A d-dimensional extension of Christoffel words*

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Abstract

In this article, we extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in arbitrary dimension that we call Christoffel graphs. Christoffel graphs when $d = 2$ correspond to well-known Christoffel words. Due to periodicity, the d-dimensional Christoffel graph can be embedded in a $(d-1)$-torus (a parallelogram when $d = 3$). We show that Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part and conjugation with their reversal. Our main result extends Pirillo's theorem (characterization of Christoffel words which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma) in arbitrary dimension. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is a translation. We show that a fully periodic subgraph of the hypercubic lattice is a translate of its flip if and only if it is a Christoffel graph.

1 Introduction

This article is a contribution to the study of discrete planes and hyperplanes in any dimension d. We study only rational hyperplanes, that is, those which are defined by an equation with rational coefficients. We extract from such an hyperplane a finite pattern that we call, for $d = 3$, a *Christoffel parallelogram*. We show that they are a generalization of Christoffel words.

Discrete planes were introduced by [Rev91] and further studied [Deb95, Fra96, ARC97, Vui99]. Recognition algorithms were proposed in [Rev95, PBDR06, FST96]. See [BCK07] for a complete review about many aspects of digital planarity, such as characterizations in arithmetic geometry, periodicity, connectivity and algorithms. Discrete planes can be seen as an union of square faces. Such stepped surface, introduced in [IO94, IO93] as a way to construct quasiperiodic tilings of

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the plane, can be generated from multidimensional continued fraction algorithms by introducing substitutions on square faces [ABEI01, ABI02].

While discrete planes are a satisfactory generalization of Sturmian words, it is still unclear what is the equivalent notion of Christoffel words in higher dimension. In [Fer07, Fig. 6.6 and 6.7], fundamental domain of rational discrete planes are constructed from the iteration of generalized substitutions on the unit cube. Recently [DV12] generalized central words to arbitrary dimension using palindromic closure. In both cases the representation is nonconvex and has a boundary like a fractal.

In this article, we propose to extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in arbitrary dimension that we call Christoffel graphs. A similar construction, called *roundwalk*, but serving a different purpose was given in [BT04] producing multidimensional words that are closely related to k-dimensional Sturmian words. Christoffel graphs when $d = 2$ correspond to Christoffel words. Due to its periods, the d-dimensional Christoffel graph can be embedded in a $(d-1)$-torus and when $d = 3$, the torus is a parallelogram. This extension is motivated by Pirillo's theorem which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is replaced by some translation. When $d = 3$, our flip corresponds to a flip in a rhombus tiling [BFRR11, BFR08, ABFJ07]. We show that these Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part (Lemma 22) and conjugation with their reversal (Corollary 25 and 26). Our main result is Theorem 36 which extends Pirillo's theorem in arbitrary dimension.

We recall in Section 2 the basic notion on Christoffel words and discrete planes. The discrete hyperplane graphs are defined in Section 3. The operation on them (flip, reversal and translation) are introduced in Section 4. We show that the flip of a Christoffel graph is a translate of itself in Section 5. This is the sufficiency of the Pirillo's theorem. In Section 6, we consider the necessity and obtain a d-dimensional Pirillo's theorem, our main result. Finally, we construct in the Section 7 in appendix, the mathematical framework for the definition of discrete hyperplanes, since we could not find explicit and complete references.

2 Christoffel words and discrete planes

2.1 Christoffel words

Recall that Christoffel words are obtained by discretizing a line segment in the plane as follows: let $(p, q) \in \mathbb{N}^2$ with $\gcd(p, q) = 1$, and let S be the line segment with endpoints $(0, 0)$ and (p, q). The word w is a *lower Christoffel word* if the path induced by w is under S and if they both delimit a polygon with no integral interior point. An *upper Christoffel word* is defined similarly, by taking the path which is above the segment. A *Christoffel word* is a lower Christoffel word. See Figure 1 and reference [BLRS08]. An astonishing result about Christoffel words is the following characteristic property given by Pirillo [Pir01].

Theorem 1 (Pirillo). *A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.*

It is even known that the two words amb and bma are conjugate by palindromes [Chu97] Theorem 3.1 (see also [BR06] Proposition 6.1): for example, the Christoffel word in Figure 1 can be factorized as a product of two palindromes, but also as a letter, a *central word* m and a last

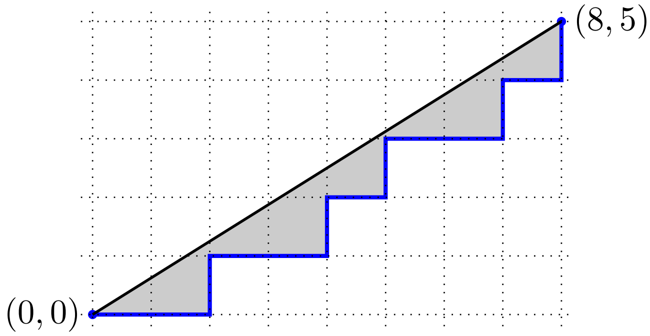


Figure 1: The lower Christoffel word $w = aabaababaabab$.

letter:

$$w = aabaa \cdot babaabab = a \cdot abaababaaba \cdot b = amb,$$

and the conjugate word w' of w obtained by exchanging of the two palindromes can also be factorized as the product of a letter, the same central word m and a last letter:

$$w' = babaabab \cdot aabaa = b \cdot abaababaaba \cdot a = bma.$$

Centrals words are the words m such that amb is a Christoffel word. They can be defined independently of Christoffel words: a word m is a *central word* if and only if for some coprime integers p and q, the length of m is $p + q - 2$ and p and q are periods of m. In this case, the Christoffel word amb is associated as above to the vector (p, q). See [CL05] for more informations and [Ber07] for fourteen different characterizations of central words. There are also some properties which are satisfied by Christoffel words but do not characterize them.

Lemma 2. *Let $w = amb$ be a Christoffel word of vector (p, q). Then,*

(i) *the central word m is a palindrome: $\widetilde{m} = m$;*

(ii) *p is a period of am and q is a period of mb;*

(iii) *the reversal $\widetilde{w}$ of a Christoffel word w is conjugate to w.*

The proof of (iii) follows from (i) and from Theorem 1. Words conjugate to their reversal were studied in [BHNR04], are product of two palindromes and are not necessarily Christoffel words. Moreover, not every palindrome is a central word. In this article, we generalize Theorem 1 to dimension 3. We also show that properties like the one enumerated in Lemma 2 hold.

2.2 Discrete planes

Given $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $\mu, \omega \in \mathbb{R}$, the *lower arithmetical discrete plane* [Rev91] $\mathcal{P}$ is the set of point $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ satisfying

$$\mu \leq a_1x_1 + a_2x_2 + a_3x_3 < \mu + \omega.$$

The parameter ω is called the (arithmetic) *width*. If $\omega = \|\vec{a}\|_1 = |a_1| + |a_2| + |a_3|$, then the discrete plane is said to be *standard*. Standard arithmetical discrete plane can be furnished with a canonical structure of a two-dimensional, connected, orientable combinatoric manifold without boundary, whose faces are quadrangles and whose vertices are points on the plane [Fra96]. See the appendix in Section 7 where we provide the mathematical framework for the definition of discrete hyperplanes $\mathcal{P}$ and *stepped surfaces* $\mathcal{S}$ [ABFJ07].

Let k be an integer such that $0 \leq k < d$. We say that $u, v \in \mathbb{Z}^d$ are *k-neighbor* if and only if

$$\|v - u\|_\infty = 1 \text{ and } \|v - u\|_1 \leq d - k.$$

In this article, we are interested in the graph representing the 2-neighboring relation for the discrete plane $\mathcal{P}$ and in general the $(d-1)$-neighboring relation for the discrete hyperplane in $\mathbb{Z}^d$. Note that $u, v \in \mathbb{Z}^d$ are $(d-1)$-neighbors if and only if their difference is $\pm e_i$ for some i such that $1 \leq i \leq d$.

3 Discrete hyperplane graphs

Let $a_1, \ldots, a_d$ be relatively prime positive integers and $s = \|\vec{a}\|_1 = \sum a_i$ be their sum. We denote $\vec{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$. We define the mapping $\mathcal{F}_{\vec{a}} : \mathbb{Z}^d \to \mathbb{Z}/s\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$ onto $\sum_i a_i x_i \bmod s$. We identify $\mathbb{Z}/s\mathbb{Z}$ and $\{0, 1, \ldots, s-1\}$. A total order on $\mathbb{Z}/s\mathbb{Z}$ is defined correspondingly; it is this order that is used in the definition of $\mathcal{H}_{\vec{a}}$ below. The map $\mathcal{F}_{\vec{a}}$ induces a $\mathbb{Z}^d$-action $x \cdot g = g + \mathcal{F}_{\vec{a}}(x)$ on the cyclic group $\mathbb{Z}/s\mathbb{Z}$, so that it is a rational case of the $\mathbb{Z}^2$-action on the torus as studied in [BV00, ABI02]. We consider $\mathbb{E}_d = \{(u, u+e_i) : u \in \mathbb{Z}^d \text{ and } 1 \leq i \leq d\}$, the set of oriented edges of the hypercubic lattice. Note that the set $\mathbb{E}_d$ also corresponds to the Cayley graph of $\mathbb{Z}^d$ with generators e_i for all i with $1 \leq i \leq d$.

3.1 The Christoffel graph $\mathcal{H}_{\vec{a}}$

The *Christoffel graph* $\mathcal{H}_{\vec{a}}$ of normal vector $\vec{a}$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\vec{a}}$:

$$\mathcal{H}_{\vec{a}} = \{(u, u + e_i) \in \mathbb{E}_d : \mathcal{F}_{\vec{a}}(u) < \mathcal{F}_{\vec{a}}(u + e_i)\}.$$

An example of the graph $\mathcal{H}_{\vec{a}}$ when $d = 2$ and $\vec{a} = (a_1, a_2) = (2, 5)$ is shown in Figure 2 (left) where the edges are represented in blue and a small red circle surrounds the origin. A first observation

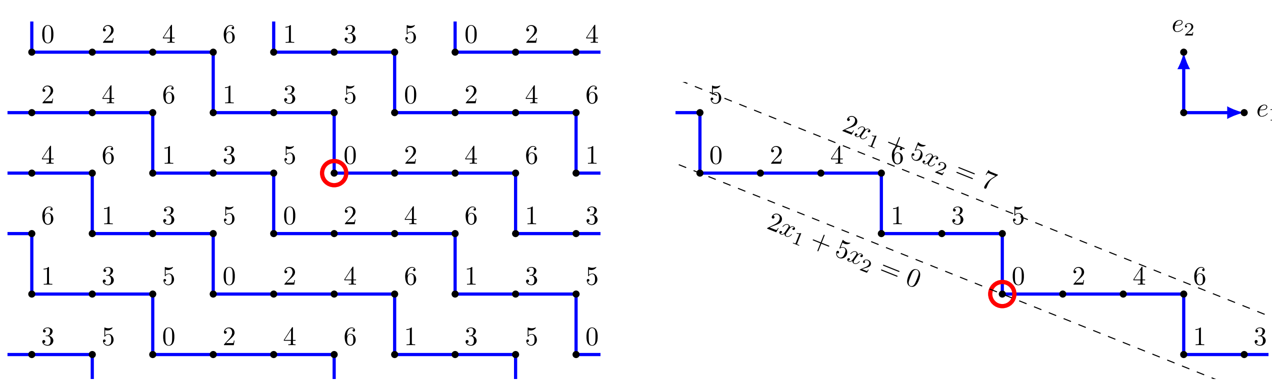


Figure 2: Left: the graph $\mathcal{H}_{\vec{a}}$ with $\vec{a} = (2, 5)$. Right: Standard discrete line $\mathcal{P}$ of normal vector $\vec{a} = (2, 5)$.

is stated in the next lemma.

Lemma 3. *The graph $\mathcal{H}_{\vec{a}}$ is invariant under the translation by the vector $\sum_{i=1}^d e_i = (1, 1, \ldots, 1)$.*

The proof is postponed at Lemma 10 where we show that the graph $\mathcal{H}_{\vec{a}}$ is invariant under all translations $t \in \operatorname{Ker} \mathcal{F}_{\vec{a}}$. Because of this invariance, a question is to find a good representent for the equivalence class $x+(1,1,\dots,1)\mathbb{Z}$ for each $x \in \mathbb{Z}^d$. It is natural to choose $\bar{x} \in x+(1,1,\dots,1)\mathbb{Z}$ such that

$$0 \leq \sum a_i \bar{x}_i < s. \tag{1}$$

If (u,v) is an edge of $\mathcal{H}_{\vec{a}}$ such that $v-u=e_i$ then $(\bar{u},\bar{v})$ is a pair of points which are (d-1)-neighbors satisfying Equation (1) and $\bar{v}-\bar{u}=e_i$. Thus the vertices satisfying Equation (1) are a set of representents for the vertices of $\mathcal{H}_{\vec{a}}$, see Figure 2 (right). Thus, each connected component of the graph $\mathcal{H}_{\vec{a}}$ corresponds exactly to a *standard discrete plane* $\mathcal{P}$ with the $(d-1)$-neighbor relation. The advantage of $\mathcal{H}_{\vec{a}}$ over the discrete hyperplane $\mathcal{P}$ is its algebraic structure. The next lemma gives an equivalent definition of the edges of the graph $\mathcal{H}_{\vec{a}}$. It will be useful in the sequel.

Let $a,b \in [0,s[$ be two integers. If $a<b$ then $]a,b]$ is a subinterval of $[0,s[$. If $a>b$ then $]a,b]=]a,s[\cup[0,b]$ is defined as the union of two subintervals of $[0,s[$.

