k=j}^{l-1}\|D^\beta(P_{x_0,k} - P_{x_0,k+1})\|_{B(x_0,N_l^{-1})}\\
&\leq \sum_{k=j}^{l-1}\|D^\beta(P_{x_0,k} - P_{x_0,k+1})\|_{B(x_0,N_{k+1}^{-1})}\\
&\leq C\sum_{k=j}^{l-1} N_k^{|\beta|}\sigma_k\\
&\leq C N_j^{|\beta|}\sigma_j.
\end{aligned}$$

□

Remark 161. Under the assumption of lemma 160, the sequence $(D^\beta P_{x_0,j}(x_0))_{j\in\mathbb{N}_0}$ ($|\beta| \leq M$) is a Cauchy sequence. Let $D^\beta f(x_0)$ denote its limit. Its value is called the *β-th Peano's* *derivative of* f *at* x_0 (see [42]), and is also called the β*-th "De la Vallée-Poussin" derivative of* f *at* x_0. It does not depend on the chosen polynomial sequence $(D^\beta P_{x_0,j}(x_0))_{j\in\mathbb{N}_0}$ satisfying (4.1). If $(D^\beta P'_{x_0,j}(x_0))_{j\in\mathbb{N}_0}$ is another polynomial sequence satisfying (4.1), then we have

$$\begin{aligned}&|D^\beta P'_{x_0,j}(x_0) - D^\beta f(x_0)|\\ &\leq |D^\beta P'_{x_0,j}(x_0) - D^\beta P_{x_0,j}(x_0)| + |D^\beta P_{x_0,j}(x_0) - D^\beta f(x_0)|.\end{aligned}$$

Using Markov's inequality, we obtain

$$\begin{aligned}&\|D^\beta(P_{x_0,j} - P'_{x_0,j})\|_{B(x_0,N_j^{-1})}\\ &\leq CN_j^{|\beta|}\|P_{x_0,j} - P'_{x_0,j}\|_{B(x_0,N_j^{-1})}\\ &\leq CN_j^{|\beta|}\left(\|P_{x_0,j} - f\|_{B(x_0,N_j^{-1})} + \|P'_{x_0,j} - f\|_{B(x_0,N_j^{-1})}\right)\\ &\leq CN_j^{|\beta|}\sigma_j \to 0 \quad \text{if } j \to +\infty,\end{aligned}$$

so the conclusion. The proof of the next result justifies its name as "derivative".

Theorem 162. (D.K., S. Nicolay) Let $M \in \mathbb{N}_0$, σ and N be two admissible sequences such that $M < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$ and $\underline{N}_1 > 1$. The following assertions are equivalent:

1. $f \in \Lambda^M_{\sigma,N}(x_0)$,
2. there exist a positive constant C and a polynomial P_{x_0} of degree less or equal to M such that
$$\sup_{|h|\leq N_j^{-1}} |f(x_0+h) - P_{x_0}(h)| \leq C\sigma_j \qquad \forall j \in \mathbb{N}_0\,.$$

Proof. The proof of $2 \Rightarrow 1$ is immediate. Let us prove that $1 \Rightarrow 2$. Let $(P_{x_0,j})_{j\in\mathbb{N}_0}$ be a sequence of polynomials of degree less or equal to M satisfying (4.1). We set

$$P_{x_0}(x) = \sum_{|\beta|\leq M} D^\beta f(x_0)\frac{(x-x_0)^\beta}{|\beta|!}$$

for $x \in B(x_0, N_0^{-1})$. We have

$$\begin{aligned}\|P_{x_0} - P_{x_0,j}\|_{B(x_0,N_j^{-1})} &= \left\|\sum_{|\beta|\leq M}\left(D^\beta f(x_0) - D^\beta P_{x_0,j}(x_0)\right)\frac{(x-x_0)^\beta}{|\beta|!}\right\|_{B(x_0,N_j^{-1})}\\ &\leq \sum_{|\beta|\leq M}|D^\beta f(x_0) - D^\beta P_{x_0,j}(x_0)|N_j^{-|\beta|}.\end{aligned}$$

Using lemma 160, we have

$$|D^\beta P_{x_0,l}(x_0) - D^\beta P_{x_0,j}(x_0)| \le C N_j^{|\beta|} \sigma_j \quad \forall j \le l,$$

and, by taking $l \to +\infty$, we find

$$|D^\beta f(x_0) - D^\beta P_{x_0,j}(x_0)| \le C N_j^{|\beta|} \sigma_j.$$

So, we obtain

$$\|P_{x_0} - P_{x_0,j}\|_{B(x_0,N_j^{-1})} \le C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

The conclusion follows immediately from

$$\begin{aligned} \|f - P_{x_0}\|_{B(x_0,N_j^{-1})} &\le \|f - P_{x_0,j}\|_{B(x_0,N_j^{-1})} + \|P_{x_0 j} - P_{x_0}\|_{B(x_0,N_j^{-1})} \\ &\le C\sigma_j \end{aligned}$$

for all $j \in \mathbb{N}_0$. □

Remark 163. Under the assumptions of theorem 162, a function f of $\Lambda^M_{\sigma,N}(x_0)$ admits a *Taylor decomposition* around the point x_0, expressed in terms of its Peano derivatives at x_0: for x in a neighbourhood of x_0, we have

$$f(x) = \sum_{|\beta| \le M} D^\beta f(x_0) \frac{(x-x_0)^\beta}{|\beta|!} + R(x - x_0),$$

where $\sup_{|h| \le N_j^{-1}} |R(h)| \le C\sigma_j$ for j sufficiently large. Under the assumptions of theorem 162, if f is M-times continuously differentiable around x_0, then the Peano derivatives at x_0 (up to the order M) coincide with classical derivatives. This justifies their name.

Remark 164. We conclude this section by a general remark concerning Peano derivatives. There exist functions which are not differentiable but admit Peano derivatives. Here is a basic example.

Let $m > 1$ and $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^{m+1} \sin(x^{-m})$ at $x \neq 0$ and $f(0) = 0$. We can check that f admits Peano derivatives up to the order m in all points of $\mathbb{R}$ but its first derivative is not continuous at 0. Moreover, Peano derivatives of f at 0 are equal to 0 up to the order m, so $f \in \Lambda^{(2^{-j(m+1)})_j,m}(0)$ by theorem 162. This space is not equal to any classical pointwise Hölder spaces. In particular, we find that $f \in \Lambda^\alpha(0)$ for all $\alpha < m+1$, but $f \notin \Lambda^{1+\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$ (otherwise the function f would be continuously differentiable according to proposition 92).

4.4 A characterization of pointwise generalized Hölder-Zygmund spaces in terms of wavelet coefficients

We proved in section 2.9 that generalized global Hölder-Zygmund spaces can be characterized in terms of wavelet coefficients. The goal of this section is to prove a similar result for pointwise Hölder spaces.

We consider Daubechies wavelets in the sequel. Moreover we suppose that such wavelets are sufficiently regular and have sufficiently many vanishing moments[3]. We let j_0 denote a natural number satisfying

$$\text{supp}\psi^i \subseteq B(0, \leq 2^{j_0}) \quad \forall i \in \{1, ..., 2^d - 1\}.$$

First, let us introduce some notations. A dyadic cube of scale $j \in \mathbb{N}_0$ is a cube that can be written as

$$\lambda = \prod_{i=1}^{d} \left[\frac{k_i}{2^j}, \frac{k_i + 1}{2^j} \right[$$

where $k = (k_1, ..., k_d) \in \mathbb{Z}^d$. We consider the indices $i \in \{1, ..., 2^d - 1\}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$ which characterize the wavelet coefficients $c^i_{j,k}$. We can suppose that i takes its value in the set $\{0,1\}^d \backslash (0, ..., 0)$ without loss of generality. We let $\lambda = \lambda(i,j,k) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + \left[0, \frac{1}{2^{j+1}}\right)^d$ denote the dyadic cube associated with the wavelet coefficient $c_\lambda = c^i_{j,k}$.

