Factor Complexity of S-adic sequences generated by the Arnoux-Rauzy-Poincaré Algorithm

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Abstract

The Arnoux-Rauzy-Poincaré multidimensional continued fraction algorithm is obtained by combining the Arnoux-Rauzy and Poincaré algorithms. It is a generalized Euclidean algorithm. Its three-dimensional linear version consists in subtracting the sum of the two smallest entries to the largest if possible (Arnoux-Rauzy step), and otherwise, in subtracting the smallest entry to the median and the median to the largest (the Poincaré step), and by performing when possible Arnoux-Rauzy steps in priority. After renormalization it provides a piecewise fractional map of the standard 2-simplex. We study here the factor complexity of its associated symbolic dynamical system, defined as an S-adic system. It is made of infinite words generated by the composition of sequences of finitely many substitutions, together with some restrictions concerning the allowed sequences of substitutions expressed in terms of a regular language. Here, the substitutions are provided by the matrices of the linear version of the algorithm. We give an upper bound for the linear growth of the factor complexity. We then deduce the convergence of the associated algorithm by unique ergodicity.

1 Introduction

Multidimensional continued fraction algorithms aim at providing good rational approximations of a given vector. There exist many different types of continued fraction algorithms. Among them, piecewise fractional ones in the sense of [Bre81, Sch00] have been widely studied whereas for their arithmetic or for their ergodic properties. The viewpoint we take here on these algorithms is issued from word combinatorics and symbolic dynamics. It is indeed possible to generate with such algorithms infinite words with prescribed letter frequencies: the letter frequency vector is indeed the vector on which the algorithm is applied. We recall that a substitution is a morphism of the free monoid that replaces letters by finite words. A piecewise fractional continued fraction algorithm produces (unimodular) matrices with non-negative entries that we consider as incidence matrices of substitutions. We then iterate these substitutions in an S-adic way, that is, as the (inverse) limit of an infinite product of substitutions (see e.g. [BD14, CN10, DLR13, Ler12]). We thus obtain an infinite word $\mathbf{u}$ of the form

$$\mathbf{u} = \lim_{n\to\infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n(a^\infty).$$

As an illustration consider the generation of Sturmian words with the classical continued fraction algorithm (see [Lot02, Fog02] for more details).

The Diophantine approximation properties of the underlying continued fraction algorithm are reflected in the generic behaviour of the balance function of the generated word $\mathbf{u}$, where the balance function counts, for each given letter, the difference between the numbers of occurrences of this letter in any two words of the same length that occur in $\mathbf{u}$. It is also closely related to the notion of symbolic discrepancy such as considered in [Ada03]. We also would like the combinatorics of the generated infinite word $\mathbf{u}$ to be "simple" in the sense that the factor complexity of $\mathbf{u}$ is expected to be of linear growth, were the factor complexity counts the number of factors of a given length.

Observe that there exist several methods for producing infinite words with prescribed letter frequencies having a linear factor complexity $p(n)$ and/or a bounded balance. The Sturmian words form a well-known family of infinite balanced words over a two-letter alphabet having a linear factor complexity ($p(n) = n+1$ for all n). Nevertheless the situation is more contrasted for words defined on alphabets having at least three letters concerning the possibility of having simultaneously prescribed letter frequencies, a linear factor complexity and a bounded balance. Typical generalizations of Sturmian words are natural codings of interval exchanges and the billiard words in the d-dimensional cube. However, billiard words have quadratic factor complexity [Bar95, Bed03] and codings of interval exchanges are not balanced [Zor97]. Other approaches were considered in digital geometry where arithmetic definitions of 3D discrete lines were proposed. The standard model of [And03] is one of them and can be encoded as a word on a three-letter alphabet. It turns out that this model corresponds to the one of billiard words [Lab12], thus also having a quadratic factor complexity in general.

The experimentations described in [BL11, Lab12] indicate that some multidimensional continued fraction algorithms generate S-adic words having a linear factor complexity and a bounded balance for almost every letter frequencies vector. In particular, Brun multidimensional continued fraction algorithm as well as the Arnoux-Rauzy-Poincaré algorithm seem to be the two best choices in terms of balance properties. In this article, we focus on the Arnoux-Rauzy-Poincaré algorithm which performs experimentally a bit better than does Brun algorithm. This algorithm (under its linear form) consists in subtracting the sum of the two smallest entries to the largest if possible and otherwise, in subtracting the smallest entry to the median and the median to the largest. In order to generate infinite words, we introduce an $\mathcal{S}$-adic system associated with the nine possible matrices of the algorithm that thus provide a set $\mathcal{S}$ of nine substitutions. Three of them are substitutions known under the name of Arnoux-Rauzy substitutions [AR91], and the other six are named Poincaré substitutions after Poincaré algorithm [Nog95]. Moreover, the execution of the Arnoux-Rauzy-Poincaré algorithm yields restrictions to the allowed infinite sequences of substitutions, expressed in terms of a regular language. We then have a bijection (up to a set of zero measure) between the infinite words in the corresponding $\mathcal{S}$-adic system and the standard 2-simplex $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 = 1\}$ (the vectors of letter frequencies). The main result of the present paper is that these words have a linear factor complexity $p(n)$.

Theorem 1 (**Factor Complexity).** *Let* $\mathbf{u}$ *be an* $\mathcal{S}$*-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector* $\mathbf{x} \in \Delta$*. Then the factor complexity of* $\mathbf{u}$ *is such that* $p(n+1) - p(n) \in \{2, 3\}$ *and* $2n + 1 \leq p(n) \leq \frac{5}{2}n + 1$ *for all* $n \geq 0$*.*

The proof relies on a careful study of bispecial factors of $\mathbf{u}$, that is, of factors having several left and right extensions in $\mathbf{u}$. We prove that weak and strong bispecial factors are alternating in the sequence (ordered by increasing length) of non-neutral bispecial factors. The restriction for the directive sequences of the $\mathcal{S}$-adic words to the regular language provided by the Arnoux-Rauzy-

Poincaré algorithm is clearly important; indeed quadratic factor complexity can be reached otherwise (see Section 4.5).

Then, by using a result of Boshernitzan [Bos85], we deduce unique ergodicity and thus, the existence of (uniform) frequency of any factor, and in particular of the letters. This also provides a combinatorial proof of convergence for this multidimensional continued fraction algorithm.

Theorem 2 (**Frequencies and Convergence).** *Let* $\mathbf{u}$ *be an* $\mathcal{S}$*-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector* $\mathbf{x} \in \Delta$*. Then the symbolic dynamical system generated by* $\mathbf{u}$ *is uniquely ergodic. As a consequence, the frequencies of factors and letters exist in* $\mathbf{u}$*, the latter being equal to the coordinates of* $\mathbf{x}$*.*

Furthermore, the Arnoux-Rauzy-Poincaré algorithm is a weakly convergent algorithm, that is, for Lebesgue almost every $\mathbf{x} \in \Delta$*, if* $(M_n)_n$ *stands for the sequence of matrices produced by the Arnoux-Rauzy-Poincaré algorithm, then one has* $\cap_n M_0 \cdots M_n(\mathbb{R}^3_+) = \mathbb{R}_+\mathbf{x}$*.*

Let us sketch the content of the present paper. The Arnoux-Rauzy-Poincaré multidimensional continued fraction algorithm is introduced in Section 2. We also define the associated Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system based on its nine substitutions (provided by its linear version) together with a rational restriction on the directive sequences of substitutions that are iterated. In Section 3, we introduce the basic notions used to compute the factor complexity, namely languages, bispecial factors and extension types. In Section 4, we study the life of bispecial factors under Arnoux-Rauzy and Poincaré substitutions (with no restriction on the order of application of substitutions). In Section 5, we prove the upper bound on the factor complexity stated in Theorem 1. The convergence of the algorithm together with unique ergodicity is lastly considered in Section 6.

This article is an extended version of [BL13]. The present paper provides the upper bound $p(n) \leq \frac{5}{2}n + 1$, whereas the upper bound in [BL13] was $p(n) \leq 3n + 1$.

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2 The Arnoux-Rauzy-Poincaré Algorithm

2.1 The algorithm

The Arnoux-Rauzy-Poincaré (**ARP**) is a multidimensional continued fraction algorithm in the sense of [Bre81, Sch00], defined by piecewise fractional maps acting on the standard 2-simplex $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$. It is a fusion algorithm such as introduced in [BL11, Lab12] which combines the two classical algorithms that are Poincaré (**P**) algorithm and Arnoux-Rauzy (**AR**) algorithm, which are respectively defined (under their linear form) in dimension 3 as follows: Poincaré algorithm acts on a triple of non-negative entries by subtracting the smallest entry to the median and the median to the largest, whereas Arnoux-Rauzy algorithm acts by subtracting the sum of the two smallest entries to the largest, when possible. Our algorithm privilegiates an Arnoux-Rauzy step if possible, otherwise it perfoms a Poincaré step.

The simplex Δ admits as vertices the vectors $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$.

In order to partition Δ, we consider the following fifteen matrices, namely

$$\begin{array}{lllll}
A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & P_{21} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & P_{31} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & H_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, & H_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & P_{12} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & P_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, & H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & H_{32} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, & P_{13} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, & P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, & H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & H_{23} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{array}$$

whose column vectors define partitions by triangles of the simplex such as illustrated at Figure 1 (left). Then, the column vectors of A_1, A_2, A_3, $P_{31}H_{31}$, $P_{13}H_{13}$, $P_{23}H_{23}$, $P_{32}H_{32}$, $P_{12}H_{12}$ and $P_{21}H_{21}$

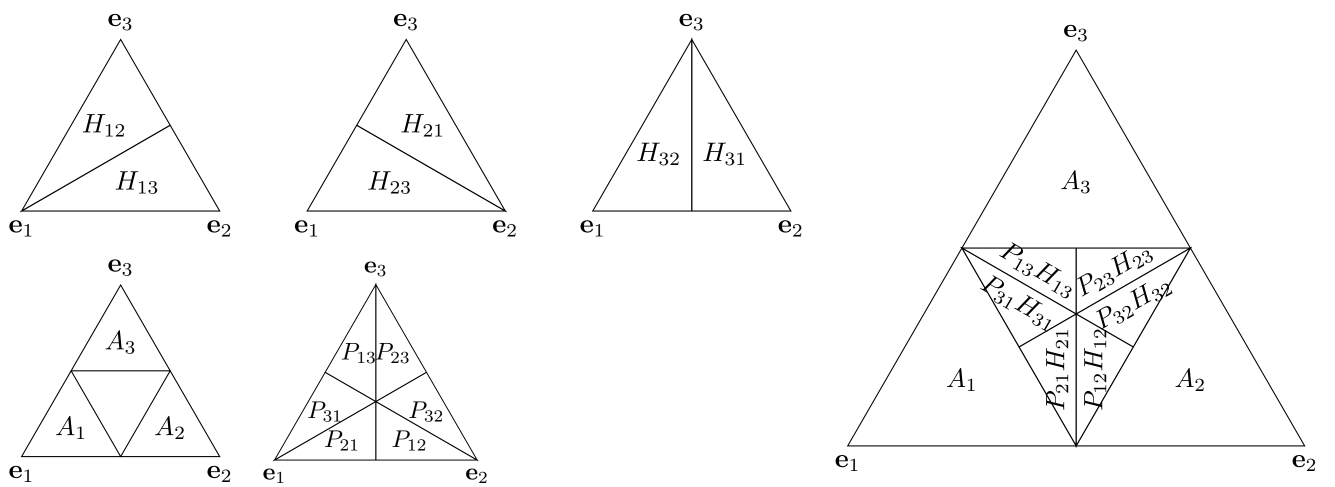

Figure 1: Left: the partition provided by three Arnoux-Rauzy matrices, the six Poincaré matrices and the six half triangles. Right: the partition of Arnoux-Rauzy-Poincaré algorithm.

describe a partition of Δ depicted in Figure 1 (right). Partitions are considered here up to a set of zero measure. This partition allows one to associate with almost every point of Δ a matrix as follows:

$$\begin{array}{rcl} M : \Delta & \to & GL(3, \mathbb{Z}) \\ \mathbf{x} & \mapsto & \begin{cases} A_k & \text{if } \mathbf{x} \in A_k\Delta, \\ P_{jk} & \text{else if } \mathbf{x} \in P_{jk}H_{jk}\Delta. \end{cases} \end{array}$$

We say that $\mathbf{x} = (x_1, x_2, x_3) \in \Delta$ is *totally irrational* if x_1, x_2, x_3 are linearly independent over $\mathbb{Q}$. When $\mathbf{x}$ is not a totally irrational vector, there might be more than one choice for the matrix $M(\mathbf{x})$ in the previous definition. Nevertheless, the matrix $M(\mathbf{x})$ is uniquely defined for a totally irrational vector.

Then, the Arnoux-Rauzy-Poincaré algorithm is defined (by renormalizing with respect to the simplex Δ) the linear map M:

$$\begin{array}{rcl} T : \Delta & \to & \Delta \\ \mathbf{x} & \mapsto & \dfrac{M(\mathbf{x})^{-1} \cdot \mathbf{x}}{\|M(\mathbf{x})^{-1} \cdot \mathbf{x}\|_1}. \end{array}$$

Each totally irrational vector $\mathbf{x} \in \Delta$ defines an orbit under the map T and a sequence of matrices $(M_n(\mathbf{x}))_{n \in \mathbb{N}}$:

$$M_0(\mathbf{x}) = \text{Id}, \quad M_n(\mathbf{x}) = M(T^{n-1}(\mathbf{x})) \text{ for all } n.$$

Example 3. *Consider* $\mathbf{x} = (1, \pi, \sqrt{2})$*. The first* 5 *points of the orbit of* $\mathbf{x}$ *under the map* T *are*

$$\mathbf{x} \in A_2\Delta, \quad T(\mathbf{x}) \in P_{13}H_{13}\Delta, \quad T^2(\mathbf{x}) \in A_2\Delta, \quad T^3(\mathbf{x}) \in A_3\Delta, \quad T^4(\mathbf{x}) \in A_1\Delta, \cdots$$

One has $M_0(\mathbf{x}) = \mathrm{Id}$*,* $M_1(\mathbf{x}) = A_2$*,* $M_2(\mathbf{x}) = P_{13}$*,* $M_3(\mathbf{x}) = A_2$*,* $M_4(\mathbf{x}) = A_3$ *and* $M_5(\mathbf{x}) = A_1$*.*

2.2 Arnoux-Rauzy-Poincaré S-adic words

We now associate with the Arnoux-Rauzy-Poincaré algorithm a finite set $\mathcal{S}$ of substitutions as well as $\mathcal{S}$-adic words.

