# Perturbation Methods

Vincent Denoël

MATH2015-1

Academic Year 2017-2018

Last update: March 14, 2018

# General Informations



## Download updates ?

http://hdl.handle.net/2268/192804
(click and accept the connection to ORBi)

## Where ? When?

Organized on Wednesday, 8:30-12:30 (2nd semester)
Room 0.33, (B37)

## Contact me?

Room: B52/3, +1/422
Phone: 04/366.29.30
Mail: v.denoel@ulg.ac.be
http://www.ssd.ulg.ac.be

## Objectives and pedagogical commitment? Exam?

http://progcours.ulg.ac.be/cocoon/cours/MATH2015-1.html
Cours 15h+15h, 3 ECTS
Written exam (85%), 2 or 3 exercises (+ evaluation during the year - quizz/homework)

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# Outline

Contents:

- **Algebraic equations**
- **Eigen value problems**
- **Asymptotic approximations**
- **Matching asymptotics**
- **Multiple Scales**

References:

E.J. Hinch, Perturbation methods, Vol. 1, Cambridge: Cambridge University Press, 1991.
S. Howison, Practical Applied Mathematics: Modelling, Analysis, Approximation, Cambridge University Press, 2005.

# Tentative Agenda



7 Feb. 2018, 8:30-12:30: Introduction - Algebraic equation
14 Feb. 2018, 9:00-12:30: Algebraic equations
21 Feb. 2018, 9:00-12:30: Eigen value problems, ODEs
28 Feb. 2018, 9:00-12:30: Asymptotic approx. in ODEs
7 Mar. 2018, 9:00-12:30: Matching asymptotics and boundary layers
14 Mar. 2018, 9:00-12.30: Method of multiple scales
21 Mar. 2018, 9:00-12:30: Method of multiple scales > VD as ESA
28 Mar. 2018, 9:00-12:30: Integrals
4/11 Apr. 2018, 8:30-12:30: Spring break

21 Feb.Quizz 1
14 Mar.Homework 1 - Due March 28th

Compliant with the Faculty calendar:
`http://www.facsa.uliege.be/upload/docs/application/pdf/2017-08/calendrier_facultaire_2017_2018.pdf`

# Concepts



- applied mathematicians .vs. mathematicians
- two complementary ways to get accurate solutions: numerical and analytical methods
- numerical methods $\equiv$ all parameters of order 1, all sizes and dimensions of the same order (ex. beam theory, lubrication theory)
- perturbation methods $\equiv$ small parameter
- what is small ? (dimensional analysis)
- very small means very large
- *finding perturbation approximations is more an art than a science*
- perturbation suggests unperturbed
- small .vs. negligible (ex. undamped oscillator)

# Regular .vs. Singular

- Solution of $x^2 + \varepsilon x - 1 = 0$
- Solution of $\varepsilon x^2 + x - 1 = 0$

Discuss:

- singular .vs. regular
- iterative approach,
- expansion,
- rescaling
- 2-term .vs. 3-term equation (probe existing solution by balancing term two-by-two - pairwise comparison)

# Other Details I

- Find the right scaling, use $\varepsilon x^2 + x - 1 = 0$ and pose $x = \delta(\varepsilon) X$
- Non-integral powers, solution of $(1-\varepsilon)x^2 - 2x + 1 = 0$

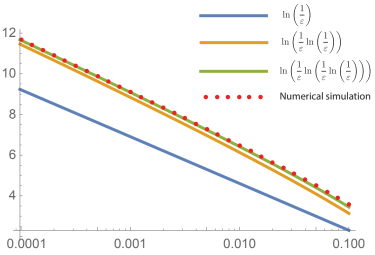
- Exercise by students : solution of $x^2 - 3x + 2 + \varepsilon = 0$
- Exercise by students : solution of $\varepsilon x^3 + x - 1 = 0$ (finish as homework)