Lemma 4. *Let $(u,v) \in \mathbb{E}_d$ such that $v=u+e_i$ for some $1 \leq i \leq d$. Then,*

$$(u,v) \in \mathcal{H}_{\vec{a}} \iff \mathcal{F}_{\vec{a}}(u) \in [0,s-a_i-1] \iff \mathcal{F}_{\vec{a}}(v) \in [a_i,s-1] \iff 0 \notin]\mathcal{F}_{\vec{a}}(u),\mathcal{F}_{\vec{a}}(v)], \tag{2}$$
$$(u,v) \notin \mathcal{H}_{\vec{a}} \iff \mathcal{F}_{\vec{a}}(u) \in [s-a_i,s-1] \iff \mathcal{F}_{\vec{a}}(v) \in [0,a_i-1] \iff 0 \in]\mathcal{F}_{\vec{a}}(u),\mathcal{F}_{\vec{a}}(v)]. \tag{3}$$

For each permutation σ of the set $\{1,2,\cdots,d\}$, there exists a *σ-path*

$$(u,u+e_{\sigma(1)}),(u+e_{\sigma(1)},u+e_{\sigma(1)}+e_{\sigma(2)}),\cdots,(u+\sum_{i=1}^{d-1} e_{\sigma(i)},u+\sum_{i=1}^{d} e_{\sigma(i)})$$

made of d edges of $\mathbb{E}_d$ going from the vertex $u \in \mathbb{Z}^d$ to the vertex $u+\sum_{i=1}^d e_i$.

Lemma 5. *All of the d edges of a σ-path but one belong to $\mathcal{H}_{\vec{a}}$.*

Proof. To each edge $(u+\sum_{i=1}^{k-1} e_{\sigma(i)},u+\sum_{i=1}^{k} e_{\sigma(i)})$ corresponds an interval (or an union of two intervals according to the above remark) $[\mathcal{F}_{\vec{a}}(u)+\sum_{i=1}^{k-1} a_{\sigma(i)},\mathcal{F}_{\vec{a}}(u)+\sum_{i=1}^{k} a_{\sigma(i)}[$. Since $\sum_{i=1}^d a_{\sigma(i)} = \sum_{i=1}^d a_i = s$, those d sets, with $1 \leq k \leq d$, are a partition of $[0,s[$. Therefore, only one of them contains 0. From Lemma 4, only one edge of the σ-path do not belong to $\mathcal{H}_{\vec{a}}$. $\square$

Let $R \subseteq \{1,2,\cdots,d\}$ and $u \in \mathbb{Z}^d$. An *hypercube graph from vertex u to vertex $u+\sum_{i\in R} e_i$* with $2^{\operatorname{Card} R}$ vertices is the subgraph of $\mathbb{E}_d$ defined by

$$\left\{\left(u+\sum_{i\in P} e_i,u+\sum_{i\in Q} e_i\right) \in \mathbb{E}_d \,\middle|\, P \subset Q \subseteq R \text{ and } \operatorname{Card} Q \setminus P = 1\right\}.$$

Each nonedge of $\mathcal{H}_{\vec{a}}$ implies the presence of a hypercube graph with 2^{d-1} vertices orthogonal and incident to it. For example, $(-e_1,0) \notin \mathcal{H}_{\vec{a}}$ and $(0,e_2) \in \mathcal{H}_{\vec{a}}$ when $\vec{a}=(2,5)$. This is proved in the next lemma.

Lemma 6. *If $(u,v) \in \mathbb{E}_d \setminus \mathcal{H}_{\vec{a}}$, then the hypercube graph from vertex v to vertex $u+\sum_{i=1}^d e_i$ with 2^{d-1} vertices is a subgraph of $\mathcal{H}_{\vec{a}}$.*

Proof. From Lemma 5, the last $d-1$ edges of every σ-path starting with the edge (u,v) and ending in $u+\sum_{i=1}^d e_i$ are in $\mathcal{H}_{\vec{a}}$. The set of last $d-1$ edges of these paths generates an hypercube graph from vertex v to vertex $u+\sum_{i=1}^d e_i$. $\square$

A line containing some point $x \in \mathbb{Z}^d$ parallel to e_i in the hypercubic lattice $\mathbb{E}_d$ is a set

$$L_{x,i} = \{(x + ke_i, x + (k + 1)e_i) : k \in \mathbb{Z}\} \subset \mathbb{E}_d.$$

The intersection $L_{x,i} \cap \mathcal{H}_{\vec{a}}$ of such a line with a discrete hyperplane graph $\mathcal{H}_{\vec{a}}$ is made of consecutive edges and nonedges. The next Lemma states that Christoffel words appear in this sequence.

Lemma 7. *The sequence of consecutive edges and nonedges in $L_{x,i} \cap \mathcal{H}_{\vec{a}}$ is periodic and the period is a Christoffel word.*

Proof. Each subset $L_{x,i} \cap \mathcal{H}_{\vec{a}}$ can be described by the subgroup of $\mathbb{Z}/s\mathbb{Z}$ generated by $\mathcal{F}_{\vec{a}}(e_i)$, i.e.,

$$(x + ke_i, x + (k + 1)e_i) \in L_{x,i} \cap \mathcal{H}_{\vec{a}} \iff 0 \leq \mathcal{F}_{\vec{a}}(x + ke_i) < s - a_i$$

This corresponds to the well-known construction of Christoffel words from the labelling of Cayley graphs of $\mathbb{Z}/s\mathbb{Z}$ with the generator a_i [BLRS08, Section 1.2 Cayley graph definition]. $\square$

For example, in the discrete hyperplane graph $H_{(2,5)}$ shown in Figure 2, coding an edge by the letter a and a nonedge by letter b, we get the periods *aaabaab* and *abbabbb* for the lines $L_{x,i} \cap \mathcal{H}_{\vec{a}}$ for $i = 1, 2$ respectively. Both are Christoffel words.

Definition 8 (Image)**.** *Let $f : \mathbb{Z}^d \to S$ be an homomorphism of $\mathbb{Z}$-module. For some subset of edges $X \subseteq \mathbb{E}_d$, we define the image by f of the edges X by*

$$f(X) = \{(f(u), f(v)) \mid (u, v) \in X\}.$$

This definition allows to define the graphs $I_{\vec{a}}$ and $\mathcal{G}_{\vec{a}}$ as projections of $\mathcal{H}_{\vec{a}}$ in the sections below.

3.2 The graph $I_{\vec{a}}$

Let π be the orthogonal projection from $\mathbb{R}^d$ onto the hyperplane $\mathcal{D}$ of equation $\sum x_i = 0$. Its restriction to the stepped surface $\mathcal{S}$ of the discrete plane $\mathcal{P}$ of normal vector $\vec{a} \in \mathbb{Z}^d$ is a bijection onto $\mathcal{D}$. It maps $\mathcal{P}$, the integral points in $\mathcal{S}$, onto a lattice L [ABI02, section 2.2] in $\mathcal{D}$ spanned by the vectors $h_i = \pi(e_i)$; they satisfy $\sum_i h_i = 0$. Note that $\pi(\mathbb{Z}^d)$ is also equal to L, since each point in $\mathbb{Z}^d$ is congruent to some point in $\mathcal{P}$ modulo the kernel of the projection. We may identify the set L and $\mathbb{Z}^d/(1, 1, \cdots, 1)\mathbb{Z}$, since two integral points are projected by π onto the same point if and only if their difference is a multiple of the vector $(1, 1, \ldots, 1)$ and since this multiple is necessarily an integral multiple. Since $\mathcal{F}_{\vec{a}}(1, 1, \ldots, 1) = 0$, the mapping $\mathcal{F}_{\vec{a}}$ induces a mapping $\mathcal{F}'_{\vec{a}} : L \to \mathbb{Z}/s\mathbb{Z}$. We have the following commuting diagram:

$$\begin{tikzcd}
\mathbb{Z}^d \arrow[rr, "\mathcal{F}_{\vec{a}}"] \arrow[dr, "\pi"'] & & \mathbb{Z}/s\mathbb{Z} \\
& L = \mathbb{Z}^d/(1,1,\ldots,1)\mathbb{Z} \arrow[ur, "\mathcal{F}'_{\vec{a}}"'] &
\end{tikzcd}$$

We consider the directed graph whose set of edges is $I_{\vec{a}} = \pi(\mathcal{H}_{\vec{a}})$. The graphs $I_{\vec{a}}$ for $\vec{a} = (a_1, a_2) = (2, 5)$ and $\vec{a} = (a_1, a_2, a_3) = (2, 3, 5)$ are shown in Figure 3. Note that the orientation of an edge is redundant when $d = 3$, since each edge is oriented as one of the vector h_i.

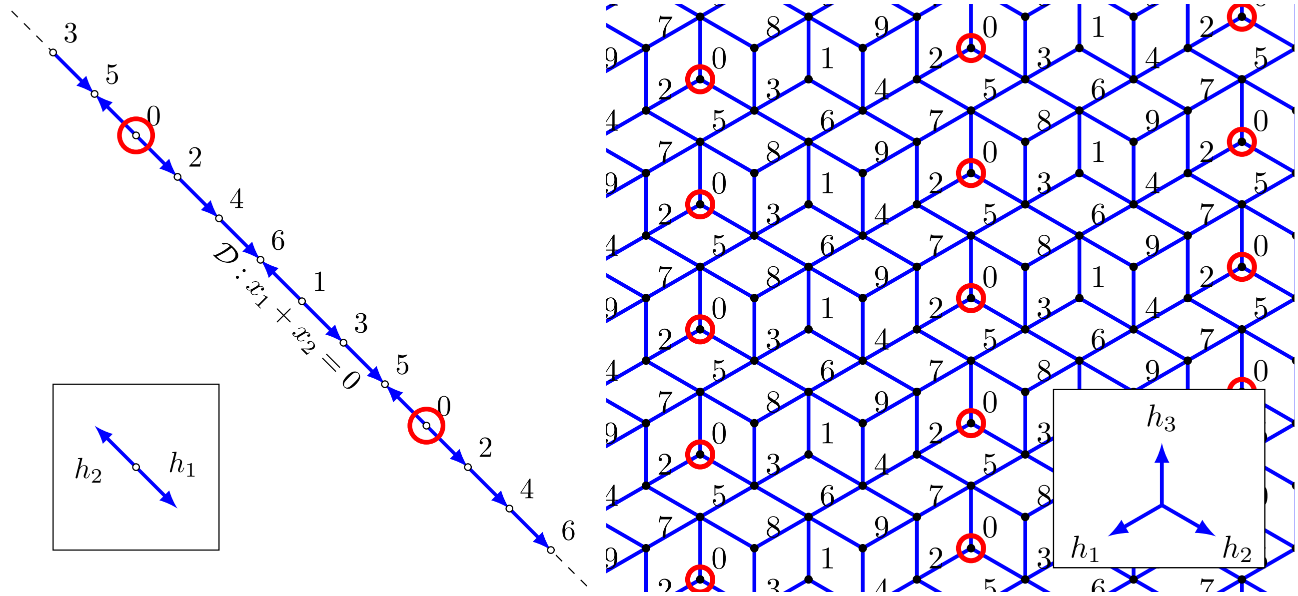


Figure 3: Left: the graph $I_{\vec{a}}$ when $\vec{a} = (2, 5)$. Right: the graph $I_{\vec{a}}$ when $\vec{a} = (2, 3, 5)$. The label at each vertex is its image under $\mathcal{F}'_{\vec{a}}$.

Lemma 6 can be seen on $I_{\vec{a}}$ when $d = 3$ by the fact that each nonedge is the short diagonal of a rhombus. For example, if $\vec{a} = (2, 3, 5)$ then $(-h_1, 0) \notin I_{\vec{a}}$. From the lemma, the paths $(0, h_2), (h_2, h_2 + h_3)$ and $(0, h_3), (h_3, h_2 + h_3)$ are in $I_{\vec{a}}$.

The next lemma is a generalization of the fact that $I_{\vec{a}}$ is a tiling of rhombus when $d = 3$ proved in [Fra96] and [ABI02]. Indeed, each rhombus is the projection under π of one of three types of square in $\mathbb{R}^3$. Below, the projection under π of the convex hull of the 2^k vertices of a k-dimensional hypercube graph in $\mathbb{E}_d$ is called a *k-dimensional parallelotope*. The edges of such a parallelotope have equal length. When $d = 3$, a $(d-1)$-dimensional parallelotope is a rhombus.

Proposition 9. *The graph $I_{\vec{a}}$ produces a tiling of $\mathcal{D}$ by d types of $(d-1)$-dimensional parallelotopes.*

In the following proof the fractional part of a real number $x \in \mathbb{R}$ is denoted by $\{x\} = x - \lfloor x \rfloor$.