We first start with some terminology. We consider a finite set of *letters* $\mathcal{A}$, called *alphabet*. Here $\mathcal{A} = \{1, 2, 3\}$. A (finite) *word* is an element of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$. The unique word of length 0 is the *empty word* and we let it be denoted as ε. We let the set of all (finite) words over $\mathcal{A}$ be denoted by $\mathcal{A}^*$. With the concatenation of words as product operation, $\mathcal{A}^*$ is the free monoid with ε as identity element. A substitution on the alphabet $\mathcal{A}$ is a non-erasing morphism of the free monoid wich replaces letters by words. Let σ be a substitution. Its *incidence matrix* (also called *abelianized matrix*) $M_\sigma = (m_{i,j})_{1 \le i,j \le d}$ is defined as the square matrix whose entry of index (i, j) is equal to the number of occurrences of the letter i in $\sigma(j)$. If a word u can be factorized as pvs, with $p, v, s \in \mathcal{A}^*$, then we say that p is a *prefix*, v is a *factor* and s is a *suffix* of u. The factor v is said *proper* if p and s are non-empty. This notion extends to any infinite word $\mathbf{u}$. The set $\mathcal{A}^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on each copy of $\mathcal{A}$; this topology is induced by the following distance: for two distinct infinite words $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{A}^{\mathbb{N}}$, $\mathrm{d}(\mathbf{u}, \mathbf{v}) = 2^{-\min\{n \in \mathbb{N} \mid u_n \neq v_n\}}$.

The infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is said to admit an *S-adic representation* if there exist a finite set S of substitutions defined on the alphabet $\mathcal{A}$, a sequence $s = (\sigma_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ of substitutions that all belong to S, and $(a_n)_{n \in \mathbb{N}}$ a sequence of letters in $\mathcal{A}$ such that

$$\mathbf{u} = \lim_{n \to \infty} \sigma_0 \sigma_1 \cdots \sigma_n(a_n^\infty),$$

with the notation a_n^∞ standing for the infinite constant word taking the value a_n. The word $\mathbf{u}$ is said to be S-adic, and the sequence s is called the *directive sequence*. We will use the following notation: for all $m \in \mathbb{N}$

$$\mathbf{u}^{(m)} = \lim_{n \to \infty} \sigma_m \sigma_{m+1} \cdots \sigma_n(a_n^\infty).$$

An S-adic expansion with directive sequence $(\sigma_n)_{n \in \mathbb{N}}$ is said *weakly primitive* if, for each n, there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive, that is, its incidence matrix has only positive entries. If an infinite word $\mathbf{u}$ admits a weakly primitive S-adic representation, then it is *uniformly recurrent*, that is, all its factors occur infinitely often and with bounded gaps [Dur03]. An infinite word $\mathbf{u}$ is said *recurrent* if all its factors occur infinitely often in $\mathbf{u}$. For more on S-adic words, see [BD14, CN10, DLR13, Ler12].

We now associate substitutions with the matrices defining the Arnoux-Rauzy-Poincaré algorithm. Let i, j, k be such that $\{i, j, k\} = \{1, 2, 3\}$. A *Poincaré substitution* is a substitution of the form $\pi_{jk} : i \mapsto ijk, j \mapsto jk, k \mapsto k$. An *Arnoux-Rauzy substitution* is given by $\alpha_k : i \mapsto ik, j \mapsto jk, k \mapsto k$. For each $\{i, j, k\} = \{1, 2, 3\}$, P_{jk} is the incidence matrix of the substitution π_{jk} and A_k is the incidence matrix of α_k. There are thus 6 Poincaré and 3 distinct Arnoux-Rauzy substitutions:

$$\pi_{23} = \begin{cases} 1 \mapsto 123 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}, \quad \pi_{13} = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 213 \\ 3 \mapsto 3 \end{cases}, \quad \alpha_3 = \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases},$$
$$\pi_{12} = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 312 \end{cases}, \quad \pi_{32} = \begin{cases} 1 \mapsto 132 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases}, \quad \alpha_2 = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases},$$
$$\pi_{31} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases}, \quad \pi_{21} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 321 \end{cases}, \quad \alpha_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases}.$$

Let

$$\mathcal{S} := \{\alpha_1, \alpha_2, \alpha_3, \pi_{12}, \pi_{13}, \pi_{21}, \pi_{23}, \pi_{31}, \pi_{32}\}.$$

We also denote by $\mathcal{S}_\alpha$, $\mathcal{S}_\pi$, respectively, the following sets of substitutions:

$$\mathcal{S}_\alpha = \{\alpha_1, \alpha_2, \alpha_3\},\ \mathcal{S}_\pi = \{\pi_{12}, \pi_{13}, \pi_{23}, \pi_{21}, \pi_{31}, \pi_{32}\},\ \text{with } \mathcal{S} = \mathcal{S}_\alpha \cup \mathcal{S}_\pi.$$

The substitutions in $\mathcal{S}$ are such that for any letter $i \in \{1,2,3\}$, $\sigma(i)$ admits i as a prefix. This yields the convergence of any $\mathcal{S}$-adic representation in $\mathcal{A}^{\mathbb{N}}$ if the sequence of letters $(a_n)_n$ is constant. More precisely, for any sequence of substitutions $(\sigma_n)_n$ with values in $\mathcal{S}$ and for every letter $a \in \{1,2,3\}$ then the following limit exists

$$\lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a^\infty).$$

Definition 4 (**Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word).** *An* Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word *is an infinite word of the form*

$$\mathbf{u} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a^\infty),$$

where $a \in \mathcal{A}$ and $\sigma_n \in \mathcal{S}$ for all $n \geq 0$. Its directive sequence is the sequence $s = (\sigma_n)_n$.

2.3 The Arnoux-Rauzy-Poincaré S-adic system

The aim of this section is to associate with the Arnoux-Rauzy-Poincaré algorithm an $\mathcal{S}$-adic symbolic dynamical system by taking into account the restrictions provided by the algorithm which is not complete but Markovian. We first recall the definition of an S-adic system. An *S-adic system* is obtained by adding restrictions on the set of allowed directive sequences: it is given by a finite directed strongly connected graph $\mathcal{G}$ labeled by the substitutions, with each infinite path giving rise to a directive sequence [BD14].

The partition of Δ allows to associate with almost any point of Δ a substitution of $\mathcal{S}$:

$$\begin{aligned} \sigma : \Delta &\to \mathcal{S} \\ \mathbf{x} &\mapsto \begin{cases} \alpha_k & \text{if } \mathbf{x} \in A_k\Delta, \\ \pi_{jk} & \text{else if } \mathbf{x} \in P_{jk}H_{jk}\Delta, \end{cases} \end{aligned}$$

and a directive sequence $s = (\sigma_n)_n$ with $\sigma_n = \sigma(T^n(\mathbf{x}))$ for all n. Observe that the substitution $\sigma(\mathbf{x})$ has for incidence matrix $M(\mathbf{x})$ such as defined in Section 2.1.

Definition 5. *An* $\mathcal{S}$-adic word $\mathbf{u}$ generated by the Arnoux-Rauzy-Poincaré algorithm applied to the totally irrational vector $\mathbf{x} \in \Delta$ *is an infinite word of the form*

$$\mathbf{u} = \lim_{n\to\infty} \left(\sigma(\mathbf{x}) \cdot \sigma(T(\mathbf{x})) \cdot \sigma(T^2(\mathbf{x})) \cdot \ldots \cdot \sigma(T^{n-1}(\mathbf{x}))\right)(a^\infty)$$

where $a \in \{1,2,3\}$. Its directive sequence is the sequence $s = (\sigma_n)_n$ with $\sigma_n = \sigma(T^n(\mathbf{x}))$ for all n.

Let us show that the factors of the directive sequences produced by the Arnoux-Rauzy-Poincaré algorithm belong to a rational language strictly included in $\mathcal{S}^*$. We consider the automaton $\mathcal{G} = (Q, \mathcal{S}, \delta, I, F)$ defined by the states

$$Q = \{\Delta, H_{12}, H_{13}, H_{21}, H_{23}, H_{31}, H_{32}\},$$

the alphabet $\mathcal{S}$, with the transitions $\delta \subset Q \times \mathcal{S} \times Q$ being defined by

$$\delta = \bigcup_{\{i,j,k\}=\{1,2,3\}} \{(\Delta, \alpha_k, \Delta), (\Delta, \pi_{jk}, H_{jk}), (H_{jk}, \alpha_j, H_{jk}), (H_{jk}, \alpha_i, \Delta), (H_{jk}, \pi_{ij}, H_{ij}), (H_{jk}, \pi_{ki}, H_{ki}), (H_{jk}, \pi_{ji}, H_{ji})\},$$

and with initial state $I = \{\Delta\}$ and final state $F = Q$ (see Figure 2). We consider the $\mathcal{S}$-adic system associated with the regular language $\mathcal{L}(\mathcal{G})$. This language corresponds to directive sequences

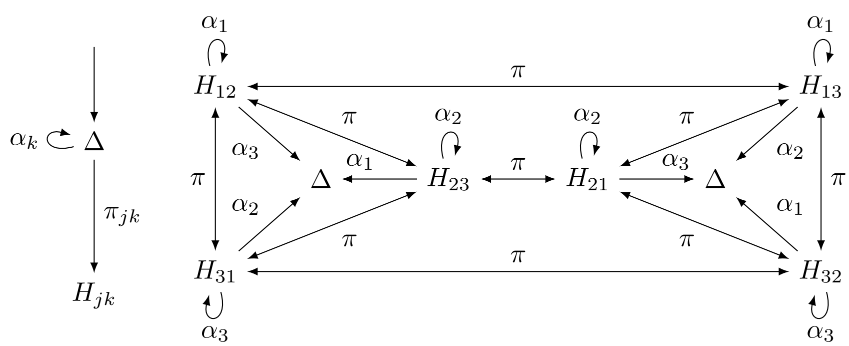

Figure 2: The deterministic automaton $\mathcal{G}$. To avoid crossing arrows, the initial state Δ is drawn at three places. The indices of π transitions are not written since they are determined by the indices of the arrival state: $\xrightarrow{\pi} H_{jk}$ means $\xrightarrow{\pi_{jk}} H_{jk}$.

for which the sequence of incidence matrices is generated by the execution of the Arnoux-Rauzy-Poincaré algorithm.

Proposition 6 (**ARP regular language).** *The set of directive sequences produced by the Arnoux-Rauzy-Poincaré algorithm is included in the set of labeled infinite paths in the automaton $\mathcal{G}$.*

The proof of the proposition is provided in the appendix.

Remark 7. *We can even prove that the closure of the set of directive sequences produced by the Arnoux-Rauzy-Poincaré algorithm is equal to the set $X_{\mathcal{G}}$ of labeled infinite paths starting in the automaton $\mathcal{G}$, as a consequence of the convergence of the algorithm proved in Section 6. Let $\Sigma\colon \Delta \to X_{\mathcal{G}}$ be the map that associates with a (totally irrational vector) $\mathbf{x}$ the directive sequence $(\sigma_n)_n$ where $\sigma_n = \sigma(T^n(\mathbf{x}))$ for all n. One has the following diagram and measure-theoretical isomorphism, where Σ is a.e. one-to-one and where the shift associates with the label of an infinite path the label of the path deprived of its first edge:*

$$\begin{array}{ccc} \Delta & \xrightarrow{T} & \Delta \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ X_{\mathcal{G}} & \xrightarrow[\text{shift}]{} & X_{\mathcal{G}} \end{array}$$

We now can define the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system from the multidimensional continued fraction algorithm.

Definition 8 (**Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system).** *The* Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system *is the set of $\mathcal{S}$-adic words*

$$\mathbf{u} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a^\infty),$$

whose directive sequence $(\sigma_n)_n$ is an infinite path in $\mathcal{G}$. We distinguish three types of directive sequences together with some restrictions on the chosen letter a:

1. *if $(\sigma)_n \in \mathcal{S}^*\{\alpha_k\}^{\mathbb{N}}$, for $k \in \{1,2,3\}$, then $a = k$ (Type 1);*
2. *else if $(\sigma)_n \in \mathcal{S}^*\{\alpha_k, \alpha_j\}^{\mathbb{N}}$, then $a \in \{j,k\}$, for some $\{i,j,k\} = \{1,2,3\}$ (Type 2);*
3. *otherwise, take any $a \in \{1,2,3\}$ (Type 3).*

The requirements in this definition concerning the choice of the letter a will be clearer with Proposition 13 below: they aim at working with recurrent words which will be used in the computation of the factor complexity function. According to Proposition 6, any $\mathcal{S}$-adic word $\mathbf{u}$ generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector $\mathbf{x} \in \Delta$, according to Definition 5, belongs to the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system. Furthermore, they correspond to Type 3 in Definition 8.

Remark 9. *We stress the following terminology: by Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word, we mean an $\mathcal{S}$-adic word with no other restriction on the directive sequence that the fact that it belongs to $\mathcal{S}^{\mathbb{N}}$ (see Definition 4), whereas for a word in the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system, the restrictions of Proposition 6 are taken into account.*

Example 10. *We continue Example 3. The word generated by the Arnoux-Rauzy-Poincaré algorithm applied to $\mathbf{x} = (1, \pi, \sqrt{2})$ is:*

$$\begin{aligned}\mathbf{u} &= \alpha_2\pi_{13}\alpha_2\alpha_3\alpha_1\pi_{31}\pi_{23}\pi_{31}\pi_{12}(\alpha_3)^8\alpha_1(\alpha_2)^6\pi_{21}\alpha_3\alpha_3\alpha_1\pi_{32}\cdots(1)\\ &= 1232212323221232212323221232123221232322123232212321232212323\cdots\end{aligned}$$

Note that the substitutions shown on the above line determine the prefix of $\mathbf{u}$ of length 1453060. The first prefixes are

$$\alpha_2(1) = 12, \quad \alpha_2\pi_{13}(1) = 1232, \quad \alpha_2\pi_{13}\alpha_2(1) = 123221232, \quad \alpha_2\pi_{13}\alpha_2\alpha_3(1) = 1232212323221232.$$

Observe that due to its S-adic construction, the infinite word $\mathbf{u}$ can be decomposed on three-block codes (that is, on codes consisting of three finite words) in many ways:

$$\begin{aligned}&12322|1232322|12322|1232322|1232|12322|1232322|1232322|\cdots\\ &12|32|2|12|32|32|2|12|32|2|12|32|32|2|12|32|12|\cdots\\ &123|22123|23|22123|22123|23|22123|2123|22123|23|22123|\cdots\end{aligned}$$

The blocks are in each case respectively $\{12322, 1232322, 1232\}$, $\{12, 2, 32\}$ and $\{23, 22123, 2123\}$ (they are obtained as $\sigma_1\cdots\sigma_n(i)$, for $i = 1,2,3$, or else, as return words on the letter 1 in $\mathbf{u}$, where a return word on 1 is a finite word v that does not contain the letter 1, but that is such that $v1$ is a factor of $\mathbf{u}$). For comparison, the billiard word of direction $(1, \pi, \sqrt{2})$ starting at $(0,0,0)$ is:

$$232123221232231223212322132223122321232232122321232213223212 3\cdots$$

It has quadratic factor complexity. It cannot be decomposed on a three-factor code (its has too much return words on each letter).