Definition 165. The *wavelet leaders* are defined by

$$d_\lambda = \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|.$$

If $f \in L^\infty(\mathbb{R}^d)$, the wavelet leaders are finite because

$$|c_\lambda| \leq 2^{dj} \int_{\mathbb{R}^d} |f(x)||\psi_\lambda(x)| dx \leq C \|f\|_{L^\infty(\mathbb{R}^d)}.$$

Definition 166. Two dyadic cubes λ_1 and λ_2 are said to be adjacent if they have the same scale and if $\text{dist}(\lambda_1, \lambda_2) = 0$ (so, a dyadic cube is adjacent to itself). Let $\lambda_j(x_0)$ denote the dyadic cube of side 2^{-j} containing x_0 and 3λ denote the set of the 3^d dyadic cubes which are adjacent to λ; then

$$d_j(x_0) = \sup_{\lambda' \in 3\lambda_j(x_0))} d_{\lambda'}.$$

The characterization of generalized pointwise Hölder spaces is expressed in terms of $d_j(x_0)$ ($j \in \mathbb{N}$). The factor 3 in the definition of $d_j(x_0)$ is needed for technical requirements, so that the characterization remains true even for limit cases.

The wavelet characterization needs an assumption on the global regularity of the considered functions. This assumption is a little more restrictive than continuity on $\mathbb{R}^d$. It is expressed by the following definition.

[3] Using the notations from the sequel, we suppose that the wavelets belong to the space $C^{M+1}(\mathbb{R}^d)$ and have $M + 1$ vanishing moments.

Definition 167. A function f is said to be *uniformly Hölder* if there exists $\varepsilon > 0$ such that $f \in \Lambda^{\varepsilon}(\mathbb{R}^d)$.

Remark 168. One can easily check that a function f is uniformly Hölder if and only if there exists an admissible sequence σ such that $\underline{s}(\sigma^{-1}) > 0$ and $f \in \Lambda^{\sigma,M}(\mathbb{R}^d)$ for a certain natural number M.

Theorem 169. (D.K., S. Nicolay) Let $M \in \mathbb{N}_0$, $x_0 \in \mathbb{R}^d$ and σ be an admissible sequence. If $f \in \Lambda^{\sigma,M}(x_0)$, then there exist $C > 0$ and $J \in \mathbb{N}_0$ such that

$$d_j(x_0) \leq C\sigma_j \qquad \forall j \geq J. \tag{4.3}$$

Conversely, let us suppose that $\sigma_j \to 0$ if $j \to +\infty$. If f is uniformly Hölder and if (4.3) is satisfied, then $f \in \Lambda^{\sigma',M}(x_0)$ where σ' is a new admissible sequence defined by $\sigma'_j = \sigma_j |\log_2(\sigma_j)|$ ($j \in \mathbb{N}_0$) and M is a natural number satisfying $M + 1 > \overline{s}(\sigma^{-1})$.

Proof. Let us suppose that $f \in \Lambda^{\sigma,M}(x_0)$, $k_0 \in \mathbb{N}_0$ be such that $2^{\max(j_0;d)+3} + 4d \leq 2^{k_0}$ and let also $j \geq k_0 + 1$ and $\lambda = \lambda(i, j', k') \subset 3\lambda_j(x_0)$ (in particular we have $j' \geq j - 2$). We find

$$\begin{aligned}
|c_\lambda| &= \left| 2^{dj'} \int_{\mathbb{R}^d} f(x)\psi^i(2^{j'}x - k')dx \right| \\
&= \left| 2^{dj'} \int_{\mathbb{R}^d} (f(x) - P_{x_0,j-k_0}(x))\psi^i(2^{j'}x - k')dx \right| \\
&= \left| 2^{dj'} \int_{B(\frac{k'}{2^{j'}}, 2^{j_0 - j'})} (f(x) - P_{x_0,j-k_0}(x))\psi^i(2^{j'}x - k')dx \right| \\
&\leq 2^{dj'} \int_{B(x_0, 2^{-(j-k_0)})} |f(x) - P_{x_0,j-k_0}(x)||\psi^i(2^{j'}x - k')|dx \\
&\leq C\sigma_{j-k_0} 2^{dj'} \int_{\mathbb{R}^d} |\psi^i(2^{j'}x - k')|dx \leq C\sigma_j.
\end{aligned}$$

Conversely, let us suppose that the wavelet coefficients satisfy (4.3) and that there exists $\varepsilon > 0$ such that $f \in \Lambda^{\varepsilon}(\mathbb{R}^d)$. We proved in theorem 115 that the functions f_j ($j \in \mathbb{N}_0 \cup \{-1\}$) have the same regularity as wavelets and that $f = \sum_{j=-1}^{+\infty} f_j$ uniformly on $\mathbb{R}^d$. Let us set

$$P_{x_0,J}(x - x_0) := \sum_{|\beta| \leq M} \frac{(x - x_0)^\beta}{|\beta|!} \sum_{j=-1}^{J} D^\beta f_j(x_0).$$

This is a polynomial of degree less or equal to M. Let n_d be a natural number such that, for each dyadic number $\frac{k}{2^j}$ ($k \in \mathbb{Z}^d$, $j \in \mathbb{N}_0$), for each $R \geq 2^{-j}$ and $x \in \mathbb{R}^d$ such that $\frac{k}{2^j} \in B(x, R)$, the dyadic cube $\frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^j}[^d$ is included in the ball $B(x, 2^{n_d}R)$. Let m_d be a natural number such that each ball $B(x, 2^{-j})$ ($x \in \mathbb{R}^d$, $j \in \mathbb{N}_0$) is included in a dyadic cube (whose vertices have dyadic coordinates) with a side length $2^{m_d}2^{-j}$. Finally, let $J \geq \sup\{J', j_0 + n_d + m_d + 1\}$, where $J' \in \mathbb{N}_0$ is such that $\sigma_j < 1$ for all $j \geq J'$. We have

$$\|f - P_{x_0,J}\|_{B(x_0,2^{-J})} \leq \sum_{j=-1}^{J} \|f_j(x) - \sum_{|\beta|\leq M} \frac{(x-x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{B(x_0,2^{-J})} + \sum_{j=J+1}^{+\infty} \|f_j\|_{B(x_0,2^{-J})}.$$

Let $J_1 \in \mathbb{N}_0$ denote the unique natural number satisfying $2^{-\varepsilon J_1} \leq \sigma_J < 2^{-\varepsilon(J_1-1)}$.

