Diametral Dimension of topological vector spaces

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Topological invariant

It is a map τ on the class of tvs (or a subclass of the class of tvs) such that

$$X \cong Y \implies \tau(X) = \tau(Y)$$

or

$$\tau(X) \neq \tau(Y) \implies X \ncong Y.$$

Topological invariants

Another notion

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- Interest : determining the diametral dimension of S^{ν} spaces

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Diametral Dimension

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- Interest : determining the diametral dimension of S^ν spaces
- Every S^ν (not pseudoconvex) has the same diametral dimension !

Kolmogorov's diameters

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Definition

The n^{th} Kolmogorov's diameter of U in respect with V is

$$\delta_n(U, V) = \inf\{\delta > 0 : \exists F \in \mathcal{L}_n(E) \textit{ such that } U \subset \delta V + F\}.$$

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- $T : E \to F$ linear $\implies \delta_n(T(U), T(V)) \leq \delta_n(U, V)$
- $T : E \to F$ isomorphism of vector spaces
  $\Rightarrow \delta_n(T(U), T(V)) = \delta_n(U, V)$

The definition

Let E be a *topological* vector space and $\mathcal{V}(E)$ be a basis of 0-neighbourhood in E.

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Definition

The diametral dimension of E is the set

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Remark : the definition does not depend on the choice of the basis.

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Theorem

The diametral dimension is a topological invariant on the class of topological vector spaces.

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Some properties/examples

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Warning !

Diametral dimension is <u>not</u> complete !

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- The class of power series spaces
- Dragilev's class

Admissible spaces

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A Banach space $(l, \|.\|_l)$ of complex sequences is *admissible* if

- $\forall \xi \in l_\infty,\ \eta \in l,\ \eta\xi \in l$ and $\|\eta\xi\|_l \leq \|\xi\|_{l_\infty}\|\eta\|_l$
- $e_k := (\delta_{k,n})_{n\in\mathbb{N}_0} \in l$ and $\|e_k\|_l = 1$

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Examples

- $l_1 := \{\xi \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n\in\mathbb{N}} |\xi_n| < \infty\}$ with the norm $\|\xi\|_{l_1} = \sum_{n\in\mathbb{N}} |\xi_n|$

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- $l_p := \{\xi \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n\in\mathbb{N}} |\xi_n|^p < \infty\}$ with the norm $\|\xi\|_{l_p} = \left(\sum_{n\in\mathbb{N}} |\xi_n|^p\right)^{1/p}$ if $p \geq 1$

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- $(l_\infty, \|.\|_{l_\infty})$ and $(c_0, \|.\|_{l_\infty})$

Köthe sets and Köthe spaces

Definition

$A \subset \mathbb{C}^{\mathbb{N}_0}$ is a *Köthe set* if

- $\forall \alpha \in A, \ n \in \mathbb{N}_0, \ \alpha_n \geq 0,$
- $\forall n \in \mathbb{N}_0, \ \exists \alpha \in A$ such that $\alpha_n > 0,$
- $\forall \alpha, \beta \in A, \ \exists \gamma \in A$ such that $\sup\{\alpha_n, \beta_n\} \leq \gamma_n$

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Definition

Köthe sequence space :

$$\lambda^l(A) := \{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall \alpha \in A, \alpha\xi \in l\}$$

with the topology defined by the semi-norms $p^l_\alpha : \xi \mapsto \|\alpha\xi\|_l$.

Köthe spaces

Remark

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Examples/applications

- If $O = \left\{(e^{-n/k})_{n\in\mathbb{N}_0} : k \in \mathbb{N}\right\}$, then $\mathcal{O}(D(0,1)) \cong \lambda^{l_1}(O)$
- If $O' = \left\{(e^{nk})_{n\in\mathbb{N}_0} : k \in \mathbb{N}_0\right\}$, then $\mathcal{O}(\mathbb{C}) \cong \lambda^{l_1}(O')$

The idea : Taylor's development $f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$

Regular Köthe spaces

A particular case

$\lambda^l(A)$ is *regular* if

- $\forall k, n \in \mathbb{N}_0,\ a_k(n) > 0$
- $\forall k, n \in \mathbb{N}_0,\ a_k(n) \leq a_{k+1}(n)$,
- $\forall k \in \mathbb{N}_0$, the sequence $\left(\dfrac{a_k(n)}{a_{k+1}(n)}\right)_{n\in\mathbb{N}_0}$ is decreasing.

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Then...

Theorem

$$\Delta(\lambda^I(A)) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}\, \exists m \in \mathbb{N} \textit{ such that } \left(\frac{a_k(n)}{a_{k+m}(n)}\xi_n\right)_{n\in\mathbb{N}_0} \in c_0\right\}$$

Köthe spaces

Corollaries

After some developments,

- $\Delta(\mathcal{O}(D(0,1))) = \bigcap_{k\in\mathbb{N}} \{\xi \in \mathbb{C}^{\mathbb{N}_0} : (\xi_n e^{-n/k})_{n\in\mathbb{N}_0} \in l_\infty\}$
- $\Delta(\mathcal{O}(\mathbb{C})) = \bigcup_{k\in\mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \left(\xi_n e^{-kn}\right)_{n\in\mathbb{N}_0} \in l_\infty\right\}$

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So $(e^n)_{n\in\mathbb{N}_0} \in \Delta(\mathcal{O}(\mathbb{C})) \setminus \Delta(\mathcal{O}(D(0,1)))$

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So $(e^n)_{n\in\mathbb{N}_0} \in \Delta(\mathcal{O}(\mathbb{C})) \setminus \Delta(\mathcal{O}(D(0,1)))$

$$\implies \mathcal{O}(D(0,1)) \not\cong \mathcal{O}(\mathbb{C})!$$

Schwartz and nuclear spaces

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TFAE :

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Theorem

TFAE :

- E is nuclear
- $\forall p > 0$, $((n+1)^p)_{n\in\mathbb{N}_0} \in \Delta(E)$
- $\exists p > 0$ such that $((n+1)^p)_{n\in\mathbb{N}_0} \in \Delta(E)$

S^ν spaces and Diametral Dimension

It can be shown that

$$\begin{aligned}\Delta(S^\nu) &= \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \lim_{n\to+\infty}(\xi_n(n+1)^{-s}) = 0 \ \forall s > 0\right\} \\ &= \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \lim_{n\to+\infty}(\xi_n(n+1)^{-1/m}) = 0 \ \forall m \in \mathbb{N}_0\right\}\end{aligned}$$

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- $l_\infty \subset \Delta(S^\nu) \implies S^\nu$ spaces are Schwartz
- $((n+1)^2)_n \notin \Delta(S^\nu) \implies S^\nu$ spaces are not nuclear

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Question : are S^ν spaces isomorphic ?

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