J Optim Theory Appl (2009) 141: 495–512
DOI 10.1007/s10957-008-9482-3

Analytic Solution for Fuel-Optimal Reconfiguration in Relative Motion

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Published online: 7 January 2009

Abstract The current paper presents simple and general analytic solutions to the optimal reconfiguration of multiple satellites governed by a variety of linear dynamic equations. The calculus of variations is used to analytically find optimal trajectories and controls. Unlike what has been determined from previous research, the inverse of the fundamental matrix associated with the dynamic equations is not required for the general solution in the current study if a basic feature in the state equations is met. This feature is very common due to the fact that most relative motion equations are represented in the LVLH frame. The method suggested not only reduces the amount of calculations required, but also allows predicting the explicit form of optimal solutions in advance without having to solve the problem. It is illustrated that the optimal thrust vector is a function of the fundamental matrix of the given state equations, and other quantities, such as the cost function and the state vector during the reconfiguration, can be concisely represented as well. The analytic solutions developed in the current paper can be applied to most reconfiguration problems in linearized relative motions. Numerical simulations confirm the brevity and accuracy of the general analytic solutions developed in the current paper.

Communicated by D.G. Hull

This work was supported by the Korea Science and Engineering Foundation (KOSEF) through the National Research Lab. Program funded by the Ministry of Education, Science and Technology (No. M10600000282-06J0000-28210).

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Keywords Satellite formation flying · Fuel-optimal reconfigurations · Continuous low-level thrust · Fundamental matrix · Linearized relative equations of motion

1 Introduction

Satellite formation flying is used when a group of satellites need to perform a unified space mission together. Many future space missions will necessitate the use of formation flying technology for multiple satellites. Among the technologies associated with satellites in a formation, a fuel-optimal reconfiguration (relocation) problem is of great concern, as it is directly related to the traditional rendezvous problem. Various researchers have investigated this problem extensively. Yang et al. [1] developed a hybrid optimization algorithm to employ a distributed computational architecture. As well, Kong and Miller [2] used the calculus of variations approach to handle the traditional rendezvous problem. Richards et al. [3] proposed the use of Mixed Integer Linear Programming (MILP) techniques to find a solution in the presence of avoidance constraints. Campbell [4] presented an algorithm for the reconfiguration problem based upon Hamilton-Jacobi-Bellman optimality to generate a set of maneuvers to move from an initial stable formation to a final stable formation. Aoude et al. [5] presented a two-stage path planning technique for designing reconfiguration maneuvers with various path constraints. They combined a Rapidly-exploring Random Trees (RRT) algorithm with a Gauss pseudospectral method. These studies mentioned have utilized numerical methods. Compared to analytic methods, these numerical methods allow us to conduct more accurate and practical analyses. However, it is necessary for analytic solutions to be found because they provide insight into the feedback controller, and thus they are easily applied to formation flying, if they can be uncovered. Also, for the actual on-board control system, it is preferable to have analytic solutions since they significantly reduce the computational loads. In addition, a use of analytic solutions makes it possible to distinguish the other different perturbations, thus providing an opportunity to make a smart decision about which perturbations are responsible for the different phenomena [6]. Vaddi et al. [7] proposed an analytic two-impulse solution using Gauss's variational equations. This algorithm is based on the circular reference orbit described by the Hill-Clohessy-Wiltshire (HCW) equations [8]. Palmer [9] presented an elegant analytic solution for the problem by representing the continuous and variable thrust acceleration in a Fourier series with a period equal to the maneuver time. He used the Parseval's theorem [10] to make the infinite sum into a closed form. However, this analytic solution is limited to formation flying in only a circular or near-circular orbit because the HCW equations are used. Using a similar approach, Cho et al. [11] extended the previous result and obtained a solution to general-elliptic-orbit cases. In [11], the Tschauner-Hempel (TH) equations [12] were used to describe relative motion in an eccentric orbit and its fundamental matrix given by Yamanaka and Ankersen [13] was used. Scott and Spencer [14] chose the calculus of variations to obtain an analytic solution. They brought in the adjoint system to find an optimal thrust vector. This solution applies only to circular-reference-orbit case. Sharma et al. [15] solved the problem for non-zero eccentricity, and also included the effects of nonlinear differential gravity. As shown by the results of the studies in [9, 11, 14, 15], the solutions need the inverse of the fundamental matrix; consequently, it is very difficult to get an analytic solution. Hence, it is very challenging to overcome this difficulty and simplify the procedure in order to get the analytic solutions.

In the current paper, the fuel-optimal reconfiguration problem is analytically solved for the general linear system with quadratic cost assuming unbounded low-thrust burn throughout the transfer time. The calculus of variations approach is used, as proposed by Refs. [14, 15]. But unlike these previous studies, the analytic solution in the current paper does not require calculating the inverse of the fundamental matrix associated with the given state equations, but instead it shows that the optimal thrusts (i.e. solutions) are the functions of this original fundamental matrix. The method proposed in the current study significantly reduces calculations and readily applies to general linear cases. Once dynamic equations that can be analytically solved are given, the optimal solutions are immediately found by the succinct and elegant method suggested in the current study. Furthermore, the state vector as well as the cost function for desired configuration also can be analytically obtained.

The analysis in the current paper is summarized as follows. In Sect. 2, a basic feature in state equations is figured out, which appears frequently in many linearized relative equations of motion. The reason of this characteristic is also explained. In Sect. 3, with this feature having been mentioned in Sect. 2, it is illustrated that the optimal control vector is generally a function of the fundamental matrix associated with the given state equations, and the inverse of this fundamental matrix is not generally required for the analytic optimal solution. Compared to previous researches [9, 11, 14, 15], the method proposed in the current paper not only reduces the amount of calculations, but also makes it possible to extract the explicit form of the optimal solution in advance without solving the problem. In Sect. 4, the solutions obtained are applied to fuel-optimal reconfiguration in a circular orbit. This verification demonstrates the brevity of the method and the accuracy of the analytic solution presented in the current paper. The conclusions are discussed in Sect. 5.