Proof. Each real point $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ of the hyperplane $\mathcal{D}$ is contained in a $(d-1)$-simplex with vertices $\{\pi(u) + \sum_{i=1}^{k} h_{\sigma(i)} : 0 \leq k \leq d-1\}$ for $u = (\lfloor x_1 \rfloor, \cdots, \lfloor x_d \rfloor) \in \mathbb{Z}^d$ and permutation σ of $\{1, 2, \cdots, d\}$ such that $\{x_{\sigma(1)}\} \geq \{x_{\sigma(2)}\} \geq \cdots \geq \{x_{\sigma(d)}\}$. We illustrate this on an example. Suppose $d = 4$ and x is such that σ is the identity permutation on $\{1, 2, 3, 4\}$. We have

$$x = \left\{ \begin{array}{l} u \\ +\{x_1\}e_1 \\ +\{x_2\}e_2 \\ +\{x_3\}e_3 \\ +\{x_4\}e_4 \end{array} \right. = \left\{ \begin{array}{l} u \\ +(\{x_1\} - \{x_2\})\, e_1 \\ +(\{x_2\} - \{x_3\})\,(e_1 + e_2) \\ +(\{x_3\} - \{x_4\})\,(e_1 + e_2 + e_3) \\ +\{x_4\}(e_1 + e_2 + e_3 + e_4) \end{array} \right. = \left\{ \begin{array}{l} (1 - \{x_1\})\, u \\ +(\{x_1\} - \{x_2\})\,(u + e_1) \\ +(\{x_2\} - \{x_3\})\,(u + e_1 + e_2) \\ +(\{x_3\} - \{x_4\})\,(u + e_1 + e_2 + e_3) \\ +\{x_4\}(u + e_1 + e_2 + e_3 + e_4). \end{array} \right.$$

Therefore x is in the convex hull of the 3-simplex with vertices $\{\pi(u), \pi(u) + h_1, \pi(u) + h_1 + h_2, \pi(u) + h_1 + h_2 + h_3\}$ since

$$x = \pi(x) = \left\{ \begin{array}{l} (1 + \{x_4\} - \{x_1\})\pi(u) \\ +(\{x_1\} - \{x_2\})(\pi(u) + h_1) \\ +(\{x_2\} - \{x_3\})(\pi(u) + h_1 + h_2) \\ +(\{x_3\} - \{x_4\})(\pi(u) + h_1 + h_2 + h_3). \end{array} \right.$$

Consider the σ-path in $\mathbb{E}_d$ starting at vertex u and ending at $u+\sum_{i=1}^d e_i$. From Lemma 5, there is an edge (u', v') of the σ-path that is not in $\mathcal{H}_{\vec{a}}$. From Lemma 6, the hypercube graph from vertex v' to vertex $u'+\sum_{i=1}^d e_i$ with 2^{d-1} vertices is a subgraph of $\mathcal{H}_{\vec{a}}$. Therefore, the hypercube graph (projected in $\pi(\mathbb{E}_d)$) going from vertex $\pi(v')$ to vertex $\pi(u')$ with 2^{d-1} vertices is a subgraph of $I_{\vec{a}}$. The convex hull of this graph contains the $(d-1)$-simplex with vertices $\{\pi(u) + \sum_{i=1}^k h_{\sigma(i)} : 0 \leq k \leq d-1\}$ which in turns contains x. Therefore the point x of the hyperplane $\mathcal{D}$ is contained in the image under π of the convex hull of a $(d-1)$-dimensional hypercube graph in $\mathbb{E}_d$. For almost every x, the inequalities $\{x_{\sigma(1)}\} > \{x_{\sigma(2)}\} > \cdots > \{x_{\sigma(d)}\}$ are strict and the parallelotope is unique. We conclude that $\mathcal{D}$ is tiled by $(d-1)$-dimensional parallelotopes. $\square$

3.3 Kernel of $\mathcal{F}_{\vec{a}}$ and $\mathcal{F}'_{\vec{a}}$

Lemma 10. *The discrete hyperplane graph $\mathcal{H}_{\vec{a}}$ is invariant under any translation $t \in \operatorname{Ker} \mathcal{F}_{\vec{a}}$.*

Proof. Let $u \in \mathbb{Z}^d$ and $t \in \operatorname{Ker} \mathcal{F}_{\vec{a}}$. We have $\mathcal{F}_{\vec{a}}(u+t) = \mathcal{F}_{\vec{a}}(u) + \mathcal{F}_{\vec{a}}(t) = \mathcal{F}_{\vec{a}}(u)$. From Lemma 4, $(u, u+e_i) \in \mathcal{H}_{\vec{a}}$ if and only if $\mathcal{F}_{\vec{a}}(u) \in [0, s-a_i-1]$ if and only if $\mathcal{F}_{\vec{a}}(u+t) \in [0, s-a_i-1]$ if and only if $(u+t, u+e_i+t) \in \mathcal{H}_{\vec{a}}$. $\square$

We can find generators of the kernel of $\mathcal{F}_{\vec{a}}$ when $d = 3$.

Proposition 11. *If $d = 3$, the kernel of $\mathcal{F}_{\vec{a}}$ is*

$$\operatorname{Ker} \mathcal{F}_{\vec{a}} = \langle (a_3, 0, -a_1), (0, a_3, -a_2), (a_2, -a_1, 0), (1, 1, 1) \rangle.$$

The result is based on the following well-known lemma.

Lemma 12. *Let K be a subgroup of $\mathbb{Z}^n$ generated by the rows of a $s \times n$ matrix $M \in \mathbb{Z}^{s\times n}$ of rank n. The index $[\mathbb{Z}^n : K]$ is equal to the* gcd *of the n-minors of the matrix M.*

Proof. By the rank condition, we have $s \geq n$. Suppose first that M is in diagonal form; that is, the diagonal elements of M are $d_1, \ldots, d_n$ and that the other elements are 0; by the rank condition, the d_i are all nonzero. Then the subgroup is $K = d_1\mathbb{Z} \times \cdots \times d_n\mathbb{Z}$, the quotient group is $\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$, and therefore the index is $d_1 \cdots d_n$. Moreover the only nonzero n-minor is $d_1 \cdots d_n$.

In the general case, it is well-known that the matrix M may be brought into diagonal form by row and column operations within $\mathbb{Z}^{s\times n}$; moreover, these operations do not change the subgroup, up to change of basis in $\mathbb{Z}^n$; and finally, the gcd of the n-minors is invariant under these operations. Thus the general case follows from the diagonal case. $\square$

Proof. (of the proposition) The $\supseteq$ part. The kernel of $F = \mathcal{F}_{\vec{a}}$ contains the four vectors, because

$$\begin{aligned}
F(a_3, 0, -a_1) &= ac + b0 + a_3(-a_1) = ac - ca = 0,\\
F(0, a_3, -a_2) &= a0 + bc + a_3(-a_2) = bc - cb = 0,\\
F(a_2, -a_1, 0) &= ab + a_2(-a_1) + c0 = ab - ba = 0.
\end{aligned}$$

and

$$F(1, 1, 1) = m(a_1 + a_2 + a_3) = 0.$$

The $\subseteq$ part. Let $K = \langle (a_3, 0, -a_1), (0, a_3, -a_2), (a_2, -a_1, 0), (1, 1, 1) \rangle$. K is a subgroup of $\mathbb{Z}^3$. By showing that the index $[\mathbb{Z}^3 : K]$ is exactly the size $a_1 + a_2 + a_3$ of the image of F, we conclude that $K = \operatorname{Ker} \mathcal{F}_{\vec{a}}$. The subgroup K is generated by the lines of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \\ a_2 & -a_1 & 0 \end{pmatrix}.$$

From Lemma 12, the index is equal to the gcd of the four 3-minors of the matrix M:

$$\det \begin{pmatrix} 1 & 1 & 1 \\ a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \end{pmatrix} = a_3(a_1 + a_2 + a_3), \qquad \det \begin{pmatrix} 1 & 1 & 1 \\ a_3 & 0 & -a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} = -a_1(a_1 + a_2 + a_3)$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & a_3 & -a_2 \\ a_2 & -a_1 & 0 \end{pmatrix} = -a_2(a_1 + a_2 + a_3), \quad \det \begin{pmatrix} a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \\ a_2 & -a_1 & 0 \end{pmatrix} = 0.$$

That is the index is

$$[\mathbb{Z}^3 : K] = \gcd(a_3(a_1 + a_2 + a_3), -a_1(a_1 + a_2 + a_3), -a_2(a_1 + a_2 + a_3), 0) = a_1 + a_2 + a_3. \qquad \square$$

Corollary 13. *The kernel of $\mathcal{F}'_{\vec{a}}$ is spanned by the vectors $a_3h_1 - a_1h_3, a_3h_2 - a_2h_3, a_2h_1 - a_1h_2$.*

Proof. This is because $\pi(\operatorname{Ker} \mathcal{F}_{\vec{a}}) = \operatorname{Ker} \mathcal{F}'_{\vec{a}}$. Indeed, $\mathcal{F}_{\vec{a}} = \mathcal{F}'_{\vec{a}} \circ \pi$ and $\pi(1, 1, 1) = 0$. $\square$

It is likely that the proposition and its corollary have an evident extension to any dimension. The proof requires a higher minor calculation. We leave this to the interested reader.

3.4 The graph $\mathcal{G}_{\vec{a}}$

Let $d \geq 2$ be an integer and $\vec{a}$ as before. The graph $\mathcal{G}_{\vec{a}}$ of normal vector $\vec{a} \in \mathbb{Z}^d$ is the directed graph $\mathcal{G}_{\vec{a}} = \mathcal{F}_{\vec{a}}(\mathcal{H}_{\vec{a}})$. It is also equal to

$$\mathcal{G}_{\vec{a}} = \{(k, k + a_i) \mid k \in \mathbb{Z}/s\mathbb{Z}, 1 \leq i \leq d \text{ and } k < k + a_i\}.$$

Two examples are shown at Figure 4. Since the graph $\mathcal{G}_{\vec{a}}$ is isomorphic to the quotients $\mathcal{H}_{\vec{a}}/\operatorname{Ker} \mathcal{F}_{\vec{a}}$

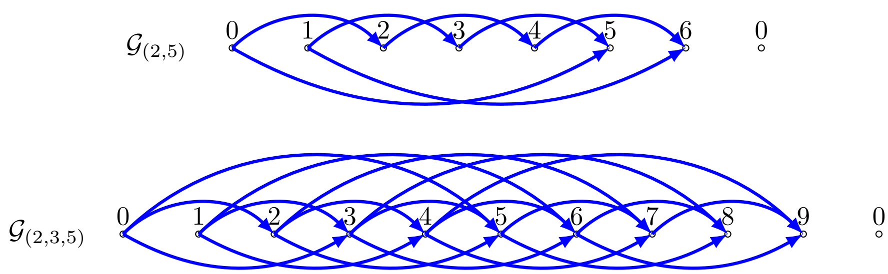


Figure 4: The Christoffel graphs $\mathcal{G}_{\vec{a}}$ for $\vec{a} = (2, 5)$ and $\vec{a} = (2, 3, 5)$.

(and also $I_{\vec{a}}/\operatorname{Ker} \mathcal{F}'_{\vec{a}}$), we call it *Christoffel graph* as well, because $\mathcal{G}_{\vec{a}}$ is the part of $\mathcal{H}_{\vec{a}}$ in its

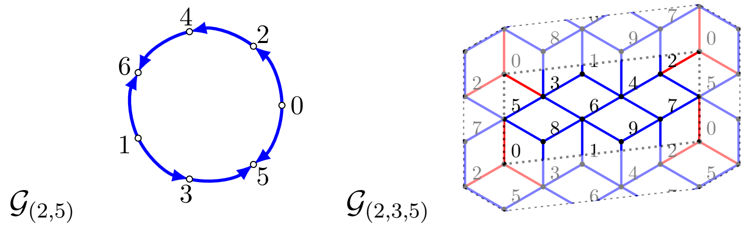


Figure 5: The Christoffel graphs $\mathcal{G}_{\vec{a}}$ for $\vec{a} = (2,5)$ and $\vec{a} = (2,3,5)$ can be embedded in the torus $\mathcal{D}/\mathrm{Ker}\,\mathcal{F}'_{\vec{a}}$.

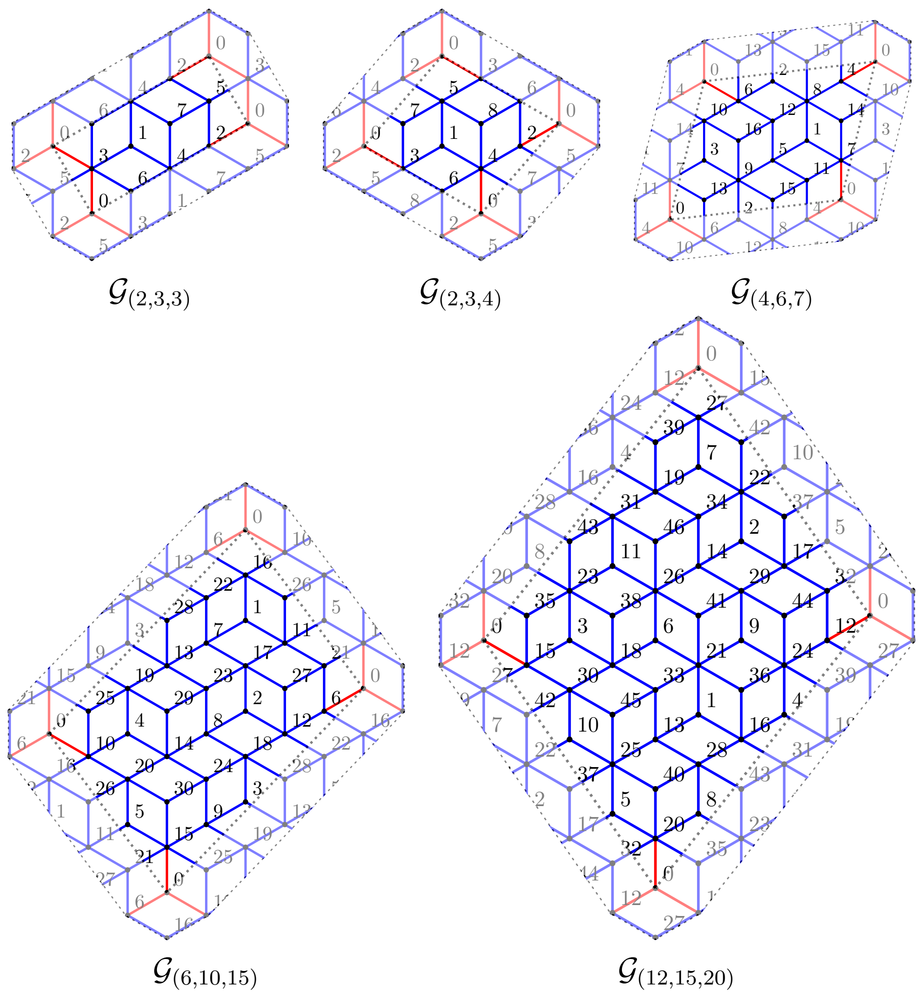


Figure 6: Some Christoffel graphs in dimension $d = 3$. The body is blue, legs are in red.

fundamental domain. The graph $\mathcal{G}_{\vec{a}}$ can be embedded in a torus. Indeed, the graph $I_{\vec{a}}$ lives in the diagonal plane $\mathcal{D} \simeq \mathbb{R}^{d-1}$ and is invariant under the group $\mathrm{Ker}\,\mathcal{F}'_{\vec{a}}$. The quotient $\mathcal{D}/\mathrm{Ker}\,\mathcal{F}'_{\vec{a}}$ is a torus and contains the graph $I_{\vec{a}}/\mathrm{Ker}\,\mathcal{F}'_{\vec{a}} \simeq \mathcal{G}_{\vec{a}}$ (see Figure 5).

The vertices of the Christoffel graph $\mathcal{G}_{\vec{a}}$ and their image under the function $\mathcal{F}_{\vec{a}}$ corresponds to what is called *roundwalk* in [BT04]. Their contribution allows to construct larger and larger domain of roundwalks by iteration of extension rules.