1. Firstly, we study the term

$$\sum_{j=-1}^{J} \|f_j(x) - \sum_{|\beta|\leq M} \frac{(x-x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{B(x_0,2^{-J})}.$$

Let $j \leq J$. By the Taylor decomposition, we find

$$\|f_j(x) - \sum_{|\beta|\leq M} \frac{(x-x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{B(x_0,2^{-J})} \leq C 2^{-J(M+1)} \sup_{|\beta|=M+1} \|D^\beta f_j\|_{B(x_0,2^{-J})}.$$

If $|\beta| = M+1$, we prove that we have

$$\|D^\beta f_j\|_{B(x_0,2^{-J})} \leq C\sigma_j 2^{j(M+1)} \sup_i \|\sum_{k\in\mathbb{Z}^d} D^\beta \psi^i(2^j x - k)\|_{B(x_0,2^{-J})}.$$

Indeed, if $x \in B(x_0, 2^{-J})$, we have

$$|D^\beta f_j(x)| \leq \sum_i \sum_{k\in\mathbb{Z}^d} |c^i_{j,k}| 2^{j(M+1)} |D^\beta \psi^i(2^j x - k)| = \sum_i \sum_{\substack{k\in\mathbb{Z}^d \\ k2^{-j}\in B(x,2^{-(j-j_0)})}} |c^i_{j,k}| 2^{j(M+1)} |D^\beta \psi^i(2^j x - k)|.$$

Each wavelet coefficient $c^i_{j,k} = c_\lambda$ in the last sum is such that the associated dyadic cube is included in $B(x, 2^{-(j-j_0-n_d)})$. If $j \geq j_0 + n_d + m_d + 1$, then we have

$$|c^i_{j,k}| \leq C\sigma_{j-j_0-n_d-m_d-1} \leq C\sigma_j.$$

Otherwise, we have $|c^i_{j,k}| \leq C \leq C'\sigma_j$ because the function f is uniformly Hölder, where $C' = \sup\{C/\sigma_j : j \in \{0, ..., j_0 + n_d + m_d\}\}$. This proves the announced inequality. This also implies

$$\|D^\beta f_j\|_{B(x_0,2^{-J})} \leq C\sigma_j 2^{j(M+1)},$$

because $D^\beta\psi^i$ has compact support and is continuous on $\mathbb{R}^d$. We obtain

$$\sum_{j=-1}^{J}\|f_j(x)-\sum_{|\beta|\leq M}\frac{(x-x_0)^\beta}{|\beta|!}D^\beta f_j(x_0)\|_{B(x_0,2^{-J})}\leq C2^{-J(M+1)}\sum_{j=-1}^{J}\sigma_j 2^{j(M+1)}$$
$$\leq C\sigma_J.$$

2. We consider the term

$$\sum_{j=J_1+1}^{+\infty}\|f_j\|_{B(x_0,2^{-J})}.$$

Using the proof of theorem 115, we have

$$\begin{aligned}\sum_{j=J_1+1}^{+\infty}\|f_j\|_{B(x_0,2^{-J})} &\leq \sum_{j=J_1+1}^{+\infty}\|f_j\|_{\mathbb{R}^d}\\ &\leq C\sum_{j=J_1+1}^{+\infty}2^{-\varepsilon j}\\ &\leq C2^{-\varepsilon J_1}\\ &\leq C\sigma_J.\end{aligned}$$

3. Let us consider the term given by

$$\sum_{j=J+1}^{J_1}\|f_j\|_{B(x_0,2^{-J})}.$$

Let $j\in\{J+1,...,J_1\}$ and $x\in B(x_0,2^{-J})$. We have

$$|f_j(x)|\leq\sum_i\sum_{\substack{k\in\mathbb{Z}^d\\ k2^{-j}\in B(x,2^{-(j-j_0)})}}|c_{j,k}^i\psi^i(2^jx-k)|. \tag{4.4}$$

If $j\geq J+j_0+n_d$, then the wavelet coefficients in (4.4) are such that the associated dyadic cubes satisfy

$$\lambda=\lambda(i,j,k)\subseteq B(x,2^{-(j-j_0-n_d)})\subseteq B(x_0,2^{-(J-1)})$$

and

$$|c_{j,k}^i|=|c_\lambda|\leq C\sigma_{J-m_d-1}\leq C\sigma_J.$$

If $j<J+j_0+n_d$, then we have

$$\lambda(i,j,k)\subseteq B(x,2^{-(j-j_0-n_d)})\subseteq B(x_0,2^{-(j-j_0-n_d-1)}),$$

which implies

$$|c^i_{j,k}| = |c_\lambda| \leq C\sigma_{j-j_0-n_d-m_d-1} \leq C\sigma_j \leq C\sigma_J .$$

Finally, we find

$$\begin{aligned}\sum_{j=J+1}^{J_1} \|f_j\|_{B(x_0,2^{-J})} &\leq CJ_1\sigma_J \\ &\leq C|\log_2(\sigma_J)|\sigma_J,\end{aligned}$$

which proves the result.

□

Remark 170. Because $\sigma'_j \to 0$ if $\sigma_j \to 0$, the previous result always gives information on the pointwise regularity of f at x_0 (under the assumptions of the previous theorem).

4.5 A characterization of pointwise generalized Hölder-Zygmund spaces in terms of the convolution product

We have seen in section 2.4 that global generalized Hölder-Zygmund spaces can be characterized through approximations with a convolution product of their own elements with a smooth function. The aim of this section is to prove a similar characterization for pointwise spaces.

We have the following result, using the same proof as in lemma 78.

Lemma 171. *Let $N \in \mathbb{N}_0$, $\rho \in D(\mathbb{R}^d)$, $\sigma = (\sigma_j)_{j\in\mathbb{N}^*}$ be an admissible sequence and $f \in L^1_{loc}(\mathbb{R}^d)$ be a function satisfying*

$$\sup_{k\geq j} \|f \star \rho_{2^{-k}} - f\|_{L^\infty(B(x_0,2^{-j}))} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0 .$$

For all $\beta \in \mathbb{N}_0^d$ such that $|\beta| \leq N$, we have

$$\|D^\beta \left(f \star \rho_{2^{-j}} - f \star \rho_{2^{-(j-1)}}\right)\|_{B(x_0,2^{-j})} \leq C2^{jN}\sigma_j \qquad \forall j \in \mathbb{N}^* .$$

The characterization of pointwise Hölder spaces in terms of the convolution product can be written in the following way (the notation Φ_j refers to $\Phi_{2^{-j}}$).

Theorem 172. (D.K., S. Nicolay) Let $M \in \mathbb{N}_0$ and σ be an admissible sequence. If $f \in \Lambda^{\sigma,M}(x_0)$, then there exists a function $\Phi \in D(\mathbb{R}^d)$ such that

$$\sup_{k\geq j} \|f - f \star \Phi_k\|_{B(x_0,2^{-j})} \leq C\sigma_j, \quad \forall j \in \mathbb{N}_0 . \tag{4.5}$$

Conversely, let us suppose that $\sigma \to 0$. If the function $f \in L^\infty(\mathbb{R}^d)$ satisfies (4.5) and

$$\sup_{j\in\mathbb{N}_0}\left(2^{\alpha j}\sup_{k\geq j}\|f\star\Phi_k - f\|_{L^\infty}\right) < +\infty$$

for a certain $\alpha > 0$, then $f \in \Lambda^{\sigma,M}(x_0)$ for all natural number $M \in \mathbb{N}_0$ such that $M+1 > \overline{s}(\sigma^{-1})$.

