Which level in Mathematics ? And for what kind of pupils ? How does research in mathematics education enlighten these questions ?

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Introduction

One specific question about the articulation between the mathematics taught in secondary school and the mathematics taught at the university

Mainly « articulation », secondarily « transition »

Errors concerning spatial analytical geometry

- For pupils : $y - 2x + 1 = 0$ ($y = 2x - 1$); $x = 0$ are equations of lines, because their previous knowledge is a « very first experience »
- Equation is, for them, a label attached to a geometrical object rather than a constraint characterizing points of a geometrical locus
- The equation of a line is : $ax + by + cz + d = 0$!

  Surprise : two characteristic equations for a line

Errors concerning spatial analytical geometry

The following system represents a line that does not include the origin while the triplet (0, 0, 0) verifies it when the parameter k is 2 :

$$x = 3k - 6$$
$$y = -k + 2$$
$$z = 2k - 4$$

Such learning difficulties seem to be independent from the teaching level : secondary schools, universities, future teachers

Defective teaching or lack of study ?

- At the university : subordination of analytical geometry to linear algebra
- Parametric and cartesian registers are a consequence of vector theory on condition this theorem : *Any vector space E, with a finite dimension on a vector field K is isomorphic to the space K^n of the coordinates, in relation to a given base of E, n a natural number*
- Efficacy of this theorical organisation

Defective teaching or lack of study ?

- The standardized transposition in secondary level education looks like this organisation but is hiding some points that are considered too difficult for the pupils (« mathematical organisation with holes in it », E. Rouy)
- Transition from the vector expression to the corresponding equalities on components is understood as a « trick »
- Questions from pupils about the relevance of the vector model

Defective teaching or lack of study ?

- Thus, how spatial analytical geometry is taught at the level of secondary education may be suspected, either when we look at it from theoretical viewpoint, or when we want to know how it is « justified » to build an algebraic model from geometrical objects
- This teaching seems to result in a kind of « low level learning » reduced to skills acquisition
- Can't we imagine another type of teaching that would precisely focus first on this model of geometrical objects?

Another didactical engineering on algebraic modelisation of geometrical objects

What could be the set of points of space which coordinates verify $y = - 3/2x + 2$?

- First interpretation in terms of line
- Idea of « dénoting » is absent

Does the point (4, -3, 10) belong to this set ?

- Debate between students : does the absence of ‘z’ in the equation mean that z has to be 0 or that z may be replaced by any value ?
- Form $y = -3/2x + 3 + 0z$ is not spontaneous, for unusual denoting

Another didactical engineering on algebraic modelisation of geometrical objects

- Idea that the set could be a plan generated by the movement of the line, or formed by a « stack » of parallel lines, or composed with points tha could be projected on this line

- For many students the equation “y=-3/2x+3” remains yet associated to “a line that moves” : “*in the plan Oxy, z is zero and the equation is y=-3/2x+3+0z. If I put the line at the position 1 for z, I get a line which height is 1 : y=-3/2x+3+1z. When z is 2, I get a line which height is 2 and y=-3/2x+3+2z, and so on…*”

Distinct geometrical objects should correspond to distinct algebraic “writings” !

And the same equation has now various meanings depending on the context : a line or a plan …

Another didactical engineering on algebraic modelisation of geometrical objects

What could be the equation of the plan Oxy ?

- Some students do not simply understand the question
- Others ones mention the equality « z = 0 » but complete it with a supplementary information : *“ax+by = d and z = 0 where a, b, c et d are real numbers. There are two tied equations. The first one tells that x et y may have any value and the second one tells that z is 0. The coordinates x and y are free […]*

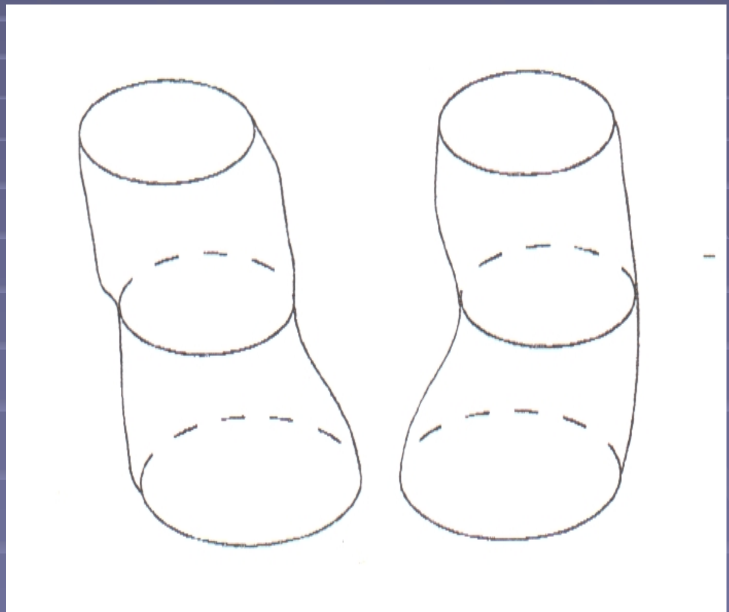
Students aim to express the freedom of variables as well as their constraints by the algebraic expressions !