2 Basic Characteristic in the Satellite Relative Motions

Satellite formation flying consists of a group of satellites working together to perform a unified space mission. There are two kinds of satellites in the artificial formation flying. The main one is called the chief satellite, and the others are the deputy satellites. The deputy satellites surround the central chief satellite. The orbit of the chief is defined as the reference orbit, whereas the deputies revolve along an orbit that is relative to the reference orbit. The dynamics of relative motion should be known exactly in order to describe the relative motion and to perform reconfiguration maneuvers in formation flying. In an inertial coordinate, the relative motion between a chief and its deputies is considered rather than the separate motion of each satellite. For this reason, a reference orbit is needed to appropriately express the relative motion. The local vertical, local horizontal (LVLH) reference coordinate is used as one of the rotating reference coordinates. The origin of the LVLH coordinate is the position of the chief. The $x(t)$ axis lies in the radial direction, the $y(t)$ axis is in the along-track direction, and the $z(t)$ axis along the orbital angular momentum vector completes a right-handed coordinate system. Relative position vector $r=[x\ \ y\ \ z]^{\mathrm{T}}$ and relative velocity vector $v=[\dot{x}\ \ \dot{y}\ \ \dot{z}]^{\mathrm{T}}$ are defined as the difference of position and velocity between the chief and deputy, respectively. Linearized equations of relative motion have been introduced in many literatures.

Before deriving general analytic solutions, the dynamics of linearized relative motions is reviewed in this section. A certain basic feature is found in many equations of relative motion. It is also shown that this characteristic is naturally implicated in the relative motions. The insight into the basic characteristic is used to derive general analytic solutions in the next section. For some cases, the fundamental matrices associated with given state equations are also provided in this section. The fundamental matrix is used to establish analytic solutions to the given system.

Let us consider general 3-dimensional state equations for linear, time, or any other independent variable-varying systems which are of this form:

$$\xi'(\alpha)=A(\alpha)\xi(\alpha)+B(\alpha)u(\alpha), \tag{1}$$

where α is used as an independent variable and the prime ($'$) represents differentiation with respect to α. In addition, $v=r'$, $\xi=[r^{\mathrm{T}}\ \ v^{\mathrm{T}}]^{\mathrm{T}}$, $u(\alpha)$ is a 3×1 control vector (either actual or pseudo), and

$$A(\alpha)=\begin{bmatrix}0_{3\times 3} & I_{3\times 3}\\ A_1(\alpha) & A_2(\alpha)\end{bmatrix},\qquad B(\alpha)=\begin{bmatrix}0_{3\times 3}\\ b(\alpha)I_{3\times 3}\end{bmatrix}. \tag{2}$$

For simpler calculations, a basic feature in the state equations is set:

The matrices $A_1(\alpha)$ and $A_2(\alpha)$ are related by $A_1(\alpha)-A_1^{\mathrm{T}}(\alpha)=A_2'(\alpha)$.

This feature also indicates that *the matrix $A_2(\alpha)$ is skew-symmetric*; that is, $A_2^{\mathrm{T}}(\alpha)=-A_2(\alpha)$. In the event that the matrix $A_2(\alpha)$ is constant, the matrix $A_1(\alpha)$ is symmetric; that is, $A_1^{\mathrm{T}}(\alpha)=A_1(\alpha)$. This feature may seem very limited, or that it can only be applied to special cases; however, many linearized equations that describe relative motion satisfy this feature. The following seven examples guarantee this assertion.

Example 2.1 First, the well-known HCW equations [8] for circular reference orbit and nonperturbative relative motions are:

$$\begin{bmatrix}\dot{r}\\ \dot{v}\end{bmatrix}=\begin{bmatrix}0_{3\times 3} & I_{3\times 3}\\ A_1 & A_2\end{bmatrix}\begin{bmatrix}r\\ v\end{bmatrix}, \tag{3}$$

where

$$A_1=\begin{bmatrix}3n^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -n^2\end{bmatrix},\qquad A_2=\begin{bmatrix}0 & 2n & 0\\ -2n & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{4}$$

where n is the constant mean motion of the reference orbit. It is self-evident that A_2 is constant as well as skew-symmetric and that A_1 is symmetric. The fundamental matrix associated with the HCW equations is given by [16]:

$$\Phi = \begin{bmatrix} 4-3\cos nt & 0 & 0 & \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ 6(\sin nt - nt) & 1 & 0 & \frac{2}{n}(\cos nt - 1) & \frac{4}{n}\sin nt - 3t & 0 \\ 0 & 0 & \cos nt & 0 & 0 & \frac{1}{n}\sin nt \\ 3n\sin nt & 0 & 0 & \cos nt & 2\sin nt & 0 \\ 6n(\cos nt - 1) & 0 & 0 & -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & -n\sin nt & 0 & 0 & \cos nt \end{bmatrix} \tag{5}$$

Example 2.2 The Schweighart and Sedwick model [17], which modifies the HCW equations in order to include the second zonal harmonic J_2 effect, has the following homogeneous form:

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} (3+5s)n^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -q^2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 2n\sqrt{1+s} & 0 \\ -2n\sqrt{1+s} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

Here, n, q, s are all constants that are outcomes of the J_2 effects on the reference orbit and the deputy's orbit. The Schweighart and Sedwick model also satisfies the basic feature. The fundamental matrix associated with this model is calculated by using the following equation:

$$\Phi = \begin{bmatrix} \frac{1}{1-s}\begin{bmatrix} 4(1+s) \\ -(3+5s)\cos\gamma t \end{bmatrix} & 0 & 0 & \frac{1}{n\sqrt{1-s}}\sin\gamma t & \frac{2\sqrt{1+s}}{n(1-s)}[1-\cos\gamma t] & 0 \\ \frac{2\sqrt{1+s}(3+5s)}{(1-s)\sqrt{1-s}}[\sin\gamma t - \gamma t] & 1 & 0 & \frac{2\sqrt{1+s}}{n(1-s)}[\cos\gamma t - 1] & \frac{1}{1-s}\begin{bmatrix} \frac{4(1+s)}{n\sqrt{1-s}}\sin\gamma t \\ -(3+5s)t \end{bmatrix} & 0 \\ 0 & 0 & \cos qt & 0 & 0 & \frac{1}{q}\sin qt \\ \frac{n(3+5s)}{\sqrt{1-s}}\sin\gamma t & 0 & 0 & \cos\gamma t & \frac{2\sqrt{1+s}}{\sqrt{1-s}}\sin\gamma t & 0 \\ \frac{2n\sqrt{1+s}(3+5s)}{1-s}[\cos\gamma t - 1] & 0 & 0 & -\frac{2\sqrt{1+s}}{\sqrt{1-s}}\sin\gamma t & \frac{1}{1-s}\begin{bmatrix} 4(1+s)\cos\gamma t \\ -(3+5s) \end{bmatrix} & 0 \\ 0 & 0 & -q\sin qt & 0 & 0 & \cos qt \end{bmatrix}, \tag{7}$$

where $\gamma \triangleq n\sqrt{1-s}$.