Proof. Let $f \in \Lambda^{\sigma,M}(x_0)$ and Φ be the function defined in section 2.2. Using a similar proof as in proposition 77, we find

$$f\star\Phi_k(x) - f(x) = C\int \Delta_{2^{-k}t}^{M'+1} f(x)\rho(t)dt, \quad \forall x \in \mathbb{R}^d,$$

for a natural number $M' \geq M$. This proves equality (4.5). Let us prove the converse part. By assumption, there exists $\alpha < 1$ such that $f \in \Lambda^\alpha(\mathbb{R}^d)$. If $(f_j)_j$ denotes the sequence of functions which are defined in proposition 80, we have

$$f = \sum_{j=1}^{+\infty} f_j \quad \text{uniformly on } \mathbb{R}^d,$$

because f is uniformly Hölder. So, we also have

$$\Delta_h^{M+1} f = \sum_{j=1}^{+\infty}\Delta_h^{M+1} f_j \quad \text{uniformly on } \mathbb{R}^d, \text{ for all } h \in \mathbb{R}^d.$$

Let $n_0 \in \mathbb{N}_0$ such that $M+1 \leq 2^{n_0}$, $h \in \mathbb{R}^d$ such that $|h| \leq 2^{-(j+n_0)}$, and $j_0 \in \mathbb{N}_0$ such that $2^{-(j_0+1)\alpha} \leq \sigma_j < 2^{-j_0\alpha}$. We have

$$\begin{aligned}\|\Delta_h^{M+1} f\|_{B(x_0,2^{-j})} \leq & \sum_{k=1}^{j-1}\|\Delta_h^{M+1} f_k\|_{B(x_0,2^{-j})} + \|\sum_{k=j}^{j_0}\Delta_h^{M+1} f_k\|_{B(x_0,2^{-j})} \\ & + \sum_{k=j_0+1}^{+\infty}\|\Delta_h^{M+1} f_k\|_{B(x_0,2^{-j})}.\end{aligned}$$

1. We find

$$\begin{aligned}\sum_{k=1}^{j-1}\|\Delta_h^{M+1} f_k\|_{B(x_0,2^{-j})} &\leq \sum_{k=1}^{j-1} C|h|^{M+1}\sup_{|\beta|=M+1}\|D^\beta f_k\|_{B(x_0,2^{-(j-1)})} \\ &\leq C2^{-j(M+1)}\sum_{k=1}^{j-1}2^{k(M+1)}\sigma_k \\ &\leq C\sigma_j,\end{aligned}$$

   using the mean value theorem for the first inequality, lemma 171 for the second one, and the assumption $M+1 > \overline{s}(\sigma^{-1})$ in the last one.

2. We have

$$\begin{aligned}\|\sum_{k=j}^{j_0}\Delta_h^{M+1}f_k\|_{B(x_0,2^{-j})} &= \|\Delta_h^{M+1}(f\star\Phi_{j_0}-f\star\Phi_{j-1})\|_{B(x_0,2^{-j})}\\ &\leq C\|f\star\Phi_{j_0}-f\star\Phi_{j-1})\|_{B(x_0,2^{-(j-1)})}\\ &\leq C\left(\|f\star\Phi_{j_0}-f\|_{\mathbb{R}^d}+\|f-f\star\Phi_{j-1})\|_{B(x_0,2^{-(j-1)})}\right)\\ &\leq C(2^{-j_0\alpha}+\sigma_{j-1})\leq C\sigma_j.\end{aligned}$$

3. Finally, as $f\in\Lambda^\alpha(\mathbb{R}^d)$, we have

$$\begin{aligned}\sum_{k=j_0+1}^{+\infty}\|\Delta_h^{M+1}f_k\|_{B(x_0,2^{-j})} &\leq C\sum_{k=j_0+1}^{+\infty}\|f_k\|_{\mathbb{R}^d}\\ &\leq C\sum_{k=j_0+1}^{+\infty}2^{-k\alpha}\\ &\leq C2^{-j_0\alpha}\leq C\sigma_j.\end{aligned}$$

The conclusion follows from

$$\sup_{|h|\leq 2^{-(j+n_0)}}\|\Delta_h^{M+1}f\|_{B(x_0,2^{-(j+n_0)})}\leq\sup_{|h|\leq 2^{-(j+n_0)}}\|\Delta_h^{M+1}f\|_{B(x_0,2^{-j})}\leq C\sigma_j\leq C\sigma_{j+n_0}.$$

□

Remark 173. The two following results, which are slight modifications of lemma 171 and theorem 172, can be proved similarly.

Lemma 174. *Let $N\in\mathbb{N}_0$, $\rho\in D(\mathbb{R}^d)$ be a function such that its support is included in the ball $B(0,1/4)$, $\sigma=(\sigma_j)_{j\in\mathbb{N}^*}$ be an admissible sequence and $f\in L^1_{loc}(\mathbb{R}^d)$ be a function satisfying*

$$\|f\star\rho_{2^{-j}}-f\|_{L^\infty(B(x_0,2^{-j}))}\leq C\sigma_j\quad\forall j\in\mathbb{N}^*\,.$$

Then, for all $\beta\in\mathbb{N}_0^d$ such that $|\beta|\leq N$, we have

$$\|D^\beta\left(f\star\rho_{2^{-j}}-f\star\rho_{2^{-(j-1)}}\right)\|_{B(x_0,2^{-(j+1)})}\leq C2^{jN}\sigma_j\qquad\forall j\in\mathbb{N}^*\,.$$

Theorem 175. *Let $M\in\mathbb{N}_0$ and σ be an admissible sequence. If $f\in\Lambda^{\sigma,M}(x_0)$, then there exists a function $\Phi\in D(\mathbb{R}^d)$ whose support is included in the ball $B(0,1/4)$ and such that*

$$\|f-f\star\Phi_j\|_{B(x_0,2^{-j})}\leq C\sigma_j,\quad\forall j\in\mathbb{N}_0\,.\tag{4.6}$$

Conversely, let us suppose that $\sigma\to 0$. If the function $f\in L^\infty(\mathbb{R}^d)$ satisfies (4.6) and

$$\sup_{j\in\mathbb{N}_0}\left(2^{\alpha j}\sup_{k\geq j}\|f\star\Phi_k-f\|_{L^\infty}\right)<+\infty$$

for some $\alpha>0$ (for some function $\Phi\in D(\mathbb{R}^d)$ with compact support included in the ball $B(0,1/4)$), then $f\in\Lambda^{\sigma,M}(x_0)$ for all $M\in\mathbb{N}_0$ such that $M+1>\overline{s}(\sigma^{-1})$.

4.6 An application of generalized Hölder-Zygmund spaces: the Takagi function

In 1903, T. Takagi published an example of a continuous but nowhere differentiable function ([115]). Other authors rediscovered this function later under different forms, which inspired and fascinated many mathematicians (this is still the case nowadays). This function has many singular properties. As the Weierstrass function, the Takagi function is continuous but nowhere differentiable. It is used as a tool in many mathematical areas, such as classical real analysis, multifractal analysis, combinatorics and number theory. For example, let us cite its use in the characterization of zero sets of continuous nowhere differentiable functions ([87, 107]) and its use as a key element for solving the binary digital sum problem ([38, 123]).

Let us recall its formal definition.

Definition 176. The *Takagi function* is defined by

$$T(x) = \sum_{n=0}^{+\infty} \frac{1}{2^n}\phi(2^n x), \quad x \in [0,1],$$

where $\phi(x) = \mathrm{dist}(x, \mathbb{Z})$. Figure 4.2 represents the graphic of T.

The goal of this section is to determine the global and the pointwise regularity of the Takagi function.

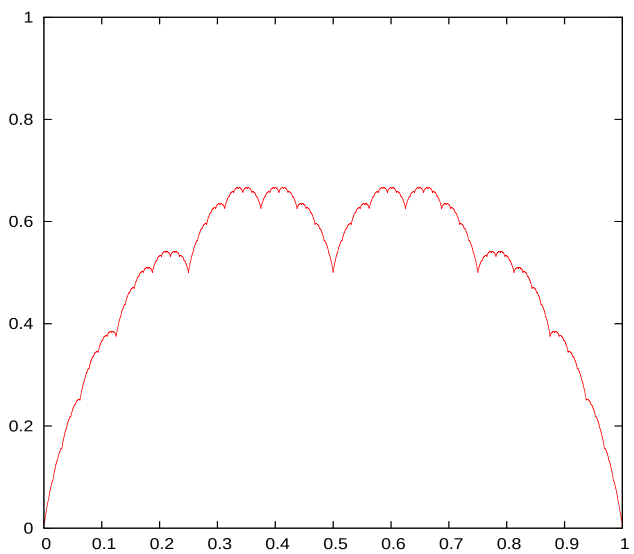

Figure 4.2: Graphic of the Takagi function T defined on $[0, 1]$.