The Christoffel graph has a natural representation inside $I_{\vec{a}}$. We define this for $d = 3$, leaving the generalizations for elsewhere. Recall that the lattice L, defined in Section 3, is a free abelian group of rank 2, spanned by the 3 vectors h_1, h_2, h_3 with $h_1 + h_2 + h_3 = 0$. Moreover, the homomorphism $\mathcal{F}'_{\vec{a}} : L \to \mathbb{Z}/s\mathbb{Z}$ maps h_i onto a_i, with $s = a_1 + a_2 + a_3$, and the a_i are relatively prime; therefore, the mapping is surjective.

Choose some parallelogram in the plane $\mathcal{D}$ which is a fundamental domain for its discrete subgroup $\operatorname{Ker}\mathcal{F}'_{\vec{a}}$. We may assume that O is a vertex of this parallelogram. Then $\mathcal{F}'_{\vec{a}}$ induces a bijection between $\mathbb{Z}/s\mathbb{Z}$ and the integral points inside the parallelogram, excluding those lying on the two edges not containing O. It is such a parallelogram, with the part of the edges of $I_{\vec{a}}$ which lie inside him, that we may call a *Christoffel parallelogram*. This we may consider as the generalization in dimension 3 of Christoffel words. Such a parallelogram tiles the plane $\mathcal{D}$ and completely codes the graph $I_{\vec{a}}$. Furthermore it is in bijection with the Christoffel graph, as is easily verified. Examples are seen in Figure 6.

Remark 14. *The graphs $\mathcal{H}_{\vec{a}}, I_{\vec{a}}, \mathcal{G}_{\vec{a}}$ are compatible, in the sense that $I_{\vec{a}}$ is the image under π of $\mathcal{H}_{\vec{a}}$, $\mathcal{G}_{\vec{a}}$ is the image under $F_{\vec{a}}$ of $\mathcal{H}_{\vec{a}}$ and also the image of $I_{\vec{a}}$ under $F'_{\vec{a}}$.*

$$\begin{tikzcd}
\mathcal{H}_{\vec{a}} \arrow[rr, "\mathcal{F}_{\vec{a}}"] \arrow[dr, "\pi"'] & & \mathcal{G}_{\vec{a}} \\
& I_{\vec{a}} \arrow[ur, "\mathcal{F}'_{\vec{a}}"'] &
\end{tikzcd}$$

3.5 The graph $\mathcal{H}_{\vec{a},\omega}$

In this section, we extend the definition of Christoffel graphs to discrete plane such that the width ω is smaller than $s = \|\vec{a}\|_1 = \sum a_i$ where $\vec{a} \in \mathbb{N}^d$ is a vector of relatively prime positive integers as before. We consider only width ω such that s/ω is a positive integer strictly smaller than the dimension d: $0 < s/\omega < d$. We define the mapping $\mathcal{F}_{\vec{a},\omega} : \mathbb{Z}^d \to \mathbb{Z}/\omega\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$ onto $\sum_i a_i x_i \bmod \omega$. We identify $\mathbb{Z}/\omega\mathbb{Z}$ and $\{0, 1, \cdots, \omega - 1\}$. A total order on $\mathbb{Z}/\omega\mathbb{Z}$ is defined correspondingly. The *Christoffel graph of normal vector $\vec{a} \in \mathbb{N}^d$ of width ω* is the subset of edges $\mathcal{H}_{\vec{a},\omega} \subseteq \mathbb{E}_d$ defined by

$$\mathcal{H}_{\vec{a},\omega} = \{(u, v) \in \mathbb{E}_d \mid \mathcal{F}_{\vec{a},\omega}(u) < \mathcal{F}_{\vec{a},\omega}(v)\}.$$

This graph is related but does not correspond exactly to discrete plane of width ω. In fact, $\mathcal{H}_{\vec{a},\omega}$ can be obtained by the superposition of s/ω discrete plane of width ω. The definition of $\mathcal{H}_{\vec{a},\omega}$ is motivated by Pirillo's theorem, because this is what allows to generalize Pirillo's theorem in arbitrary dimension (see Theorem 36). Of course if $\omega = s$, then $\mathcal{H}_{\vec{a},\omega} = \mathcal{H}_{\vec{a}}$ is the Christoffel graph of normal vector $\vec{a}$. Also, if $d = 2$ then $s = \omega$. If $d = 3$, then either $\omega = s$ or $\omega = s/2$. If $d = 4$, then either $\omega = s$, $\omega = s/2$ or $\omega = s/3$ and so on for $d \geq 5$. If s is a prime number, then $\omega = s$.

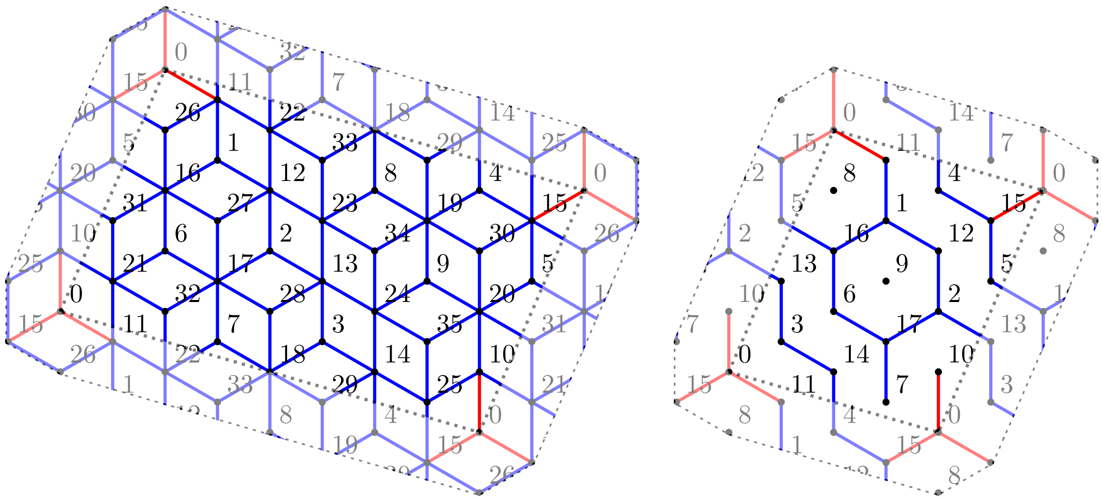
As earlier, we define the projected graphs $\mathcal{G}_{\vec{a},\omega} := \mathcal{F}_{\vec{a},\omega}(\mathcal{H}_{\vec{a},\omega})$ and $I_{\vec{a},\omega} := \pi(\mathcal{H}_{\vec{a},\omega})$. The Christoffel graph $\mathcal{G}_{\vec{a},\omega}$ for the vector $\vec{a} = (15, 11, 10)$ of width $\omega = s = 36$ is shown at Figure 7 (left). The Christoffel graph $\mathcal{G}_{\vec{a},\omega}$ for the vector $\vec{a} = (15, 11, 10)$ of width $\omega = 18 = s/2$ is shown at Figure 7 (right) and a larger part is shown at Figure 8.

The next lemma gives an equivalent definition of the edges of the graph $\mathcal{H}_{\vec{a},\omega}$.

Lemma 15. *Let $(u, v) \in \mathbb{E}_d$ such that $v - u = e_i$ for some $1 \leq i \leq d$. Then,*

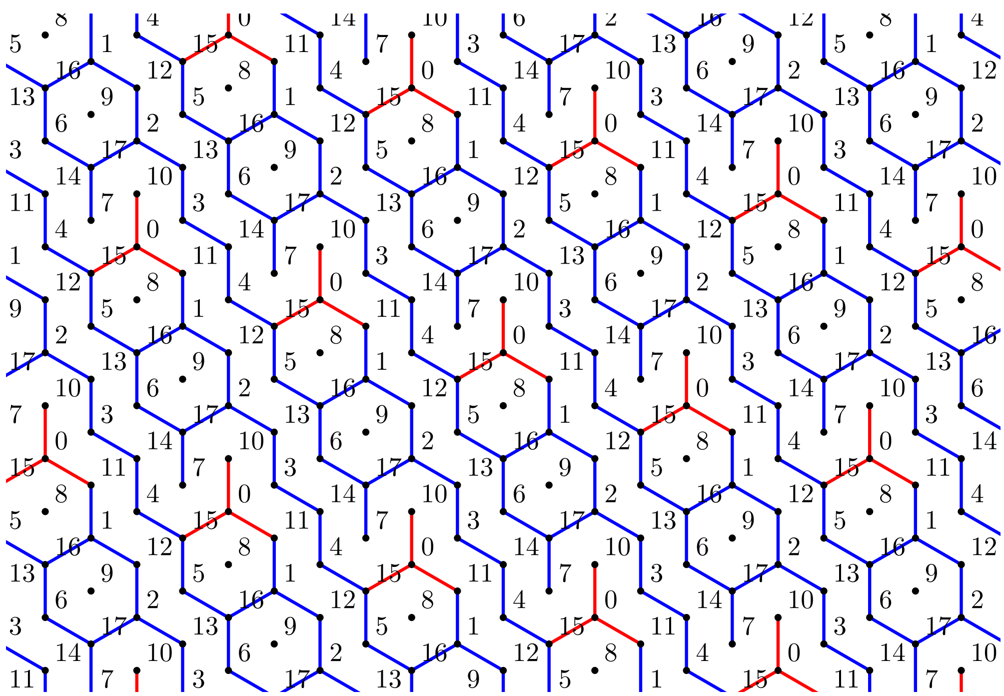
$$(u, v) \in \mathcal{H}_{\vec{a},\omega} \iff \mathcal{F}_{\vec{a},\omega}(u) \in [0, \omega - a_i - 1] \iff \mathcal{F}_{\vec{a},\omega}(v) \in [a_i, \omega - 1], \tag{4}$$

$$(u, v) \notin \mathcal{H}_{\vec{a},\omega} \iff \mathcal{F}_{\vec{a},\omega}(u) \in [\omega - a_i, \omega - 1] \iff \mathcal{F}_{\vec{a},\omega}(v) \in [0, a_i - 1]. \tag{5}$$



$\mathcal{G}_{(15,11,10)} = \mathcal{G}_{(15,11,10),36}$ $\qquad$ $\mathcal{G}_{(15,11,10),18}$

Figure 7: Left: the Christoffel graph $\mathcal{G}_{\vec{a}}$ for the vector $\vec{a} = (15, 11, 10)$. Right: the Christoffel graph $\mathcal{G}_{\vec{a},\omega}$ of width $\omega = 18$ for the vector $\vec{a} = (15, 11, 10)$.



$I_{(15,11,10),18}$

Figure 8: The Christoffel graph $I_{\vec{a},\omega}$ of width $\omega = 18$ for the vector $\vec{a} = (15, 11, 10)$. It corresponds to the union of two discrete planes of width ω.

4 Flip, reversal and translation

In this short section, we define the flip, reversal and translate of set of edges. We define the operations for set of edges $X \subseteq \mathbb{E}_d$ but they extend naturally to set of edges of the form $\pi(X)$ and $\mathcal{F}_{\vec{a}}(X)$ (see Definition 20 below). In order to define the flip operation, we need to define the edges incident to zero.

Definition 16 (edges of $\mathbb{E}_d$ incident to zero). *Let $d \geq 2$ be an integer and $\vec{a} \in \mathbb{Z}^d$ be a vector of relatively prime positive integers. The set of* edges of $\mathbb{E}_d$ incident to zero *is*

$$\mathcal{Q} = \{(u, v) \in \mathbb{E}_d : \mathcal{F}_{\vec{a}}(u) = 0 \text{ or } \mathcal{F}_{\vec{a}}(v) = 0\}.$$

Definition 17 (body, legs). *Let $X \subseteq \mathbb{E}_d$. The set $X \setminus \mathcal{Q}$ is the* body *and the edges of $X \cap \mathcal{Q}$ are the* legs *of X.*

See Figure 6 where the legs of graphs $\mathcal{G}_{\vec{a}}$ are represented in red, and the body in black. The FLIP is an operation which generalizes the function $amb \mapsto bma$ defined for Christoffel words. While we define the flip on graphs, it can also be seen as a flip in a rhombus tiling when $d = 3$ [BFRR11, BFR08, ABFJ07].

Definition 18 (FLIP). *For a subset of edges $X \subseteq \mathbb{E}_d$, we define the* FLIP *operation which exchanges edges incident to zero:*

$$\textsc{flip} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

We see that $\textsc{flip}(X)$ exchanges the legs of X and keeps the body of X invariant. If $(u, v) \in \mathbb{E}_d$, then the reversal edge $(-v, -u) \in \mathbb{E}_d$ is also an edge of the hypercubic lattice and similarly for the translated edge $(u + t, v + t) \in \mathbb{E}_d$ for all $t \in \mathbb{Z}^d$. The reversal and translate operations extend on subsets of edges as follows:

Definition 19 (Reversal, Translate). *Let $X \subseteq \mathbb{E}_d$ be a subset of edges. We define the* reversal $-X$ *of X and the* translate $X + t$, *for some $t \in \mathbb{Z}^d$, of X as*

$$-X = \{(-v, -u) \mid (u, v) \in X\} \qquad \text{and} \qquad X + t = \{(u + t, v + t) \mid (u, v) \in X\}.$$

Definition 20 (FLIP, Reversal, Translate). *Let $X \subseteq \mathbb{E}_d$. The flip, reversal and translate of set of edges of the form $\pi(X)$ and $\mathcal{F}_{\vec{a}}(X)$ are defined naturally by commutativity:*

$$\begin{array}{lll} \textsc{flip}(\pi(X)) := \pi(\textsc{flip}(X)), & -(\pi(X)) := \pi(-X), & \pi(X) + \pi(t) := \pi(X + t), \\ \textsc{flip}(\mathcal{F}_{\vec{a}}(X)) := \mathcal{F}_{\vec{a}}(\textsc{flip}(X)), & -(\mathcal{F}_{\vec{a}}(X)) := \mathcal{F}_{\vec{a}}(-X), & \mathcal{F}_{\vec{a}}(X) + \mathcal{F}_{\vec{a}}(t) := \mathcal{F}_{\vec{a}}(X + t). \end{array}$$

Therefore, statements proven for $\mathcal{H}_{\vec{a}}$ using flip, reversal and translate operations are also true for $I_{\vec{a}}$ and $\mathcal{G}_{\vec{a}}$. For example, the goal of next section is to show that $\mathcal{H}_{\vec{a}} + t = \textsc{flip}(\mathcal{H}_{\vec{a}})$ for some $t \in \mathbb{Z}^d$. If such an equation is satisfied for $\mathcal{H}_{\vec{a}}$, it is clear from Definition 20 that $I_{\vec{a}} + \pi(t) = \textsc{flip}(I_{\vec{a}})$ and $\mathcal{G}_{\vec{a}} + \mathcal{F}_{\vec{a}}(t) = \textsc{flip}(\mathcal{G}_{\vec{a}})$.