Another didactical engineering on algebraic modelisation of geometrical objects

- We can see here the learning difficulties that are neglected when we deduce the cartesian and parametric register from the vector register
- Without working on their pertinence as models for geometric objects, these difficulties may appear again and again during the exercices

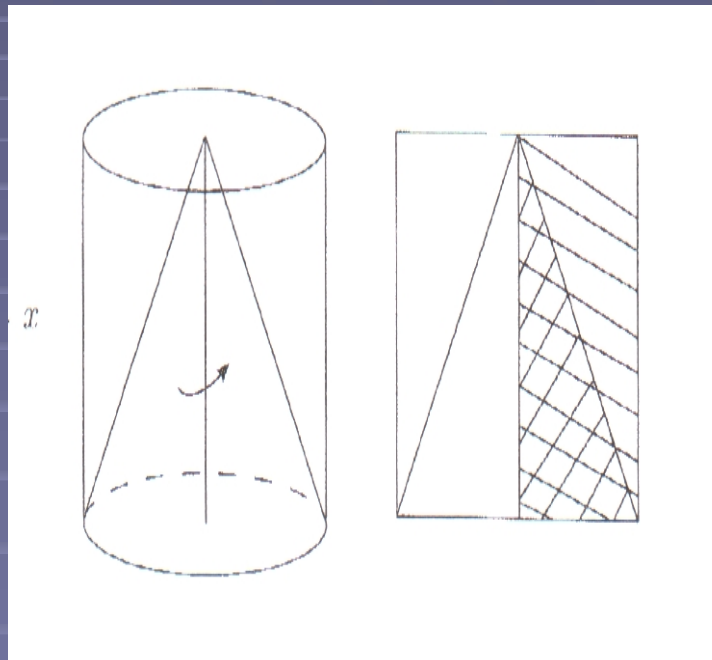
Necessity of working on the modelling stage in analysis

Cavalieri deduces a ratio between the volumes of two solids placed between two parallel plans from the invariability of the ratio between the areas of their « indivisibles »



Necessity of working on the modelling stage in analysis

Obstacle of « dimensions heterogeneity » : to make same comparisons in the case of revolution solids, for example



Necessity of working on the modelling stage in analysis

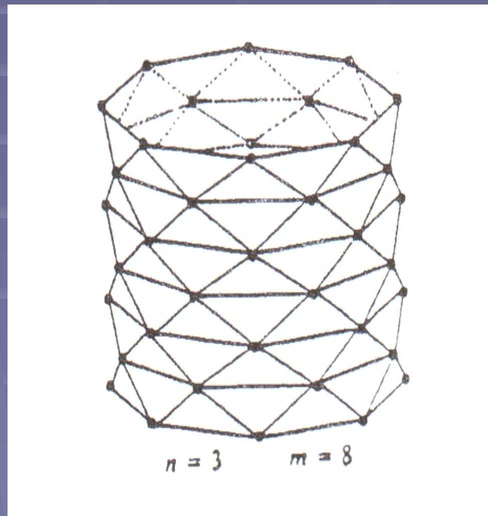
- Magnitude’s measures are not considered as specific concepts having their own « life », but as a way to translate properties of the magnitudes themselves
- This learning obstacle reveals a general position towards mathematics, the empirical positivism : concepts are exact reflect of objects in the « physical » world and not human intellectual elaborations

Necessity of working on the modelling stage in analysis

Examples of empirical positivism :

- To think tangent as ‘limit’ of secants
- To think that the instantaneous velocity cannot exist as it cannot be exactly assessed by means of observations and physical measures
- To suspect the exact equality of curvilinear area with the limit of a sequence of rectangles « reducing » into segments

- To define surface’s area as being the limit of the sum of areas of triangles of an inscribed polyhedral surface Counter-example of Schwarz, 1883



In other domains

- Relative numbers : are not « discovered » in the nature but « invented »
- Causal and chronologist conceptions of the conditional probability explained by the difficulty of reasoning without reference to precise contexts

To ameliorate the teaching of analysis in the secondary school

To make the difference between two kinds of mathematical projects corresponding to two historical stages :