Example 2.3 The Ross model [18] is also a linearized model that shows relative motion in the presence of the J_2 disturbance in a circular reference orbit, and it has the following matrices $A_1(t)$ and A_2:

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix},$$

where

$$A_1 = -(K + P(t)), \qquad A_2 = -C,$$

$$K = \begin{bmatrix} -3n^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n^2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -2n & 0 \\ 2n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P(t) = n^2 J_R \begin{bmatrix} 12\sin^2 i \sin^2 nt - 4 & -4\sin^2 i \sin 2nt & -4\sin 2i \sin nt \\ -4\sin^2 i \sin 2nt & 1 + \sin^2 i(2 - 7\sin^2 nt) & \sin 2i \cos nt \\ -4\sin 2i \sin nt & \sin 2i \cos nt & 3 - \sin^2 i(2 + 5\sin^2 nt) \end{bmatrix}, \tag{8}$$

where n and i are the constant mean motion and inclination of a reference orbit, respectively, and $J_R \triangleq \frac{3J_2R_e^2}{2R_0^2}$. R_e is the mean radius of the Earth, and R_0 is a constant radius of a reference orbit. Then, A_1 is symmetric, and A_2 is constant as well as skew-symmetric.

Example 2.4 To describe the relative motion in elliptic orbits, the Tschauner-Hempel (TH) equations [12] are used:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -2 \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} - \begin{bmatrix} -\dot{\theta}^2 & 0 & 0 \\ 0 & -\dot{\theta}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$- \begin{bmatrix} 0 & -\ddot{\theta} & 0 \\ \ddot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\rho(\theta)^3}{\Gamma^4} \begin{bmatrix} 2x \\ -y \\ -z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}. \tag{9}$$

In this equation, $\theta(t)$ and e refer to the true anomaly and the eccentricity of the chief's orbit; respectively, $\rho(\theta) \triangleq 1 + e\cos\theta$ and $\Gamma \triangleq h^{3/2}/GM$ are defined, where h is the magnitude of the orbital angular momentum of the chief satellite, G is the universal gravitational constant, and M is the mass of the central body: Earth. The dot $(\cdot)$ represents the differentiation with respect to time (t), and it is assumed that the unconstrained thrust vector $[T_x(t)\ T_y(t)\ T_z(t)]^{\mathrm{T}}$ can be continuously applied in the desired directions during the maneuver. For brevity, when changing the independent variable from time t to true anomaly θ, the following transformation is considered:

$$[\tilde{x} \quad \tilde{y} \quad \tilde{z}]^{\mathrm{T}} = \dot{\theta}^{1/2}[x \quad y \quad z]^{\mathrm{T}},$$

$$\tilde{u} = [\tilde{u}_x \quad \tilde{u}_y \quad \tilde{u}_z]^{\mathrm{T}} = [T_x \quad T_y \quad T_z]^{\mathrm{T}}/\dot{\theta}^{3/2}.$$

The tildes are used to represent pseudovalues. For a completely analytic analysis, the TH equations are transformed into (10). Use of the same procedure as that derived by Humi [19] makes (9) very simple,

$$\tilde{\xi}' = \tilde{A}(\theta)\tilde{\xi} + \tilde{B}(\theta)\tilde{u}.$$

In this equation, the prime (′) represents differentiation with respect to true anomaly and

$$\begin{aligned}
&\tilde{\xi} = [\tilde{r}^{\mathrm{T}} \quad \tilde{v}^{\mathrm{T}}]^{\mathrm{T}} = [\tilde{x} \quad \tilde{y} \quad \tilde{z} \quad \tilde{x}' \quad \tilde{y}' \quad \tilde{z}']^{\mathrm{T}},\\
&\tilde{A}(\theta) = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ \tilde{A}_1 & \tilde{A}_2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0_{3\times 3} \\ I_{3\times 3} \end{bmatrix},\\
&\tilde{A}_1 = \begin{bmatrix} 3/\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{10}$$

It is found that the TH equations [12] are nonautonomous [20], but they also satisfy the basic feature. It is noticed that $\tilde{A}_2$ is constant and skew-symmetric and that $\tilde{A}_1(\theta)$ is symmetric. Yamanaka and Ankersen [13] provide the fundamental matrix associated with the matrix $\tilde{A}$,

$$\tilde{\Phi} = \begin{bmatrix} 0 & -c & 0 & -s & 3es\Omega - 2 & 0 \\ 1 & s(1+1/\rho) & 0 & -c(1+1/\rho) & 3\rho^2\Omega & 0 \\ 0 & 0 & c/\rho & 0 & 0 & s/\rho \\ 0 & -c' & 0 & -s' & 3e(s'\Omega + s/\rho^2) & 0 \\ 0 & 2c-e & 0 & 2s & 3(1-2es\Omega) & 0 \\ 0 & 0 & -s/\rho & 0 & 0 & c/\rho \end{bmatrix}. \tag{11}$$

In this equation, $s \triangleq \rho\sin\theta$, $c \triangleq \rho\cos\theta$, and

$$\Omega \triangleq \frac{1}{\Gamma^2}(t - t_0) = \int_{\theta_0}^{\theta} \frac{1}{\rho(\varphi)^2} d\varphi,$$

where t_0 and θ_0 are the time and true anomaly when thrusters start to fire.

