# Towards a Simplified Stiff String Model for the Torque and Drag Problem

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*RÉSUMÉ. L'industrie pétrolière recourt régulièrement à des trains de tiges de plusieurs kilomètres de long pour transmettre les efforts de forage nécessaires pour atteindre des réservoirs d'hydrocarbures profonds. L'évaluation de la perte d'énergie le long du train de tiges, connu comme problème de "torque and drag", joue un rôle essentiel dans la planification et le forage des puits puisque le frottement apparaissant aux contacts entre le train de tiges et les parois du puits peuvent considérablement augmenter les coûts ou, même être un facteur limitant dans certaines configurations. L'identification du nombre de contacts, ainsi que leurs positions et longueures constitue la préoccupation principale de ce problème. Les non-linéarités associées aux grands déplacements du train de tiges et à la condition de non-pénétration, ainsi qu'au nombre a priori inconnu de contacts rendent l'utilisation d'outils numériques classiques plutôt inefficace. Pour contourner ces difficultés, un modèle simplifié est dérivé. Celui-ci partitionne le train de tiges en une succession de segments, soit en contact continu avec la parois du puits soit libres entre leurs extrémités ; l'originalité de la formulation proposée réside dans la description de la déviation du train de tiges comme une perturbation de l'axe du puits.*

*ABSTRACT. The petroleum industry relies on several kilometer long drillstrings to transmit the axial force and torque necessary to drill the rock formations and reach deep hydrocarbon reservoirs. The assessment of the energy loss along the drillstring, known as the torque and drag problem, plays an essential role in well planning and drilling as the friction appearing at the contacts between the drillstring and the borehole may dramatically increase the costs or, even, be a limiting factor in some configurations. The identification of the number of contacts as well as their positions and extents constitutes the central concern of this question. The nonlinearities associated with the large deflection of the drillstring and the non-penetration condition as well as the a priori unknown number of contacts, however, make the use of conventional numerical tools rather inefficient. To circumvent these difficulties, a simplified stiff string model is derived. Partitioning the global problem in segments of drillstring either in continuous contact with the borehole or free of contact between their extremities, the originality of the proposed formulation lies in the descripion of the drillstring deflection as perturbation of the well-path.*

*MOTS-CLÉS : forage, puits, train de tige, contact*

*KEYWORDS: torque and drag (T&D), soft/stiff string model, contact, drillstring, borehole*

## 1. Introduction

The interest for a complete *torque and drag (T&D)* model has been growing since the early eighties, shortly after the commercial viability of deviated wells was recognised by the oil and gas industry. Specifically, the necessity to drill deeper and more complex well profiles as well as the realisation of extended reach boreholes prompted the development of means to evaluate the transmission of forces from the rig to the bit. This question indeed plays an essential role in drilling and well design as the friction appearing at the contacts between the drillstring and the borehole may dramatically increase the costs or, possibly, be a limiting factor in some configurations. As the extremely low data-rates achieved through mud-pulse telemetry prevent the transmission of comprehensive information and measures from the bit to the rig, the drilling conditions at the bottom of the borehole generally remain mostly unknown until the drillstring is pulled out the well. Hence, to track a predefined well-path and reach the intended target while limiting the borehole tortuosity, the driller relies on its training as well as the accuracy of models predicting the drilling tendency [MAR 14]. Oilfield drilling operations being extremely costly, the accurate prediction as well as the real-time simulation and analysis of the forces involved therefore became a crucial issue for the oil and gas industry in the last decades.

The earliest contribution towards torque and drag modelling is due to Johancsick *et al.* [JOH 84], who proposed a *soft string model* that ignores the bending stiffness of the drillstring as well as the clearance between the borehole and the drillstring. This simple model, resulting in a continuous contact along the entire drillstring, however loses its accuracy as the borehole (micro and macro) tortuosity increases and the trajectories complicate. To overcome these shortcomings, *stiff string models* accounting for the drillstring bending stiffness have later been developed. Considerably more complex, these models resort to advanced numerical methods with built-in management of the contacts to accurately account for the effects of hole size, clearance and post-buckling behaviour. Conventional tools such as the finite element method however generally result in computational times unsuitable for real-time simulations or prove to be ineffective in describing continuous interactions with the borehole. Although the significance of the drillstring bending stiffness in torque and drag analysis is commonly acknowledged, the substantial complexity associated with the stiff string model led the industry to favour its soft string counterpart until relatively recently. Following an approach originated in [DEN 11] and [HUY 15b], we propose a novel stiff string model that hinges on *(i)* describing the drillstring deformed configuration by means of its deflection relative to the borehole axis, and *(ii)* partitioning the global problem in a sequence of drillstring segments either in continuous contact with the borehole or free of contact between their extremities. Exploiting the relatively small deflection of the drillstring with resect to the borehole axis, the drillstring configuration is further interpreted as a perturbation of the well-path. This approach reduces the global burden associated with the model and trivialises the detection of new contacts.