# Logarithms and Trigonometric I





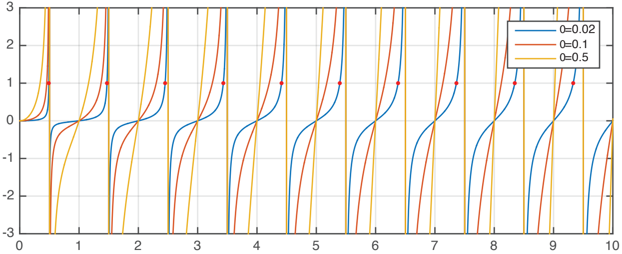
Logarithms, illustrate with $x\,e^{-x} = \varepsilon$ (use iteration)




- Proposed exercise @ home : $x^2 - \varepsilon e^x - 1 = 0$

# Logarithms and Trigonometric II

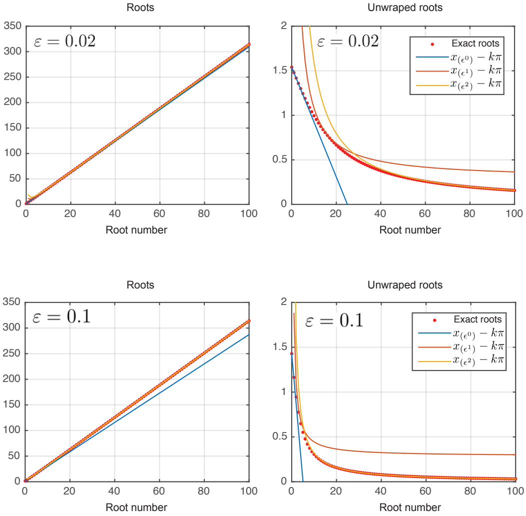
- Trigonometric equation : solution of $x \tan x = \frac{1}{\varepsilon}$




- To find the right balance is not easy (?)
- Do not expect to be able to develop a function in the neighborhood of a singular point

# Logarithms and Trigonometric III





# Logarithms and Trigonometric IV



- Proposed exercise @ home : $\tan x = \varepsilon x$

# Eigenvalue Problems

- Solve $\mathbf{Ax} + \varepsilon \mathbf{b}(\mathbf{x}) = \lambda \mathbf{x}$

**(Sept 2016)** The dimensionless stiffness and mass matrices of a lightpole with a small mass at its top are given by

$$\mathbf{K} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad ; \quad \mathbf{M} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1+\varepsilon \end{pmatrix}.$$

Assuming $\varepsilon \ll 1$, develop a perturbation method to determine 2nd-order-accurate approximations of the two eigen frequencies $\omega_1$ and $\omega_2$ of this structure. Remember eigen frequencies and mode shapes are determined by $\left(\mathbf{K} - \mathbf{M}\omega_i^2\right)\mathbf{x}_i = 0$.

# Regular Perturbations of ODEs/PDEs





Perturbation of

- a term/factor in the equation,
- a boundary condition,
- the position of a boundary condition

- The projectile equation

$$\frac{d^2x}{dt^2} = \frac{-gR^2}{(x+R)^2}$$

- Exercise by students : projectile with a small friction

$$\frac{d^2y}{d\tau^2} + \varepsilon\frac{dy}{d\tau} = -1 \qquad ; \quad y(0) = 0; y'(0) = 1$$

- (An oscillator with a small stiffness) -> Remove (?)
- Potential outside a nearly spherical body
- Flow past a nearly circular cylinder
- Exercise by students : Nearly uniform inertial flow past a cylinder

Boundary/Edge/Skin Layers

Consider $\varepsilon y'' + y' + y = 0$, with boundary conditions $y(0) = a$ and $y(1) = b$.

- Discuss unperturbed equation
- Determine scaling in boundary layers
- Introduce rescaled coordinates
- Solve for inner and outer solutions (regular ansatz)
- Use B.C. together with inner solutions & use matching to determine remaining constants of integration

Other exercises.