It is known that[4] $T \in \Lambda^\alpha(\mathbb{R})$ for $\alpha \in]0,1[$ but $T \notin \Lambda^{\alpha,(2^{-j})_j}(\mathbb{R})$ because T is not Lipschitz ([110]). However, there exist some points $x \in [0,1]$ called *slow points* such that $T \in \Lambda^{\alpha,(2^{-j})_j}(x)$ for $\alpha \in]0,1[$ ([1]). In [10], it is shown that $T \in \Lambda^{\alpha,\sigma}(\mathbb{R})$ where $\sigma_j = 2^{-j}j$ ($j \in \mathbb{N}^*$) and $T \notin \Lambda^1(\mathbb{R})$ for $\alpha \in]0,1[$. Moreover, the sequence σ is the best estimate for T to be an element of a generalized Hölder-Zygmund space.

This example shows that generalized Hölder-Zygmund spaces can provide interesting additional information besides classical Hölder-Zygmund spaces. It also shows that pointwise generalized Hölder-Zygmund spaces complete the study of the global regularity in some cases.

A global survey about the Takagi function is done in [5].

4.7 Generalized pointwise Hölder exponent

The goal of this section is to give some conditions on admissible sequences that lead to embedded generalized pointwise Hölder spaces. If we have embedded pointwise spaces, we can define an Hölder exponent which gives information on the pointwise regularity at x_0. Our aim is to prove an analogous result to proposition 132 obtained in section 3.1. However, we need to completely change the ideas behind the proofs.

Notation 177. Let $M \in \mathbb{N}_0$. Let $P^{(M)}$ denote the part of degree less or equal to M of a polynomial P.

First, let us consider the following result.

Proposition 178. *Let $M \in \mathbb{N}^*$ and σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > M$. We have*

$$\Lambda^{\sigma,M}(x_0) \subseteq \Lambda^{(2^{-jM})_j,M-1}(x_0).$$

Proof. Let P_{x_0} be the polynomial given by theorem 162. It is given by

$$P_{x_0}(x) = \sum_{|\beta|\leq M} D^\beta f(x_0)\frac{(x-x_0)^\beta}{|\beta|!}.$$

Let us note that

$$\left| f(x_0+h) - \sum_{|\beta|\leq M-1} D^\beta f(x_0)\frac{h^\beta}{|\beta|!}\right| \leq C\sigma_j + \left|\sum_{|\beta|=M} D^\beta f(x_0)\frac{h^\beta}{|\beta|!}\right|$$

for all $|h| \leq 2^{-j}$. We have

$$\inf_{P\in\mathbb{P}_{M-1}} \|f-P\|_{B(x_0,2^{-j})} \leq \|f - P_{x_0}^{(M-1)}\|_{B(x_0,2^{-j})} \leq C(\sigma_j + 2^{-jM}) \leq C' 2^{-jM},$$

because $2^{jM}\sigma_j \to 0$. □

[4] We can consider that the Takagi function is defined on $\mathbb{R}$ by multiplying it with an adequate regular function with compact support. For example, one can use $\rho \in D(\mathbb{R})$ such that $\rho = 1$ on $[\varepsilon, 1-\varepsilon]$ (where $\varepsilon > 0$ is sufficiently small) and equal to 0 outside of $]0,1[$).

This result is interesting but does not give enough accurate information. This is a consequence of the assumption expressed in terms of Boyd indices among other things[5]. The two following results give more information than proposition 178.

Proposition 179. *Let $f \in \Lambda^{\sigma,M}(x_0)$ and $(P_{x_0,j})_{j\in\mathbb{N}_0}$ be a sequence of polynomials associated with the function f (given by the definition of the space $\Lambda^{\sigma,M}(x_0)$).*

1. *If $2^M d_1 < 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j + 2^{-jM}\right) \quad \forall j \in \mathbb{N}_0\,.$$
2. *If $2^M d_1 > 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j + 2^{-jM}(2^M d_1)^j\right) \quad \forall j \in \mathbb{N}_0\,.$$
3. *If $2^M d_1 = 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j + 2^{-jM} j\right) \quad \forall j \in \mathbb{N}_0\,.$$

Proposition 180. *Let $f \in \Lambda^{\sigma,M}(x_0)$ and $(P_{x_0,j})_{j\in\mathbb{N}_0}$ be a sequence of polynomials associated to the function f (given by the definition of the space $\Lambda^{\sigma,M}(x_0)$).*

1. *If $2^M d_0 < 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j (2^M d_0)^{-j} + 2^{-jM}\right) \quad \forall j \in \mathbb{N}_0\,.$$
2. *If $2^M d_0 > 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j + 2^{-jM}\right) \quad \forall j \in \mathbb{N}_0\,.$$
3. *If $2^M d_0 = 1$, then*
$$\|f - P^{(M-1)}_{x_0,j}\|_{B(x_0,2^{-j})} \leq C\left(\sigma_j j + 2^{-jM}\right) \quad \forall j \in \mathbb{N}_0\,.$$

These two results are mainly a consequence of the following lemma:

Lemma 181. *Let $f \in \Lambda^{\sigma,M}(x_0)$ and $(P_{x_0,j})_{j\in\mathbb{N}_0}$ be the sequence of polynomials associated to the function f. If $a^\beta_{x_0,j}$ denotes the β^{th} coefficient of the polynomial $P_{x_0,j}$, then we have*
$$\sup_{|\beta|=M} |a^\beta_{x_0,j}| \leq C\left(\sum_{k=1}^{j-1}(2^M d_1)^k + 1\right), \quad \forall j \in \mathbb{N}_0,$$
and
$$\sup_{|\beta|=M} |a^\beta_{x_0,j}| \leq C\left(\sigma_j d_0^{-j}\sum_{k=1}^{j-1}(2^M d_0)^k + 1\right), \quad \forall j \in \mathbb{N}_0\,.$$

[5]Let us remind that in section 3.1, we obtained two types of results for global spaces: the first one is expressed in terms of Boyd indices and the second one is expressed in terms of the factors d_0 and d_1 (given by definition of admissible sequences). The first one is a weaker result.

Proof of lemma 181. For all $|\beta| \leq M$, we have, using Markov's inequality (4.2),

$$\begin{aligned}
&\|D^\beta(P_{x_0,j} - P_{x_0,j+1})\|_{B(x_0,2^{-(j+1)})} \\
&\leq C2^{jM}\|P_{x_0,j} - P_{x_0,j+1}\|_{B(x_0,2^{-(j+1)})} \\
&\leq C2^{jM}\left(\|P_{x_0,j} - f\|_{B(x_0,2^{-j})} + \|P_{x_0,j+1} - f\|_{B(x_0,2^{-(j+1)})}\right) \\
&\leq C2^{jM}\sigma_j,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|D^\beta(P_{x_0,1} - P_{x_0,j})\|_{B(x_0,2^{-j})} \\
&\leq \sum_{k=1}^{j-1} \|D^\beta(P_{x_0,k} - P_{x_0,k+1})\|_{B(x_0,2^{-j})} \\
&\leq \sum_{k=1}^{j-1} \|D^\beta(P_{x_0,k} - P_{x_0,k+1})\|_{B(x_0,2^{-(k+1)})} \\
&\leq C\sum_{k=1}^{j-1} 2^{kM}\sigma_k \leq C\sigma_0 \sum_{k=1}^{j-1}(2^M d_1)^k,
\end{aligned}$$

for all $j \in \mathbb{N}_0$. Let $\beta \in \mathbb{N}_0^d$ such that $|\beta| = M$. We have

$$\begin{aligned}
\|D^\beta(P_{x_0,j} - P_{x_0,1})\|_{B(x_0,2^{-j})} &= \beta!|a^\beta_{x_0,j} - a^\beta_{x_0,1}| \\
&\geq |a^\beta_{x_0,j}|\beta! - |a^\beta_{x_0,1}|\beta!.
\end{aligned}$$