Example 2.5 Théron et al. [21] modified the TH equations [12] in order to take into account the J_2 effect for an elliptic reference orbit, which has the following form:

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix},$$

where

$$
\begin{aligned}
A_1 &= -\frac{n^2\rho^3(\theta)}{(1-e^2)^3}\begin{bmatrix} -3-e\cos\theta & 2e\sin\theta & 0 \\ -2e\sin\theta & -e\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} + k_{J_2}\frac{n^2\rho^5(\theta)}{(1-e^2)^5} \\
&\quad \times \begin{bmatrix} 4-12\sin^2 i\sin\tilde{\theta} & 8\sin^2 i\sin\tilde{\theta}\cos\tilde{\theta} & 8\sin i\cos i\sin\tilde{\theta} \\ 8\sin^2 i\sin\tilde{\theta}\cos\tilde{\theta} & 4-7\sin^2 i\cos^2\tilde{\theta}-5\cos^2 i & -2\sin i\cos i\cos\tilde{\theta} \\ 8\sin i\cos i\sin\tilde{\theta} & -2\sin i\cos i\cos\tilde{\theta} & 4-7\cos^2 i-5\sin^2 i\cos^2\tilde{\theta} \end{bmatrix}, \\
A_2 &= -\frac{n\rho^2(\theta)}{(1-e^2)^{3/2}}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{12}
$$

Here, e, n, θ, i are the eccentricity, mean motion, true anomaly, and inclination of the reference orbit, respectively. In addition, $k_{J_2} \triangleq \frac{3}{2}J_2(\frac{R_e}{a})^2$ and $\tilde{\theta} \triangleq \omega + \theta$ are defined, where R_e is the mean radius of the Earth, and a and ω are the semimajor axis and the argument of the perigee of the reference orbit, respectively. It can be shown that $A_1 - A_1^{\mathrm{T}} = \dot{A}_2$.

Example 2.6 The sixth example is that of the linearized equations including quadratic atmospheric drag proposed by Carter and Humi [22]. They use normalized units and the independent variable is the true anomaly of the reference orbit. A simple version of these equations has the following form:

$$
\begin{bmatrix} r' \\ v' \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix}, \qquad v = r',
$$

and

$$
A_1 = \begin{bmatrix} 3(1+4\chi^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13}
$$

where χ is a constant, including drag and the geometry of the satellite. Also, A_1 is symmetric and A_2 is a constant and skew-symmetric matrix. The fundamental matrix associated with these equations is

$$
\Phi = \begin{bmatrix}
\frac{1}{1-12\chi^2}\left[4-3(1+4\chi^2)\cos\tilde{\gamma}\theta\right] & 0 & 0 & \frac{1}{\sqrt{1-12\chi^2}}\sin\tilde{\gamma}\theta & \frac{2}{1-12\chi^2}[1-\cos\tilde{\gamma}\theta] & 0 \\
\frac{6(1+4\chi^2)}{(1-12\chi^2)\sqrt{1-12\chi^2}}[\sin\tilde{\gamma}\theta - \tilde{\gamma}\theta] & 1 & 0 & \frac{2}{1-12\chi^2}[\cos\tilde{\gamma}\theta - 1] & \frac{1}{1-12\chi^2}\begin{bmatrix} \frac{4}{\sqrt{1-12\chi^2}}\sin\tilde{\gamma}\theta \\ -3(1+4\chi^2)\theta \end{bmatrix} & 0 \\
0 & 0 & \cos\theta & 0 & 0 & \sin\theta \\
\frac{3(1+4\chi^2)}{\sqrt{1-12\chi^2}}\sin\tilde{\gamma}\theta & 0 & 0 & \cos\tilde{\gamma}\theta & \frac{2}{\sqrt{1-12\chi^2}}\sin\tilde{\gamma}\theta & 0 \\
\frac{6(1+4\chi^2)}{1-12\chi^2}\left[\cos\tilde{\gamma}\theta - 1\right] & 0 & 0 & -\frac{2}{\sqrt{1-12\chi^2}}\sin\tilde{\gamma}\theta & \frac{1}{1-12\chi^2}\begin{bmatrix} 4\cos\tilde{\gamma}\theta \\ -3(1+4\chi^2) \end{bmatrix} & 0 \\
0 & 0 & -\sin\theta & 0 & 0 & \cos\theta
\end{bmatrix}, \tag{14}
$$

where $\tilde{\gamma} \triangleq \sqrt{1-12\chi^2}$.

Example 2.7 Luquette and Sanner [23] proposed relative motion equations based on restricted three-body dynamics, which are not limited to circular cases and have the following form:

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix},$$

where

$$A_1 = -(c_1 + c_2) I_{3\times 3} + 3c_1 \hat{r}_E(t)\hat{r}_E^{\mathrm{T}}(t) + 3c_2 \hat{r}_S(t)\hat{r}_S^{\mathrm{T}}(t) + \dot{W} - W^2, \qquad A_2 = -2W^{\mathrm{T}}. \tag{15}$$

Here, $\hat{r}_E(t)$ is the unit vector from the Earth to the chief satellite in the rotating frame, $\hat{r}_S(t)$ is the unit vector from the Sun to the chief satellite in the rotating frame, and

$$c_1 = \frac{GM_E}{\|r_E\|}, \qquad c_2 = \frac{GM_S}{\|r_S\|}, \qquad W = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix},$$

where G is the gravitational constant, and M_E and M_S are the masses of the Earth and the Sun, respectively. The rotating frame is rotating with respect to the inertial frame at the angular velocity $[\omega_x\ \omega_y\ \omega_z]^{\mathrm{T}}$. It is not difficult to show that the matrices A_1 and A_2 satisfy the feature mentioned.