2.4 Totally irrational vectors and weak primitivity

The next lemma provides a characterization of weakly primitive $\mathcal{S}$-adic expansions. Indeed weak primitivity fails if and only if the directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ contains finitely many Poincaré substitutions and takes ultimately at most two values (that thus are Arnoux-Rauzy substitutions).

Lemma 11. *Let $\mathbf{u} = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_n(a_n^\infty)$ be an $\mathcal{S}$-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to the vector $\mathbf{x} = (x_1, x_2, x_3) \in \Delta$. Its associated $\mathcal{S}$-adic expansion is weakly primitive if and only if*

$$(\sigma_n)_{n\in\mathbb{N}} \notin \mathcal{S}^* \cdot \Big(\{\alpha_1,\alpha_2\}^{\mathbb{N}} \cup \{\alpha_1,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_2,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_1\}^{\mathbb{N}} \cup \{\alpha_2\}^{\mathbb{N}} \cup \{\alpha_3\}^{\mathbb{N}}\Big).$$

Proof. If $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^*\cdot\Big(\{\alpha_1,\alpha_2\}^{\mathbb{N}} \cup \{\alpha_1,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_2,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_1\}^{\mathbb{N}} \cup \{\alpha_2\}^{\mathbb{N}} \cup \{\alpha_3\}^{\mathbb{N}}\Big)$, then it is easily seen that $(\sigma_n)_{n\in\mathbb{N}}$ is not weakly primitive.

Now, let $(\sigma_n)_{n\in\mathbb{N}}$ be the directive sequence of an $\mathcal{S}$-adic expansion which is not weakly primitive in the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system. Being not weakly primitive means that there exists m such that for all p with $m \le p$ the substitution $\sigma_m \cdots \sigma_p$ is not positive, that is, one of the entries of its incidence matrix is zero. Moreover, for all p and r such that $m \le p \le r$ the incidence matrix of the substitution $\sigma_p \cdots \sigma_r$ is not positive.

Note that since the incidence matrix of every substitution in $\mathcal{S}$ has entries 1 on the diagonal, the positivity of entries is preserved by left and right multiplication. Therefore, if $\sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}^*$ is positive, then φ is positive for every $\varphi \in \mathcal{S}^*\sigma_1\mathcal{S}^*\sigma_2\mathcal{S}^*\cdots\mathcal{S}^*\sigma_n\mathcal{S}^*$.

Assume first that $(\sigma_n)_{n\ge m}$ contains no Poincaré substitution. If $(\sigma_n)_{n\ge m}$ contains three distinct Arnoux-Rauzy substitutions, there are some values of p and r with $m \le p \le r$ such that $\sigma_p\cdots\sigma_r$ contains three distinct Arnoux-Rauzy substitutions. One verifies that $\alpha_i\alpha_j\alpha_k$ is positive for all possible values of i, j, k with $\{i,j,k\} = \{1,2,3\}$. Then $\sigma_p\cdots\sigma_r$ is positive which is a contradiction. Therefore, we conclude that $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^* \cdot \Big(\{\alpha_1,\alpha_2\}^{\mathbb{N}} \cup \{\alpha_1,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_2,\alpha_3\}^{\mathbb{N}}\Big)$.

Assume $(\sigma_n)_{n\ge m}$ contains at least one Poincaré substitution. We may suppose that $(\sigma_n)_{n\ge p}$ starts with a Poincaré substitution $\sigma_p = \pi_{jk}$ for $p \ge m$. Since $(\sigma_n)_{n\ge p} \in \mathcal{L}(\mathcal{G})$, then

$$(\sigma_n)_{n\ge p} \in \Big(\pi_{jk}\alpha_j^\infty\Big) \cup \Big(\pi_{jk}\alpha_j^t\{\alpha_i,\pi_{ki},\pi_{ji}\}\mathcal{S}^{\mathbb{N}}\Big) \cup \Big(\pi_{jk}\alpha_j^t\pi_{ij}\alpha_i^\infty\Big) \cup \Big(\pi_{jk}\alpha_j^t\pi_{ij}\alpha_i^s\{\alpha_k,\pi_{jk},\pi_{ik},\pi_{ki}\}\mathcal{S}^{\mathbb{N}}\Big),$$

for some non-negative integers s and t and $\{i,j,k\} = \{1,2,3\}$. But $\pi_{jk}\alpha_i$, $\pi_{jk}\pi_{ki}$ and $\pi_{jk}\pi_{ji}$ are positive. Also $\pi_{jk}\pi_{ij}\alpha_k$, $\pi_{jk}\pi_{ij}\pi_{jk}$, $\pi_{jk}\pi_{ij}\pi_{ik}$ and $\pi_{jk}\pi_{ij}\pi_{ki}$ are positive. Therefore,

$$(\sigma_n)_{n\ge p} \in \Big(\pi_{jk}\alpha_j^\infty\Big) \cup \Big(\pi_{jk}\alpha_j^t\pi_{ij}\alpha_i^\infty\Big)$$

and we have shown that $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^* \cdot \Big(\{\alpha_1\}^{\mathbb{N}} \cup \{\alpha_2\}^{\mathbb{N}} \cup \{\alpha_3\}^{\mathbb{N}}\Big)$. □

Proposition 12. *Let $\mathbf{u}$ be an $\mathcal{S}$-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to the totally irrational vector $\mathbf{x} \in \Delta$. Then the associated S-adic expansion is weakly primitive. In particular, $\mathbf{u}^{(m)}$ is of Type 3, uniformly recurrent and proper, for all m.*

Proof. The conclusion follows from Lemma 11 by noticing that if

$$(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^* \cdot \Big(\{\alpha_1,\alpha_2\}^{\mathbb{N}} \cup \{\alpha_1,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_2,\alpha_3\}^{\mathbb{N}} \cup \{\alpha_1\}^{\mathbb{N}} \cup \{\alpha_2\}^{\mathbb{N}} \cup \{\alpha_3\}^{\mathbb{N}}\Big)$$

then $\mathbf{x}$ cannot be totally irrational. □

Observe that not every word of the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system is uniformly recurrent. Nevertheless, one easily checks that words of this system are all recurrent.

Proposition 13. *Any infinite word* $\mathbf{u}$ *in the Arnoux-Rauzy-Poincaré system is recurrent as well as* $\mathbf{u}^{(m)}$ *for any* m.

Example 14. *The infinite word* $\alpha_1^\infty(2^\infty) = 21^\infty$ *is not recurrent whereas* $\alpha_1^\infty(1^\infty) = 1^\infty$ *is recurrent.*

The restriction of the infinite words under study to the case where each letter always appears as proper factor will also be useful to prove the main result of this article.

Definition 15 (Proper word). *A word* $\mathbf{u} \in \{1,2,3\}^{\mathbb{N}}$ *is said* proper *if each letter* $i \in \{1,2,3\}$ *is a proper factor of* $\mathbf{u}$*, or equivalently, for each letter* $i \in \{1,2,3\}$*, there exists a letter* e *such that* ei *is a factor of* $\mathbf{u}$.

3 Factor complexity

In this section, we define the terminology relative to languages, bispecial factors, extension types and factor complexity. We adopt the notation of [CN10].

3.1 Language and complexity function $p(n)$

Let $\mathcal{A} = \{1,2,\ldots,d\}$ be an alphabet. The length of a word $u \in \mathcal{A}^n$ is denoted by $|u|$ and is equal to n, whereas the notation $|u|_i$ stands for the number of occurrences of the letter i in u. A *language* is a subset of the free monoid $\mathcal{A}^*$. A language L is *factorial* if for any $w \in L$, then any factor u of w belongs to L. The abelianized of a finite word $w \in \mathcal{A}^*$ is the vector

$$\overrightarrow{w} = (|w|_1, |w|_2, \ldots, |w|_d) \in \mathbb{N}^d.$$

We consider an infinite word $\mathbf{u} = u_0u_1u_2u_3\cdots \in \mathcal{A}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, $\mathcal{L}_n(\mathbf{u})$ is the set of factors of length n in $\mathbf{u}$, while $\mathcal{L}(\mathbf{u})$ is the set of all factors in $\mathbf{u}$, and is called the *language of* $\mathbf{u}$. The language of $\mathbf{u}$ is factorial. For each $n \in \mathbb{N}$, let $p_{\mathbf{u}}(n)$ be the cardinality of $\mathcal{L}_n(\mathbf{u})$. Then $p_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$ is a function called the *factor complexity function* of $\mathbf{u}$. When no confusion is possible, we omit $\mathbf{u}$ and just write p.

3.2 Bispecial Factors and Extension Types

Let w be a factor of either a recurrent infinite word or of a finite word $\mathbf{u}$. We let $E^+(w) = \{x \in \mathcal{A} \mid wx \in \mathcal{L}(\mathbf{u})\}$ denote the set of right extensions of w in $\mathbf{u}$. The *right valence* $d^+(w) = \operatorname{Card} E^+(w)$ of w (in $\mathbf{u}$) is defined as the number of distinct right extensions of w. *Left extensions* $E^-(w)$ and *left valence* $d^-(w)$ are defined in a a similar way. A factor whose right valence is at least 2 is called *right special.* A factor whose left valence is at least 2 is called *left special.* A factor which is both left and right special is called *bispecial.* The *extension type* $E_{\mathbf{u}}(w)$ of a factor w of $\mathbf{u}$ is the set of pairs (a,b) of $\mathcal{A} \times \mathcal{A}$ such that w can be extended in both directions as awb:

$$E_{\mathbf{u}}(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}(\mathbf{u})\}.$$

We also use the notation $E_{\mathbf{u}}(w)$ by $E(w)$ when the context is clear. The *bilateral multiplicity* of a factor w is the number

$$m(w) = \operatorname{Card} E(w) - d^-(w) - d^+(w) + 1.$$

We have the following fact (see e.g. [CN10, Proposition 4.5.1]) which links bilateral multiplicity to the notion of bispecial factor: let w be a factor of a recurrent infinite word such that $m(w) \neq 0$;

then, w is bispecial. A bispecial factor is said *strong* if $m(w) > 0$, *weak* if $m(w) < 0$ and *neutral* if $m(w) = 0$. A bispecial factor is *ordinary* if there exist letters $a, b \in \mathcal{A}$ such that

$$\{(a, b)\} \subseteq E(w) \subseteq (\{a\} \times \mathcal{A}) \cup (\mathcal{A} \times \{b\}) \,. \tag{1}$$

An ordinary bispecial factor is neutral, but the converse is not true for $|\mathcal{A}| > 2$. We will use this notion in particular in Section 4.4.

Lemma 16. *If a bispecial factor is ordinary, then it is neutral.*

Proof. If w is ordinary, then

$$\mathrm{Card}\, E(w) = \mathrm{Card}\, E^-(w) + \mathrm{Card}\, E^+(w) - 1$$

because $(a, b) \in E(w)$. Thus, we have $m(w) = \mathrm{Card}\, E(w) - d^-(w) - d^+(w) + 1 = 0$. □

It is convenient to represent the extension type $E(w)$ of a bispecial factor w in a graphical way. It is often represented as a bipartite graph, but we choose here a table representation: a cross ($\times$) is drawn at the intersection of row a and column b if and only if $(a, b) \in E(w)$ (see Figure 3).

	1	2	3
1		×	
2		×	
3	×	×	×

$m(w) = 0$
neutral and ordinary

	1	2	3
1		×	
2			×
3	×	×	×

$m(w) = 0$
neutral but not ordinary

	1	2	3
1		×	
2			
3			×

$m(w) = -1$
weak

	1	2	3
1			
2		×	×
3	×	×	×

$m(w) = 1$
strong

Figure 3: Examples of tables representing the extension type $E(w)$ of a bispecial factor w.

Definition 17 (Left equivalence). *Let w and w' be two bispecial factors defined on the alphabet $\mathcal{A}$. We say that their extension types are* left equivalent *if there exists a permutation τ acting on $\mathcal{A}$ such that $E(w') = \{(\tau(a), b) \mid (a, b) \in E(w)\}$.*

Right equivalence is defined similarly. Left equivalence can be interpreted on the table representation of the extension type as follows. Indeed one representation can be obtained from the other by a permutation of the rows:

$E(w) =$

	1	2	3
1			×
2			
3	×	×	×

and $E(w') =$

	1	2	3
1	×	×	×
2			×
3			

Substitutions considered in this article preserve the first letter and thus preserve the right extensions. Then, the notion of left equivalence is sufficient for our need. But in general, we have the following definition. Of course if the extension type of w and w' are left or right equivalent, then they are also equivalent. When the extension type of two words are equivalent, they share common properties. In particular, being ordinary, strong or weak is preserved under equivalence.