We thus get the first inequality of the result. To prove the second inequality, we notice that

$$\|D^\beta(P_{x_0,1} - P_{x_0,j})\|_{B(x_0,2^{-j})} \leq C\sum_{k=1}^{j-1} 2^{kM}\sigma_k \leq C\sigma_j d_0^{-j} \sum_{k=1}^{j-1}(2^M d_0)^k.$$

□

Proposition 179 is a consequence of the first inequality of lemma 181. Proposition 180 is a consequence of the second inequality of lemma 181.

Definition 182. Let $x_0 \in \mathbb{R}^d$. A family of admissible sequences $\sigma^{(\cdot)}$ is said to be x_0-*decreasing* if $\alpha < \beta$ implies $\Lambda^{\sigma^{(\beta)},\lfloor\beta\rfloor}(x_0) \subseteq \Lambda^{\sigma^{(\alpha)},\lfloor\alpha\rfloor}(x_0)$.

Corollary 183. (D.K., S. Nicolay) A family of admissible sequences $\sigma^{(\cdot)}$ is x_0-decreasing if it satisfies the following conditions:

1. if $m \leq \alpha < \beta < m+1$ for some natural number m, there exist C, $J > 0$ such that

$$\sigma_j^{(\beta)} \leq C\sigma_j^{(\alpha)} \quad \forall j \geq J,$$

2. for all $m \in \mathbb{N}^*$, at least one of the two following conditions is satisfied:

(a)
- there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$\sigma_j^{(m)} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 < 2^m d_1^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$2^{-jm}(2^m d_1^{(m)})^j \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 > 2^m d_1^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$2^{-jm} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 = 2^m d_1^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$j2^{-jm} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$

(b)
- there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$2^{-jm} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 < 2^m d_0^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$\sigma_j^{(m)} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 > 2^m d_0^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$\sigma_j^{(m)}(2^m d_0^{(m)})^{-j} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$
- if $1 = 2^m d_0^{(m)}$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exist C, $J > 0$ such that
$$j\sigma_j^{(m)} \leq C\sigma_j^{(m-\varepsilon)} \quad \forall j \geq J,$$

(where we let $d_0^{(m)}$ and $d_1^{(m)}$ denote some constants satisfying the inequality (1.2) associated to $\sigma^{(m)}$).

Remark 184. The conditions of corollary 183 imply that, for all $m \in \mathbb{N}^*$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, we have

$$2^{-jm} \leq C_\varepsilon \sigma_j^{(m-\varepsilon)} \quad \text{and} \quad \sigma_j^{(m)} \leq C_\varepsilon \sigma_j^{(m-\varepsilon)} \quad \forall j,$$

for some constant $C_\varepsilon > 0$. If the family of sequences does not even satisfy these two simplified conditions, then it is not worthwhile checking whether the conditions of the corollary are satisfied.

Remark 185. One can easily check that the sufficient conditions described in corollary 183 are equivalent to the ones obtained in proposition 132. It is remarkable that pointwise spaces maintain all main properties of global spaces, even though the proofs need to be completely adapted. One main difference is that we do not have classical derivatives as a tool in the pointwise case.

Example 186. In particular, corollary 183 can be used to prove that classical pointwise Hölder spaces are embedded: if $0 < \alpha < \beta$, we have

$$\Lambda^\beta(x_0) = \Lambda^{(2^{-j\beta})_j, \lfloor \beta \rfloor}(x_0) \subseteq \Lambda^\alpha(x_0) = \Lambda^{(2^{-j\alpha})_j, \lfloor \alpha \rfloor}(x_0).$$

As announced in the introduction, the concept of x_0-decreasing family of admissible sequences leads to the concept of generalized Hölder exponent at x_0.

Definition 187. Let $\sigma^{(\cdot)}$ be a x_0-decreasing family of admissible sequences. The *generalized Hölder exponent at* x_0 *associated with* $\sigma^{(\cdot)}$ of a function $f \in L^\infty_{loc}(\mathbb{R}^d)$ is defined by

$$h_f^{\sigma^{(\cdot)}}(x_0) = \sup\{\alpha > 0 : f \in \Lambda^{\sigma^{(\alpha)}, \lfloor \alpha \rfloor}(x_0)\}.$$

Chapter 5

Appendix

5.1 About subadditive sequences

Definition 188. A sequence $(a_n)_{n\in\mathbb{N}^*}$ of real numbers is said to be *subadditive* if it satisfies the inequality

$$a_{n+m} \leq a_n + a_m \qquad \forall n, m \in \mathbb{N}^* .$$

The following lemma is attributed to M. Fekete ([52]). We show a proof that is relatively simple here below.

Lemma 189 (Fekete (1923))**.** *For every subadditive sequence $(a_n)_{n\in\mathbb{N}^*}$, the limit $\lim_{n\to+\infty} \frac{a_n}{n}$ exists and is equal to $\inf_{n\in\mathbb{N}^*} \frac{a_n}{n}$ (the limit can be equal to $-\infty$).*

Proof. We set $\gamma = \inf\left(\frac{a_n}{n}\right)$. Let us at first suppose that $\gamma > -\infty$ (so γ is a real number). For all $\varepsilon > 0$, there exists $k \in \mathbb{N}_0$ such that $a_k \leq (\gamma+\varepsilon)k$. We also have

$$a_{nl} \leq n a_l \quad \forall n, l \in \mathbb{N}^*$$

by subadditivity, which implies

$$\frac{a_{nl}}{nl} \leq \frac{a_l}{l} \quad \forall n, l \in \mathbb{N}^* .$$

For all $n \in \mathbb{N}^*$, if $l \in \mathbb{N}_0$ is such that $n < lk$, then we have

$$\inf_{m\geq n} \frac{a_m}{m} \leq \frac{a_{lk}}{lk} \leq \gamma + \varepsilon$$

which implies $\gamma = \liminf \frac{a_n}{n}$.