5 Flipping is translating

In this section, we show that the flip of the Christoffel graph $\mathcal{H}_{\vec{a}}$ is a translate of $\mathcal{H}_{\vec{a}}$; this is a generalization of one implication of Theorem 1. We also show that the body of $\mathcal{H}_{\vec{a}}$ is symmetric and as a consequence we obtain that a Christoffel graph is a translate of its reversal. The results stated in this section are stated and proved for $\mathcal{H}_{\vec{a}}$ but they are valid for $I_{\vec{a}}$ and $\mathcal{G}_{\vec{a}}$ by Definition 20.

The following lemma describes the legs of $\mathcal{H}_{\vec{a}}$.

Lemma 21 (Legs of $\mathcal{H}_{\vec{a}}$). *An edge (u, v) is a leg of $\mathcal{H}_{\vec{a}}$ if and only if $\mathcal{F}_{\vec{a}}(u) = 0$.*

Proof. We have $(-e_i, 0) \notin \mathcal{H}_{\vec{a}}$ because there is no $u \in \mathbb{Z}^d$ such that $\mathcal{F}_{\vec{a}}(u) < \mathcal{F}_{\vec{a}}(0) = 0$. Moreover $(0, e_i) \in E$ for each i, $1 \leq i \leq d$, because $\mathcal{F}_{\vec{a}}(0) = 0 < a_i = \mathcal{F}_{\vec{a}}(e_i)$. $\square$

We now show that the body of a Christoffel graph is *symmetric*, i.e., it is equal to its reversal. This generalizes the fact that central words are palindromes.

Lemma 22. *The body of $\mathcal{H}_{\vec{a}}$ is symmetric, i.e., $-(\mathcal{H}_{\vec{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$.*

Proof. It is sufficient to prove $-(\mathcal{H}_{\vec{a}} \setminus \mathcal{Q}) \supseteq \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$, the other inclusion being equivalent, since symmetry is involutive. Let $(u, v) \in \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$. Then $v - u = e_i$ for some $1 \leq i \leq d$. Then $\mathcal{F}_{\vec{a}}(u) \in [0, s - a_i - 1]$ by Lemma 4 and $\mathcal{F}_{\vec{a}}(u) \notin \{0, s - a_i\}$ so that $\mathcal{F}_{\vec{a}}(u) \in [1, s - a_i - 1]$. Thus $\mathcal{F}_{\vec{a}}(-u) = s - \mathcal{F}_{\vec{a}}(u) \in [a_i + 1, s - 1]$. We obtain that $(-v, -u) \in \mathcal{H}_{\vec{a}}$ by Lemma 4 because $-u - (-v) = v - u = e_i$. Since $\mathcal{Q} = -\mathcal{Q}$, $(u, v) \notin \mathcal{Q}$ implies that $(-v, -u) \notin \mathcal{Q}$. We conclude $(-v, -u) \in \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$ and $(u, v) \in -(\mathcal{H}_{\vec{a}} \setminus \mathcal{Q})$. $\square$

Now we show that the reversal is equal to the flip of a Christoffel graph. This generalizes the fact that the reversal $\widetilde{amb}$ of a Christoffel word is equal to bma.

Lemma 23. *The reversal of $\mathcal{H}_{\vec{a}}$ is equal to its flip, i.e., $-\mathcal{H}_{\vec{a}} = \textsc{flip}(\mathcal{H}_{\vec{a}})$.*

Proof. For $\mathcal{H}_{\vec{a}}$, we have to show that $-\mathcal{H}_{\vec{a}} = (\mathcal{H}_{\vec{a}} \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus \mathcal{H}_{\vec{a}})$. We prove the result in two parts since $-\mathcal{H}_{\vec{a}} = ((-\mathcal{H}_{\vec{a}}) \setminus \mathcal{Q}) \cup ((-\mathcal{H}_{\vec{a}}) \cap \mathcal{Q})$ is the disjoint union of a part outside of $\mathcal{Q}$ and a part inside of $\mathcal{Q}$. Outside of $\mathcal{Q}$: since $\mathcal{Q}$ is symmetric and because $\mathcal{H}_{\vec{a}}$ is symmetric from Lemma 22, we have $-(\mathcal{H}_{\vec{a}}) \setminus \mathcal{Q} = -(\mathcal{H}_{\vec{a}}) \setminus -\mathcal{Q} = -(\mathcal{H}_{\vec{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$. Inside of $\mathcal{Q}$: we have $(-\mathcal{H}_{\vec{a}}) \cap \mathcal{Q} = -(\mathcal{H}_{\vec{a}} \cap \mathcal{Q}) = \{(u, v) \in \mathbb{E}_d : \mathcal{F}_{\vec{a}}(v) = 0\} = \mathcal{Q} \setminus \mathcal{H}_{\vec{a}}$ by Lemma 21. $\square$

Now we show that the flip of a Christoffel graph $\mathcal{H}_{\vec{a}}$ is equal to a translate of $\mathcal{H}_{\vec{a}}$. It generalizes one implication of Theorem 1. It corresponds to the fact that a Christoffel word amb is conjugate to to bma. Proposition 24 is illustrated in Figure 9 and Figure 10.

Proposition 24. *Let $t \in \mathbb{Z}^d$ be such that $\mathcal{F}_{\vec{a}}(t) = 1$. The translate by t of $\mathcal{H}_{\vec{a}}$ is equal to its flip, i.e., $\mathcal{H}_{\vec{a}} + t = \textsc{flip}(\mathcal{H}_{\vec{a}})$.*

Note that, since $\mathcal{F}_{\vec{a}}$ is surjective, there exists indeed $t \in \mathbb{Z}^d$ such that $\mathcal{F}_{\vec{a}}(t) = 1$. Also, $\mathcal{F}_{\vec{a}}(-t) = -\mathcal{F}_{\vec{a}}(t) = -1$.

Proof. We prove the result in two parts since $\mathcal{H}_{\vec{a}} + t$ is the disjoint union of a part outside of $\mathcal{Q}$ and a part inside of $\mathcal{Q}$:

$$\mathcal{H}_{\vec{a}} + t = ((\mathcal{H}_{\vec{a}} + t) \setminus \mathcal{Q}) \cup ((\mathcal{H}_{\vec{a}} + t) \cap \mathcal{Q}).$$

1. $(\mathcal{H}_{\vec{a}} + t) \setminus \mathcal{Q} \supseteq \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$. Suppose that $(u, v) \in \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$. Thus $v - u = e_i$ for some $1 \leq i \leq d$. Then, $\mathcal{F}_{\vec{a}}(u) \in [0, s - a_i - 1]$ by Lemma 4 and, since the edge is not a leg, $\mathcal{F}_{\vec{a}}(u) \notin \{0, s - a_i\}$. Hence, $\mathcal{F}_{\vec{a}}(u) \in [1, s - a_i - 1]$. Then $\mathcal{F}_{\vec{a}}(u - t) = \mathcal{F}_{\vec{a}}(u) - 1 \in [0, s - a_i - 2]$ which implies that $(u - t, v - t) \in \mathcal{H}_{\vec{a}}$. Then $(u, v) \in \mathcal{H}_{\vec{a}} + t$.

2. $(\mathcal{H}_{\vec{a}} + t) \setminus \mathcal{Q} \subseteq \mathcal{H}_{\vec{a}} \setminus \mathcal{Q}$. Let $(u + t, v + t) \in (\mathcal{H}_{\vec{a}} + t) \setminus \mathcal{Q}$ for some edge $(u, v) \in \mathcal{H}_{\vec{a}}$. Then, $\mathcal{F}_{\vec{a}}(u + t) \notin \{0, s - a_i\}$. From Lemma 4, $\mathcal{F}_{\vec{a}}(u + t) = \mathcal{F}_{\vec{a}}(u) + 1 \in [1, s - a_i]$. Therefore, $\mathcal{F}_{\vec{a}}(u + t) \in [1, s - a_i - 1]$ and we conclude that $(u + t, v + t) \in \mathcal{H}_{\vec{a}}$.

3. $(\mathcal{H}_{\vec{a}} + t) \cap \mathcal{Q} \supseteq \mathcal{Q} \setminus \mathcal{H}_{\vec{a}}$. Let $(u, v) \in \mathcal{Q} \setminus \mathcal{H}_{\vec{a}}$. Then, $\mathcal{F}_{\vec{a}}(v) = 0$ which implies that $\mathcal{F}_{\vec{a}}(v - t) = s - 1$. By Lemma 4, we have $(u - t, v - t) \in \mathcal{H}_{\vec{a}}$ so that $(u, v) \in \mathcal{H}_{\vec{a}} + t$.

4. $(\mathcal{H}_{\vec{a}} + t) \cap \mathcal{Q} \subseteq \mathcal{Q} \setminus \mathcal{H}_{\vec{a}}$. Let $(u + t, v + t) \in (\mathcal{H}_{\vec{a}} + t) \cap \mathcal{Q}$ for some edge $(u, v) \in \mathcal{H}_{\vec{a}}$. Either $\mathcal{F}_{\vec{a}}(u + t) = 0$ or $\mathcal{F}_{\vec{a}}(v + t) = 0$. If $\mathcal{F}_{\vec{a}}(u + t) = 0$, then $\mathcal{F}_{\vec{a}}(u) = s - 1$ which implies that $(u, v) \notin \mathcal{H}_{\vec{a}}$ by Lemma 4, a contradiction. One must have $\mathcal{F}_{\vec{a}}(v + t) = 0$, which implies that $(u + t, v + t) \notin \mathcal{H}_{\vec{a}}$. $\square$

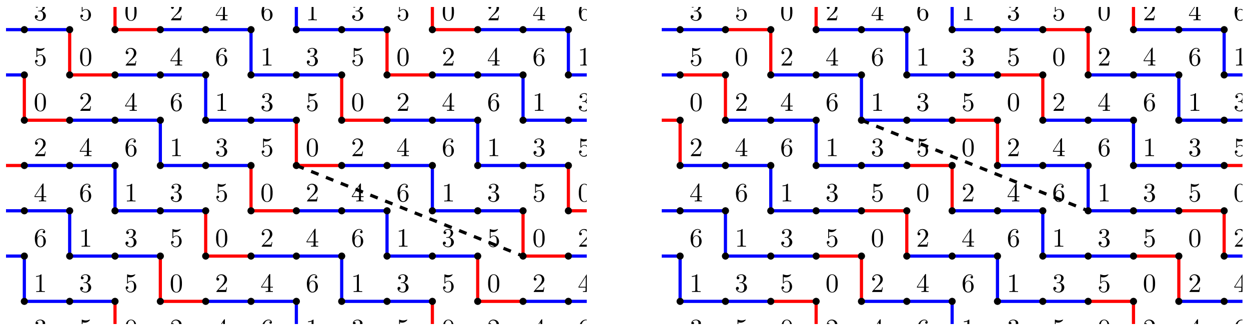


Figure 9: Left: the graph $\mathcal{H}_{\vec{a}}$ with $\vec{a} = (2, 5)$. Right: FLIP$(\mathcal{H}_{\vec{a}})$.

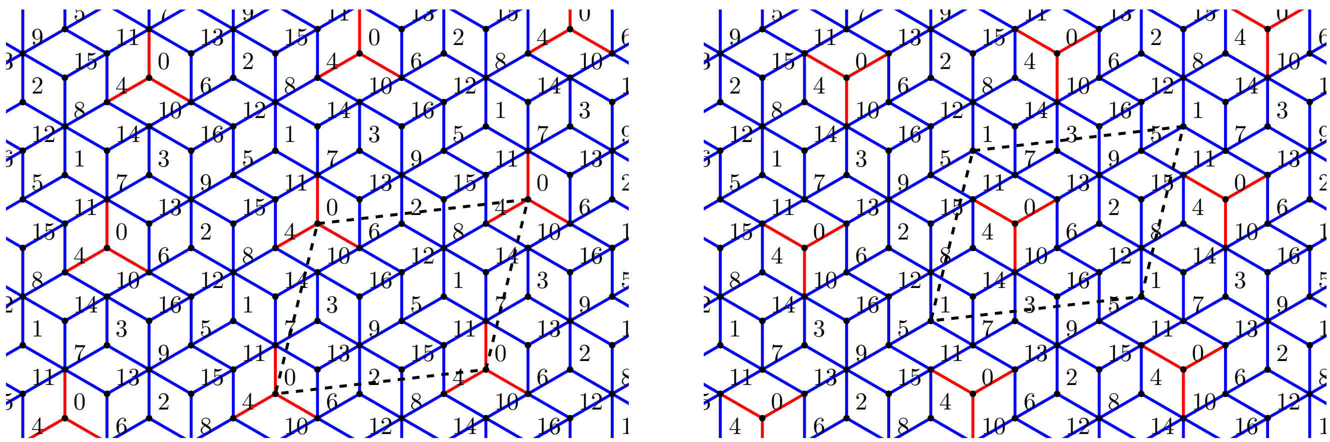


Figure 10: Left: the graph $I_{\vec{a}}$ with $\vec{a} = (4, 6, 7)$. Right: FLIP$(I_{\vec{a}})$. Consider the Christoffel parallelogram P with vertices labeled by 0 embedded in $I_{\vec{a}}$. The parallelogram P also appears in the graph FLIP$(I_{\vec{a}})$ with vertices labeled by 1.