Lemma 18. *Let w and w' be two bispecial factors such that the extension type of w and w' are equivalent, then*

- *w is ordinary (neutral, strong, weak resp.) if and only if w' is ordinary (neutral, strong, weak resp.),*
- $\mathrm{Card}E(w) = \mathrm{Card}E(w')$, $d^-(w) = d^-(w')$, $d^+(w) = d^+(w')$, $m(w) = m(w')$,
- *if the extension type of w and w' are left equivalent, then $E^+(w) = E^+(w')$,*
- *if the extension type of w and w' are right equivalent, then $E^-(w) = E^-(w')$.*

3.3 Factor Complexity

Let $p(n)$ be the factor complexity function of the infinite word $\mathbf{u}$. Two other functions derived from the factor complexity are useful, namely the sequences of *finite differences of order* 1 *and* 2 respectively of $p(n)$:

$$s(n) = p(n+1) - p(n), \tag{2}$$
$$b(n) = s(n+1) - s(n). \tag{3}$$

Of course, we have

$$p(n) = p(0) + \sum_{\ell=0}^{n-1} s(\ell), \tag{4}$$
$$s(n) = s(0) + \sum_{\ell=0}^{n-1} b(\ell). \tag{5}$$

These equations are very useful to compute the complexity function $p(n)$ when its growth is slow (for example in the case of a linear growth), since in this case functions s and b take small values. For example, we have $p(n) = n+1$ for all n ($\mathbf{u}$ is thus a Sturmian word) if and only if $s(n)$ is always equal to 1, which is also equivalent to the fact that exactly two letters occur ($p(1) = 2$, $s(0) = 1$) and that $b(n)$ always takes the value 0.

In this article, one of our main results is to show that some infinite words on a three-letter alphabet have complexity $p(n) < 3n$. In order to achieve this, we use the next lemma.

Lemma 19. *Suppose* $|\mathcal{A}| = 3$. *Then,* $p(n+1) - p(n) \in \{2, 3\}$ *if and only if* $\sum_{\ell=0}^{n-1} b(\ell) \in \{0, 1\}$. *Furthermore, if the sequence of finite differences of order* 2 *is such that*

$$(b(\ell))_\ell = 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0, -1, \ldots$$

then $\sum_{\ell=0}^{n-1} b(\ell) \in \{0, 1\}$.

Proof. Since $|\mathcal{A}| = 3$, then $p(1) = 3$ and $s(0) = p(1) - p(0) = 3 - 1 = 2$. We have

$$p(n+1) - p(n) = s(n) = s(0) + \sum_{\ell=0}^{n-1} b(\ell) = 2 + \sum_{\ell=0}^{n-1} b(\ell),$$

which yields the proof of the first statement.

The proof of the second one comes from the fact that the first non-zero term of the sequence $(b(\ell))_\ell$ is $+1$. □

The finite differences of order 1 and 2 of $p(n)$ are related to special and bispecial factors as explained in [Cas97a]. We state a weaker form (for recurrent words) of a result of [CN10]. Indeed, as we are interested in the factor complexity of some recurrent words, we do not need to consider unioccurrent or exceptional prefixes.

Theorem 20. [CN10, Theorem 4.5.4] *Let* $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ *be an infinite recurrent word. Then, for all* $n \in \mathbb{N}$*:*

$$s(n) = \sum_{w \in \mathcal{L}_n(u)} (d^+(w) - 1) = \sum_{w \in \mathcal{L}_n(u)} (d^-(w) - 1) \tag{6}$$
$$b(n) = \sum_{w \in \mathcal{L}_n(u)} m(w). \tag{7}$$

4 Bispecial Factors under Arnoux-Rauzy and Poincaré Substitutions

The goal of the next sections is to describe factors of Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic words. The key ingredient is a synchronization lemma that allows the desubstitution with respect to the substitutions in $\mathcal{S}$ (Section 4.1). As a consequence for bispecial factors, antecedents (they are uniquely defined and always bispecial) and bispecial images together with their possible extensions are described in details in Section 4.2 and 4.3, respectively. We then can consider the notion of life of a bispecial factor produced by an $\mathcal{S}$-adic expansion (Section 4.4). So far we still do not use the restrictions of Proposition 6 on the possible directive sequences in $\mathcal{S}^{\mathbb{N}}$ (they will be considered only in Section 5). Section 4.5 illustrates the fact that a quadratic factor complexity can be reached without these restrictions. We end this section with the introduction of notions of order on vectors allowing the comparison of abelianized vectors under the application of substitutions in $\mathcal{S}$ (Section 4.6).

We recall that a Poincaré substitution is of the form $\pi_{jk} : i \mapsto ijk, j \mapsto jk, k \mapsto k$. An Arnoux-Rauzy substitution is given by $\alpha_k : i \mapsto ik, j \mapsto jk, k \mapsto k$.

4.1 Synchronization lemma

From now on, the alphabet is set to $\mathcal{A} = \{1, 2, 3\}$. The next lemma describes the preimage of a factor under Arnoux-Rauzy (**AR**) and Poincaré (**P**) substitutions. Such statements are classical tools when computing the factor complexity of fixed points of substitutions.

Lemma 21 (**Synchronization**). *Let $u \in \mathcal{A}^*$ and w be a factor of $\alpha_k(u)$ for some $\{i, j, k\} = \{1, 2, 3\}$.*

(i) *If w is empty or if the first letter of w is i or j, then there exist a unique $v \in \mathcal{A}^*$ and a unique $s \in \{\varepsilon, i, j\}$ such that $w = \alpha_k(v) \cdot s$.*

(ii) *If the first letter of w is k, then there exist a unique $v \in \mathcal{A}^*$ and a unique $s \in \{\varepsilon, i, j\}$ such that $w = k \cdot \alpha_k(v) \cdot s$.*

Let $u \in \mathcal{A}^*$ and w be a factor of $\pi_{jk}(u)$ for some $\{i, j, k\} = \{1, 2, 3\}$.

(iii) *If w is empty or if the first letter of w is i, then there exist a unique $v \in \mathcal{A}^*$ and a unique $s \in \{\varepsilon, i, j, ij\}$ such that $w = \pi_{jk}(v) \cdot s$.*

(iv) *If $w = j$, then there exist a unique $v(= \varepsilon)$ such that $w = j \cdot \pi_{jk}(v)$.*

(v) *If the first letter of w is j and $|w| > 1$, then there exist a unique $v \in \mathcal{A}^*$ and a unique $s \in \{\varepsilon, i, j, ij\}$ such that $w = jk \cdot \pi_{jk}(v) \cdot s$.*

(vi) *If the first letter of w is k, then there exist a unique $v \in \mathcal{A}^*$ and a unique $s \in \{\varepsilon, i, j, ij\}$ such that $w = k \cdot \pi_{jk}(v) \cdot s$.*

Proof. The sets $\{ik, jk, k\}$ and $\{ijk, jk, k\}$ form a prefix code. □

Definition 22 (**Antecedent, extended image**). *Let $\sigma = \alpha_k$ or $\sigma = \pi_{jk}$, $u \in \mathcal{A}^*$ and w be a factor of $\sigma(u)$. We say that the* antecedent of w under σ *is the unique word v as defined by Lemma 21. If v is the antecedent of a word w, then we say that the word w is an* extended image *of v.*

Note that the antecedent is unique, but that a word v may have more than one extended image. Consider for instance $w_1 = 23\pi_{23}(11)1 = 231231231$ and $w_2 = 3\pi_{23}(11)2 = 31231232$ which are two distinct extended images of $v = 11$. This is why the situation becomes here quite intricate especiallly for bispecial factors. In fact, it happens that strong and weak bispecial words appear in pairs: the image of a neutral bispecial factor v can have two extended images that are bipsecial, with one of them being strong, and the other one being weak. For more details, see Lemma 36 and Remark 37 below.

We now consider images and antecedents of bispecial factors.

Definition 23 (**Bispecial extended image**). *Let $u \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ and v be a factor of u. We shall say that a* bispecial extended image *w of v under σ is a bispecial word of $\sigma(u)$ which is an extended image of v under σ.*

For example, let v be a bispecial factor and suppose $E(v) = \{(1,2),(2,3),(3,1),(3,2),(3,3)\}$. Then $w = 3\pi_{23}(v)$ and $w' = 23\pi_{23}(v)$ are both bispecial extended images of v under π_{23}. Indeed, we have

$$\pi_{23}\left(\{1v2, 2v3, 3v1, 3v2, 3v3\}\right) = \{123\pi_{23}(v)23, 23\pi_{23}(v)3, 3\pi_{23}(v)123, 3\pi_{23}(v)23, 3\pi_{23}(v)3\}$$

and the extension types are $E(w) = \{(2,2),(2,3),(3,1),(3,2),(3,3)\}$ and $E(w') = \{(1,2),(3,3)\}$.

The next lemma allows one to relate every bispecial factor to a shorter one and eventually to the empty word.

Lemma 24 (**Bispecial extended image growth**). *Let $\sigma = \alpha_k$ or $\sigma = \pi_{jk}$ and $w \neq \varepsilon$ be a non-empty bispecial extended image of v under σ. Then, $|v| < |w|$.*

Proof. Suppose that $\sigma = \alpha_k$ for some $k \in \{1,2,3\}$. Since w is non-empty, w starts and ends with letter k and from Lemma 21 (ii), the unique antecedent v of w is such that $w = k\alpha_k(v)$. We conclude that $|v| < |w|$.

Suppose that $\sigma = \pi_{jk}$ for some $\{i,j,k\} = \{1,2,3\}$. Since w is non-empty, w starts with letter j or k and ends with letter k. From Lemma 21 (iv) and (v), the unique antecedent v of w is such that $w = k\pi_{jk}(v)$ or $w = jk\pi_{jk}(v)$. In both cases, $|v| < |w|$. □

4.2 Arnoux-Rauzy substitutions

The case of Arnoux-Rauzy substitutions is particularly convenient to handle, both for bispecial extended images or for antecedents of bispecial factors.

Lemma 25 (**AR - Bispecial extended image**). *Let $u \in \mathcal{A}^*$ and let v be a bispecial factor of u. There is a unique bispecial extended image $w = k\alpha_k(v)$ of v in $\alpha_k(u)$.*

Proof. Let w and w' be two extended images of v under α_k. Since they are bispecial factors, one deduces from Lemma 21 that both w and w' start and end with letter k. Hence $w = k\alpha_k(v) = w'$. □

Lemma 26 (**AR - Antecedent of a bispecial**). *Let $u \in \{1,2,3\}^*$ and $w \neq \varepsilon$ be a bispecial factor of $\alpha_k(u)$. Let v be the unique antecedent of w under α_k. One has $w = k\alpha_k(v)$. Furthermore, v is bispecial and it has the same extension type $E_{\alpha_k(u)}(w) = E_u(v)$ and same multiplicity $m(w) = m(v)$ as w.*

Proof. One checks that $(a,b) \in E(v)$ if and only if $(a,b) \in E(k\alpha_k(v))$ (see Figure 4). Then $E(k\alpha_k(v)) = E(v)$. We deduce that $E^+(k\alpha_k(v)) = E^+(v)$ and $E^-(k\alpha_k(v)) = E^-(v)$. From this we conclude that $m(k\alpha_k(v)) = m(v)$. □

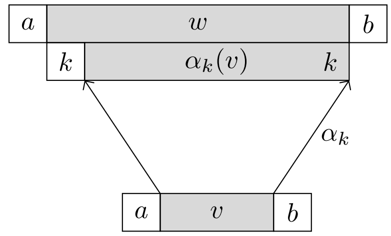

Figure 4: The preimage of the bispecial word w under α_k.

4.3 Poincaré substitutions

The case of Poincaré substitutions is more delicate to handle as already illustrated by the following result. We loose here unicity for the bispecial extended images.

Lemma 27 (**P - Bispecial extended images).** *Let i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$. Let $u \in \{1, 2, 3\}^*$ and let v be a bispecial factor of u. There are at most two distinct bispecial extended images of v under π_{jk}. They are either $k\pi_{jk}(v)$ or $jk\pi_{jk}(v)$.*

Proof. Let w be a bispecial extended image of v under π_{jk}. Since w is a bispecial factor, it must start with letter j or k and end with letter k. From Lemma 21, one gets $w \in \{jk\pi_{jk}(v), k\pi_{jk}(v)\}$. □

The "at most two" of Lemma 27 will be made more precise later in Lemma 30 where conditions will be given for when a bispecial factor has one or two bispecial extended images under a Poincaré substitution.

In order to get a similar result concerning the antecedent of a bispecial factor under Poincaré substitutions (see Lemma 29 below), we first need the following result stated for factors in general which is also used for proving Lemma 30 and 36.

Lemma 28 (**P - Extensions).** *Let i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$. Let $u \in \{1, 2, 3\}^*$ and v be a factor of u. We assume that for all $(a, b) \in E(v)$, there exists a letter e such that $eavb$ is also a factor of u. The extensions of v in u are related to the extensions of $k\pi_{jk}(v)$ and $jk\pi_{jk}(v)$ considered as factors of $\pi_{jk}(u)$:*

$$\begin{aligned}
(i, b) \in E(v) &\iff (j, b) \in E(k\pi_{jk}(v)) \quad \text{and} \quad (i, b) \in E(jk\pi_{jk}(v)),\\
(j, b) \in E(v) &\iff (j, b) \in E(k\pi_{jk}(v)) \quad \text{and} \quad (k, b) \in E(jk\pi_{jk}(v)),\\
(k, b) \in E(v) &\iff (k, b) \in E(k\pi_{jk}(v)).
\end{aligned}$$

Proof. First note that $i \notin E^-(k\pi_{jk}(v))$ and $j \notin E^-(jk\pi_{jk}(v))$. Note also that the right extensions are preserved by π_{jk} because π_{jk} preserves the first letter of words. Let $(a_0, b) \in E(v)$, $(a_1, b) \in E(k\pi_{jk}(v))$ and $(a_2, b) \in E(jk\pi_{jk}(v))$ and let us consider each case $a_0 = i$, $a_0 = j$ and $a_0 = k$ separately (see Figure 5). According to the assumption made on v, one checks that if $a_0 = i$, then $a_1 = j$ and $a_2 = i$; if $a_0 = j$, then $a_1 = j$ and $a_2 = k$; if $a_0 = k$, then $a_1 = k$. The reciprocals are also verified. □

In the next lemma, we show that bispecial factors are preserved under desubstitution by the Poincaré substitution.

Lemma 29 (**P - Antecedent of a bispecial).** *Let $u \in \{1, 2, 3\}^*$ and $w \neq \varepsilon$ be a bispecial factor of $\pi_{jk}(u)$. Let v be the unique antecedent of w under π_{jk}. One has either $w = k\pi_{jk}(v)$, or $w = jk\pi_{jk}(v)$. Furthermore, v is a bispecial factor of u.*

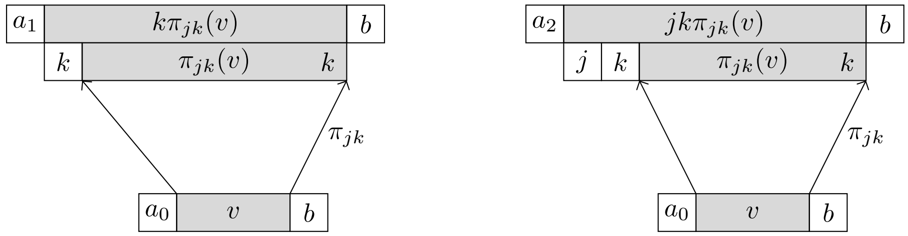

Figure 5: The preimage of $k\pi_{jk}(v)$ and $jk\pi_{jk}(v)$ under π_{jk}.