## 2. Problem definition

The determination of the position of the contacts between the drillstring and the borehole as well as the extent of these contacts constitutes a central concern of this work. As discrete and/or continuous contacts develop, a commonly adopted strategy consists in partitioning the global problem into a set of boundary value problems specifying both the position and inclination of the drillstring at its extremities [CHE 07, DEN 11, HUY 15b]. According to this *segmentation strategy*, each elementary problem corresponds to a segment of drillstring either in continuous contact with the borehole or free of contact between its extremities. These two distinct but complementary configurations essentially differ by the nature of the body force acting along the drillstring. While solely subjected to its weight in the absence of contact, the drillstring is additionally compelled to lay on the borehole surface along a continuous contact. Assuming frictionless interactions, the resulting contact pressure acts as an additional distributed body force operating normally to the wall of the borehole, its magnitude being however unknown.

The crux of the proposed stiff string model resides within these elementary problems and how they are formulated. The following sections therefore focus on this particular aspect and consider a segment of drillstring, either in continuous contact with the borehole or free of contact between its extremities, that is forced to go through two fixed points in space while being subjected to a distributed body force. The length of the drillstring spanning this elementary problem and satisfying the associated boundary conditions is however *a priori* unknown and, therefore, constitutes an inherent part of the solutions. This feature, specific to free boundary problems, combined with the boundary conditions specifying the rod location at both extremities lead to the establishment of integral constraints (namely isoperimetric constraints) on the unknown length of the rod.

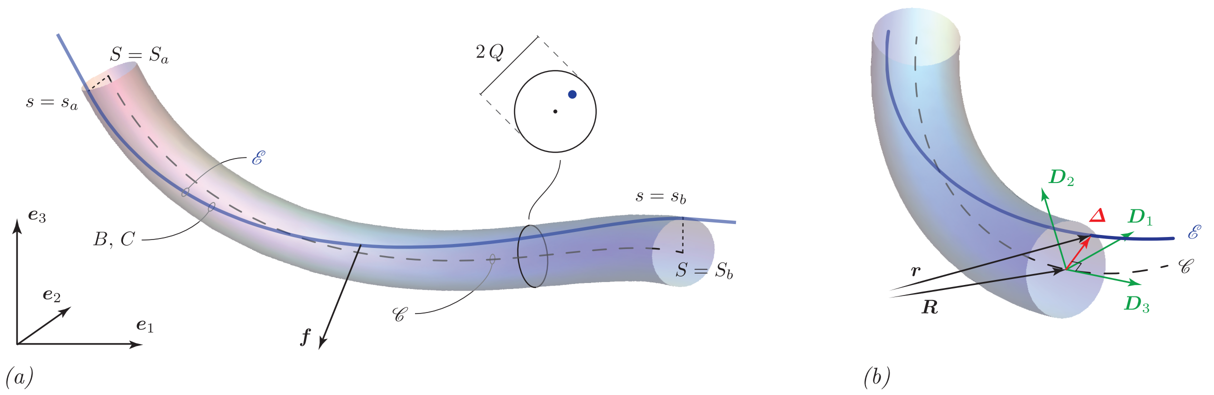


**Figure 1.** *(a) Description of an elementary problem : the drillstring, materialised by its axis $\mathscr{E}$, is subjected to the body force $\boldsymbol{f}$ between its extremities $s_a$ and $s_b$ ; it is characterised by its bending stiffness $B$ and torsional stiffness $C$. (b) Decomposition of the position vector $\boldsymbol{r}\,(s) = \boldsymbol{R}\,(S) + \boldsymbol{\Delta}\,(S)$ and description of the $\{\boldsymbol{D}_j\}$–basis attached to the borehole axis.*

### 2.1. *Governing equations*

The drillstring is henceforth assumed to behave as a one-dimensional elastic body that can undergo large deformations in space by experiencing bending and torsion. For the sake of brevity, the effects of shear and extensibility are disregarded. The borehole is idealised as a perfectly rigid tube-like surface whose axis $\mathscr{C}$ follows the borehole trajectory, *viz.* the well-path. Defining the right-handed orthonormal basis $\{\boldsymbol{e}_k\}$ for the Euclidean space $\mathbb{E}^3$, the parameterisation of this space curve is denoted by $\boldsymbol{R}\,(S) = X_j\,\boldsymbol{e}_j$ where the arc-length parameter $S$ identifies a section of the borehole. Similarly, let us define the parameterisation $\boldsymbol{r}\,(s) = x_j\,\boldsymbol{e}_j$ for the position vector of the drillstring axis. According to the inextensibility assumption, the material coordinate $s$ identifies a cross section along the drillstring independently of its configuration. To fully characterise the spatial configuration of the drillstring, one has to additionally supply the space curve $\mathscr{E}$ with a vector describing the orientation of its cross section. Defining a pair of orthonormal vectors $\boldsymbol{d}_1\,(s)$, $\boldsymbol{d}_2\,(s)$ along two fixed material lines of the cross section, the drillstring configuration is entirely defined by the two vector-valued functions

$$[s_a, s_b] \ni s \mapsto \boldsymbol{r}\,(s)\,,\; \boldsymbol{d}_1\,(s) \in \mathbb{E}^3, \tag{1}$$

with $s_a < s_b$ corresponding to the extremities $S_a < S_b$ of the elementary problem, see Figure 1(a), and such that the length of the drillstring segment is $\ell = s_b - s_a$.