Up to this point in the current study, some famous linearized equations have been provided, which describe the relative motion in various circumstances. Why is this characteristic so common in many cases of relative motion? This is related to the fact that relative equations are generally represented in the LVLH (Local Vertical, Local Horizontal) frame. For brevity, let us consider an autonomous system [20]. The well-known transformation to obtain the acceleration relative to the rotating LVLH frame is [16]:

$$(\delta\ddot{r})_R = \delta\ddot{r} - 2\omega \times (\delta\dot{r})_R - \dot{\omega} \times \delta r - \omega \times (\omega \times \delta r) \tag{16}$$

where the acceleration $(\delta\ddot{r})_R$ is relative to a rotating observer fixed in the LVLH frame, and $\delta\ddot{r}$ is the relative acceleration to an observer in an inertial frame. It is well known that the cross product is equivalent to the skew-symmetric matrix multiplication; that is, for an arbitrary 3×1 vector $a = [a_x\ a_y\ a_z]^{\mathrm{T}}$ and angular velocity vector of the LVLH frame $\omega = [\omega_x\ \omega_y\ \omega_z]^{\mathrm{T}}$,

$$\omega \times a = Wa = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} [a_x \quad a_y \quad a_z]^{\mathrm{T}} \tag{17}$$

holds, and the matrix W is skew-symmetric. In addition, we linearize $\delta\ddot{r}$ about the reference orbit $r^*(t)$; then:

$$\delta\ddot{r} = G(r^*)\delta r. \tag{18}$$

Here, the symmetric matrix $G(r)$ is calculated by [16],

$$G(r) = \frac{\partial g(r)}{\partial r},$$

where the gravitational acceleration $g(r)$, a conservative force field, comes from the original equations of motion,

$$\ddot{r} = g(r).$$

Then, (16) can be newly represented in a matrix form,

$$\begin{aligned}(\delta\ddot{r})_R &= G\delta r - 2W(\delta\dot{r})_R - \dot{W}\delta r - W^2\delta r \\ &= (G - W^2 - \dot{W})\delta r - 2W(\delta\dot{r})_R.\end{aligned} \tag{19}$$

It is noted that G and W^2 are symmetric and that W and $\dot{W}$ are skew-symmetric. Equation (19) is a linearized equation of relative motion, so the matrices A_1 and A_2 can be defined as

$$A_1 \triangleq G - W^2 - \dot{W}, \qquad A_2 \triangleq -2W. \tag{20}$$

From (20), the following relationship is obtained:

$$A_1 - A_1^{\mathrm{T}} = -\dot{A}_2^{\mathrm{T}} = \dot{A}_2. \tag{21}$$

Equation (21) is exactly the same as the basic feature, and this is why the feature appears frequently in linearized relative motions.

3 General Analytic Solutions to the Fuel-Optimal Reconfiguration Problems in Relative Motions

In this section, the analytic solutions are obtained for general fuel-optimal reconfiguration problems in relative motions. The basic feature in linearized relative motions mentioned in Sect. 2 is also used in the procedure. The general solutions provide analytic control history and analytic history of state variables, as well as cost function value. The objective is to minimize the fuel consumption during the reconfiguration; therefore, (1) is generally used as the state equation, and a cost function can be set as

$$J = \frac{k^2}{2}\int_{\alpha_0}^{\alpha_f} \{f(\beta)\}^2 u^{\mathrm{T}} u \, d\beta, \tag{22}$$

where k and f are a constant and a function, both of which are dependent on the given problem; α_0 and α_f are the moments when thrusters start to fire and turn off, respectively. β is used as an integration variable, and the Hamiltonian function is defined as

$$H \triangleq \frac{k^2}{2}\{f(\alpha)\}^2 u^{\mathrm{T}} u + \lambda_r^{\mathrm{T}} v + \lambda_v^{\mathrm{T}}(A_1 r + A_2 v + b(\alpha)u).$$

If we define the adjoint vector by

$$\lambda \triangleq \begin{bmatrix} \lambda_r \\ \lambda_v \end{bmatrix}, \tag{23}$$

then the following equations must be satisfied for the optimality conditions [24]:

$$\lambda'^{\mathrm{T}} = -\frac{\partial H}{\partial \xi}, \qquad \frac{\partial H}{\partial u} = 0. \tag{24}$$

For the system in this section, (24) yields

$$\lambda_r' = -A_1^{\mathrm{T}}\lambda_v, \qquad \lambda_v' = -A_2^{\mathrm{T}}\lambda_v - \lambda_r = A_2\lambda_v - \lambda_r, \qquad u = -\frac{1}{k^2}\frac{b(\alpha)}{\{f(\alpha)\}^2}\lambda_v. \tag{25}$$

Here, the previous feature is used. Then, the state and adjoint equations are

$$\begin{bmatrix} r' \\ v' \\ \lambda_r' \\ \lambda_v' \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ A_1 & A_2 & 0_{3\times 3} & -\frac{1}{k^2}\left\{\frac{b(\alpha)}{f(\alpha)}\right\}^2 I_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & -A_1^{\mathrm{T}} \\ 0_{3\times 3} & 0_{3\times 3} & -I_{3\times 3} & A_2 \end{bmatrix} \begin{bmatrix} r \\ v \\ \lambda_r \\ \lambda_v \end{bmatrix}. \tag{26}$$

It is assumed that the fundamental matrix (Φ) of the state equations (see (1)) is already known. The fundamental matrix (Φ) has generally the following form:

$$\Phi = \begin{bmatrix} \Phi_A \\ \Phi_A' \end{bmatrix},$$

where both Φ_A and Φ_A' are 3×6 matrices. Then, from (25) and the basic feature mentioned in Sect. 2,

$$\lambda_v'' = A_2'\lambda_v + A_2\lambda_v' - \lambda_r' = A_2\lambda_v' + A_1\lambda_v. \tag{27}$$

It is noticed that λ_v and λ_v' have Φ as their fundamental matrix, which is also the fundamental matrix of r and r', because $r'' = A_2 r' + A_1 r$, as shown in (1) and (2) without a control vector. Then,

$$\lambda_v = \Phi_A \Lambda_0, \qquad \lambda_v' = \Phi_A' \Lambda_0, \tag{28}$$

where Λ_0 is a 6×1 constant matrix determined by the boundary condition at $\alpha = \alpha_f$. Then, when substituting (28) into (25), the result is:

$$\lambda_r = A_2\lambda_v - \lambda_v' = (A_2\Phi_A - \Phi_A')\Lambda_0. \tag{29}$$

In sum, the fundamental matrix associated with the adjoint system is given by:

$$\Psi = \begin{bmatrix} A_2\Phi_A - \Phi_A' \\ \Phi_A \end{bmatrix} \triangleq \begin{bmatrix} \Phi_B \\ \Phi_A \end{bmatrix}. \tag{30}$$