Proof. The result is a direct consequence of Lemma 28. Since right extensions are preserved by π_{jk}, we only need to check that if w has at least two left extensions then so does v.

Suppose that $w = k\pi_{jk}(v)$. Remark that $i \notin E^-(w)$. Thus $j, k \in E^-(w)$ since w is bispecial. From Lemma 28, $k \in E^-(w)$ implies $k \in E^-(v)$. Also, $j \in E^-(w)$ implies that $i \in E^-(v)$ or $j \in E^-(v)$. Thus v is bispecial.

Suppose that $w = jk\pi_{jk}(v)$. Since $j \notin E^-(w)$, then $i, k \in E^-(w)$. Or course, the existence of w implicitly suppose $j \in E^-(k\pi_{jk}(v))$. Then, $i, j \in E^-(v)$. We conclude that v is bispecial. ☐

Now we want to describe more precisely under which conditions a bispecial word v has a unique bispecial extended image and provide its extension type as we were able to do in Lemma 26 for Arnoux-Rauzy substitutions. In general (see Table 1 and 2), this depends on its left extensions $E^-(v)$. However, if the left valence satisfies $d^-(v) = 2$, we deduce the unicity of the bispecial extended image as well as important information on the extension type of the extended image. Recall that the notion of left equivalence for extension types was defined in Section 3.2 in Definition 17.

Lemma 30 (P - Bispecial extended images in details). *Let i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$. Let $u \in \{1, 2, 3\}^*$ and let v be a bispecial factor of u. We assume that for all $(a, b) \in E(v)$, there exists a letter e such that $eavb$ is also a factor of u.*

(i) *If $d^-(v) = 2$, v admits a unique bispecial extended image $w \in \{k\pi_{jk}(v), jk\pi_{jk}(v)\}$ under π_{jk} and $d^-(w) = 2$. Moreover, the extension types $E(v)$ and $E(w)$ (in $\pi_{jk}(u)$) are left equivalent and are related according to Table 1.*

(ii) *If $d^-(v) = 3$, then v admits either one, or two bispecial extended images $w \in \{k\pi_{jk}(v), jk\pi_{jk}(v)\}$ under π_{jk}. In any case, $d^-(w) = 2$ and the two non-empty rows of $E(w)$ are obtained by projection of rows of $E(v)$. Furthermore, they are related according to Table 2.*

Proof. For each $a \in \{1, 2, 3\}$, let $R_a \subseteq \{1, 2, 3\}$ be such that

$$E(v) = \bigcup_{a \in \{1,2,3\}} \{a\} \times R_a.$$

The set R_a denotes the right extensions associated with the left extension $a \in E^-(v)$.
(i) If $d^-(v) = 2$, then $E^-(v)$ is equal to either $\{i, j\}$, $\{i, k\}$ or $\{j, k\}$. We proceed case by case. If $E^-(v) = \{i, j\}$, then $k\pi_{jk}(v)$ is not left special and $jk\pi_{jk}(v)$ is the unique bispecial extended image of v. If $E^-(v) = \{i, k\}$ or $\{j, k\}$, then $jk\pi_{jk}(v)$ is not left special and $k\pi_{jk}(v)$ is the unique bispecial extended image of v. This is summarized in Table 1 where the information follows from Lemma 28. In each case, the extension type $E(v)$ is left equivalent to the extension type of the unique bispecial extended image w of v. Moreover $d^-(w) = 2$.

(ii) If $d^-(v) = 3$, i.e., $E^-(v) = \{i, j, k\}$, then $E^-(k\pi_{jk}(v)) = \{j, k\}$ and $E^-(jk\pi_{jk}(v)) = \{i, k\}$. Thus, both extended images can be bispecial but their left valence is at most 2. This is summarized in Table 2. ☐

$E(v)$	$E(k\pi_{jk}(v))$	$E(jk\pi_{jk}(v))$
$(\{i\} \times R_i) \cup (\{j\} \times R_j)$	$\{j\} \times (R_i \cup R_j)$	$(\{i\} \times R_i) \cup (\{k\} \times R_j)$
$(\{i\} \times R_i) \cup (\{k\} \times R_k)$	$(\{j\} \times R_i) \cup (\{k\} \times R_k)$	$\{i\} \times R_i$
$(\{j\} \times R_j) \cup (\{k\} \times R_k)$	$(\{j\} \times R_j) \cup (\{k\} \times R_k)$	$\{k\} \times R_j$

Table 1: If $d^-(v) = 2$, then exactly one extended image of v amongst $k\pi_{jk}(v)$ and $jk\pi_{jk}(v)$ is bispecial. This only depends on the left extensions as the right extensions are preserved.

$E(v)$	$E(k\pi_{jk}(v))$	$E(jk\pi_{jk}(v))$
$(\{i\} \times R_i) \cup (\{j\} \times R_j) \cup (\{k\} \times R_k)$	$(\{j\} \times R_i \cup R_j) \cup (\{k\} \times R_k)$	$(\{i\} \times R_i) \cup (\{k\} \times R_j)$

Table 2: If $d^-(v) = 3$, then one or both extended images of v amongst $k\pi_{jk}(v)$ and $jk\pi_{jk}(v)$ are bispecial. In each case, their left valence is 2.

Note that Table 1 and 2 provide much more information than does the statement of Lemma 30 and they will be used to prove a more general result in Lemma 36. For example, in Table 2, if v is a bispecial factor such that $d^-(v) = 3$, $R_i = R_j$ and $|R_i| = |R_j| = 1$, then $jk\pi_{jk}(v)$ is a left special factor but not a right special factor, it is thus not bispecial.

4.4 Life of a bispecial factor under ARP substitutions

In this section, the life of a bispecial factor is analyzed more precisely under the application of Arnoux-Rauzy and Poincaré substitutions in the spirit of [Cas97a, Section 4.2.2] where bispecial factors are described under the image of circular morphisms. To achieve this, we need to understand exactly the left extensions which will give information about the multiplicity of the bispecial factors.

Let $\mathcal{S} = \mathcal{S}_\alpha \cup \mathcal{S}_\pi$. Let w be a factor of an infinite word Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word. Let $w_0 = w$ and w_{i+1} be the unique antecedent of w_i under σ_i for $i \geq 0$. In particular, w_1 is the antecedent of w_0 under σ_0 and w_2 is the antecedent of w_1 under σ_1. If $|w_i| > 0$, then $|w_{i+1}| < |w_i|$ by Lemma 24. There thus exists n such that $w_n = \varepsilon$.

Definition 31 (**Age, History, Life**). *Let w be a factor of an Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word. Let $w_0 = w$ and w_{i+1} be the unique antecedent of w_i under σ_i for $i \geq 0$. The smallest of the integers n for which $w_n = \varepsilon$ is called the* age *of w and is denoted as* $\operatorname{age}(w)$. *Furthermore, we say that the finite sequence $\sigma_0\sigma_1\cdots\sigma_n$ is the* history *and the sequence $(w_i)_{0\leq i\leq n}$ is the* life *of the word w.*

The above definition is illustrated in Figure 6. According to Lemma 26 and 29, all the words w_i of the history of w are bispecial factors when w is bispecial. We will consider from now on recurrent Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic words $\mathbf{u}$, with $\mathbf{u}^{(m)}$ being also recurrent, in order to apply the assumptions of Lemma 28 and 30. According to Proposition 13, note that this assumption applies in particular to all the words of the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system.

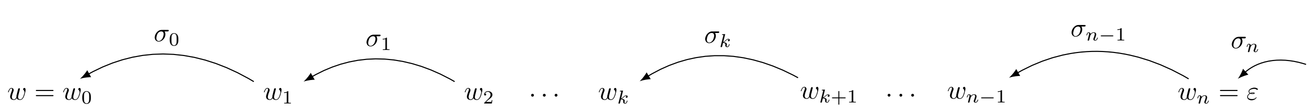

Figure 6: Life and history of a factor w.

Lemma 32. *Let* $\mathbf{u}$ *be a recurrent Arnoux-Rauzy-Poincaré* $\mathcal{S}$*-adic word such that* $\mathbf{u}^{(m)}$ *is also recurrent for all* m*. Let* $n \geq 0$ *be an integer. Let* B_n *be the set of all bispecial factors of age* n *in* $\mathbf{u}$*. Then* $\operatorname{Card} B_n \leq 2$*.*

Proof. Let $w \in B_n$, $\sigma_0\sigma_1 \cdots \sigma_n$ be its history, and let $(w_i)_{0 \leq i \leq n}$ be its life.

Suppose first that $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}_\alpha^* \mathcal{S}$, that is, the substitutions of the history of w are all Arnoux-Rauzy substitutions except possibly σ_n which may be a Poincaré substitution. From Lemma 25, w_i is the unique extended image of w_{i+1} for all $0 \leq i \leq n-1$. Then $\operatorname{Card} B_n = 1$.

Suppose now that $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}^* \pi_{jk} \mathcal{S}_\alpha^* \mathcal{S}$. Let ℓ be the largest index smaller than n of occurrence of π_{jk}, that is,

$$\sigma_0\sigma_1 \cdots \sigma_\ell \in \mathcal{S}^* \pi_{jk} \quad \text{and} \quad \sigma_{\ell+1}\sigma_{\ell+2} \cdots \sigma_n \in \mathcal{S}_\alpha^* \mathcal{S}.$$

Then, from Lemma 25, w_i is the unique extended image of w_{i+1} for all $\ell + 1 \leq i \leq n-1$. Also, from Lemma 30, $w_{\ell+1}$ has at most two extended images w_ℓ and w'_ℓ in $\{k\pi_{jk}(w_{\ell+1}), jk\pi_{jk}(w_{\ell+1})\}$. But then, $d^-(w'_\ell) = d^-(w_\ell) = 2$ (still by Lemma 30). Therefore both w'_ℓ has a unique extended image $w'_{\ell-1}$ and w_ℓ has a unique extended image $w_{\ell-1}$ (by Lemma 30 (i)). Recursively, we get $d^-(w'_i) = d^-(w_i) = 2$ for all $0 \leq i \leq \ell$, w'_i has a unique extended image w_{i-1}, and w_i has a unique extended image w_{i-1} for all $1 \leq i \leq \ell$. We thus get $\operatorname{Card} B_n \leq 2$. □

The life $(w_i)_{0 \leq i \leq n}$ of bispecial factors "starts" (when read backwards with decreasing indices) as the empty word ε at $i = n$. The word w_i for $i < n$ is then obtained as the concatenation of one or two letters concatenated with $\sigma_i(w_{i+1})$. These letters depend on the extension type $E(w_{i+1})$ and recursively on the extension type $E(w_n)$ of $w_n = \varepsilon$. Furthermore, w_n is the antecedent of w_{n-1} under σ_{n-1} and the extension type $E(w_n)$ of $w_n = \varepsilon$ depends on σ_n. Thus, it is important to understand properly what are the possible extension types of the empty word under the application of Arnoux-Rauzy and Poincaré substitutions. Below, the extension type $E(\varepsilon)$ of the empty word considered as a bispecial factor in the language of $\sigma(u)$ is denoted by $E_{\sigma(u)}(\varepsilon)$.

Lemma 33. *Let* $\mathbf{u} \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ *be a proper word. Considered as a bispecial factor of the language of the word* $\alpha_k(\mathbf{u})$*, the empty word* ε *is ordinary. Considered as a bispecial factor of the language of the word* $\pi_{jk}(\mathbf{u})$*, the empty word* ε *is neutral but not ordinary:*

$$E_{\alpha_k(\mathbf{u})}(\varepsilon) = \begin{array}{c|ccc} & i & j & k \\ \hline i & & & \times \\ j & & & \times \\ k & \times & \times & \times \end{array} \qquad \textit{and} \qquad E_{\pi_{jk}(\mathbf{u})}(\varepsilon) = \begin{array}{c|ccc} & i & j & k \\ \hline i & & \times & \\ j & & & \times \\ k & \times & \times & \times \end{array}.$$

Proof. We need to consider the set of pairs of consecutive letters appearing in the language $\alpha_k(u)$. These can be consecutive letters inside $\alpha_k(1)$, $\alpha_k(2)$ or $\alpha_k(3)$, i.e., $\{ik, jk\}$. Alternatively, it may be the last letter of a word $\alpha_k(a)$ with the first letter of a word $\alpha_k(b)$: $\{ki, kj, kk\}$.

Similarly for the language $\pi_{jk}(u)$, consecutive letters inside $\pi_{jk}(i)$, $\pi_{jk}(j)$ and $\pi_{jk}(k)$ are $\{ij, jk\}$ and pairs made of the last letter of a word $\pi_{jk}(a)$ with the first letter of a word $\pi_{jk}(b)$ are $\{ki, kj, kk\}$. □

From now on, we assume that the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic words $\mathbf{u}^{(m)}$ are all proper for all m in order to apply Lemma 33 for the bispecial factors of all ages. Note that being recurrent does not imply the fact of being proper: indeed an infinite word can be recurrent on the alphabet $\{1, 2\}$ while each letter of the alphabet $\{1, 2, 3\}$ must appear for this word to be proper.

Lemma 34. *Let* $\mathbf{u}$ *be an Arnoux-Rauzy-Poincaré* $\mathcal{S}$*-adic word such that* $\mathbf{u}^{(m)}$ *is proper and recurrent for all* m*. Let* w *be a bispecial factor of* $\mathbf{u}$*. Then* $|E(w)| \leq 5$*.*

Proof. Let $n = \mathrm{age}(w)$ and $(w_i)_i$ be the life of w. From Lemmas 26, 30 and 33, we have

$$|E(w_0)| \leq |E(w_1)| \leq |E(w_2)| \leq \cdots \leq |E(w_n)| \leq 5.$$ □

The following lemma shows that the histories of bispecial factors in a same infinite word $\mathbf{u}$ are related.