The previous proposition proves that the body of a Christoffel graph $\mathcal{H}_{\vec{a}}$ has a period. This generalizes the fact that central words of length $p + q - 2$ have periods p and q (remark that p and $-q$ is the same period mod the length of the Christoffel words $|w| = p + q$). Indeed, let P be some parallelogram and M be an inner point. Consider the 4 vectors with origin equal to one of the vertices of P and with end M. Then, for each point X in P, there is one of these vector, $\vec{v}$ say, such that the segment $[X, X + \vec{v}]$ is contained in P. We leave the verification of this to the reader. It follows that a Christoffel parallelogram may be reconstructed from the edges incident to zero by applying translations which stay completely in the parallelogram. This is completely analoguous to the fact that a central word is completely determined by its two periods.

The next result generalizes the fact that the reversal $\widetilde{w}$ of a Christoffel word w is conjugate to w. This is not a characteristic property of Christoffel words, because it is satisfied for all words that are the product of two palindromes.

Corollary 25. *Let $t \in \mathbb{Z}^d$ be such that $\mathcal{F}_{\vec{a}}(t) = 1$. Then $-\mathcal{H}_{\vec{a}} = \mathcal{H}_{\vec{a}} + t$.*

Proof. Follows from Lemma 23 and Proposition 24. □

Corollary 26. *(i) The body of a Christoffel parallelogram is symmetric with respect to its center.*

(ii) Consider a Christoffel parallelogram P embedded in $I_{\vec{a}}$. The parallelogram obtained by

symmetry of P with respect to its center appears as a translate of P within $I_{\vec{a}}$, and is also equal to the flip of P.

We thus have obtained a generalization of: (i) a central word is a palindrome; (ii) the reversal of a Christoffel word *amb* is conjugate to it, and equal to *bma*. The corollary can be checked on Figures 6, 9 and 10

6 Higher-dimensional Pirillo's theorem

In this section, we study the converse of Proposition 24. In other words, does the fact of being a translate of its flip is a characteristic property of Christoffel graphs as it is the case for Christoffel words? We show that it must a Christoffel graph $\mathcal{H}_{\vec{a},\omega}$ for some vector $\vec{a} \in \mathbb{Z}^d$ and width ω. If $d = 3$, we show that parallelograms that are translate to their flip are Christoffel parallelograms or their edge-complement.

Let K be a subgroup of $\mathbb{Z}^d$ for some integer $d \geq 2$ such that the index $[\mathbb{Z}^d : K]$ is finite and $\sum_{i=1}^{d} e_i = (1, 1, \ldots, 1) \in K$. Let $\mathcal{Q}$ be the set of *edges of $\mathbb{E}_d$ incident to zero mod K*:

$$\mathcal{Q} = \{(u, v) \in \mathbb{E}_d \mid u \in K \text{ or } v \in K\}.$$

For a subset of edges $X \subseteq \mathbb{E}_d$, we redefine the FLIP operation according to the above set $\mathcal{Q}$:

$$\textsc{flip} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

In what follows, we assume that $M \subseteq \mathbb{E}_d$ is a set of edges such that

- M is invariant for the group of translations K;
- $\textsc{flip}(M) = M + t$ for some $t \in \mathbb{Z}^d$.

If $\textsc{flip}(M) = M + t$, then for each i, $(0, e_i) \in M$ or $(-e_i, 0) \in M$ but not both. Otherwise the number of edges parallel to the vector e_i is not preserved by the FLIP and the equation can not be satisfied. Therefore, we suppose that for each i, $1 \leq i \leq d$, $(0, e_i) \in M$ and $(-e_i, 0) \notin M$. In other words, the legs of M are:

- $\mathcal{Q} \cap M = \{(u, u + e_i) \in \mathbb{E}_d \mid u \in K\}$.

The question we consider in this section is: for which set of edges $M \subseteq \mathbb{E}_d$ satisfying the above three conditions does there exist a translation $t \in \mathbb{Z}^d$ such that $\textsc{flip}(M) = M + t$ (see Figure 11).

Lemma 27. *Let $t \in \mathbb{Z}^d$ be a translation. Let $X \subseteq \mathbb{E}_d$ and $h \in \mathbb{E}_d$. We have*

(i) *If $h \in X$, then $h + t \notin \mathcal{Q}$ if and only if $h + t \in \textsc{flip}(X + t)$.*

(ii) *If $h \notin X$, then $h + t \in \mathcal{Q}$ if and only if $h + t \in \textsc{flip}(X + t)$.*

Proof. We have $\textsc{flip}(X + t) = ((X + t) \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus (X + t))$.

(i) If $h \in X$ then $h + t \in X + t$. If $h + t \notin \mathcal{Q}$, then $h + t \in (X + t) \setminus \mathcal{Q} \subseteq \textsc{flip}(X + t)$. If $h + t \in \mathcal{Q}$, then $h + t \in (X + t) \cap \mathcal{Q}$. Therefore $h + t \notin \textsc{flip}(X + t)$.

(ii) If $h \notin X$ then $h + t \notin X + t$. If $h + t \in \mathcal{Q}$, then $h + t \in \mathcal{Q} \setminus (X + t) \subseteq \textsc{flip}(X + t)$. If $h + t \notin \mathcal{Q}$, then $h + t \notin (X + t) \cup \mathcal{Q} \supseteq \textsc{flip}(X + t)$. Therefore $h + t \notin \textsc{flip}(X + t)$. $\square$

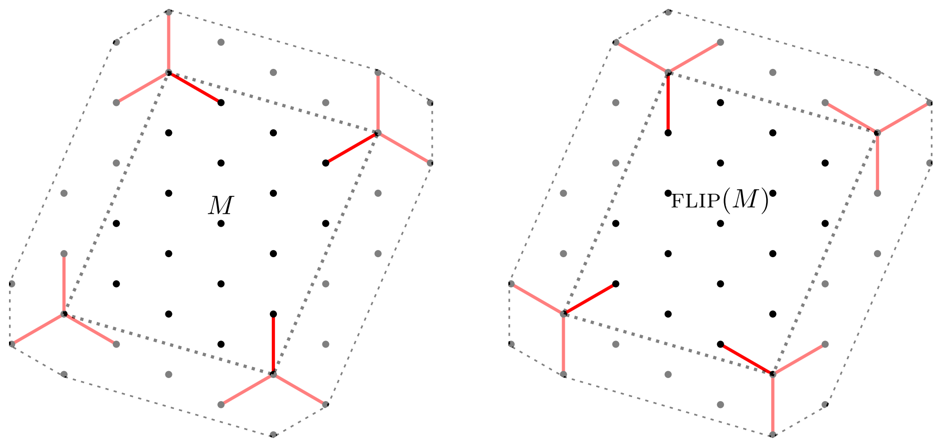


Figure 11: Left: M. Right: $\textsc{flip}(M)$ for the subgroup $K = \langle (0,4,1), (-2,0,3), (1,1,1) \rangle$.

Proposition 28. *For all i, with $1 \le i \le d$, there exists a unique integer b_i, $0 < b_i < \omega$, such that $e_i + b_i t \in K$ where ω is the order of t in the group $\mathbb{Z}^d/K$. Moreover,*

$$(0, e_i) + kt \begin{cases} \in M & \text{if} \quad 0 \le k < b_i, \\ \notin M & \text{if} \quad b_i \le k < \omega, \end{cases}$$

In the following proof, for two elements $u, u' \in \mathbb{Z}^d$ the notation $u \equiv u'$ is used when $u' - u \in K$. The notation is also used for two edges $(u, v), (u', v') \in \mathbb{E}_d$: $(u, v) \equiv (u', v')$ if and only if $u' - u = v' - v \in K$.

Proof. Let ω be the order of t in $\mathbb{Z}^d/K$. In this proof, we denote by $\vec{0}$ the zero of $\mathbb{Z}^d/K$. Thus let

$$\omega = \min\{k > 0 | kt \in K\} = \text{order}_{\mathbb{Z}^d/K}(t)$$

Consider the orbit under the translation t of the edge $h = (\vec{0}, e_i) \in \mathcal{Q} \cap M$. We have that $h + \omega t \equiv h \in \mathcal{Q} \cap M$. We want to show that there exists b_i, such that $0 < b_i < \omega$ and $h + b_i t \equiv (-e_i, \vec{0}) \in \mathcal{Q}$.

Suppose (by contradiction) that $h + kt \notin \mathcal{Q}$ for all $0 < k < \omega$. From Lemma 27 (i), $h \in M$ and $h + t \notin \mathcal{Q}$, then $h + t \in \textsc{flip}(M + t) = M$. Recursively, we have $h + kt \in \textsc{flip}(M + t) = M$ for all $0 < k < \omega$. This is summarized in the following graph:

$$\begin{array}{ccccccccccc} h & \xrightarrow{+t} & h+t & \xrightarrow{+t} & h+2t & \xrightarrow{+t} & \cdots & \xrightarrow{+t} & h+(\omega-1)t & \xrightarrow{+t} & h+\omega t \equiv h \\ & & & & & & & & & & \\ \in \mathcal{Q} \cap M & & \in M \setminus \mathcal{Q} & & \in M \setminus \mathcal{Q} & & & & \in M \setminus \mathcal{Q} & & \in \mathcal{Q} \end{array}$$

But then, $h + (\omega - 1)t \in M$ and $h + \omega t \in \mathcal{Q}$, so that $h + \omega t \notin \textsc{flip}(M + t) = M$ from Lemma 27 (i). This is a contradiction because $h + \omega t \equiv h \in M$. Hence, there must exist some b_i, $0 < b_i < \omega$ such that $h + b_i t \in \mathcal{Q}$. Since h is an edge parallel to the vector e_i, then either $h + b_i t \equiv (\vec{0}, e_i)$ or $h + b_i t \equiv (-e_i, \vec{0})$. The first option contradicts the minimality of ω. We conclude that $e_i + b_i t \equiv \vec{0}$.

The number b_i is also unique. Indeed, suppose there exist b_i and b'_i with $0 < b_i < b'_i < \omega$ such that $h + b_i t \equiv h + b'_i t \equiv (-e_i, \vec{0})$. Then $(b'_i - b_i)t = (\vec{0} + b'_i t) - (\vec{0} + b_i t) \equiv (-e_i) - (-e_i) = \vec{0}$. This contradicts the minimality of ω since $0 < b'_i - b_i < \omega$.

From the above paragraph, we have that $h + kt \notin \mathcal{Q}$ for all k such that $0 < k < b_i$ or $b_i < k < \omega$. Using Lemma 27 (i), if $h + (k-1)t \in M$ and $h + kt \notin \mathcal{Q}$, then $h + kt \in \textsc{flip}(M + t) = M$. Thus by recursion $h + kt \in M$ for all k with $0 < k < b_i$. Also $h + b_i t \equiv (-e_i, \vec{0}) \in \mathcal{Q} \setminus M$. Using Lemma 27 (ii), if $h + b_i t \notin M$ and $h + (b_i + 1)t \notin \mathcal{Q}$, then $h + (b_i + 1)t \notin \text{FLIP}(M + t) = M$. Thus by recursion $h + kt \notin M$ for all k with $b_i < k < \omega$. $\square$

Lemma 29. *$\mathbb{Z}^d/K$ is cyclic and generated by t.*

Proof. Let $u = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$. Using Proposition 28, we have

$$u = \sum x_i e_i \equiv \sum x_i(-b_i t) = -\sum(b_i x_i)t$$

Let $k = -\sum(b_i x_i) \mod \omega$. Then, $0 \le k < \omega$ and $u = kt$. $\square$

Lemma 30.

$$M = \{(0, e_i) + kt : 1 \le i \le d \text{ and } 0 \le k < b_i\} + K.$$

Proof. ($\supseteq$) If $0 \le k < b_i$, then $(0, e_i) + kt \in M$ by Proposition 28.

($\subseteq$) Let $(u, u + e_i) \in M$ with $u \in \mathbb{Z}^d$. From Lemma 29, $(u, u + e_i) = (0, e_i) + u \equiv (0, e_i) + kt$ for some integer k such that $0 \le k < \omega$. From Proposition 28, $0 \le k < b_i$. $\square$

For all i with $1 \le i \le d$, let a_i be such that $a_i + b_i = \omega$. Also let

$$\vec{b} = (b_1, b_2, \cdots, b_d) \quad \text{and} \quad \vec{a} = (a_1, a_2, \cdots, a_d)$$

We have $a_i t = (\omega - b_i)t = \omega t - b_i t \equiv e_i$. The next result shows that ω is a divisor of $\sum a_i$ and $\sum b_i$.

Lemma 31. *There exist integers q and ℓ, with $0 < q < d$ and $0 < \ell < d$, such that $\omega \cdot q = a_1 + a_2 + \cdots + a_d$ and $\omega \cdot \ell = b_1 + b_2 + \cdots + b_d$. Moreover $d = q + \ell$.*

Proof. For all $1 \le i \le d$, we have $e_i = a_i t = -b_i t$. Thus, $-(b_1+b_2+\cdots+b_d)t$ is an overall translation of $e_1 + e_2 + \cdots + e_d \in K$, i.e., the identity. Similarly, $(a_1 + a_2 + \cdots + a_d)t = e_1 + e_2 + \cdots + e_d \in K$. Therefore, the order of t $(= \omega)$ must divide both $a_1 + a_2 + \cdots + a_d$ and $b_1 + b_2 + \cdots + b_d$. Then, there exist integers q and ℓ such that $\omega \cdot q = a_1 + a_2 + \cdots + a_d$ and $\omega \cdot \ell = b_1 + b_2 + \cdots + b_d$. But $a_i < \omega$ for each i so that $a_1 + a_2 + \cdots + a_d < d\omega$ and $q < d$. Similarly, $\ell < d$. Finally, $\omega q + \omega \ell = \sum(a_i + b_i) = \omega d$ and therefore $d = q + \ell$. $\square$

If the sum of the a_i or the sum of the b_i is ω, then the next two theorems claim that M is closely related to the Christoffel graph.

Theorem 32. *(i) If $\sum a_i = \omega$, then $M = \mathcal{H}_{\vec{a}}$;*

(ii) if $\sum b_i = \omega$, then the complement $M^c = \mathbb{E}_d \setminus M$ of M is equal to $-H_{\vec{b}}$.