For all $m \in \mathbb{N}_0$, we write $m = kn + j$ where $0 \leq j < k$ so that

$$a_m = a_{nk+j} \leq a_{kn} + a_j \leq (\gamma+\varepsilon)kn + a_j$$

which implies

$$\frac{a_m}{m} \leq (\gamma+\varepsilon)\frac{kn}{m} + \sup_{0\leq j<k} \frac{a_j}{m}.$$

So, we find

$$\sup_{m\geq n'}\frac{a_m}{m}\leq\sup_{m\geq n'}\left((\gamma+\varepsilon)\frac{kn}{m}\right)+\sup_{m\geq n'}\sup_{0\leq j<k}\frac{a_j}{m}\leq\sup_{m\geq n'}\left((\gamma+\varepsilon)\frac{kn}{m}\right)+\sup_{m\geq n'}\sup_{0\leq j<k}\frac{ja_1}{m}.$$

Let us prove that $\lim_{n'\to+\infty}\sup_{m\geq n'}\sup_{0\leq j<k}\frac{ja_1}{m}\leq 0$. If $a_1\leq 0$, the result is obvious and if $a_1>0$, this results from

$$\sup_{0\leq j<k}\frac{ja_1}{m}\leq\frac{k}{nk}a_1=\frac{a_1}{n}\to 0\quad\text{if}\quad n'\to+\infty.$$

Moreover, we have

$$\lim_{n'\to+\infty}\sup_{m\geq n'}\left((\gamma+\varepsilon)\frac{kn}{m}\right)\leq\gamma+\varepsilon.$$

Indeed, if $\gamma+\varepsilon>0$, this results immediately from $\frac{kn}{m}\leq 1$. If $\gamma+\varepsilon<0$, this results from

$$(\gamma+\varepsilon)\frac{kn}{m}\leq(\gamma+\varepsilon)\frac{kn}{k(n+1)}=(\gamma+\varepsilon)\frac{n}{(n+1)}\to(\gamma+\varepsilon)\quad\text{if}\quad n'\to+\infty.$$

We have proved that

$$\limsup\frac{a_m}{m}\leq\gamma+\varepsilon=\liminf\frac{a_m}{m}+\varepsilon.$$

Now, let us consider the case $\gamma=-\infty$. Let us prove that

$$\limsup\frac{a_n}{n}=-\infty.$$

For all $N>0$, there exists $k\in\mathbb{N}_0$ such that $\frac{a_k}{k}\leq -N$. Applying the same logic as the ideas developed in the first part of the proof, we have

$$\frac{a_{kn}}{kn}\leq\frac{a_k}{k}\leq -N$$

and if we keep the notation of the decomposition $m=kn+j$ where $0\leq j<k$, we have

$$\begin{aligned}\sup_{m\geq n'}\frac{a_m}{m}&\leq\sup_{m\geq n'}(\frac{a_{nk}}{m})+\sup_{m\geq n'}(\frac{a_j}{m})\\&\leq\sup_{m\geq n'}(\frac{-Nkn}{m})+\sup_{m\geq n'}(j\frac{a_1}{m})\\&\leq\sup_{m\geq n'}(\frac{-Nn}{n+1})+\sup_{m\geq n'}(j\frac{a_1}{m}).\end{aligned}$$

If we separate the case $a_1\geq 0$ from the case $a_1<0$, we conclude that the second term of this last inequality converges to 0 if n' goes to infinity. Hence the conclusion follows. □

An analogous result to this lemma is true for the *superadditives sequences*, i.e. the sequences satisfying $a_{n+m}\geq a_n+a_m\quad\forall m,n\in\mathbb{N}^*$.

5.2 Some inequalities from S. Bernstein

Proposition 190. *Let $k \in \mathbb{N}_0$ and R_1, $R_2 \in \mathbb{R}$ such that $0 < R_1 < R_2$. There exists a positive constant C (which only depends on R_1, R_2, k and d) such that for every function $u \in L^\infty(\mathbb{R}^d)$, we have*

$$supp\mathcal{F}u \subseteq B(0, R_1\lambda) \Rightarrow \sup_{|\alpha|=k} \|D^\alpha u\|_{L^\infty} \leq C\lambda^k \|u\|_{L^\infty}, \tag{5.1}$$

$$supp\mathcal{F}u \subseteq B(0, \leq R_2\lambda)\backslash B(0, R_1\lambda) \Rightarrow C^{-1}\lambda^k\|u\|_{L^\infty} \leq \sup_{|\alpha|=k} \|D^\alpha u\|_{L^\infty} \leq C\lambda^k\|u\|_{L^\infty}. \tag{5.2}$$

5.3 About the Littlewood-Paley decomposition

The aim of this section is to prove certain results presented in section 2.8.

Proposition 191. *We have*

$$Id = S_0 + \Delta_0 + \Delta_1 + \dots,$$

with convergence in $\mathcal{S}'(\mathbb{R}^d)$.

Proof. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned}(S_0 u + \sum_{j=0}^{N} \Delta_j u)(f) &= S_{N+1}u(f)\\ &= \mathcal{F}^{-1}(\hat{\varphi}(2^{-(N+1)}\xi)\mathcal{F}u)(f)\\ &= u(\mathcal{F}(\hat{\varphi}(2^{-(N+1)}\xi)\mathcal{F}^{-1}f))\\ &= u(\mathcal{F}^{-1}(\hat{\varphi}(-2^{-(N+1)}\xi)\mathcal{F}f)).\end{aligned}$$

Let us prove that we have

$$\mathcal{F}^{-1}(\hat{\varphi}(-2^{-(N+1)}\xi)\mathcal{F}f) \to f \quad \text{in } \mathcal{S}(\mathbb{R}^d)$$

which is equivalent to

$$\hat{\varphi}(-2^{-(N+1)}\xi)\mathcal{F}f \to \mathcal{F}f \quad \text{in } \mathcal{S}(\mathbb{R}^d).$$

Let k, $M \in \mathbb{N}_0$. It is sufficient to prove that

$$\sup_{x\in\mathbb{R}^d} (1 + |x|)^M |D^\alpha \left((\hat{\varphi}(-2^{-(N+1)}x) - 1)\mathcal{F}f\right)(x)| \longrightarrow 0$$

for all $|\alpha| = k$. Let $0 \neq \beta \leq \alpha$. We have

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d} (1 + |x|)^M |D^\beta(\hat{\varphi}(-2^{-(N+1)}x) - 1)D^{\alpha-\beta}\mathcal{F}f(x)|\\ &\qquad = \sup_{x\in\mathbb{R}^d} (1 + |x|)^M |D^\beta\hat{\varphi}(-2^{-(N+1)}x)2^{-(N+1)|\beta|}D^{\alpha-\beta}\mathcal{F}f(x)|\\ &\qquad \leq C2^{-(N+1)|\beta|} \sup_{x\in\mathbb{R}^d} (1 + |x|)^M |D^{\alpha-\beta}\mathcal{F}f(x)|.\end{aligned}$$

Moreover,

$$\sup_{x\in\mathbb{R}^d}(1+|x|)^M|(\hat{\varphi}(-2^{-(N+1)}x)-1)D^\alpha\mathcal{F}f(x)|$$
$$\leq C\sup_{|x|\geq 2^N}(1+|x|)^M|D^\alpha\mathcal{F}f(x)|\longrightarrow 0 \quad \text{as } N\to+\infty.$$

By using the Leibniz formula, we obtain the desired result. □

Lemma 192. *If $f\in L^p(\mathbb{R}^d)$ where $p\in[1,+\infty]$, then the functions $S_j(f)$ and $\Delta_j(f)$ belong to the space $L^p(\mathbb{R}^d)$ and*

$$S_j(f)=2^{jd}\varphi(2^j\cdot)\star f \quad \textit{and} \quad \Delta_j(f)=2^{jd}\psi(2^j\cdot)\star f$$

for all $j\in\mathbb{Z}$.

Proof. Let us prove the result for $S_j(f)$ (the proof can easily be adapted to $\Delta_j(f)$). Since the function $2^{jd}\varphi(2^j\cdot)$ belongs to the Schwartz space, it belongs in particular to the spaces $L^q(\mathbb{R}^d)$ for all $q\in[1,+\infty]$. Using Hausdorff-Young inequalities, the function defined by $2^{jd}\varphi(2^j\cdot)\star f$ exists, belongs to $L^p(\mathbb{R}^d)$ and, by a classical property of the Fourier transform of tempered distributions, it satisfies

$$\mathcal{F}(2^{jd}\varphi(2^j\cdot)\star f)=\mathcal{F}(2^{jd}\varphi(2^j\cdot))\mathcal{F}f=\hat{\varphi}(2^{-j}\xi)\mathcal{F}f.$$

□

5.4 Some reminders about wavelets

In this section, we recall the concept of wavelets as well as some of their basic properties.