Lemma 35. *Let $\mathbf{u}$ be an Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word such that $\mathbf{u}^{(m)}$ is proper and recurrent for all m. Let w and z be two bispecial factors of $\mathbf{u}$.*

(i) *If $\mathrm{age}(w) < \mathrm{age}(z)$, then the history of w is a prefix of the one of z.*

(ii) *If $\mathrm{age}(w) = \mathrm{age}(z)$, then w and z have the same history.*

Proof. Statement (i) follows from the definition and statement (ii) follows from (i). □

In the next lemma, we describe exactly what are the bispecial factors associated with each possible history. We recall that there are at most two bispecial factors of the same age for a given history according to Lemma 32. It has the same history as w according to Lemma 35.

Lemma 36. *Let $\mathbf{u}$ be an Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word such that $\mathbf{u}^{(m)}$ is proper and recurrent for all m. Let w be a bispecial factor of $\mathbf{u}$ and let $n = \mathrm{age}(w)$. Let w' be the other bispecial factor of the same age as w if it exists. Then the common history $\sigma_0\sigma_1\cdots\sigma_n$ of w and w' determines the left valence, the multiplicity and the extension type of both w and w'. More precisely, the multiplicity and the extension type are described in Table 3, whereas extension types are provided in Figures 7, 8, 9 and 10.*

$\sigma_0\sigma_1\cdots\sigma_n \in$	$d^-(w)$	$m(w)$	ordinary	$d^-(w')$	$m(w')$	ordinary
$\mathcal{S}_\alpha^*\mathcal{S}_\alpha$	3	0	yes			
$\mathcal{S}_\alpha^*\mathcal{S}_\pi$	3	0	no			
$\mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_k\}$	2	0	yes			
$\mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_i,\alpha_j\}$	2	0	yes	2	0	yes
$\mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\pi_{ji},\pi_{ki},\pi_{ij},\pi_{kj}\}$	2	0	yes	2	0	yes
$\mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\pi_{jk},\pi_{ik}\}$	2	+1	no	2	−1	no

Table 3: Left valence and multiplicity for the (at most two) bispecial factors of the same age.

Remark 37. *Recall that the occurrence of strong and weak bispecial factors has an impact on the factor complexity. According to Lemma 36, strong and weak bispecial words appear in pairs under the application of Poincaré substitutions each time π_{jk} is followed by either π_{jk} or π_{ik} for $\{i,j,k\} = \{1,2,3\}$ with possibly some Arnoux-Rauzy substitutions α_k, $k \in \{1,2,3\}$, in between.*

Proof. In the following proof, elements (j,k) of $E(w)$ are noted jk for short. We refer below to the lines of Table 3.

Line 1. Assume $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}_\alpha^*\mathcal{S}_\alpha$. According to Lemma 26, the extension type is preserved by Arnoux-Rauzy substitutions, which yields $E(w) = E_{\sigma_n}(\varepsilon)$, so that $d^-(w) = 3$. Moreover, since $\sigma_n \in \mathcal{S}_\alpha$, then $E_{\sigma_n}(\varepsilon)$ is ordinary and the multiplicity is $m(w) = 0$ (by Lemma 33). Also, the bispecial extended images are unique under the application of each substitution $\sigma_i \in \mathcal{S}_\alpha$, by Lemma 25.

Line 2. Assume $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}_\alpha^*\mathcal{S}_\pi$. The proof is the same as for Line 1 except that the extension type of the empty word $E_{\sigma_n}(\varepsilon)$ is not ordinary because $\sigma_n \in \mathcal{S}_\pi$ (by Lemma 33).

Line 3-6. We assume $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\mathcal{S}$. Let ℓ be the largest index of occurrence smaller than n of π_{jk}, that is,

$$\sigma_0\sigma_1\cdots\sigma_\ell \in \mathcal{S}^*\,\pi_{jk} \quad \text{and} \quad \sigma_{\ell+1}\sigma_{\ell+2}\cdots\sigma_n \in \mathcal{S}_\alpha^*\mathcal{S}.$$

The bispecial antecedent of w under the substitution $\sigma_0\sigma_1\cdots\sigma_{\ell-1} \in \mathcal{S}^*$ is w_ℓ, and $w_{\ell+1}$ is the bispecial antecedent of w_ℓ under the substitution $\sigma_\ell = \pi_{jk}$. Since $\sigma_{\ell+1}\sigma_{\ell+2}\cdots\sigma_n \in \mathcal{S}_\alpha^*\mathcal{S}$, then $d^-(w_{\ell+1}) = 3$ and $w_{\ell+1}$ has two extended images under $\sigma_\ell = \pi_{jk}$. Moreover, let w'_ℓ, with $w'_\ell \neq w_\ell$, be the other extended image of $w_{\ell+1}$. One has $w_\ell, w'_\ell \in \{k\pi_{jk}(w_{\ell+1}),\ jk\pi_{jk}(w_{\ell+1})\}$. Note that the factor w'_ℓ may be bispecial or not (see e.g. the proof of the case of Line 3 below). The end of the proof for lines 3-6 follows the same pattern. In fact, the first part $\sigma_0\sigma_1\cdots\sigma_{\ell-1} \in \mathcal{S}^*$ is always applied on a bispecial factor w_ℓ or w'_ℓ with left valence satisfying $d^-(w_\ell) = d^-(w'_\ell) = 2$. Therefore, from Lemma 30 (i) the extension types of $w = w_0$ and w_ℓ are left-equivalent. Similarly, the extension types of $w' = w'_0$ and w'_ℓ are left-equivalent (where the w'_i are inductively defined as extended images). From Lemma 18, the multiplicity, the left valence and the fact of being strong, weak or ordinary is preserved by left-equivalence. Below, we suppose $w_\ell = k\pi_{jk}(w_{\ell+1})$ and $w'_\ell = jk\pi_{jk}(w_{\ell+1})$.

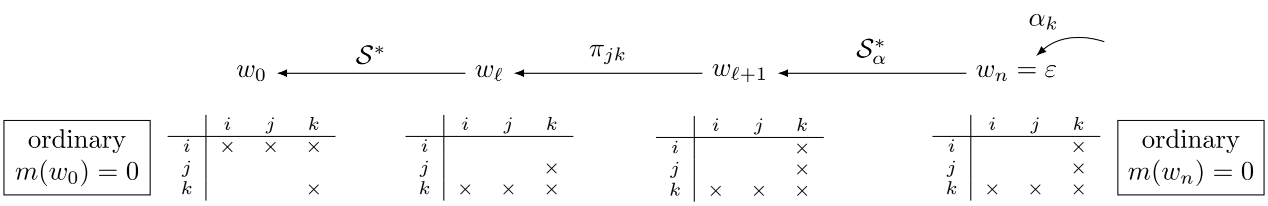

Figure 7: Life of a bispecial word if $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_k\}$.

Line 3. We assume $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_k\}$ (see Figure 7). If $\sigma_n = \alpha_k$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\alpha_k(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ik, jk, ki, kj, kk\}$, $E(w_\ell) = \{jk, ki, kj, kk\}$ and $E(w'_\ell) = \{ik, kk\}$. Then w_ℓ is bispecial ordinary and w'_ℓ is not bispecial.

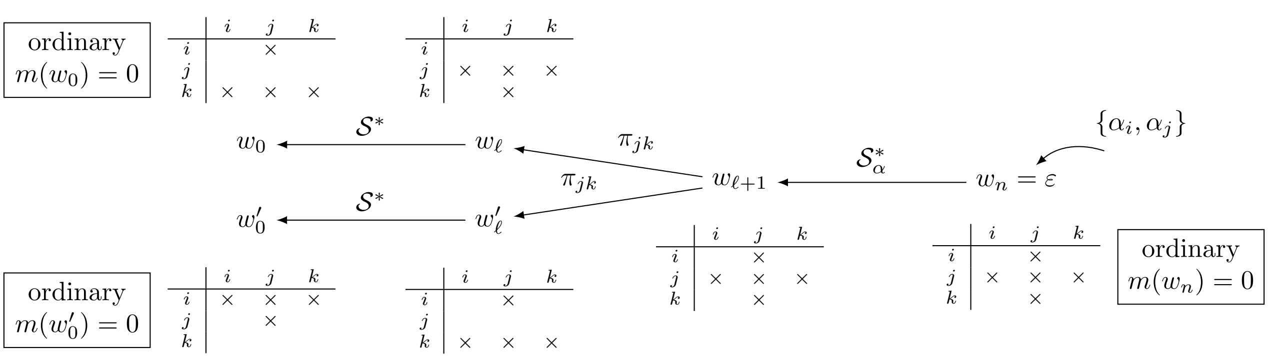

Figure 8: Life of a bispecial word if $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_i,\alpha_j\}$. The extension types depicted represent the case $\sigma_n = \alpha_j$.

Line 4. Assume $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\{\alpha_i,\alpha_j\}$ (see Figure 8). If $\sigma_n = \alpha_i$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\alpha_i(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ii, ij, ik, ji, ki\}$, $E(w_\ell) = \{ji, jj, jk, ki\}$, and $E(w'_\ell) = \{ii, ij, ik, ki\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

If $\sigma_n = \alpha_j$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\alpha_j(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ij, ji, jj, jk, kj\}$, $E(w_\ell) = \{ji, jj, jk, kj\}$ and $E(w'_\ell) = \{ij, ki, kj, kk\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

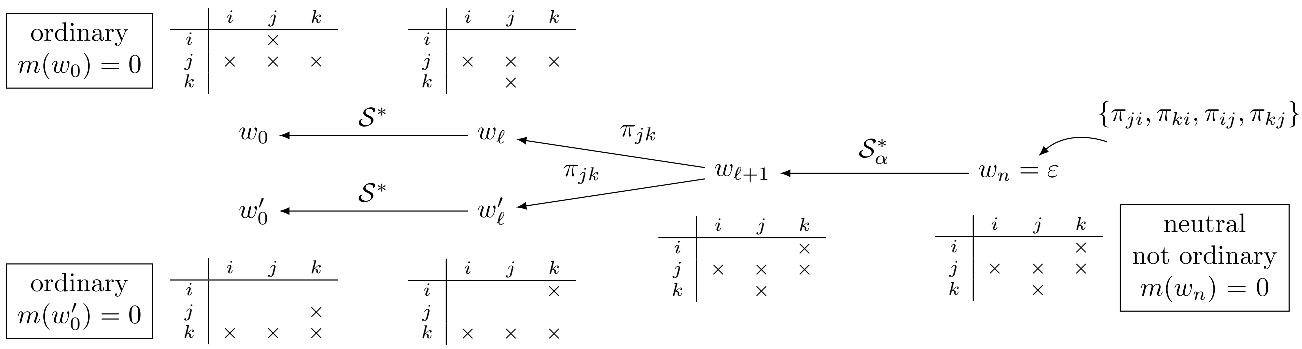

Figure 9: Life of a bispecial word if $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}^* \pi_{jk} \mathcal{S}^*_\alpha \{\pi_{ji}, \pi_{ki}, \pi_{ij}, \pi_{kj}\}$. The extension types depicted represent the case $\sigma_n = \pi_{kj}$.

Line 5. Assume $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}^* \pi_{jk} \mathcal{S}^*_\alpha \{\pi_{ji}, \pi_{ki}, \pi_{ij}, \pi_{kj}\}$ (see Figure 9). If $\sigma_n = \pi_{ji}$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\pi_{ji}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ii, ij, ik, ji, kj\}$, $E(w_\ell) = \{ji, jj, jk, kj\}$ and $E(w'_\ell) = \{ii, ij, ik, ki\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

If $\sigma_n = \pi_{ki}$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\pi_{ki}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ii, ij, ik, jk, ki\}$, $E(w_\ell) = \{ji, jj, jk, ki\}$ and $E(w'_\ell) = \{ii, ij, ik, kk\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

If $\sigma_n = \pi_{ij}$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\pi_{ij}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ij, ji, jj, jk, ki\}$, $E(w_\ell) = \{jk, jj, jk, ki\}$ and $E(w'_\ell) = \{ij, ki, kj, kk\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

If $\sigma_n = \pi_{kj}$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\pi_{kj}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ik, ji, jj, jk, kj\}$, $E(w_\ell) = \{ji, jj, jk, kj\}$ and $E(w'_\ell) = \{ik, ki, kj, kk\}$. Then w_ℓ and w'_ℓ are both bispecial ordinary.

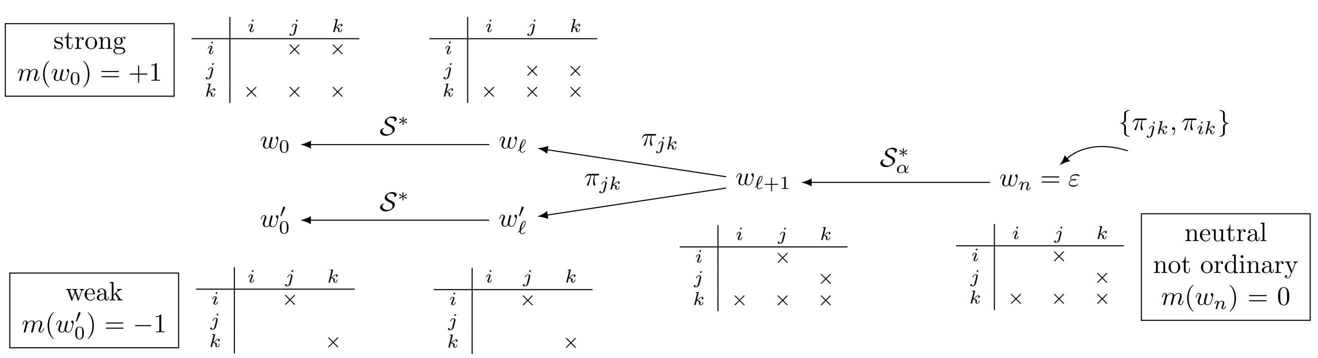

Figure 10: Life of a bispecial word if $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}^* \pi_{jk} \mathcal{S}^*_\alpha \{\pi_{jk}, \pi_{ik}\}$. The extension types shown represent the case $\sigma_n = \pi_{jk}$.

Line 6. Assume $\sigma_0\sigma_1 \cdots \sigma_n \in \mathcal{S}^* \pi_{jk} \mathcal{S}^*_\alpha \{\pi_{jk}, \pi_{ik}\}$ (see Figure 10). If $\sigma_n = \pi_{jk}$, then $E(w_{\ell+1}) = E_{\sigma_n(u)}(\varepsilon) = E_{\pi_{jk}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ij, jk, ki, kj, kk\}$,

$E(w_\ell) = \{jj, jk, ki, kj, kk\}$ and $E(w'_\ell) = \{ij, kk\}$. Then w_ℓ is bispecial strong and w'_ℓ is bispecial weak.