The cross section attitude is then described by its normal $\boldsymbol{d}_3\,(s) = \boldsymbol{d}_1 \times \boldsymbol{d}_2$ such that the resulting triplet of *directors* $\{\boldsymbol{d}_j\,(s)\}$ constitutes, for each cross section $s$, an orthonormal basis in which the deformed state of the drillstring is naturally described. The unshearability and inextensibility assumptions imply that the director $\boldsymbol{d}_3$ coincides with the tangent vector to the space curve $\mathscr{E}$, that is $\mathrm{d}\boldsymbol{r}/\mathrm{d}s = \boldsymbol{d}_3$. The kinematic of this material frame satisfies the skew symmetric relation

$$\frac{\mathrm{d}\boldsymbol{d}_j}{\mathrm{d}s} = \boldsymbol{u} \times \boldsymbol{d}_j, \tag{2}$$

where the twist vector $\boldsymbol{u}\,(s) = u_j\,\boldsymbol{d}_j$ relates the rotation of the directors to the strains : $u_1$ and $u_2$ measure the local material curvature and $u_3$ is the twist density. These strain variables are related to the internal moment $\boldsymbol{M}\,(s) = M_j\,\boldsymbol{d}_j$ by means of the constitutive equation

$$\boldsymbol{M}\,(s) = B\,(u_1\,\boldsymbol{d}_1 + u_2\,\boldsymbol{d}_2) + C\,u_3\,\boldsymbol{d}_3, \tag{3}$$

with the bending stiffness $B$ and torsional stiffness $C$ of the drillstring. Finally, denoting by $\boldsymbol{F}\,(s) = F_j\,\boldsymbol{d}_j$ the internal force, the conservation of linear and angular momenta yields

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s} + \boldsymbol{f} = 0, \qquad\qquad \frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}s} + \boldsymbol{d}_3 \times \boldsymbol{F} = 0, \tag{4}$$

where $\boldsymbol{f}\,(s) = -w\,\boldsymbol{e}_3 + \boldsymbol{p}$ is the body force per unit reference length, which embodies the drillstring weight $w\,(s)$ and the potential contact pressure $\boldsymbol{p}\,(s)$ along continuous contacts.

### 2.2. *Shortcomings*

The system (2-4) constitutes the set of equations that governs the drillstring deflection under prescribed boundary conditions. The two-point boundary value problem under consideration necessitates the imposition of the drillstring location at both extremities $s_a$ and $s_b$ of the domain. However, as the parametric coordinates of its axis $x_j(s)$ in the absolute reference frame $\{\boldsymbol{e}_j\}$ read

$$x_j(s) = x_j(s_a) + \int_{s_a}^{s} \boldsymbol{d}_3 \cdot \boldsymbol{e}_j \,\mathrm{d}s, \tag{5}$$

these boundary conditions reduce to a set of integral constraints on the unknown length of the drillstring segment $\ell$. These stiff constraints require the use of involved numerical techniques that contribute to the numerical burden associated with a conventional formulation.

The assessment of the unilateral contact condition constitutes a further source of difficulties. To ensure that the drillstring remains either free of contact or in continuous contact along the elementary boundary value problem under consideration, this constraint necessitates the evaluation of the distance between the drillstring $\mathscr{E}$ and the borehole axis $\mathscr{C}$. However, these two curves being naturally parameterized by distinct curvilinear coordinates, the evaluation of this distance reveals to be computationally intensive as it requires the identification of the mapping between these coordinates.

## 3. Description of the model

The proposed approach hinges on describing the drillstring deformed configuration by means of its relative position about the well-path. By analogy with the director basis, one may arbitrarily define a triplet $\{\boldsymbol{D}_j(S)\}$ constituting a right-handed orthonormal basis for each cross section $S$ along the borehole trajectory and such that $\boldsymbol{D}_3 = \mathrm{d}\boldsymbol{R}/\mathrm{d}S$ is the unit vector tangent to $\mathscr{C}$. Specifically, this adapted frame may be defined so that it constitutes a Bishop frame [BIS 75] whose kinematics along the well-path satisfies $\mathrm{d}\boldsymbol{D}_j/\mathrm{d}S = \boldsymbol{U} \times \boldsymbol{D}_j$ with Darboux vector $\boldsymbol{U}(S) = U_1\,\boldsymbol{D}_1 + U_2\,\boldsymbol{D}_2$. Accordingly, the space curve $\mathscr{E}$ materialising the drillstring axis may be reparameterized as

$$\boldsymbol{r}(s(S)) = \boldsymbol{R}(S) + \boldsymbol{\Delta}(S), \tag{6}$$

where the *eccentricity vector* $\boldsymbol{\Delta}(S) = \Delta_1\,\boldsymbol{D}_1 + \Delta_2\,\boldsymbol{D}_2$ is a measure of the drillstring relative position in the borehole cross section of abscissa $S$, see Figure 1(b). This expression defines the mapping $s(S)$ between the curvilinear coordinate $S$ along the well-path $\mathscr{C}$ and the material coordinate $s$ parameterising the drillstring $\mathscr{E}$. It is shown next that, for most practical circumstances, the drift between these two coordinates is negligible.