- « magnitudes modelling » (articulated with « functional modelling »)
- elaboration of the mathematical analysis as an autonomous and deductive discipline

« magnitudes modelling »

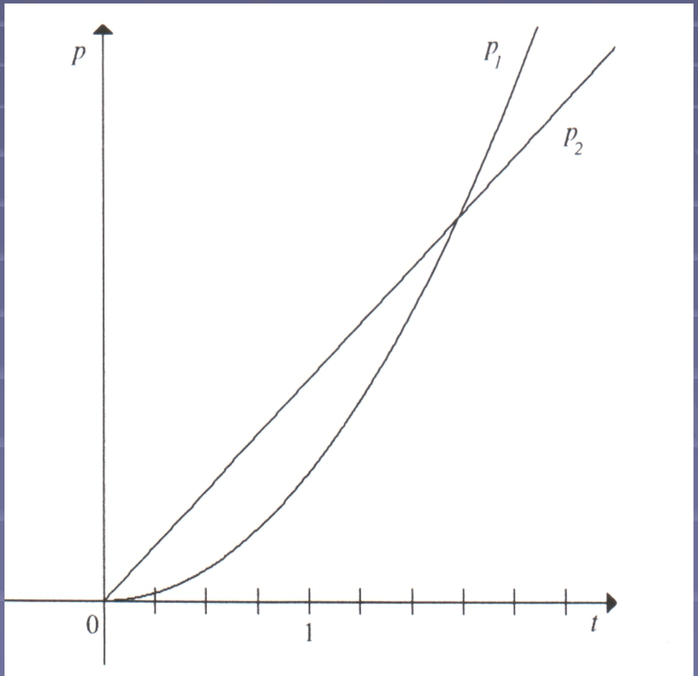
- Determination of curvilinear areas and volumes, of variable velocities and tangents; questions of magnitude’s optimisation
- Magnitudes are not really defined and are used as kinds of « préconstruit » (pre-conceived)
- Calculation of limits is in an embryonic form as the technique consisting in suppressing specific terms
- The technological discourse validates that such a calculation technique provides the « exact » value of curvilinear area or instantaneous velocity, …

The instantaneous velocity

At which instant do two mobile particles have the same velocity ?

They move on a straight path, on of them (p_2) with a constant velocity

The law of the other (p_1) will be first a quadratic function and then a third degree function



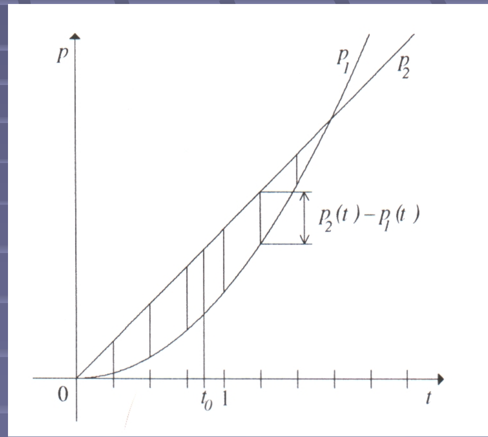
The instantaneous velocity

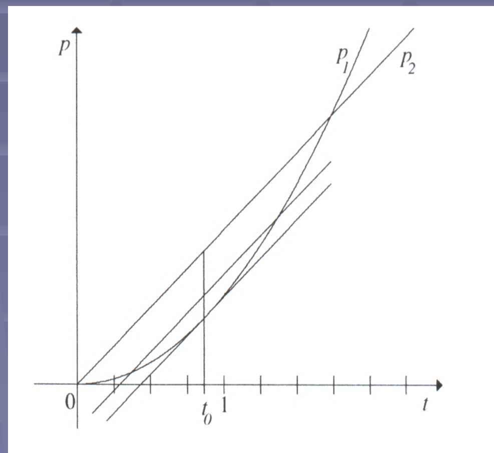
- Strategie 1

  To determine which value of t corresponds to the maximum gap between the two curves

- Strategie 2

  To determine for which value of t a line having the same slope of p_2 will cross the curve in only a point