If $\sigma_n = \pi_{ik}$, then $E(w_{\ell+1}) = E_{\sigma_n}(u)(\varepsilon) = E_{\pi_{ik}(u)}(\varepsilon)$. Then from Lemma 30 (ii) and Table 2, we have $E(w_{\ell+1}) = \{ik, ji, ki, kj, kk\}$, $E(w_\ell) = \{ji, jk, ki, kj, kk\}$ and $E(w'_\ell) = \{ik, ki\}$. Then w_ℓ is bispecial strong and w'_ℓ is bispecial weak. □

4.5 Quadratic complexity is achievable

According to Remark 37, each time π_{jk} and π_{ik} are found one next to the other in a certain $\mathcal{S}$-adic sequence, a new pair of strong and weak bispecial factors is created (see Lemma 36) and the length of a newly created weak bispecial factor can be larger than the length of an older strong bispecial factor. Therefore, the complexity can increase by more than 3, i.e., $p(n+1) - p(n) > 3$ for some values of n. Let us illustrate it on the following example. Let

$$\begin{aligned} u &= \pi_{23}\pi_{23}\pi_{13}\pi_{23}\pi_{23}\alpha_1\alpha_3\alpha_2(1) \\ &= 12323332331232333323312323333331232333323\cdots \end{aligned}$$

The bispecial factors of u of age ≤ 5 and their life are shown in Figure 11. We see that some weak bispecial factors are longer than older strong bispecial factors. Because of this fact and of Equation (7), the non-zero values of the sequence $(b(n))_n$ do not alternate in the set $\{+1, -1\}$. Therefore, there are values of n for which $s(n) > 3$. The complete computation of $b(n)$, $s(n)$ and $p(n)$ for $n \leq 10$ is given in Table 4. The complexity $p(n)$ of the finite word u satisfies $p(n+1) - p(n) = 4$ for some values of n and $p(n) > 3n + 1$ for n such that $7 \leq n \leq 17$ (recall Equations (2), (3), (4), (5), (7) and in particular that $s(n) = 2 + \sum_{\ell=0}^{n-1} b(\ell)$ when the size of alphabet is 3).

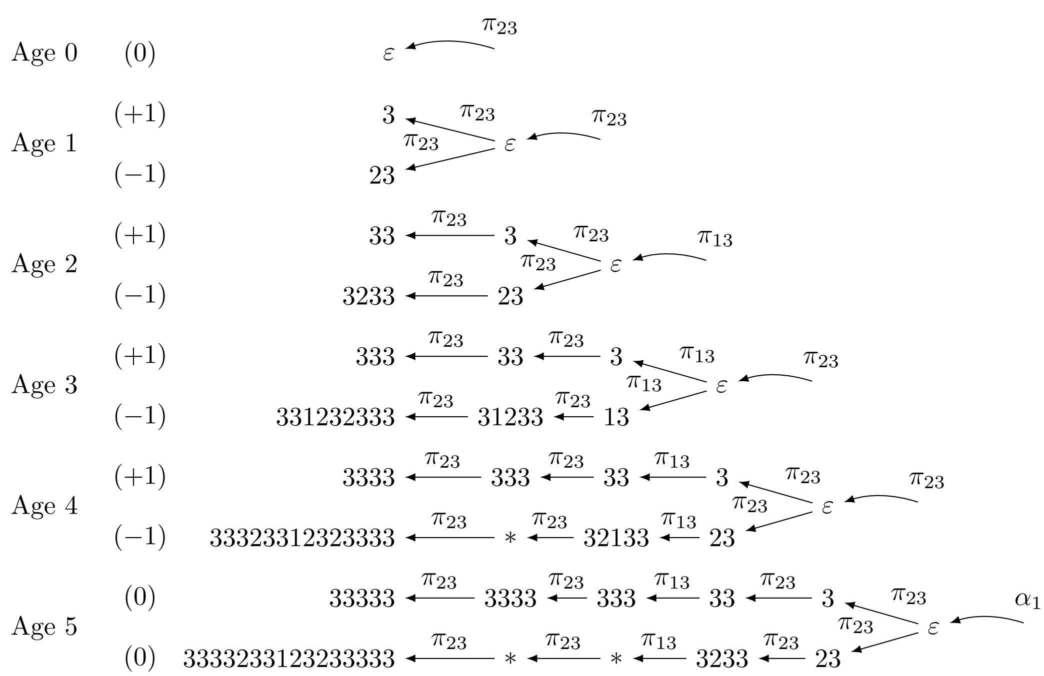

Figure 11: The young bispecial factors of $u = \pi_{23}\pi_{23}\pi_{13}\pi_{23}\pi_{23}\alpha_1\alpha_3\alpha_2(1)$ of age ≤ 5 and their life. The factors 3, 33, 333, 3333 are strong while 33333 is neutral. For the age k equal to 2 and 3, the weak factor of age k is longer than the strong factor of age $k+1$. The quantity under parenthesis indicates the value of m.

n	$\sum_{\substack{w\in\mathcal{L}_n(u)\\ w \text{ is strong}}} m(w)$	$\sum_{\substack{w\in\mathcal{L}_n(u)\\ w \text{ is weak}}} m(w)$	$b(n)$	$s(n)$	$p(n)$	$3n+1$
0	0	0	0	2	1	1
1	+1	0	1	2	3	4
2	+1	−1	0	3	5	7
3	+1	0	1	3	8	10
4	+1	−1	0	4	11	13
5	0	0	0	4	15	16
6	0	0	0	4	19	19
7	0	0	0	4	23	22
8	0	0	0	4	27	25
9	0	−1	−1	4	31	28
10	0	0	0	3	35	31

Table 4: The lengths of strong bispecial factors of u are 1, 2, 3, 4 while the lengths of weak bispecial factors of u are 2, 4, 9, 14. Since there are two more strong bispecial factors of length $\leq n$ than the number of weak bispecial factors of length $\leq n$ for all n such that $3 \leq n \leq 8$, then $s(n) = 4$ for each n with $4 \leq n \leq 9$. For example, $s(4) = p(5) - p(4) = s(0) + \sum_{\ell=0}^{3} b(\ell) = 4$. Moreover, $p(7) = 23 > 22 = 3 \cdot 7 + 1$.

In fact the complexity can get higher. It follows from Theorem 4.7.66 of [CN10, p. 214] that the fixed point of $\pi_{23}\pi_{13}$ starting with letter 1 has a quadratic factor complexity because it has infinitely many distinct factors, namely the factors 3^n, that are bounded (in fact fixed) under $\pi_{23}\pi_{13}$.

4.6 Partial and strict partial order on $\mathbb{R}^3$

In this section, we consider two distinct partial orders on $\mathbb{R}^3$ and consider how these partial orders are preserved by the application of Arnoux-Rauzy and Poincaré substitutions. The results allow the understanding of the growth of bispecial factors and are used in the proof of Theorem 1 in the next section.

Let $\overrightarrow{u} = (u_1, u_2, u_3), \overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{N}^3$ be two abelianized vectors (for two words u, v). We define $<$ as the strict partial order (irreflexive, transitive and thus asymmetric) defined coordinate per coordinate on $\mathbb{N}^3$ by:

$$\overrightarrow{u} < \overrightarrow{v} \iff u_1 < v_1 \quad \text{and} \quad u_2 < v_2 \quad \text{and} \quad u_3 < v_3.$$

Also, we define $\leq$ as the partial order (reflexive, transitive and antisymmetric) defined coordinate per coordinate on $\mathbb{N}^3$:

$$\overrightarrow{u} \leq \overrightarrow{v} \iff u_1 \leq v_1 \quad \text{and} \quad u_2 \leq v_2 \quad \text{and} \quad u_3 \leq v_3.$$

Moreover, we say that the inequality $\overrightarrow{u} \leq \overrightarrow{v}$ is *strict on the index* i if $u_i < v_i$. Note that $\leq$ is not the reflexive closure of $<$ since it includes more relations.

The next lemma shows that the relation $<$ is preserved by Arnoux-Rauzy and Poincaré substitutions and that some stronger conditions are satisfied. These stronger conditions are used to show at Lemma 39 that the relation $<$ is also preserved for extended images of factors. In the next lemma and the next sections, we fix $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$.

Lemma 38. *Let $v, v' \in \mathcal{A}^*$ be such that $\overrightarrow{v} < \overrightarrow{v'}$. For all $\{i, j, k\} = \{1, 2, 3\}$,*

(i) $\overrightarrow{\alpha_k(v)} + 2\mathbf{e}_k < \overrightarrow{\alpha_k(v')}$,

(ii) $\overrightarrow{\pi_{jk}(v)} + \mathbf{e}_j + 2\mathbf{e}_k < \overrightarrow{\pi_{jk}(v')}$.

In particular, if $\overrightarrow{v} < \overrightarrow{v'}$ *then* $\overrightarrow{\alpha_k(v)} < \overrightarrow{\alpha_k(v')}$ *and* $\overrightarrow{\pi_{jk}(v)} < \overrightarrow{\pi_{jk}(v')}$.

The proof is in the appendix.

The next lemma shows that the relation $<$ is preserved by Arnoux-Rauzy and Poincaré substitutions from a pair of factors to their extended images.

Lemma 39. *Let* $\sigma \in \mathcal{S}$. *Let* $v, v', w, w' \in \mathcal{A}^*$ *and suppose* w *(resp.* w'*) is an extended image of* v *(resp.* v'*) under* σ. *If* $\overrightarrow{v} < \overrightarrow{v'}$, *then* $\overrightarrow{w} < \overrightarrow{w'}$.

The proof is in the appendix.

Remark 40. *The previous lemma is false for the order* $\leq$. *Indeed* π_{jk} *does not preserve the relation* $\leq$ *for extended images. For example, if* $v = \varepsilon$ *and* $v' = 3$, *then* $\overrightarrow{v} = (0,0,0) \leq (0,0,1) = \overrightarrow{v'}$ *but*

$$\overrightarrow{13\pi_{13}(v)} = \overrightarrow{13} = (1,0,1) \not\leq (0,0,2) = \overrightarrow{33} = \overrightarrow{3\pi_{13}(v')},$$

and this may even lead after some more substitutions to an inversion of the order:

$$\overrightarrow{3\pi_{23}(13)} = \overrightarrow{31233} = (1,1,3) \geq (0,0,3) = \overrightarrow{333} = \overrightarrow{3\pi_{23}(33)}.$$

This example can be seen between age 3 *and* 4 *in Figure 11.*

5 Proof of Theorem 1

We now consider $\mathcal{S}$-adic words $\mathbf{u}$ generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector $\mathbf{x} \in \Delta$. By Proposition 12, $\mathbf{u}^{(m)}$ is proper and uniformly recurrent for all m so the hypothesis introduced in the previous section is satisfied. Such sequences are in the Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic system (Type 3), that is, we take into account the restrictions on the directive sequences provided by Proposition 6. The examples in Section 4.5 show that Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic sequences can lead in general to quadratic factor complexity. Nevertheless we show that the factor complexity $p(n)$ of $\mathcal{S}$-adic words $\mathbf{u}$ generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector satisfy $p(n+1) - p(n) \in \{2, 3\}$. Thus, their factor complexity is bounded below and above, that is, $2n + 1 \leq p(n) \leq 3n + 1$ for all n. In fact, we even prove that $p(n+1) - p(n)$ is equal to 2 more often than it is equal to 3 which implies that $p(n) \leq \frac{5}{2}n + 1$. More precisely, we will show that strong and weak bispecial words alternate when the length increases in Section 5.1. We then consider more closely the lengths of consecutive values of 2 and 3 in the sequence $(p(n+1) - p(n))_n$ in Section 5.2. By making use of Lemma 19 together with Lemma 44 (see Figure 14), we will be able to prove Theorem 1 in Section 5.3.

5.1 Alternance of strong and weak bispecial factors

We first gather the lemmas required in the proof (see Section 5.3) of the fact that the $\mathcal{S}$-adic words $\mathbf{u}$ (with $\mathbf{u}^{(m)}$ recurrent for all m) such that $\sigma_k \sigma_{k+1} \cdots \sigma_\ell \in \mathcal{L}(\mathcal{G})$ (for all k, ℓ) provide words that satisfy $p(n+1) - p(n) \in \{2, 3\}$.

Restricted to the language of the automaton $\mathcal{G}$, illustrated in Figure 2, the history of a strong or weak bispecial factor necessarily contains Arnoux-Rauzy substitutions.

Lemma 41. *Let* $\mathbf{u} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a_n)$ *be an* $\mathcal{S}$*-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector* $\mathbf{x} \in \Delta$*. Let* w *be a bispecial factor of* $\mathbf{u}$ *and let* $n = \text{age}(w)$*.*

If w *is weak or strong and the history of* w *is in the regular language* $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{L}(\mathcal{G})$*, then*

$$\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\{\alpha_j\}^*\,\alpha_i\,\mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$$

for some $\{i,j,k\} = \{1,2,3\}$*.*

Proof. From Lemma 36, we have

$$\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\,\mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$$

for some $\{i,j,k\} = \{1,2,3\}$. Let $p \in \mathcal{S}^*\,\pi_{jk}$ and $q \in \mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$ such that $pq = \sigma_0\sigma_1\sigma_2\cdots\sigma_n$. The word p starts at the initial state Δ and ends in the state H_{jk}, the word q starts from the state H_{jk} and ends in state H_{jk} or H_{ik} (see Figure 12). In the automaton $\mathcal{G}$, the possible transitions issued

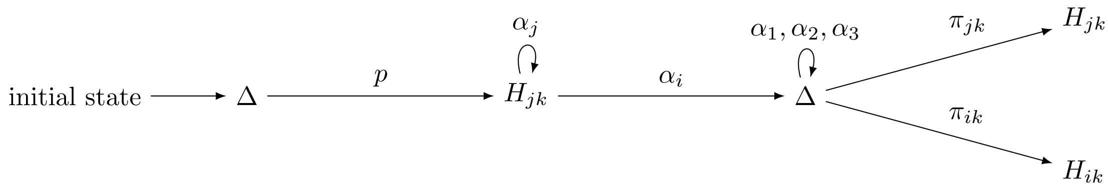

Figure 12: The subautomaton of $\mathcal{G}$ describing a path $\sigma_0\sigma_1\cdots\sigma_n = pq$ such that $p \in \mathcal{S}^*\,\pi_{jk}$ and $q \in \mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$.

from state H_{jk} are π_{ij}, π_{ji}, π_{ki}, α_j and α_i where only α_j (looping on state H_{jk}) and α_i (going to state Δ) are allowed by $q \in \mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$. Once in state Δ, q allows loops for each symbol in $\mathcal{S}_\alpha$, and finally the transitions π_{jk} or π_{ik} (see Figure 12). It follows from this that

$$q \in \{\alpha_j\}^*\,\alpha_i\,\mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$$

which was to be proved. □

Lemma 42. *Let* w *be a bispecial factor of an Arnoux-Rauzy-Poincaré* $\mathcal{S}$*-adic word. If for some* $\{i,j,k\} = \{1,2,3\}$*,*

$$\text{history}(w) \in \pi_{jk}\,\mathcal{S}^*\alpha_i\,\mathcal{S}^*\mathcal{S},$$

then $\overrightarrow{w} \geq (1,1,1)$*.*

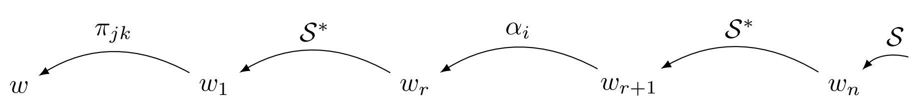

Figure 13: We suppose here that $\text{history}(w) \in \pi_{jk}\,\mathcal{S}^*\alpha_i\,\mathcal{S}^*\mathcal{S}$.