### 3.1. *Directors and strain variables*

The curvilinear coordinate $S$, the twist vector $\boldsymbol{u}(s)$ and the Darboux vector $\boldsymbol{U}(S)$ are scaled by the known length $L = S_b - S_a$ of the elementary problem under consideration, leading to the introduction of the dimensionless curvilinear coordinate $\xi = (S - S_a)/L$ and vectors

$$\boldsymbol{\omega}(\xi) = L\,\boldsymbol{u}(s(\xi)), \qquad \boldsymbol{\mathcal{U}}(\xi) = L\,\boldsymbol{U}(S(\xi)). \tag{7}$$

Arguing that the drillstring remains within close range of the well-path, the eccentricity vector is scaled by the clearance $Q$ between the drillstring and the borehole, *viz.* $\delta_i(\xi) = \Delta_i(S(\xi))/Q$. Differentiating equation (6) therefore yields

$$\boldsymbol{d}_3(\xi) = \mathcal{J}_1\left[\delta_1'\,\boldsymbol{D}_1 + \delta_2'\,\boldsymbol{D}_2 + \left(\varepsilon^{-1} - \delta_1\,\mathcal{U}_2 + \delta_2\,\mathcal{U}_1\right)\boldsymbol{D}_3\right], \tag{8}$$

where primes denote differentiation with respect to the dimensionless coordinate $\xi$ and the Jacobian of the mapping reads [HUY 15b]

$$\mathcal{J}_1(\xi) = \frac{1}{\sqrt{\delta_1'^2 + \delta_2'^2 + \left(\varepsilon^{-1} - \delta_1\,\mathcal{U}_2 + \delta_2\,\mathcal{U}_1\right)^2}}, \tag{9}$$

with the scaled clearance $\varepsilon = Q/L$. In general, the well-path $\mathscr{C}$ is only slightly tortuous and its scaled curvature $\mathcal{K} = \sqrt{\mathcal{U}_1^2 + \mathcal{U}_2^2}$ is relatively small. Hence, the products $\varepsilon\,\mathcal{U}_1$ and $\varepsilon\,\mathcal{U}_2$ may be considered comparatively smaller such that, for $\varepsilon \to 0$, we may assume $\varepsilon\,\mathcal{U}_1 = \mathcal{O}(\varepsilon^2)$ and $\varepsilon\,\mathcal{U}_2 = \mathcal{O}(\varepsilon^2)$. Considering only the leading order terms, expression (9) for the Jacobian therefore reads $\mathcal{J}_1(\xi) = 1 + \mathcal{O}(\varepsilon^2)$ and the distinction between the curvilinear coordinates $s$ and $S$ is seen to be negligible.

Accordingly, equation (8) reduces to

$$\boldsymbol{d}_3(\xi) = \boldsymbol{D}_3 + \varepsilon\,(\delta_1'\,\boldsymbol{D}_1 + \delta_2'\,\boldsymbol{D}_2) + \mathcal{O}\left(\varepsilon^2\right), \tag{10}$$

as $\varepsilon \to 0$, which emphasises the description of the drillstring configuration as a small perturbation of the well-path. The orientation of the pair $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ may be thus described as

$$\boldsymbol{d}_1(\xi) = \cos\varphi\,\boldsymbol{D}_1 + \sin\varphi\,\boldsymbol{D}_2 - \varepsilon\,(\delta_1'\cos\varphi + \delta_2'\,\sin\varphi)\,\boldsymbol{D}_3 + \mathcal{O}\left(\varepsilon^2\right), \tag{11}$$

$$\boldsymbol{d}_2(\xi) = -\sin\varphi\,\boldsymbol{D}_1 + \cos\varphi\,\boldsymbol{D}_2 + \varepsilon\,(\delta_1'\sin\varphi - \delta_2'\,\cos\varphi)\,\boldsymbol{D}_3 + \mathcal{O}\left(\varepsilon^2\right), \tag{12}$$

where the *twist angle* $\varphi(\xi)$ measures the rotation of the drillstring cross section about the director $\boldsymbol{d}_3$. In view of the kinematic relation (2), differentiation of expression (10–12) and projection in the director basis yields

$$\omega_1(\xi) = \left(\mathcal{U}_1 - \varepsilon\,\overline{\delta}_2''\right)\cos\varphi + \left(\mathcal{U}_2 + \varepsilon\,\overline{\delta}_1''\right)\sin\varphi + \mathcal{O}\left(\varepsilon^2\right), \tag{13}$$

$$\omega_2(\xi) = -\left(\mathcal{U}_1 - \varepsilon\,\overline{\delta}_2''\right)\sin\varphi + \left(\mathcal{U}_2 + \varepsilon\,\overline{\delta}_1''\right)\cos\varphi + \mathcal{O}\left(\varepsilon^2\right), \tag{14}$$

$$\omega_3(\xi) = \varphi' + \mathcal{O}\left(\varepsilon^2\right), \tag{15}$$

for the scaled strain variables.