Since A_2, Φ_A, Φ_A' are already known, the fundamental matrix Ψ is easily obtained. Then,

$$\begin{aligned} \Psi^{\mathrm{T}}\Phi &= [-\Phi_A^{\mathrm{T}}A_2 - \Phi_A'^{\mathrm{T}} \quad \Phi_A^{\mathrm{T}}] \begin{bmatrix} \Phi_A \\ \Phi_A' \end{bmatrix} = -\Phi_A^{\mathrm{T}}A_2\Phi_A - \Phi_A'^{\mathrm{T}}\Phi_A + \Phi_A^{\mathrm{T}}\Phi_A' \\ &= \Phi_A^{\mathrm{T}}\Phi_A' - (\Phi_A^{\mathrm{T}}\Phi_A')^{\mathrm{T}} - \Phi_A^{\mathrm{T}}A_2\Phi_A = C, \end{aligned} \tag{31}$$

where C is a constant matrix [25]. It is noted that, since $\Phi_A^{\mathrm{T}}\Phi'_A - (\Phi_A^{\mathrm{T}}\Phi'_A)^{\mathrm{T}}$ and $\Phi_A^{\mathrm{T}} A_2 \Phi_A$ are skew-symmetric, C is also skew-symmetric. Therefore, all diagonal elements are zero, and only lower or upper triangular elements are needed. The matrix C can be analytically obtained from (31) or inserting any value of α, for example $\alpha = 0$, yields the same results because C is constant and independent of α. Once the matrix C is determined from (31), then the inverse of Φ is readily obtained,

$$\Phi^{-1} = C^{-1}\Psi^{\mathrm{T}} = C^{-1}[\,\Phi_B^{\mathrm{T}} \quad \Phi_A^{\mathrm{T}}\,]. \tag{32}$$

The optimal thrust functions are from (25):

$$u(\alpha) = -\frac{1}{k^2}\frac{b(\alpha)}{\{f(\alpha)\}^2}\lambda_v = -\frac{1}{k^2}\frac{b(\alpha)}{\{f(\alpha)\}^2}\Phi_A\Lambda_0. \tag{33}$$

The solutions are functions of the upper half of the fundamental matrix, which is needed only to determine a constant matrix Λ_0. The solution of (1) is [26]:

$$\xi(\alpha) = \Phi(\alpha)\Phi^{-1}(\alpha_0)\xi(\alpha_0) + \Phi(\alpha)\int_{\alpha_0}^{\alpha}\Phi^{-1}(\beta)B(\beta)u(\beta)d\beta. \tag{34}$$

Considering (32) and (33), the function in the integration becomes

$$\begin{aligned}\Phi^{-1}Bu &= C^{-1}[\,\Phi_B^{\mathrm{T}} \quad \Phi_A^{\mathrm{T}}\,]\begin{bmatrix} 0_{3\times 3} \\ b(\alpha)I_{3\times 3}\end{bmatrix}\left(-\frac{1}{k^2}\right)\frac{b(\alpha)}{\{f(\alpha)\}^2}\Phi_A\Lambda_0 \\ &= -\frac{1}{k^2}C^{-1}\left(\left\{\frac{b(\alpha)}{f(\alpha)}\right\}^2\Phi_A^{\mathrm{T}}\Phi_A\right)\Lambda_0.\end{aligned} \tag{35}$$

By substituting (35) into (34), and setting $\alpha = \alpha_f$, the 6×1 matrix Λ_0 is calculated as follows:

$$\Lambda_0 = -k^2 S_f^{-1} C K, \tag{36}$$

where

$$S(\alpha) \triangleq \int_{\alpha_0}^{\alpha}\left\{\frac{b(\beta)}{f(\beta)}\right\}^2\Phi_A^{\mathrm{T}}\Phi_A d\beta, \qquad S_f \triangleq S(\alpha_f), \qquad K = \Phi_f^{-1}\xi_f - \Phi_0^{-1}\xi_0, \tag{37}$$

where $\xi_0 \triangleq \xi(\alpha_0)$, $\xi_f \triangleq \xi(\alpha_f)$, the subscripts 0 and f refer to the values at $\alpha = \alpha_0$ and $\alpha = \alpha_f$, respectively. It is noted that the 6×6 matrix S is symmetric. Then, the cost function (22) can be simplified because of the constant matrix K and the definition of S_f,

$$\begin{aligned}J &= \frac{k^2}{2}\int_{\alpha_0}^{\alpha_f}\{f(\beta)\}^2 u^{\mathrm{T}}u\,d\beta = \frac{k^2}{2}\int_{\alpha_0}^{\alpha_f}\left\{\frac{b(\beta)}{f(\beta)}\right\}^2 K^{\mathrm{T}}C^{\mathrm{T}}S_f^{-1}\Phi_A^{\mathrm{T}}\Phi_A S_f^{-1}CK\,d\beta \\ &= \frac{k^2}{2}K^{\mathrm{T}}C^{\mathrm{T}}S_f^{-1}CK.\end{aligned} \tag{38}$$

In sum, the optimal thrust vector is a function of the upper half (Φ_A) of the fundamental matrix associated with the state equations and its inverse is not needed. All results are represented succinctly in the following form:

$$S(\alpha) = \int_{\alpha_0}^{\alpha} \left\{ \frac{b(\beta)}{f(\beta)} \right\}^2 \Phi_A^{\mathrm{T}} \Phi_A d\beta, \qquad K = \Phi_f^{-1} \xi_f - \Phi_0^{-1} \xi_0,$$
$$u(\alpha) = \frac{b(\alpha)}{\{f(\alpha)\}^2} \Phi_A S_f^{-1} C K, \qquad J = \frac{k^2}{2} K^{\mathrm{T}} C^{\mathrm{T}} S_f^{-1} C K, \tag{39}$$
$$\xi = \Phi(\Phi_0^{-1}\xi_0 + C^{-1} S S_f^{-1} C K)$$

where $\Phi = \left[\begin{smallmatrix}\Phi_A \\ \Phi'_A\end{smallmatrix}\right]$, $C = \Phi_A^{\mathrm{T}} \Phi'_A - (\Phi_A^{\mathrm{T}} \Phi'_A)^{\mathrm{T}} - \Phi_A^{\mathrm{T}} A_2 \Phi_A$, and $S_f = S(\alpha_f)$. The initial (ξ_0) and final (ξ_f) states are given, and K is defined in (37). K contains the inverse of the fundamental matrix (Φ), but both Φ_0 and Φ_f are constant so their inverses are easily evaluated. Since the fundamental matrix (Φ) has already been revealed, the symmetric matrix S is readily calculated, and the constant matrix C is given in (31). Then, the cost function, J, the optimal control vector, u, and the state variables during the reconfiguration, ξ, can be solved in a completely analytic way. In previous research studies conducted [14, 15], S contains the inverse of the fundamental matrix associated with the original system, which makes the required computations much more complex. Furthermore, (39) in the current study reveals the fact that the optimal control vector is a function of the fundamental matrix so the form of solutions can be established without time-consuming calculations, while previous results [15] do not present explicit forms of the solutions due to the complexity of the inverse of the fundamental matrix. Equation (39) yields exactly the same results obtained in [9], which will be shown in Sect. 4.

