

Introduction

Bohmian mechanics is an alternative approach to quantum mechanics which is **deterministic** and **non-local**, i.e. particles follow definite and predictable trajectories that depend on the configuration of the whole universe. In this work, we investigate the emergence of chaos in Bohmian trajectories for **stationary wave functions** [1].

Questions

Does $\psi(\mathbf{r}, \mathbf{x})$ allow for chaos in 3d or in 2d many-particle systems ?
 Entanglement and complexity of the wave function $\stackrel{?}{\leftrightarrow}$ chaos

Bohmian mechanics

Actual point-particles of position $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$ moving under the influence of the wave function [2]

- Wave function ψ satisfies the usual Schrödinger equation
- **Particles trajectories** are determined by the guiding equation

$$\frac{d\mathbf{r}}{dt} = v^\psi(\mathbf{r}, t)$$

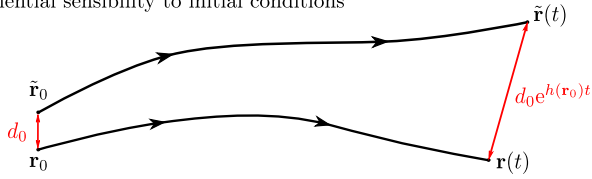
with velocity field $v^\psi = (v_1^\psi, \dots, v_n^\psi)$ given by ($\hbar = 1$)

$$\mathbf{v}_k^\psi(\mathbf{r}, t) = \frac{1}{m_k} \Im \left(\frac{\nabla_k \psi(\mathbf{r}, t)}{\psi(\mathbf{r}, t)} \right)$$

Can lead to the same predictions as standard quantum mechanics

Characterizing chaotic trajectories

Exponential sensibility to initial conditions



Quantifying chaos : **Lyapunov exponent** [3]

$$h(\mathbf{r}_0) = \lim_{t \rightarrow \infty} \left(\lim_{|d(t_0)| \rightarrow 0} \frac{1}{t} \ln \left(\frac{|d(t)|}{|d_0|} \right) \right)$$

Poincaré-Bendixon theorem [3]

Chaos only possible if $n - c \geq 3$ with n the number of degrees of freedom and c the number of constants of motion

Constants of motion for stationary wave functions

We consider

$$\psi(\mathbf{r}, t) = e^{iEt} \phi(\mathbf{r}) = e^{iEt} \sum_i c_i \phi_i(\mathbf{r}) \Rightarrow v^\psi(\mathbf{r}, \mathbf{x})$$

$c_i \in \mathbb{C}$ and $\phi_i(\mathbf{r})$ are **degenerate eigenstates** of the system Hamiltonian and constitute an **orthogonal basis** of the Hilbert space

Eigenstates in spherical coordinates

$$\phi_i(\mathbf{r}) = f_i(r) g_i(\theta) e^{im_i \varphi}$$

with f_i and g_i real functions

Example : Common eigenstates of \hat{L}^2 and \hat{L}_z for a 3d harmonic oscillator

Results : Stationary states allow for chaos

Symmetries of the wave function \Rightarrow constant of motion

Example : The state

$$\phi(\mathbf{r}) = \phi_a(\mathbf{r}) + \phi_b(\mathbf{r}) = f_a(r) g_a(\theta) \chi_a(\varphi) + f_b(r) g_b(\theta) \chi_b(\varphi)$$

admits the constant of motion $C = F(r) - G(\theta)$ with $dF(r)/dr = f(r)$, $dG(\theta)/d\theta = g(\theta)$ and

$$f(r) = \frac{1}{r^2} \frac{f_a f_b}{f_a \partial_r f_b - f_b \partial_r f_a}, \quad g(\theta) = \frac{g_a g_b}{g_a \partial_\theta g_b - g_b \partial_\theta g_a}$$

Energy \neq constant of motion

The energy of the system is **the same for any initial conditions** and thus **does not constrain** the Bohmian trajectories.