### 3.2. *Free of contact segments*

Considering the previous definitions and denoting the characteristic force $F^\star = B/L^2$, the scaling leads to the introduction of the following vector fields

$$\boldsymbol{\sigma}(\xi) = L\,\boldsymbol{f}(S(\xi))/F^\star, \qquad \boldsymbol{\mathcal{F}}(\xi) = \boldsymbol{F}(S(\xi))/F^\star, \qquad \boldsymbol{\mathcal{M}}(\xi) = \boldsymbol{M}(S(\xi))/L\,F^\star, \tag{16}$$

for the scaled body force, internal force and internal moment, respectively. According to this scaling and the constitutive relation (3), the dimensionless strain variables are analogous to the components of the scaled internal moment in the director basis. Relation (3) therefore becomes

$$\boldsymbol{\mathcal{M}}(\xi) = \omega_1\,\boldsymbol{d}_1 + \omega_2\,\boldsymbol{d}_2 + (1+\nu)^{-1}\,\varphi'\,\boldsymbol{d}_3, \tag{17}$$

where $\varphi(\xi) = \varphi(S(\xi))$ and $\nu = B/C - 1$ is Poisson's ratio. Introducing the following notation for the shear components of the internal force and body force

$$\widetilde{\mathcal{F}}_1(\xi) = \mathcal{F}_1\,\cos\varphi - \mathcal{F}_2\,\sin\varphi, \qquad \widetilde{\sigma}_1(\xi) = \sigma_1\,\cos\varphi - \sigma_2\,\sin\varphi, \tag{18}$$

$$\widetilde{\mathcal{F}}_2(\xi) = \mathcal{F}_1\,\sin\varphi + \mathcal{F}_2\,\cos\varphi, \qquad \widetilde{\sigma}_2(\xi) = \sigma_1\,\sin\varphi + \sigma_2\,\cos\varphi, \tag{19}$$

where $\mathcal{F}_j(\xi) = \boldsymbol{\mathcal{F}}\cdot\boldsymbol{d}_j$ and $\sigma_j(\xi) = \boldsymbol{\sigma}\cdot\boldsymbol{d}_j$, the projection of the equilibrium equations (4) in the director basis yields

$$\widetilde{\mathcal{F}}_1' + \mathcal{F}_3\,(\mathcal{U}_2 + \varepsilon\,\delta_1'') + \widetilde{\sigma}_1 = \mathcal{O}\left(\varepsilon^2\right), \tag{20}$$

$$\widetilde{\mathcal{F}}_2' - \mathcal{F}_3\,(\mathcal{U}_1 - \varepsilon\,\delta_2'') + \widetilde{\sigma}_2 = \mathcal{O}\left(\varepsilon^2\right), \tag{21}$$

$$\mathcal{F}_3' + \widetilde{\mathcal{F}}_2\,(\mathcal{U}_1 - \varepsilon\,\delta_2'') - \widetilde{\mathcal{F}}_1\,(\mathcal{U}_2 + \varepsilon\,\delta_1'') + \sigma_3 = \mathcal{O}\left(\varepsilon^2\right), \tag{22}$$

$$\varepsilon\,\delta_1''' - \frac{\varphi'}{1+\nu}\,(\mathcal{U}_1 - \varepsilon\,\delta_2'') + \mathcal{U}_2' + \widetilde{\mathcal{F}}_1 = \mathcal{O}\left(\varepsilon^2\right), \tag{23}$$

$$\varepsilon\,\delta_2''' - \frac{\varphi'}{1+\nu}\,(\mathcal{U}_2 + \varepsilon\,\delta_2'') - \mathcal{U}_1' + \widetilde{\mathcal{F}}_2 = \mathcal{O}\left(\varepsilon^2\right), \tag{24}$$

$$\varphi'' = \mathcal{O}\left(\varepsilon^2\right), \tag{25}$$

for $\varepsilon \to 0$. Note that, along free of contact segments, the body force reduces to its sole nonzero component $\boldsymbol{\sigma} = -\lambda\, \boldsymbol{e}_3$ with the scaled weight per unit reference length $\lambda = L\, w/F^{\star}$.