The wavelet bases in $L^2(\mathbb{R}^d)$ can be obtained from a function ϕ and from 2^d-1 functions $\psi^1,...,\psi^{2^d-1}$, which are all assumed to be sufficiently regular. The function ϕ is often called the scaling function or "father" wavelet, and the functions ψ^i are often called the "mother" wavelets. We define the following functions by translations and dilatations of functions ϕ and ψ^i:

$$\phi_k(x)=\phi(x-k), \quad \psi^i_{j,k}(x)=\psi^i(2^jx-k)$$

for all $i\in\{1,...,2^d-1\}$, $j\in\mathbb{N}_0$, $k\in\mathbb{Z}^d$.

Under some conditions, the set $\{\phi_k\}\cup\{2^{jd/2}\psi^i_{j,k}\}_{i,j,k}$ defines an orthonormal basis of $L^2(\mathbb{R}^d)$ and forms what is called a *multiresolution analysis* or a *multiresolution approximation* ([37, 93]). We recall this concept here below:

Definition 193. A sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ is a *multiresolution analysis* if the six following properties are satisfied:

1. the space V_j is invariant by translation proportional to the scale 2^{-j}: for all $j\in\mathbb{Z}$, $k\in\mathbb{Z}^d$, the function f belongs to the space V_j if and only if $f(.-2^{-j}k)$ also belongs to the space V_j;

2. for all $j \in \mathbb{Z}$, we have $V_j \subset V_{j+1}$;

3. for all $j \in \mathbb{Z}$, the function f belongs to the space V_{j+1} if and only if $f(./2)$ belongs to the space V_j;

4. we have
$$\bigcap_{j=-\infty}^{+\infty} V_j = \{0\};$$

5. we have
$$\overline{\bigcup_{j=-\infty}^{+\infty} V_j} = L^2(\mathbb{R}^d);$$

6. there exists a function θ such that the sequence $\{\theta(. - k)\}_{k\in\mathbb{Z}^d}$ is a Riesz basis of the space V_0, i.e. it is a sequence of elements of V_0 such that there exist C_1, $C_2 > 0$ such that, for every sequence of scalars $(\alpha_k)_{k\in\mathbb{Z}^d} \in l^2(\mathbb{Z})$, we have
$$C_1 \left(\sum_k |\alpha_k|^2\right)^{1/2} \leq \| \sum_k \alpha_k \theta(. - k)\|_{L^2(\mathbb{R}^d)} \leq C_2 \left(\sum_k |\alpha_k|^2\right)^{1/2}, \quad (5.3)$$
and the vector space of finite sums $\sum_k \alpha_k \theta(. - k)$ (on which the inequality (5.3) is tested) is dense in V_0.

If the functions ϕ and ψ^i, which constitute the multiresolution analysis, belong to $C^r(\mathbb{R}^d)$ and if the derivatives of order less or equal to r are rapidly decreasing, we say that the multiresolution analysis is of regularity r ([64]). We define the *wavelet coefficients* of a function f of $L^2(\mathbb{R}^d)$ by
$$C_k = \int_{\mathbb{R}^d} f(x)\phi_k(x)dx, \quad c^i_{j,k} = 2^{jd} \int_{\mathbb{R}^d} f(x)\psi^i_{j,k}(x)dx \quad (5.4)$$
for all $j \in \mathbb{N}_0$, $i \in \{1, ..., 2^d - 1\}$, $k \in \mathbb{Z}^d$.

Hence, any function f of $L^2(\mathbb{R}^d)$ can be decomposed in the basis in the following way:
$$f = \sum_{k\in\mathbb{Z}^d} C_k\phi_k + \sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} c^i_{j,k}\psi^i_{j,k}. \quad (5.5)$$
For more information concerning the multiresolution analysis and the wavelet decomposition, the reader can refer to [37, 89, 93].

Under some classical regularity conditions on the multiresolution analysis, the formulae (5.4) and (5.5) are still true in a more general framework than $L^2(\mathbb{R}^d)$ ([93]). The formulae should be understood as a dual product between sufficiently regular functions (the wavelets) and distributions (f in our case). Let us recall that if the multiresolution analysis is of regularity r, the wavelets have a corresponding number of vanishing moments ([93]):

$$\text{if } |\alpha| < r, \text{ we have } \int_{\mathbb{R}^d} \psi^i(x) x^\alpha dx = 0.$$

Therefore, if the wavelets belong to the Schwartz class, all their moments vanish.

5.5 Some reminders about the interpolation theory

In this section, we recall some of the classical concepts of the interpolation theory.

In the sequel, we consider two Banach spaces A_0 and A_1, which are continuously embedded in a topological vector space V, so that spaces $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined Banach spaces. Let us define the operator J for all $t > 0$ and $a \in A_0 \cap A_1$ by

$$J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

Let us give the definition of the J-method of interpolation.

Definition 194. Let $0 < \theta < 1$ and $1 \leq q \leq +\infty$. We define the *interpolation space* $[A_0, A_1]_{\theta,q,J}$ in the following way: we say that a belongs to $[A_0, A_1]_{\theta,q,J}$ if a can be written as $a = \sum_{j\in\mathbb{Z}} u_j$ with convergence in A_0+A_1, where $u_j \in A_0 \cap A_1$ and $(2^{-j\theta} J(2^j, u_j))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

Let us define the operator K for all $t > 0$ and $a \in A_0 + A_1$ by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

Let us give the definition of the K-method of interpolation.

Definition 195. Let $0 < \theta < 1$ and $1 \leq q \leq +\infty$. The *interpolation space* $[A_0, A_1]_{\theta,q,K}$ is defined in the following way: we say that a belongs to $[A_0, A_1]_{\theta,q,K}$ if $a \in A_0 + A_1$ and $(2^{-j\theta} K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

Let us recall the definition of the Sobolev spaces.

Definition 196. Let $p \in [1, +\infty]$ and $m \in \mathbb{N}_0$. The *Sobolev space* $W^p_m(\mathbb{R}^d)$ is defined by

$$W^p_m(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : D^\alpha f \in L^p(\mathbb{R}^d) \quad \forall |\alpha| \leq m\}.$$

We define a norm on this space by

$$\|f\|_{W^p_m} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p}.$$

The derivatives in this definition should be understood in a weak sense.

Remark 197. Even if the Sobolev spaces are defined in terms of derivatives in the weak sense, it is still possible to link them with the classical concept of derivatives up to a certain order. Indeed, the result below links Sobolev spaces and classical Hölder spaces together, and is generally attributed to Morrey ([114, 116]).

Proposition 198. *Let* $p \in]d, +\infty]$*; we have* $W_1^p(\mathbb{R}^d) \subset \Lambda^\gamma(\mathbb{R}^d)$ *where* $\gamma = 1 - \frac{d}{p}$*, and*

$$|f(x) - f(y)| \leq C|x - y|^\gamma \|f\|_{W_1^p} \qquad \forall f \in W_1^p(\mathbb{R}^d).$$

As a consequence, if $f \in W_m^\infty(\mathbb{R}^d)$, then the function f can be modified on a negligible set so that it belongs to the space $C^{m-1}(\mathbb{R}^d)$, and its derivatives $D^\alpha f$ for $|\alpha| = m - 1$ are differentiable almost everywhere on $\mathbb{R}^d$. Moreover, proposition 198 gives a result similar to the mean value theorem for these almost everywhere differentiable functions.

For more information about the classical theory of interpolation spaces, the reader can refer to [17, 86].

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