Chaos and complexity of the wave function

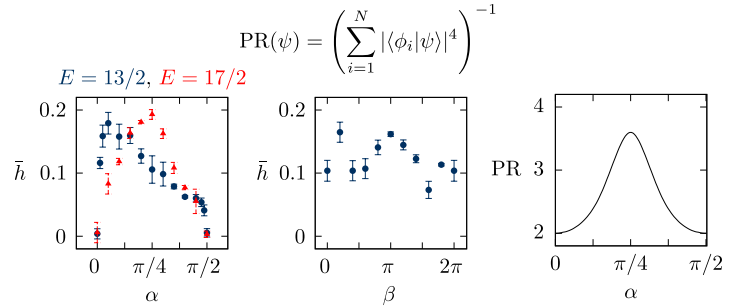
Stationary wave functions of the 3d harmonic oscillator ($n = 3$)

$$\begin{aligned} \phi(\mathbf{r}; \alpha, \beta) &= \sum_{i=1}^4 |c_i(\alpha)| e^{i\phi_i(\beta)} \phi_i(\mathbf{r}) \\ &= \mathcal{N}(\alpha) \left(\cos \alpha \phi_1(\mathbf{r}) + e^{i(\beta+\pi/3)} \sin \alpha \phi_2(\mathbf{r}) \right. \\ &\quad \left. + e^{i(2\pi \cos \beta + \pi/5)} \cos^2 \alpha \phi_3(\mathbf{r}) + e^{i(-2\beta+\pi/7)} \sin^2 \alpha \phi_4(\mathbf{r}) \right) \end{aligned}$$

$\alpha = 0$ or $\pi/2 \Rightarrow$ constant of motion \Rightarrow **no chaos**

Quantifying the complexity of the wave function

Measure of the degree of superposition using the **participation ratio** [4]



Average of the Lyapunov exponent \bar{h} as a function of α and β

Chaos and entanglement

W entanglement class (3 particles, $n = 6$)

$$\begin{aligned} \phi_W^{3p}(\alpha, \beta, \gamma) &= \sqrt{\alpha} \phi_{a,1} \phi_{a,2} \phi_{b,3} + \sqrt{\beta} \phi_{a,1} \phi_{b,2} \phi_{a,3} \\ &\quad + \sqrt{\gamma} \phi_{b,1} \phi_{a,2} \phi_{a,3} + \sqrt{\delta} \phi_{a,1} \phi_{a,2} \phi_{a,3} \end{aligned}$$

$\phi_{i,k}(r_k, \varphi_k)$ ($i = a, b$ and $k = 1, 2, 3$) degenerate eigenstates of the 2d harmonic oscillator, $\alpha, \beta, \gamma \geq 1$ and $\delta = 1 - (\alpha + \beta + \gamma) \geq 0$

Example : $\phi_W^{3p}(1/3, 1/3, 1/3)$

1 constant of motion (r_1, r_2, r_3) and $h \approx 0.11$

Entanglement measures

- **Geometric entanglement E_G** [5]

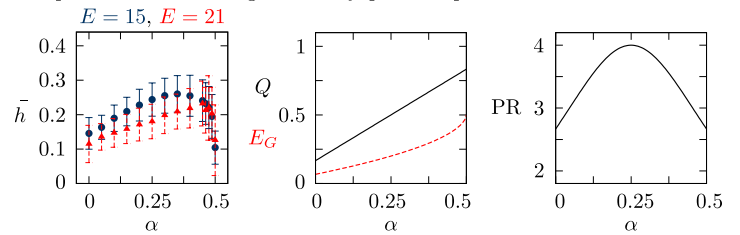
Distance to the set of separable states

$$E_G(\phi_W^{3p}(\alpha, 1/2 - \alpha, 1/2)) = 1/2 \text{ for all values of } \alpha$$

- **Meyer-Wallach Q** [6]

Average of the linear entropies of each qubit

Comparison with average of the Lyapunov exponent \bar{h} :



Conclusion

Stationary states \Rightarrow **chaotic Bohmian trajectories**

provided the effective number of degrees of freedom $n - c$ is greater than 3

Complexity of the wave function, entanglement and chaos

- Participation ratio is a good indicator of the amount of chaos
- Meyer-Wallach measure of entanglement displays the best qualitative agreement with the amount of chaos

However, PR and Q do not depend on the intrinsic complexity of basis states which can modify the amount of chaos

[1] A. Cesa, J. Martin, W. Struyve, arXiv:1603.01387 (2016).

[2] D. Bohm, Phys. Rev. **85**, 166 (1952).

[3] A. J. Lichtenberg, M. A. Leiberman, Regular and chaotic dynamics, second edition, Springer, New York (1992).

[4] L. Viola, W.G. Brown, J. Phys. A **40**, 8109 (2007).

[5] T.-C. Wei, P.M. Goldbart, Phys. Rev. A **68**, 042307 (2003).

[6] D.A. Meyer, N.R. Wallach, J. Math. Phys. **43**, 4273 (2002).