This mixed order system of nonlinear differential equations in the six unknowns $\left\{\varphi, \delta_1, \delta_2, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \mathcal{F}_3\right\}$ requires the specification of eleventh constants of integration. In the context under consideration, that is within a segmentation strategy, one extremity of the drillstring may be considered as clamped while the other moves freely through a frictionless sliding sleeve under the combined action of known axial force and twisting moment [HUY 15b, HUY ed]. The corresponding two-point boundary value problem is therefore subjected to the boundary conditions

$$\left\{\varphi\left(0\right), \delta_i\left(0\right), \delta_i'\left(0\right)\right\}, \qquad \text{and} \qquad \left\{\varphi'\left(1\right), \delta_i\left(1\right), \delta_i'\left(1\right), \widetilde{\mathcal{F}}_3\left(1\right)\right\}, \tag{26}$$

with $i = 1, 2$. As an essential outcome of the proposed formulation, the isoperimetric constraints on the unknown length of the rod (5) disappear and the system of equations (20–25) with the boundary conditions (26) constitute a classical boundary value problem (as opposed to a free boundary problem.) Additionally, the description of the drillstring deformed configuration through its relative position with respect to the borehole axis provides a straightforward means to detect the appearance of new contacts and, therefore, trivialises the assessment of the unilateral contact condition.

### 3.3. *Continuous contact segments*

Alternatively, although the magnitude of the eccentricity vector is known along continuous contact problems, the magnitude of the reaction pressure $\rho\left(\xi\right)$ included in the body force $\boldsymbol{\sigma}\left(\xi\right)$ and acting normally to the constraint surface is *a priori* unknown. The nonlinear boundary value problem (20–26) is therefore supplemented by the relation

$$\delta_1^2 + \delta_2^2 = 1, \tag{27}$$

to close the formulation and ensure the continuous contact along the whole domain. Although the system (20–25) with the constraint (27) and the boundary conditions (26) is well posed and can therefore be solved, it is convenient to adapt the governing equations to account for the particular nature of the elementary problem under consideration. This indeed allows to streamline its resolution and eliminate the algebraic component, *viz.* the reaction pressure, from the resulting system of differential-algebraic equations [HUY 15a, HUY 15b].

In view of constraint (27), we define the polar angle $\beta\left(\xi\right)$ satisfying $\tan\beta = \delta_2/\delta_1$ to describe the angular position of the drillstring axis in the borehole cross section. The notion of unit normal $\boldsymbol{N}\left(\xi\right) = \cos\beta\, \boldsymbol{D}_1 + \sin\beta\, \boldsymbol{D}_2$ to the borehole surface along the drillstring axis $\mathscr{E}$ naturally follows and the shear forces are decomposed into their normal $\mathcal{F}_n = \boldsymbol{\mathcal{F}} \cdot \boldsymbol{N}$ and geodesic $\mathcal{F}_g = \boldsymbol{\mathcal{F}} \cdot \left(\boldsymbol{N} \times \boldsymbol{d}_3\right)$ components, that is their normal and tangential components with respect to the borehole surface

$$\mathcal{F}_n\left(\xi\right) = \mathcal{F}_1\, \sin\left(\beta - \varphi\right) + \mathcal{F}_2\, \cos\left(\beta - \varphi\right), \qquad \mathcal{F}_g\left(\xi\right) = \mathcal{F}_1\, \cos\left(\beta - \varphi\right) - \mathcal{F}_2\, \sin\left(\beta - \varphi\right). \tag{28}$$

The normal $\lambda_n = -\lambda\, \boldsymbol{e}_3 \cdot \boldsymbol{N}$, geodesic $\lambda_g = -\lambda\, \boldsymbol{e}_3 \cdot \left(\boldsymbol{N} \times \boldsymbol{d}_3\right)$ and axial $\lambda_3 = -\lambda\, \boldsymbol{e}_3 \cdot \boldsymbol{d}_3$ components of the drillstring weight per unit reference length are similarly introduced such that, upon projection of the governing equations (4) in the orthonormal frame $\left\{\boldsymbol{N} \times \boldsymbol{d}_3, \boldsymbol{N}, \boldsymbol{d}_3\right\}$, we obtain the following mixed order system of nonlinear differential equations

$$\mathcal{F}_g' + \beta'\, \mathcal{F}_n - \left(\mathcal{U}_1\, \cos\beta + \mathcal{U}_2\, \sin\beta - \varepsilon\, \beta''\right) \mathcal{F}_3 + \lambda_g = \mathcal{O}\left(\varepsilon^2\right), \tag{29}$$

$$\mathcal{F}_3' + \left(\mathcal{U}_1\, \cos\beta + \mathcal{U}_2\, \sin\beta - \varepsilon\, \beta''\right) \mathcal{F}_g + \left(\mathcal{U}_1\, \sin\beta - \mathcal{U}_2\, \cos\beta + \varepsilon\, \beta'^2\right) \mathcal{F}_n + \lambda_3 = \mathcal{O}\left(\varepsilon^2\right), \tag{30}$$