Consider $\varepsilon y'' - y' + e^x = 0$, with boundary conditions $y(0) = 0$ and $y(1) = 0$ (B.L. on right side).

Solution: $y(x) = e^x - 1 - (e-1)\, e^{-\frac{1-x}{\varepsilon}} + \mathrm{ord}(\varepsilon)$

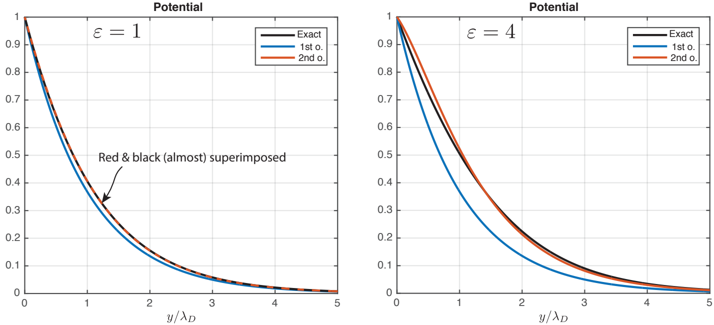
Consider $\varepsilon y'' + 2y' + 2y = 0$ , with boundary conditions $y(0) = 0$ and $y(1) = 1$.

Consider $\varepsilon y'' + \sqrt{x} y' + y = 0$ , with boundary conditions $y(0) = 0$ and $y(1) = 1$. (Practice with finding the right scaling)

Solution: $y(x) = e^2 \left( 1 - \frac{\Gamma\left(\frac{2}{3}, \frac{2}{3}\frac{x^{3/2}}{\varepsilon}\right)}{\Gamma\left(\frac{2}{3}\right)} \right)$

# Singlular Perturbations of ODEs/PDEs III

Debye layer $\varepsilon\psi'' = 1 - \mathrm{e}^{-\varepsilon\psi}$, $\psi(0) = 1$, $\psi(+\infty) = 0$




$$\psi\left(\frac{y}{\lambda_D}\right) = \mathrm{e}^{-\frac{y}{\lambda_D}} + \frac{\varepsilon}{6}\left(\mathrm{e}^{-\frac{y}{\lambda_D}} - \mathrm{e}^{-\frac{2y}{\lambda_D}}\right) \qquad \varepsilon = \frac{\zeta z}{k_B T}$$

# Singluar Perturbations of ODEs/PDEs IV

## Homework (2017):

$$\varepsilon y''(x) + \frac{y'(x)}{x} + y(x) = 0$$

on the domain $x \in [1, 2]$, for $\varepsilon \ll 1$ and with the boundary conditions $y(1) = 0$ and $y(2) = \varepsilon$.

## Homework (2018):

$$\varepsilon^2 y''(x) + \varepsilon x y'(x) - y = -e^x$$

on the domain $x \in [0, 1]$, for $\varepsilon \ll 1$ and with $y(0) = 2$, $y(1) = 1$.

# Singlular Perturbations of ODEs/PDEs V



- where is the boundary layer ?
- $\varepsilon^2 y'' + xy' - \varepsilon y = 0$, with boundary conditions $y(-1) = 0$ and $y(1) = 1$ (B.L. *inside* the domain).
- develop higher order approximations & match at higher orders (use other intermediate scale, not just half-way)
- the cable with the small boundary layer
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# Multiple Scales

- Introduction with the shallow waves equation
- The Duffing oscillator, oscillator with pure cubic stiffness
- The van der Pol oscillator
- Synchronization under small forcing

# Integrals

- Local / Global contributions
- Illustrate with multiple timescale spectral analysis

# Lectures complémentaires I



- E.J. Hinch, Perturbation methods, Vol. 1, Cambridge: Cambridge University Press, 1991.
- S. Howison, Practical Applied Mathematics: Modelling, Analysis, Approximation, Cambridge University Press, 2005.