Proof. (i) For all $u = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$, we have $u = kt$ with

$$k = \sum(-b_i x_i) \bmod \omega = \sum(a_i - \omega)x_i \bmod \omega = \sum a_i x_i \bmod \sum a_i = \mathcal{F}_{\vec{a}}(u)$$

We want to show that $M = \mathcal{H}_{\vec{a}}$. We have that $(u, u+e_i) = (0, e_i)+kt \in M$ if and only if $0 \le k < b_i$ if and only if $\mathcal{F}_{\vec{a}}(u) \in [0, \omega - a_i - 1]$ if and only if $(u, u + e_i) \in \mathcal{H}_{\vec{a}}$.

(ii) For all $u = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$, we have $u \equiv kt$ with

$$k = \sum(-b_i x_i) \bmod \omega = -\sum b_i x_i \bmod \sum b_i = -\mathcal{F}_{\vec{b}}(u)$$

We want to show that $M^c = -\mathcal{H}_{\vec{b}}$. We have that $(u, v) = (u, u + e_i) = (0, e_i) + kt \in \mathbb{E}_d \setminus M$ if and only if $b_i \le k < \omega$ if and only if $\mathcal{F}_{\vec{b}}(-u) \in [b_i, \omega - 1]$ if and only if $(-u - e_i, -u) \in \mathcal{H}_{\vec{b}}$ if and only if $(u, v) \in -\mathcal{H}_{\vec{b}}$. $\square$

Corollary 33. *Let $d = 3$. M is the Christoffel graph $\mathcal{H}_{\vec{a}}$ or M is the complement of the reversal of the Christoffel graph $\mathcal{H}_{\vec{b}}$.*

Note that the complement of the reversal is equal to the reversal of the complement.

Proof. From Lemma 31 there exist integers $0 < q < 3$ and $0 < \ell < 3$ such that $\omega \cdot q = a_1 + a_2 + a_3$ and $\omega \cdot \ell = b_1 + b_2 + b_3$. Therefore, there are two cases, either $q = 1$ and $\ell = 2$ or $q = 2$ and $\ell = 1$. If $q = 1$, then Theorem 32 (i) applies. Therefore, M is a Christoffel graph $M = \mathcal{H}_{\vec{a}}$ for the vector $\vec{a} = (a_1, a_2, a_3)$. If $\ell = 1$, then Theorem 32 (ii) applies. Therefore, the complement of M is the reversal of a Christoffel graph. More precisely, $M^c = -\mathcal{H}_{\vec{b}}$ for the vector $\vec{b} = (b_1, b_2, b_3)$. $\square$

The previous result has also a counterpart in the triangular lattice L.

Corollary 34. *Let $M' \subset \pi(\mathbb{E}_d)$ such that $\textsc{flip}(M' + t') = M'$ for some $t' \in L$, that is invariant under some subgroup of finite index of L and that satisfies $\mathcal{Q} \cap M = \{(0, h_1), (0, h_2), \ldots, (0, h_d)\} + K$. If $d = 3$, then M' is equal to a graph $I_{\vec{a}}$ or to the reversal of its edge-complement.*

Proof. All we have to do is to lift M' to a set $M \subset \mathbb{E}_d$ using the projection $\pi : \mathbb{R}^d \to \mathcal{D}$ in such a way that $M = \textsc{flip}(M + t)$, with $\pi(t) = t'$, and to show that M is invariant under the subgroup $K = \pi^{-1}(K')$ and satisfies $\mathcal{Q} \cap M = \{(0, e_1), (0, e_2), \ldots, (0, e_d)\} + K$. Then the corollary follows from the previous one. The details are left to the reader. $\square$

Finally, we give the similar result for Christoffel parallelograms. We consider some parallelogram P whose vertices are in L, and edges (which are in $\mathbb{E}'_d$) within it; these edges must be torally compatible, in the sense that such an edge hits some edge of the parallelogram, then it reappears on the opposite edge of the parallelogram. Such a parallelogram defines a subgroup of finite index K' of L (spanned by the edges of the parallelogram) and tiles the whole hyperplane D. We say that $\textsc{flip}(P) = P + t'$, for some $t' \in L$, if $P + t'$ is the parallelogram obtained by flipping the edges of P incident to zero mod K.

Corollary 35. *Under the previous hypothesis, P is a Christoffel parallelogram or the reversal of its edge-complement.*

Proof. It suffices to verify that P defines a subset M' of $\pi(\mathbb{E}_d)$ satisfying the hypothesis of the previous corollary. $\square$

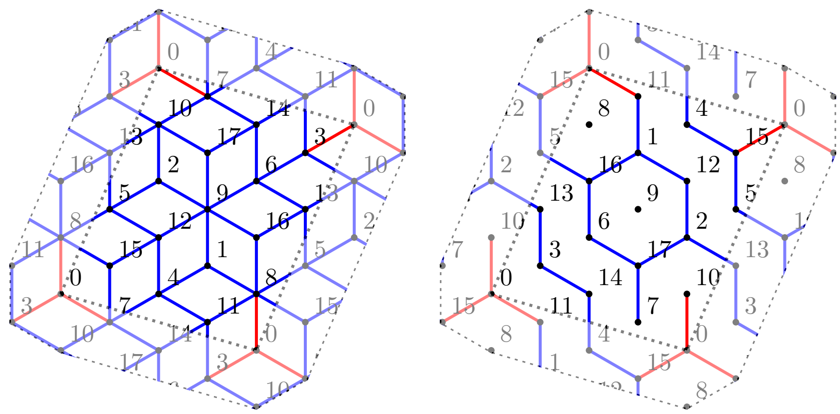
The result is illustrated in Figure 12.

We are now ready for the main result of this article which generalizes Pirillo's theorem (Theorem 1) to arbitrary dimension: a graph $M \subseteq \mathbb{E}_d$ is a translate of its flip if and only if it is a Christoffel graph of width ω.

Theorem 36 (**d-dimensional Pirillo's theorem**)**.** *Let K be a subgroup of finite index of $\mathbb{Z}^d$. Let $M \subseteq \mathbb{E}_d$ be a subset of edges invariant for the group of translations K such that the edges of M incident to zero mod K are $\mathcal{Q} \cap M = \{(0, e_i) \mid 1 \leq i \leq d\} + K$. There exists $t \in \mathbb{Z}^d$ such that $M = \textsc{flip}(M + t)$ if and only if $M = \mathcal{H}_{\vec{a},\omega}$ is the Christoffel graph of normal vector $\vec{a}$ and width ω.*

Proof. Suppose $M = \textsc{flip}(M + t)$ for some $t \in \mathbb{Z}^d$. From Lemma 29, for all $u = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$ there exists an integer k such that $u = kt$ with

$$k = \sum(-b_i x_i) \bmod \omega = \sum(a_i - \omega)x_i \bmod \omega = \sum a_i x_i \bmod \omega = \mathcal{F}_{\vec{a},\omega}(u).$$



$\mathcal{H}_{(3,7,8)}$ $\qquad$ $\mathcal{H}_{(15,11,10),18}$

Figure 12: Left: the Christoffel graph $\mathcal{H}_{\vec{a}}$ for the vector $\vec{a} = (3, 7, 8)$. It satisfies the equation $M = \text{FLIP}(M+t)$ for the translation vector $t = e_3 - e_2$. Right: the complement of the reversal of the Christoffel graph for the vector $\vec{b} = (3, 7, 8)$, i.e. $-\mathcal{H}^c_{\vec{b}}$. It corresponds to the Christoffel graph $\mathcal{H}_{\vec{a},\omega}$ for the vector $\vec{a} = (15, 11, 10)$ and width $\omega = 18$. It satisfies the equation $M = \text{FLIP}(M + t)$ for the translation vector $t = e_2 - e_3$. They represent the only two possibilities for a pattern M satisfying $M = \text{FLIP}(M + t)$ when $d = 3$ and K is the subgroup of $\mathbb{Z}^3$ given by $\langle(0, 4, 1), (-2, 0, 3), (1, 1, 1)\rangle$.

We want to show that $M = \mathcal{H}_{\vec{a},\omega}$. We have that $(u, u + e_i) = (0, e_i) + kt \in M$ if and only if $0 \leq k < b_i$ if and only if $\mathcal{F}_{\vec{a},\omega}(u) \in [0, \omega - a_i - 1]$ if and only if $(u, u + e_i) \in \mathcal{H}_{\vec{a},\omega}$ from Lemma 15.

Reciprocally, suppose $\mathcal{H}_{\vec{a},\omega}$ is the Christoffel graph of normal vector $\vec{a}$ of width ω. We can show that $\mathcal{H}_{\vec{a},\omega} + t = \text{FLIP}(\mathcal{H}_{\vec{a},\omega})$ where $t \in \mathbb{Z}^d$ is such that $\mathcal{F}_{\vec{a},\omega}(t) = 1$. The proof goes along the same lines as Proposition 24 using Lemma 15 instead of Lemma 4. $\square$

7 Appendix: Discrete planes

In this section, we show some results on standard discrete planes. Discrete planes were introduced in [Rev91] and standard discrete planes were further studied in [Fra96]. The projection of a standard discrete plane gives a tiling of $\mathcal{D}$ by three kinds of rhombus [BV00] thus yielding a coding of it by $\mathbb{Z}^2$-actions by rotations on the unit circle [ABI02, ABFJ07]. Our construction of the discretized hyperplane is equivalent, for the dimension 3, to that in [ABI02]. Our point of view is slightly different from the classical one; inspired by the 2-dimensional case (discrete lines), we define a discrete hyperplane by "what the observer sees": the observer is at $-\infty$ in the direction $(1, 1, \ldots, 1)$ and he looks towards the "transparent" hyperplane all the unit hypercubes which are located on the other side. This may be modelled mathematically; all the results are intuitively clear, but require a proof. We prove them, since we could not find precise references. We recover some known results.

Imagine the d-dimensional space filled with unit hypercubes with opaque faces. Consider a transverse hyperplane generated by its integer points (formally, of equation $\sum a_i x_i = 0$, $a_i > 0$ coprime integers). As an observer, we install in the open half-space H_- bounded by the plane. Then, we remove all the cubes in this half-space, including the cubes intersecting this half-space; in other words, we keep only the cubes contained in H_+. Figure 13 illustrates this construction for $d = 2$. For $d = 3$, when we look towards H_+ parallely to the vector $(1, 1, 1)$, then we see something like in Figure 14.

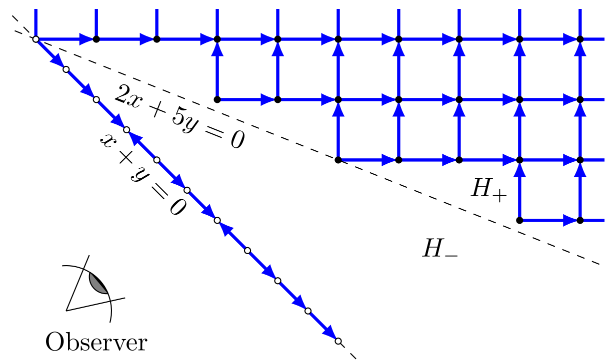


Figure 13: Observation in dimension 2. What the observer sees can be projected parallel to the vector $(1,1)$ on the line $x + y = 0$.

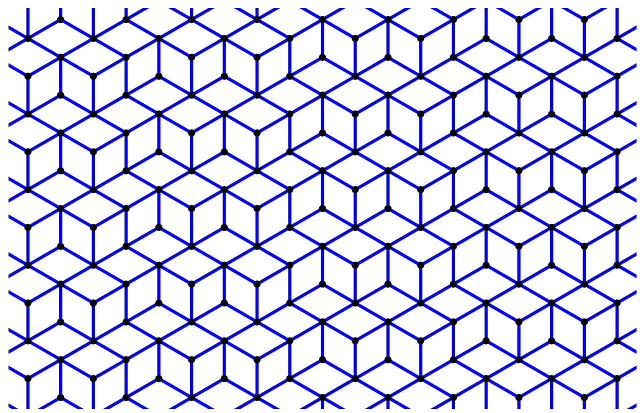


Figure 14: What the observer sees in dimension 3. The surface of cubes was projected parallel to the vector $(1,1,1)$ on the plane $x + y + z = 0$.

Let s be the sum $s = \sum_i a_i$. We denote $\vec{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$. The complement of H has two connected components H_- and H_+, where the first is determined by the inequation $\sum_i a_i x_i < 0$. We consider the unit cubes in $\mathbb{R}^d$ and their facets. Such a facet is a subset of $\mathbb{R}^d$ of the form $M + \sum_{j \neq i} [0,1] e_j$, for some coordinate $i \in \{1, \ldots, d\}$ and some integral point $M \in \mathbb{Z}^d$. Denote by $\mathcal{C}$ the standard unit cube.

Consider the unit hypercubes that are contained in the closed half-space $H \cup H_+$ and their facets; denote by $\mathcal{U}_+$ the union of all these facets. Note that a unit cube $M + \mathcal{C}$ ($M \in \mathbb{Z}^d$) is contained in $H \cup H_+$ if and only if $M \in H \cup H_+$ if and only if $\sum_j a_j m_j \geq 0$.

We say that a point M in $\mathbb{R}^d$ is *visible* if the open half-line $M +]-\infty, 0[(1, 1, \ldots, 1)$ does not contain any point in $\mathcal{U}_+$. Intuitively, this means that, all facets being opaque, that an observer located at infinity in the direction of the vector $-(1, 1, \ldots, 1)$ can see this point M, because no point in $\mathcal{U}_+$ hides this point.

Now, we consider the set of visible points which belong to $\mathcal{U}_+$. This we may call the *discretized hyperplane associated to* H. Intuitively, it is the set of facets that the observer can sees, as is explained in the introduction.

We characterize now the discretized hyperplane. For this, we denote by R the following subset of $\mathbb{R}^d$: $R = \{(x_i) \mid 0 \leq \sum_i a_i x_i < s\}$. Note that $R \subset H \cup H_+$.