$$\varepsilon\, \beta''' - \left(\mathcal{U}_1'\, \cos\beta + \mathcal{U}_2'\, \sin\beta + \varepsilon\, \beta'^3\right) + \frac{\varphi'}{1+\nu} \left(\mathcal{U}_1\, \sin\beta - \mathcal{U}_2\, \cos\beta + \varepsilon \beta'^2\right) + \mathcal{F}_g = \mathcal{O}\left(\varepsilon^2\right), \tag{31}$$

$$\varphi'' = \mathcal{O}\left(\varepsilon^2\right), \tag{32}$$

for $\varepsilon \to 0$, while the normal component of the internal force and the reaction pressure are given by

$$\mathcal{F}_n\left(\xi\right) = 3\, \varepsilon\, \beta'\, \beta'' + \mathcal{U}_1'\, \sin\beta - \mathcal{U}_2'\, \cos\beta + \frac{\varphi'}{1+\nu} \left(\mathcal{U}_1\, \cos\beta + \mathcal{U}_2\, \sin\beta - \varepsilon \beta''\right) + \mathcal{O}\left(\varepsilon^2\right), \tag{33}$$

$$\rho\left(\xi\right) = -\mathcal{F}_n' + \beta'\, \mathcal{F}_g + \left(\mathcal{U}_1\, \sin\beta - \mathcal{U}_2\, \cos\beta + \varepsilon\, \beta'^2\right) \mathcal{F}_3 - \lambda_n + \mathcal{O}\left(\varepsilon^2\right), \tag{34}$$

respectively.

The set of ordinary differential equations (29–32) in the four unknowns $\{\varphi, \beta, \mathcal{F}_g, \mathcal{F}_3\}$ requires the specification of seven constants of integration. The counterpart to the clamped-sliding sleeve boundary conditions (26) reads

$$\Big\{\varphi(0), \beta(0), \beta'(0)\Big\}, \qquad \text{and} \qquad \{\varphi'(1), \beta(1), \beta'(1), \mathcal{F}_3(1)\}, \tag{35}$$

such that the integral constraints on the unknown length of the rod (5) are again suppressed and the system of equations (29–32) with the boundary conditions (35) constitutes a classical boundary value problem. This formulation, leading to an explicit representation of the drillstring axis, further reduces the dimensions of system to be solved by providing, through relation (34), an *a posteriori* expression for the reaction pressure.

### 3.4. *Segmentation strategy and computational model*

As already emphasised, the nub of the segmentation strategy is the determination of the positions of the contacts between the drillstring and the borehole as well as the extent of these contacts. The methodology leading to the identification of the contact pattern satisfying the non-penetration condition as well as the global boundary conditions, and preserving the integrity of the drillstring may be systematised in an algorithm involving three nested loops. This procedure is outlined here, details of its implementation may be found in [HUY 15b].

The **outer loop**, concerned with the identification of the sequence of elementary problems satisfying the unilateral contact condition, establishes the number of contacts and their nature, that is either discrete or continuous. As this procedure cannot be cast in a conventional form involving a finite set of unknowns and objective functions that must be minimised or zeroed, a collection of verifications is established to ensure the correctness of the contact pattern. The verification of the contact pattern is twofold. Firstly, the non-penetration condition requires that new contacts be created whenever the magnitude of the eccentricity vector $\sqrt{\delta_1^2 + \delta_2^2}$ exceeds the clearance $\varepsilon$ of the borehole, while the rod curvature at existing discrete contacts complies with the borehole local geometry. Secondly, reaction forces at discrete contacts and reaction pressures along continuous contacts must not be tensile.

For a given sequence, the **median loop** determines the position and inclination of the connections that ascertain the continuity of the drillstring between elementary problems. Although the clamped-sliding sleeve boundary conditions (26,35) imposed at each elementary problem guarantee the continuity of both the space curve $\mathscr{E}$ and its tangent, the drillstring curvature and, therefore, the bending moment may present jump discontinuities at these connections. In the absence of concentrated body couple or other singularities at the connections, these discontinuities are however not physical. Similarly, these boundary conditions fail to maintain a smoothness of the rotation angle sufficient to ensure the continuity of the directors along the drillstring. Finally, the component $\mathcal{F}_g$ of the shear forces may also present unrealistic jump discontinuities at these connections. Resorting to a shooting method, this loop is concerned with the determination of the rod position, inclination and rotation angle at the connections that reestablishes the integrity of the drillstring between elementary problems.

The median loop requires the evaluation of the rod configuration along the global problem and, therefore, yield yet an additional **inner loop** that resolves the sequence of elementary problems. To ensure the correct transmission of the axial force and twisting moment, this resolution is conducted sequentially by propagating the solution from on extremity of the global problem to the other. To solve numerically the mixed order systems of nonlinear ordinary differential equations (20–25) and (29–32) associated with the boundary conditions (26) and (35), respectively, a collocation method has been implemented. For reasons of efficiency, stability, and flexibility in order and continuity, *B*-splines are chosen as basis functions while collocation is applied at Gaussian points [ASC 79].