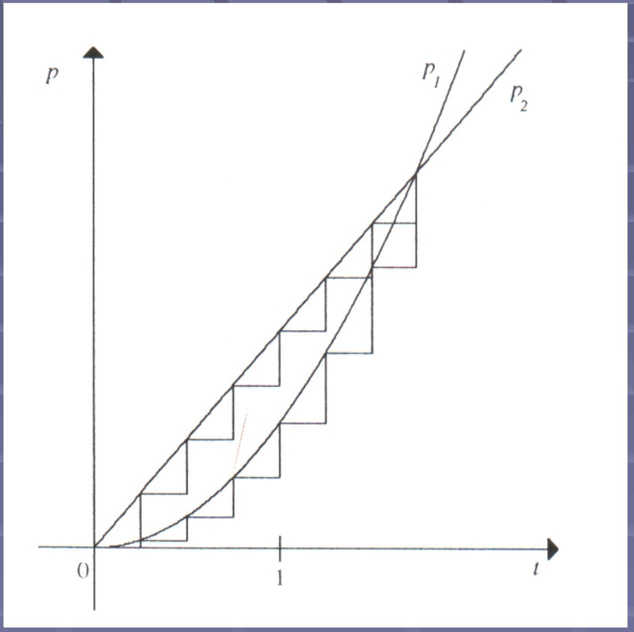




The instantaneous velocity

- Strategie 3

  To determine the time interval for which the two mobiles have the same average velocity by reducing the interval's size, which leads to the following calculation :



The instantaneous velocity

Si l'on appelle Δp_1 la hauteur de la contremarche de p_1 et Δp_2 la hauteur de la contremarche de p_2 entre les instants t et $t + \Delta t$. On a

$$v_{m_1} = \frac{\Delta p_1}{\Delta t} = \frac{2t \cdot \Delta t + (\Delta t)^2}{\Delta t}$$

et

$$v_{m_2} = \frac{\Delta p_2}{\Delta t} = \frac{\sqrt{3} \cdot \Delta t}{\Delta t}.$$

En écrivant que les vitesses moyennes sont égales, on trouve

$$2t + \Delta t = \sqrt{3}.$$

En posant $\Delta t = 0$, on retrouve le même résultat que précédemment.

- In the case of t^2 function, pupils are able to check themselves that the three strategies provide the same result which gives credibility to the third despite its 'strange' nature
- The two first strategies become unfeasible in the case of the t^3 law motion
- Embryonic definition of instantaneous velocity

Casting the calculus in a deductive mould

- To require a validation without any geometric and cinematic argument
- Limit concept : « proof-generated » concept
- Derivative and integral are defined of the limit concept
- Magnitudes defined by means of the limit concept : one is sure that the calculation of limit provides the exact value !

Casting the calculus in a deductive mould

Such a universe may be introduced by degrees to pupils of secondary schools :

One may ask the question of the conditions for the existence of a vertical asymptote $x = a$ for any function $f(x)$ expressed as a ratio : is it enough if the denominator is 0 when $x = a$?

New hypotheses, such as the continuity, will emerge from counter-examples accompanied with a reasoning handling an embryonic form of the quantified definition of limit

Two kinds of mathematical organisations

- « Modelling organisation » : activity that consists in setting up first mathematical model aiming at dealing with questions related to informal objects, within mathematics or within other contexts
- « Deductive organisation » : activity that consists in the elaboration of a deductive framework for the concepts and properties issued from the first activity

Two kinds of mathematical organisations

- Correspond to two levels of mathematical rationality : the impossibility to recognize the modelling organisation is detrimental for teaching (E. Rouy)
- This does not mean that secondary schools teaching should deal only with the modelling organisation while university teaching could work with deductive work
- It is important to explain students when one is passing from a level to another one and how it modifies the « game rules »

In order to initiate a debate

« Je crois que tout simplement dans le secondaire j'ai vu la limite et la dérivée comme des techniques. Je savais très bien dériver, je ne me trompais pas mais la signification profonde de la dérivée, je ne l'avais pas perçue. Je pense que la maturité de l'élève est telle que c'est une notion sur laquelle il faut revenir après. Je ne vois pas de problème à dire : on a donné la définition, on a surtout insisté sur la technique de calcul parce que c'est à la portée des élèves à cet âge-là et puis en premier bac, on revient sur la notion en disant : attention, voilà ce qu'il y a en plus. Même en bio, je reviens dessus en disant : c'est un taux de variation instantané particulier. Et ça, dans le secondaire, on ne l'a pas vu mais il ne fallait peut-être pas le voir. C'est à nous à le faire » (professeur 1er BAC)

In order to initiate a debate

- ✓ *« Je pense que, dans le secondaire, les élèves n'ont aucun intérêt, aucun désir de maîtriser les dérivées »* (professeur d'université)
- ✓ *« Les élèves qui arrivent du secondaire ne réfléchissent pas : ils appliquent des procédures »* (professeur d'université)
- ✓ *« On nous dit qu'il faut évaluer selon trois compétences : connaître, appliquer et résoudre des problèmes. Mais, il vaut mieux mettre le maximum de points pour la deuxième rubrique si l'on veut ne pas avoir trop d'échecs »* (professeur du secondaire)
- ✓ *« Tout ce qu'on nous demande, c'est de préparer les élèves à bien calculer pour la suite »* (professeur du secondaire)