Denote by $\mathcal{S}$ the union of the facets that are contained in R. In other words,

$$\mathcal{S} = \bigcup_{M\in\mathbb{Z}^d, 1\le i\le d, M+\sum_{j\ne i}[0,1]e_j\subset R} (M + \sum_{j\ne i}[0,1]e_j).$$

Observe that the condition $M + \sum_{j\ne i}[0,1]e_j \subset R$ is equivalent to: $\sum_j a_j m_j \ge 0$ and $a_i m_i + \sum_{j\ne i} a_j(m_j + 1) < s$. Note also that $\mathcal{S} \subset \mathcal{U}_+$.

Theorem 37. *The discretized hyperplane is equal to $\mathcal{S}$.*

Observe that if we project $\mathcal{S}$ onto the hyperplane perpendicular to the vector $(1, 1, \ldots, 1)$, we obtain exactly what the observer sees.

An example of this, for $d = 3$, is given in Figure 14. This observation motivates the introduction of the graph $I_{\vec{a}}$ in Section 3.2.

We first give an simple characterization of $\mathcal{S}$.

Proposition 38. *Let $X = (x_i) \in \mathbb{R}^d$. Then X is in $\mathcal{S}$ if and only if the three conditions below hold:*
(i) some coordinate of X is an integer;
(ii)$\sum_i a_i\lfloor x_i\rfloor \ge 0$;
(iii) $\sum_i a_i\lceil x_i\rceil < s$.

We recover Proposition 1 of [ABI02].

Corollary 39. *Let $X = (x_i) \in \mathbb{Z}^d$. Then X is in $\mathcal{S}$ if and only if $0 \le \sum_i a_i x_i < s$.*

Proof of the proposition. Suppose that $X \in \mathcal{S}$. Then $X \in M + \sum_{j\ne i}[0,1]e_j \subset \mathcal{S}$ and the coordinates m_j of M are integers. Thus, by an observation made previously, $0 \le \sum_j a_j m_j \le \sum_j a_j\lfloor x_j\rfloor$, since $x_j = m_j + \theta_j$, with $0 \le \theta_j \le 1$ and $\theta_i = 0$. Moreover, $\lceil x_i\rceil = m_i$, and $\lceil x_j\rceil \le m_j + 1$ if $j \ne i$. Thus, $\sum_j a_j\lceil x_j\rceil \le a_i m_i + \sum_{j\ne i}(m_j + 1) < s$, by the same observation.

Conversely, suppose that the three conditions of the proposition hold. Without restricting the generality (by permutation of the coordinates), we may assume that for some $i \in \{1, \ldots, d\}$, one has $x_i, \ldots, x_i \in \mathbb{Z}$ and $x_{i+1}, \ldots, x_d \notin \mathbb{Z}$. Let $0 \le p \le i$ be maximum subject to the condition $\sum_{j\le p} a_j(x_j - 1) + \sum_{j>p} a_j\lfloor x_j\rfloor \ge 0$ (note that p exists since the inequality is satisfied for $p = 0$). Suppose that $p = i$; then $\sum_j a_j\lceil x_j\rceil = \sum_{j\le i} a_j x_j + \sum_{j>i} a_j(\lfloor x_j\rfloor + 1) = a_1 + \cdots + a_d + \sum_{j\le i} a_j(x_j - 1) + \sum_{j>i} a_j\lfloor x_j\rfloor \ge a_1 + \cdots + a_d$ (since $p = i$) $= s$; thus we obtain a contradiction with condition (iii).

Thus $p < i$ and $p + 1 \le i$. We have by maximality the inequality $\sum_{j\le p+1} a_j(x_j - 1) + \sum_{j>p+1} a_j\lfloor x_j\rfloor < 0$. Let $M = (m_j) = (x_1 - 1, \ldots, x_p - 1, \lfloor x_{p+1}\rfloor, \ldots, \lfloor x_d\rfloor \in \mathbb{Z}^d$. We have $\sum_j a_j m_j \ge 0$ (by definition of p) and $a_{p+1}m_{p+1} + \sum_{j\ne p+1} a_j(m_j + 1) = \sum_{j\le p}(a_j(x_j - 1) + a_j) + a_{p+1}(x_{p+1} - 1) + a_{p+1} + \sum_{j>p+1}(a_j\lfloor x_j\rfloor + a_j) = s + \sum_{j\le p+1} a_j(x_j - 1) + \sum_{j>p+1} a_j\lfloor x_j\rfloor < s$, by the previous inequality. Thus $M + \sum_{j\ne i}[0,1]e_j \subset \mathcal{S}$, by the observation made above. Moreover $X \in M + \sum_{j\ne i}[0,1]e_j$ since $p + 1 \le i$. □

Corollary 40. *For each point X in $\mathbb{R}^d$, there is a unique point Y in $\mathcal{S}$ such that XY is parallel to the vector* $(1, 1, \ldots, 1)$.

Denote by f the function such that $Y = f(X)$ with the notations of the corollary. This function is a kind of projection onto $\mathcal{S}$, parallely to the vector $(1, 1, \ldots,)$. Denote also by $t(X)$ the real-valued function defined by $X = f(X) + t(X)(1, 1, \ldots, 1)$, and by $t(X) = 0$ if and only if $X \in \mathcal{S}$.

Proof. We prove first unicity. By contradiction: we have $Y, Z \in \mathcal{S}$ and $Z = Y + t(1, 1, \dots, 1)$ with $t > 0$. Then $z_i = y_i + t$. Thus $\lceil z_i \rceil \geq \lfloor y_i \rfloor + 1$. Hence $\sum_i a_i \lceil z_i \rceil \geq s + \sum_i a_i \lfloor y_i \rfloor$. Since by the proposition, applied to Y, the last sum is ≥ 0, we obtain $\sum_i a_i \lceil z_i \rceil \geq s$, which contradicts the proposition, applied to Z.

We prove now the existence of Y. We may assume that $L(X) = \sum_i a_i \lfloor x_i \rfloor \geq 0$, by adding to X some positive multiple of $(1, 1, \dots, 1)$ if necessary. We prove existence of Y by induction on the sum $U(X) = \sum_i a_i \lceil x_i \rceil$.

Let $\epsilon = \min_i (x_i - \lfloor x_i \rfloor)$. Observe that if we replace X by $X - \epsilon(1, 1, \dots, 1)$, then $L(X)$ does not change, $U(X)$ does not increase and moreover some x_i is now an integer.

If $U(X)$ is $< s$, this observation implies the existence of Y.

Suppose now that $U(X) \geq s$. By the observation, we may assume that at least one of the x_i is an integer. Without restricting the generality, we may also assume that $x_1, \dots, x_i \in \mathbb{Z}$ and that $x_{i+1}, \dots, x_d \notin \mathbb{Z}$, with $i \geq 1$.

If $i = d$, then the x_j are all integers, $L(X) = U(X)$, we replace X by $X - (1, 1, \dots, 1)$ and we conclude by induction, since $L(X)$ is replaced by $L(X) - s$.

Suppose now that $i < d$. Let $\epsilon = \min_{j>i}(x_j - \lfloor x_j \rfloor)$; then $\epsilon > 0$. We have $s \leq \sum_j a_j \lceil x_j \rceil = \sum_{j \leq i} a_j x_j + \sum_{j>i} a_j (\lfloor x_j \rfloor + 1) = L(X) + a_{i+1} + \ldots + a_d$, hence $L(X) \geq a_1 + \cdots + a_i$. Note that $\sum_j a_j (\lfloor x_j - \epsilon \rfloor) = \sum_{j \leq i} a_j (x_j - 1) + \sum_{j>i} a_j \lfloor x_j \rfloor = L(X) - a_1 - \cdots - a_i \geq 0$. We replace X by $X - \epsilon(1, 1, \dots, 1)$, and we may conclude by induction, since $U(X)$ strictly decreases and since $L(X)$ remains ≥ 0. ☐

Proof. (of the theorem) Let X be a point on the discretized hyperplane associated to H. Suppose that $t(X) > 0$. Then $X = f(X) + t(X)(1, 1, \dots, 1)$ so that X is hidden by $f(X)$: formally, $f(X)$ is on the open half-line $X +]-\infty, 0[(1, 1, \dots, 1)$ and since $f(X)$ is in $\mathcal{S}$, it is a point in $\mathcal{U}_+$. We conclude that we must have $t(X) \leq 0$. Suppose that $t(X) < 0$. We know that X is in $\mathcal{U}_+$, so that X belongs to a hypercube $M + \mathcal{C}$ with $\sum_j a_j m_j \geq 0$, and therefore $x_j \geq m_j$. Let $Y = f(X)$. Then $X = Y + t(X)(1, 1, \dots, 1)$ so that $y_j > x_j \geq m_j$ which implies $\sum_j a_j \lceil y_j \rceil \geq \sum_j a_j (m_j + 1) \geq s$, a contradiction with Proposition 38. Thus $t(X) = 0$ and $X \in \mathcal{S}$.

Conversely suppose that $X \in \mathcal{S}$. Suppose that X is not on the discretized hyperplane associated to H. This implies that there is some point $Y \in \mathcal{U}_+$ on the open half-line $X +]-\infty, 0[(1, 1, \dots, 1)$. We have $Y \in M + \mathcal{C}$ with $\sum_j a_j m_j \geq 0$. Thus $x_j > y_j \geq m_j$ which implies that $\sum_j a_j \lceil x_j \rceil \geq \sum_j a_j (m_j + 1) \geq s$, a contradiction with Proposition 38. ☐

Corollary 41. *Let $d \geq 2$. Let $M \in \mathcal{S} \cap \mathbb{Z}^d$. Let $i = 1, 2, \dots, d$ and $N = M + e_i$.*

(i) $N \in \mathcal{S}$ if and only if $\sum_j a_j n_j < s$; in this case, the segment $M + [0, 1]e_i$ is contained in $\mathcal{S}$.

(ii) If $N \notin \mathcal{S}$, then the only point in $(M + [0, 1]e_i) \cap \mathcal{S}$ is M.

Proof. The fact that $N \in \mathcal{S}$ if and only if $\sum_j a_j n_j < s$ is a consequence of the proposition.

Suppose that $N \in \mathcal{S}$ and let X be on the segment $M + [0, 1]e_i$. Then $0 \leq \sum_j a_j m_j \leq \sum_j a_j \lfloor x_j \rfloor$ and $\sum_j a_j \lceil x_j \rceil \leq \sum_j a_j n_j < s$. Thus the corollary follows from the proposition.

Suppose no that $N \notin \mathcal{S}$ and let X be on this segment. Since $0 \leq \sum_j a_j m_j$, we have also $0 \leq \sum_j a_j n_j$. Since $N \notin \mathcal{S}$, we must have $\sum_j a_j n_j \geq s$. Moreover, if $X \neq M$, we have $\lceil x_j \rceil = n_j$, so that $\sum_j a_j \lceil x_j \rceil \geq s$ and $X \notin \mathcal{S}$. ☐

The next result, which is not needed in this article, is of independent interest, and intuitively clear (but it requires a proof).

Proposition 42. *The function $f : X \mapsto Y$, with the notations of Corollary 40, is continuous. The open set $\mathbb{R}^d \setminus \mathcal{S}$ has two connected components.*

Lemma 43. *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^d$such that for each X in $\mathbb{R}^d$, there is a unique Y in $\mathcal{S}$ such that XY is parallel to $(1, 1, \ldots, 1)$. If the mapping $X \mapsto Y$ is bounded, then it is continuous.*

Proof. Recall that a bounded sequence in $\mathbb{R}^d$, converges if for any two convergent subsequences, they have the same limit. Let (X_n) be a sequence in $\mathbb{R}^d$, with limit l. It is enough to show that $(f(X_n))$ converges; note that this sequence is bounded. Consider two subsequences of (X_n) such that their images under f have limits, l_1 and l_2 say. Since $\mathcal{S}$ is closed, $l_1, l_2 \in \mathcal{S}$. Let $\epsilon > 0$. For n large enough, $\mid X_n - l \mid < \epsilon$; hence $f(X_n)$ is in the open cylinder of diameter ϵ and with axis the line $l + (1, 1, \ldots, 1)$. This implies that l_1, l_2 are in this cylinder and consequently, ϵ being arbitrary, l_1, l_2 are on the previous line. By unicity, $l_1 = l_2$ $(= f(l))$. We conclude using the remark at the beginning of the proof. □

Proof. (of the proposition) The mapping f is continuous: by the lemma, it is enough to show that $\mathcal{S}$ is closed and that the mapping is bounded. Since each convergent sequence is contained in some compact set, it is enough to show that for each compact set K, $K \cap \mathcal{S}$ is closed; but this is clear, since the latter set is the union of finitely many $K \cap F$, F facet of unit hypercube. The mapping is bounded since its image is between the two hyperplanes of equations $\sum_i a_i x_i = 0$ and $\sum_i a_i x_i = s$, so that the image of each bounded set is contained in a cylinder of axis parallel to $(1, 1, \ldots)$ and limited by these two hyperplanes.

Now, we show that the set $\mathbb{R}^d \setminus \mathcal{S}$ has two connected components. Note that for each point X, one has $X = f(X) + t(X)(1, 1, \ldots, 1)$ for some continuous real-valued function t. Since $f(1, 1, \ldots, 1) = (0, 0, \ldots, 0) = f(-1, -1, \ldots, -1)$, one has $t(1, 1, \ldots, 1) = 1$ and $t(-1, -1, \ldots, -1) = -1$. Moreover $t(X) = 0$ if and only if $X \in \mathcal{S}$. Thus $t(\mathbb{R}^d \setminus \mathcal{S})$ is not connected and neither is $\mathbb{R}^d \setminus \mathcal{S}$.

Now, if $t(X) > 0$, one may connect X by a piece of the line $X + \mathbb{R}(1, 1, \ldots, 1)$ to a point of the half-space $\sum_i a_i x_i > 0$ and this implies that the set of points X with $t(X) > 0$ is connected. Similarly, the set of points with $t(X) < 0$ is connected, and $\mathbb{R}^d) \setminus \mathcal{S}$ has therefore two connected components. □

We recover Proposition 2 of [ABI02] and Proposition 4 of [ABFJ07].

Corollary 44. *The restriction of f to $\mathcal{D}$ is a homeomorphism of $\mathcal{D}$ onto $\mathcal{S}$.*

Proof. Indeed, the inverse mapping is the projection onto the hyperplane $\mathcal{D}$ parallely to the vector $(1, 1, \ldots, 1)$. □

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