## 4. Application

As an illustration of the proposed model, let us consider a segment of a helical borehole characterised by its constant curvature and torsion $\kappa L = \tau L = \pi/\sqrt{2}$. For the sake of simplicity, we consider a fictitious, weightless, drillstring that is centred on the borehole axis at its extremities. The borehole is further assumed to have a constant diameter with clearance $\varepsilon = 1/20$. Figure 2 depicts the buckling of the drillstring for a twisting moment $\mathcal{M}_3(0) = -1$ and various values of the axial force $\mathcal{F}_3(0) = \mathcal{T}$. The proposed stiff string model is compared to the exact solution. The discrepancy between the solutions is seen to be reasonable and the small inclination approximation

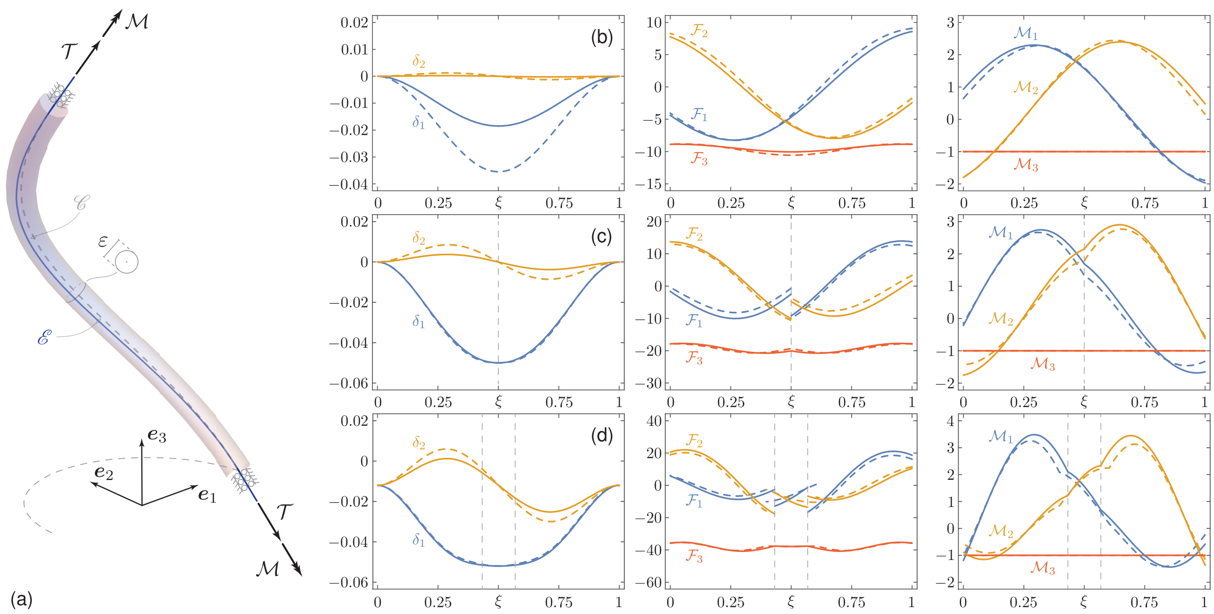

**Figure 2.** *Buckling of an elastic, weightless, drillstring inside a helical borehole. Results obtained with the proposed stiff string model (——) are compared to the exact solution (- - -) for* $\mathcal{M} = -1$ *and (c)* $\mathcal{T}/|\mathcal{T}_0| = -1.25$*, (c)* $\mathcal{T}/|\mathcal{T}_0| = -2.5$*, (d)* $\mathcal{T}/|\mathcal{T}_0| = -5$*. Vertical dotted lines mark the positions of discrete contacts.*

to only slightly affect the rod overall response of the drillstring ; the profiles of the eccentricity vector, internal force and moment indeed remaining sensibly similar.

## 5. Conclusion

The simplified, yet versatile, stiff string model described in the present paper was motivated by the need to accurately and efficiently tackle the interactions between boreholes and drillstrings. The proposed formulation hinges on the segmentation of the global problem into simpler elementary problems, corresponding to drillstring segments either in continuous contact with the borehole or free of contact between their extremities. The originality of the proposed formulation, which resolves in one stroke a series of issues that afflict conventional approaches (isoperimetric constraints, detection of contacts, continuous interactions with the borehole, etc.), lies in the introduction of the eccentricity vector and the description of the drillstring configuration as a small perturbation of the borehole axis.

Although, in the present context, restricted to quasi-static solutions of the governing equations, the proposed formulation is readily extendable to incorporate the drillstring dynamic by adding inertial terms to the equilibrium equations (4) and defining the eccentricity vector as time-dependent. More importantly, the assumption of frictionless interaction between the drillstring and the borehole surface may be relaxed and friction forces introduced through the components $(\sigma_g, \sigma_3)$ of the body forces, however requiring to deal with evolutive problems as solutions become history dependent.

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