4 Application and Verification of the General Solutions

In this section, as an example, the general solution derived in Sect. 3 is applied to fuel-optimal reconfiguration in a circular orbit. This example illustrates that the general solutions obtained in the current study can be readily applied to many relative motions. A fuel-optimal reconfiguration problem in circular orbits is the first case in Sect. 2. Therefore, the state equations and the fundamental matrix in Sect. 2 are used for this example. In this case, the independent variable is time and $b(t) = 1$. The cost function is set as:

$$J = \frac{1}{2} \int_0^{t_f} u^{\mathrm{T}} u d\tau, \tag{40}$$

so $k = 1$ and $f(t) = 1$. The fundamental matrix associated with the HCW equations is well known [16] and also given in (5). Then, from (39),

$$S = \int_0^t \Phi_A^{\mathrm{T}} \Phi_A d\tau, \qquad K = \Phi_f^{-1} \xi_f - \Phi_0^{-1} \xi_0,$$
$$u(t) = \Phi_A S_f^{-1} C K, \qquad J = \frac{1}{2} K^{\mathrm{T}} C^{\mathrm{T}} S_f^{-1} C K, \tag{41}$$
$$\xi = \begin{bmatrix} r \\ v \end{bmatrix} = \Phi(\Phi_0^{-1}\xi_0 + C^{-1} S S_f^{-1} C K).$$

In this case, a skew-symmetric constant matrix C is

$$C = \Phi_A^{\mathrm{T}}\dot{\Phi}_A - (\Phi_A^{\mathrm{T}}\dot{\Phi}_A)^{\mathrm{T}} - \Phi_A^{\mathrm{T}} A_2 \Phi_A = \begin{bmatrix} 0 & -2n & 0 & 1 & 0 & 0 \\ 2n & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

In addition, the symmetric matrix S is

$$S = \begin{bmatrix} S_{11} & S_{12} & 0 & S_{14} & S_{15} & 0 \\ S_{21} & S_{22} & 0 & S_{24} & S_{25} & 0 \\ 0 & 0 & S_{33} & 0 & 0 & S_{36} \\ S_{41} & S_{42} & 0 & S_{44} & S_{45} & 0 \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & S_{63} & 0 & 0 & S_{66} \end{bmatrix} \tag{43}$$

and each element is

$$\begin{aligned}
S_{11} &= \frac{13}{2n^2}t - \frac{8}{n^3}\sin nt + \frac{3}{4n^3}\sin 2nt, \\
S_{12} = S_{21} &= \frac{3}{n}t^2 - \frac{6}{n^2}t\sin nt - \frac{3}{2n^3}\cos 2nt, \\
S_{14} = S_{41} &= \frac{1}{4n^2}(16\cos nt - 3\cos 2nt), \\
S_{15} = S_{51} &= -\frac{11}{n}t + \frac{14}{n^2}\sin nt - \frac{3}{2n^2}\sin 2nt, \\
S_{22} &= \frac{14}{n^2}t - \frac{32}{n^3}\sin nt + 3t^3 - \frac{3}{n^3}\sin 2nt + \frac{24}{n^2}t\cos nt, \\
S_{24} = S_{42} &= \frac{6}{n}t\cos nt + \frac{5}{n}t - \frac{8}{n^2}\sin nt - \frac{3}{2n^2}\sin 2nt, \\
S_{25} = S_{52} &= -\frac{9}{2}t^2 + \frac{12}{n}t\sin nt + \frac{4}{n^2}\cos nt + \frac{3}{n^2}\cos 2nt, \\
S_{33} &= -\frac{1}{4n^3}(\sin 2nt - 2nt), \qquad S_{36} = S_{63} = \frac{1}{4n^2}\cos 2nt, \\
S_{44} &= \frac{5}{2}t - \frac{3}{4n}\sin 2nt, \qquad S_{45} = S_{54} = \frac{3}{n}\cos nt\,(\cos nt - 2), \\
S_{55} &= \frac{1}{n}(19nt - 24\sin nt + 3\sin 2nt), \qquad S_{66} = \frac{1}{4n}(2nt + \sin 2nt).
\end{aligned} \tag{44}$$

The results, such as the optimal control vector u, the cost function J, and the state variables during the maneuver ξ, are exactly the same as (30), (34)–(36), (44), (45), (49)–(51) in [9]. Thus, it is shown that the general analytic solutions in the current paper can easily lead to the solutions found in previous literature.

As an example, the same problem as in [9] is considered. The chief satellite moves in a circular orbit of radius 7000 km about the center of the Earth, and its mean motion, n, is 0.00107801 sec^{-1}. A transfer time, t_f, is 691.8 sec, and two deputy satellites flying in formation, designated as Sat 1 and Sat 2, are considered. Sat 1 has the following initial and final conditions:

$$\xi_{0,\mathrm{Sat1}} = [-200\ (\mathrm{m}) \quad -200\ (\mathrm{m}) \quad -10\ (\mathrm{m}) \quad 0\ (\mathrm{m/s}) \quad 0.431203\ (\mathrm{m/s}) \quad 0\ (\mathrm{m/s})]^{\mathrm{T}}, \tag{45a}$$

$$\xi_{f,\mathrm{Sat1}} = [200\ (\mathrm{m}) \quad -200\ (\mathrm{m}) \quad 10\ (\mathrm{m}) \quad 0\ (\mathrm{m/s}) \quad -0.431203\ (\mathrm{m/s}) \quad 0\ (\mathrm{m/s})]^{\mathrm{T}}. \tag{45b}$$

Sat 2's initial and final conditions are

$$\xi_{0,\mathrm{Sat2}} = [200\ (\mathrm{m}) \quad 200\ (\mathrm{m}) \quad 10\ (\mathrm{m}) \quad 0\ (\mathrm{m/s}) \quad -0.431203\ (\mathrm{m/s}) \quad 0\ (\mathrm{m/s})]^{\mathrm{T}}, \tag{46a}$$

$$\xi_{f,\mathrm{Sat2}} = [-200\ (\mathrm{m}) \quad 200\ (\mathrm{m}) \quad -10\ (\mathrm{m}) \quad 0\ (\mathrm{m/s}) \quad 0.431203\ (\mathrm{m/s}) \quad 0\ (\mathrm{m/s})]^{\mathrm{T}}. \tag{46b}$$

From (41), (42), (43), the matrices K, C, S are determined, and the cost function J has a value of $4.99798 \times 10^{-3}\ \mathrm{m}^2/\mathrm{s}^3$ in common for each deputy satellite. The optimal control vector u, which is a function of the upper half (Φ_A) of the fundamental matrix, is immediately calculated from (41) or (33),

$$u(t) = \Phi_A S_f^{-1} C K = -\Phi_A \Lambda_0$$
$$= -\begin{bmatrix} \lambda_1(4 - 3\cos nt) + \frac{\lambda_4}{n}\sin nt + \frac{2\lambda_5}{n}(1 - \cos nt) \\ 6\lambda_1(\sin nt - nt) + \lambda_2 + \frac{2\lambda_4}{n}(\cos nt - 1) + \lambda_5(\frac{4}{n}\sin nt - 3t) \\ \lambda_3 \cos nt + \frac{\lambda_6}{n}\sin nt \end{bmatrix},$$

where $\Lambda_0 = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6]^{\mathrm{T}}$ and $\lambda_1 = -4.52240 \times 10^{-3}$, $\lambda_2 = -1.13478 \times 10^{-3}$, $\lambda_3 = -2.34571 \times 10^{-4}$, $\lambda_4 = 1.24626 \times 10^{-5}$, $\lambda_5 = 9.75035 \times 10^{-6}$, and $\lambda_6 = 6.46421 \times 10^{-7}$ for Sat 1. Numerical simulations are shown graphically in Figs. 1 and 2. Figure 1 shows the optimal trajectories ($\boldsymbol{\xi}$) of Sat 1 and Sat 2; this was obtained from (41) during the reconfigurations on the $y - x$ plane (i.e., $z = 0$ plane). Sat 1 is moved from its position in front of the plane of the page to behind, and Sat 2 is reversed about the page. Squares and circles represent initial and final positions, respectively. In Fig. 2 the thrust functions for each of the three thrusters for Sat 1 are shown. Figures 1 and 2 are exactly the same as those in [9].

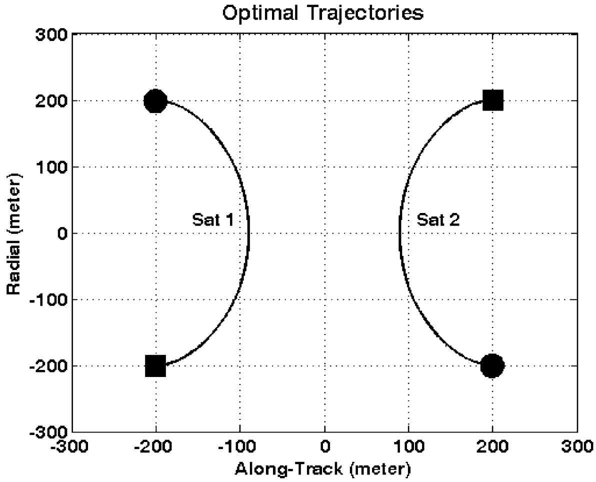

Fig. 1 Optimal reconfiguration trajectories of two deputy satellites during the reconfigurations in the $y-x$ plane for a circular reference orbit. The *squares* and *circles* represent initial and final positions, respectively

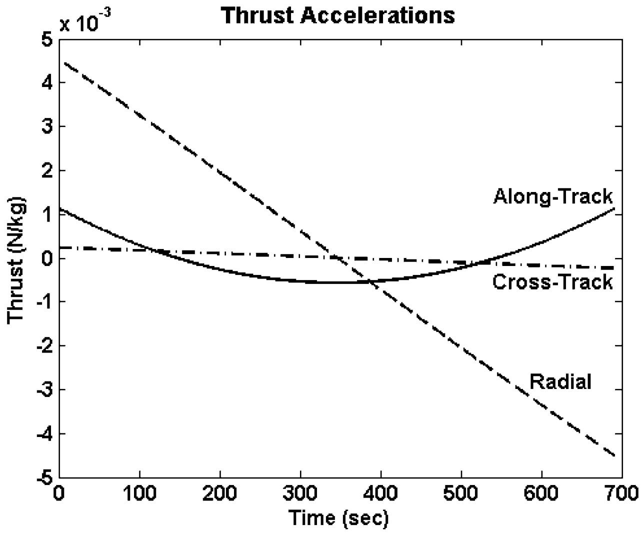

Fig. 2 Thrust profiles of Sat 1 for each of the three thrusters for a circular reference orbit

5 Conclusions

A new analytic method for solving reconfiguration problems in relative motion is presented, which does not need the inverse of the fundamental matrix. If the linear dynamic equations are represented in a rotating LVLH frame, then the basic feature shown in the current study is guaranteed in general. If this basic feature is met, the state and adjoint systems hold half of the fundamental matrix in common. By nature, optimal thrust functions contain the components of the fundamental matrix of the original dynamic equations. Once this fundamental matrix is found, the optimal thrust accelerations, cost function, and states during the maneuver are much more easily calculated than previous research has shown. Thus, optimal reconfiguration problems can be easily solved in many linearized cases of relative motion. The results obtained in the current paper allow one to predict the explicit form of the optimal solutions in advance without solving a given problem, thus significantly reduce the amount of calculations required. These results have been verified for brevity and to prove the accuracy of the analytic solutions.

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