Proof. Let w_1 be the ancestor of w under π_{jk}. Let r and n be integers such that $1 \leq r < n = \text{age}(w)$ and w_{r+1} is the ancestor of w_r under substitution α_i as depicted in Figure 13. We have that $\overrightarrow{w_n} = (0,0,0)$. Also, $\overrightarrow{w_{r+1}} \geq (0,0,0)$ but $w_r = i\alpha_i(w_{r+1})$ contains at least one occurence of the letter i. Then, w_1 also contains at least one occurrence of the letter i. Therefore $\overrightarrow{w} \geq (1,1,1)$, because π_{jk} maps i to ijk. □

In order to prove that $p(n+1)-p(n) \in \{2,3\}$ for Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic words $\mathbf{u}$ such that $\mathbf{u}^{(m)}$ is proper and recurrent for all m, it is sufficient that strong and weak bispecial words alternate when the length increases because of Lemma 19. More precisely, if z_1 and z_3 are two strong (with multiplicity $+1$) bispecial factors of a word $\mathbf{u}$ such that $|z_1| < |z_3|$, then there exists a weak (with multiplicity -1) bispecial factor z_2 such that $|z_1| < |z_2| \leq |z_3|$. Note that the notion of alternance was also used to prove Theorem 4.11.2 in [CN10, p. 238].

Lemma 43. *Let $\mathbf{u}$ be an Arnoux-Rauzy-Poincaré $\mathcal{S}$-adic word such that $\mathbf{u}^{(m)}$ is proper and recurrent for all m. Let z^+ and z^- be two bispecial factors of $\mathbf{u}$ of the same age. Suppose that z^- is weak and z^+ is strong. Then $|z^+| < |z^-|$.*

Proof. In this proof, we denote by $\overrightarrow{z^+} \leq_j \overrightarrow{z^-}$ when $\overrightarrow{z^+} \leq \overrightarrow{z^-}$ is strict on the coordinate $j \in \{1,2,3\}$.

We prove by induction on the age of bispecial factors that $\overrightarrow{z^+} \leq_j \overrightarrow{z^-}$ is strict on at least one coordinate $j \in \{1,2,3\}$ with $j \in E^-(z^+)$.

Let us prove the base step of the induction. Suppose that z^+ and z^- have a common neutral bispecial antecedent v thus under the substitution π_{jk} for some $\{i,j,k\} = \{1,2,3\}$. Then, $z^+ = k\pi_{jk}(v)$ and $z^- = jk\pi_{jk}(v)$ so that $\overrightarrow{z^+} \leq_j \overrightarrow{z^-}$ is strict on the coordinate j. Moreover $E^-(z^+) = \{j,k\}$ and $E^-(z^-) = \{i,k\}$ and hence $j \in E^-(z^+)$.

Suppose now that z_h^+ and z_h^- are two respectively strong and weak bispecial factors of a word u of the same age such that $\overrightarrow{z_h^+} \leq_k \overrightarrow{z_h^-}$ is strict on at least one coordinate $k \in E^-(z_h^+)$. Let z_{h-1}^+ and z_{h-1}^- be respectively the unique bispecial extended images of z_h^+ and z_h^- under the application of σ_{h-1}. We want to show the following implication for proving the induction:

$$\overrightarrow{z_h^+} \leq_k \overrightarrow{z_h^-} \text{ and } k \in E^-(z_h^+) \implies \text{there exists } j \text{ such that } \overrightarrow{z_{h-1}^+} \leq_j \overrightarrow{z_{h-1}^-} \text{ and } j \in E^-(z_{h-1}^+).$$

Since the letters prepended to the left of bispecial extended images depend on the left extensions by Table 1, if $E^-(z_h^-) = E^-(z_h^+)$, it is clear that $E^-(z_{h-1}^-) = E^-(z_{h-1}^+)$ and $\overrightarrow{z_{h-1}^+} \leq_j \overrightarrow{z_{h-1}^-}$ is strict for some letter $j \in E^-(z_{h-1}^+)$. Suppose now that $E^-(z_h^-) \neq E^-(z_h^+)$ and suppose without lost of generality that $E^-(z_h^+) = \{2,3\}$ and $E^-(z_h^-) = \{1,3\}$. The possible cases depending on σ_{h-1} are described in the following table:

σ_{h-1}	z_{h-1}^+	$E^-(z_{h-1}^+)$	z_{h-1}^-	$E^-(z_{h-1}^-)$	if $k=2$ $\{j \mid \overrightarrow{z_{h-1}^+} \leq_j \overrightarrow{z_{h-1}^-}\}$	if $k=3$ $\{j \mid \overrightarrow{z_{h-1}^+} \leq_j \overrightarrow{z_{h-1}^-}\}$
α_1	$1\alpha_1(z_h^+)$	$\{2,3\}$	$1\alpha_1(z_h^-)$	$\{1,3\}$	$\{1,2\}$	$\{1,3\}$
α_2	$2\alpha_2(z_h^+)$	$\{2,3\}$	$2\alpha_2(z_h^-)$	$\{1,3\}$	$\{2\}$	$\{2,3\}$
α_3	$3\alpha_3(z_h^+)$	$\{2,3\}$	$3\alpha_3(z_h^-)$	$\{1,3\}$	$\{2,3\}$	$\{3\}$
π_{12}	$2\pi_{12}(z_h^+)$	$\{1,2\}$	$12\pi_{12}(z_h^-)$	$\{2,3\}$	$\{1,2\}$	$\{1,2,3\}$
π_{32}	$2\pi_{32}(z_h^+)$	$\{2,3\}$	$32\pi_{32}(z_h^-)$	$\{1,2\}$	$\{2,3\}$	$\{2,3\}$
π_{13}	$3\pi_{13}(z_h^+)$	$\{1,3\}$	$3\pi_{13}(z_h^-)$	$\{1,3\}$	$\{1,2,3\}$	$\{3\}$
π_{23}	$3\pi_{23}(z_h^+)$	$\{2,3\}$	$3\pi_{23}(z_h^-)$	$\{2,3\}$	$\{2,3\}$	$\{3\}$
π_{21}	$21\pi_{21}(z_h^+)$	$\{1,3\}$	$1\pi_{21}(z_h^-)$	$\{1,2\}$	$\{1\}$	$\{1,3\}$
π_{31}	$31\pi_{31}(z_h^+)$	$\{1,2\}$	$1\pi_{31}(z_h^-)$	$\{1,3\}$	$\{1,2\}$	$\{1\}$

We check that for all nine possible values of $\sigma_{h-1} \in \mathcal{S}$, we always have that $\overrightarrow{z_{h-1}^+} \leq_j \overrightarrow{z_{h-1}^-}$ is strict for some $j \in E^-(z_{h-1}^+)$. Since we proved $\overrightarrow{z^+} \leq_j \overrightarrow{z^-}$ is strict on at least one coordinate j, then we conclude that $|z^+| < |z^-|$. □

Lemma 44. *Let* $\mathbf{u} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a_n)$ *be an* $\mathcal{S}$*-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector* $\mathbf{x} \in \Delta$*. Let* z^- *and* w^+ *be two bispecial factors of* $\mathbf{u}$ *such that* z^- *is weak,* w^+ *is strong, and the history of both* z^- *and* w^+ *are in the regular language* $\sigma_0\sigma_1\cdots\sigma_n \in \mathcal{L}(\mathcal{G})$*. If* $\text{age}(z^-) < \text{age}(w^+)$*, then* $|z^-| < |w^+|$*.*

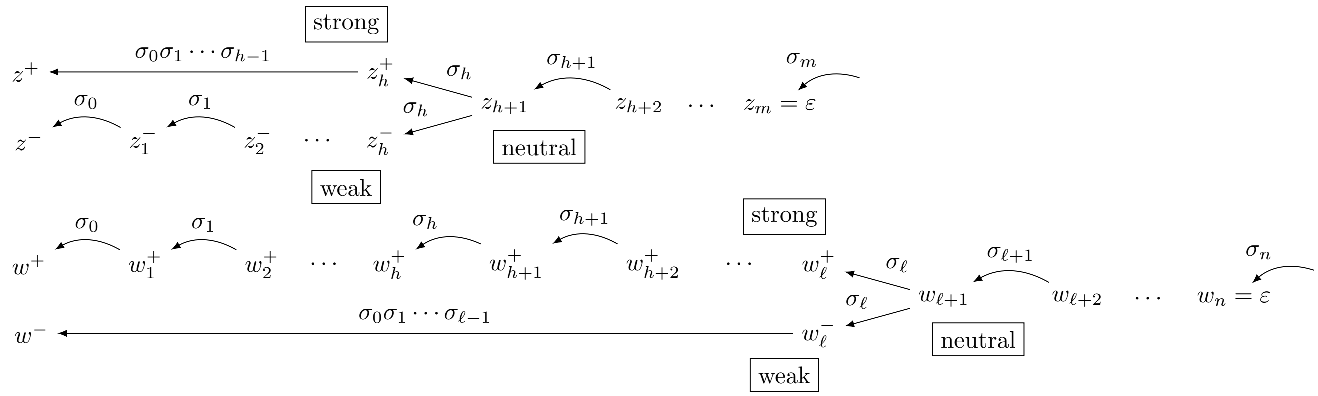

Figure 14: Lifes of two pairs of strong and weak bispecial factors: z^+, z^- and w^+, w^-.

Proof. In this proof, we denote a bispecial factor w as w^+ if it is strong, w^- if it is weak, and with no sign if it is neutral. Let $m = \text{age}(z^-)$, $z_0^- = z^-$ and z_{i+1} be the unique antecedent of z_i under σ_i for $0 \le i \le m-1$. Let $n = \text{age}(w^+)$, $w_0^+ = w^+$ and w_{i+1} be the unique antecedent of w_i under σ_i for $0 \le i \le n-1$. From Lemma 41, we have

$$\begin{aligned}\text{history}(w^+) &= \sigma_0\sigma_1\cdots\sigma_n \in \mathcal{S}^*\,\pi_{jk}\{\alpha_j\}^*\,\alpha_i\,\mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\},\\ \text{history}(z^-) &= \sigma_0\sigma_1\cdots\sigma_m \in \mathcal{S}^*\,\pi_{j'k'}\{\alpha_{j'}\}^*\,\alpha_{i'}\,\mathcal{S}_\alpha^*\,\{\pi_{i'k'},\pi_{j'k'}\},\end{aligned}$$

for some $\{i,j,k\} = \{1,2,3\}$ and some other values of $\{i',j',k'\} = \{1,2,3\}$. We want to show that $\overrightarrow{w^+} > \overrightarrow{z^-}$ in order to conclude that $|w^+| > |z^-|$. Let ℓ (h resp.) be the largest integer such that w_ℓ^+ (z_h^- resp.) is strong (weak resp.). The situation is illustrated in Figure 14.

We have $m \le \ell$. Indeed, suppose on the contrary that $m > \ell$. We know that $\sigma_m \in \mathcal{S}_\pi$. Also, $\sigma_i \in \mathcal{S}_\alpha$ for all $\ell+1 \le i \le n-1$. This implies that $m \ge n$ which is a contradiction. Hence, $h < m \le \ell < n$. Since $\text{history}(w_\ell^+) \in \pi_{jk}\{\alpha_j\}^*\,\alpha_i\,\mathcal{S}_\alpha^*\,\{\pi_{ik},\pi_{jk}\}$, from Lemma 42 we have that $\overrightarrow{w_\ell^+} \ge (1,1,1)$. Then,

$$\overrightarrow{z_m} = (0,0,0) < (1,1,1) \le \overrightarrow{w_\ell^+} \le \overrightarrow{w_m^+}.$$

Using induction and Lemma 39, we obtain that $\overrightarrow{z^-} < \overrightarrow{w^+}$. Then, $|w^+| > |z^-|$. □

5.2 Ranges of 2 and 3 in the sequence $(p(n+1) - p(n))_n$

The next lemma, whose proof requires a deeper understanding of the abelianized vectors of bispecial factors, will allow us in Section 5.3 to get a more precise information concerning the alternance of weak and strong bispecial factors.

Lemma 45. *Let* $\mathbf{u}$ *be an* $\mathcal{S}$*-adic word generated by the Arnoux-Rauzy-Poincaré algorithm applied to a totally irrational vector* $\mathbf{x} \in \Delta$*. Let* w^+ *and* w^- *be two bispecial factors of the same age of* $\mathbf{u}$ *such that* w^+ *is strong and* w^- *is weak. If there exists a younger weak bispecial factor* z^- *of* $\mathbf{u}$*, i.e.,* $\text{age}(z^-) < \text{age}(w^+) = \text{age}(w^-)$*, then* $|w^+| - |z^-| > |w^-| - |w^+|$*. If there is no younger weak bispecial factor, then* $|w^+| \ge |w^-| - |w^+|$*.*

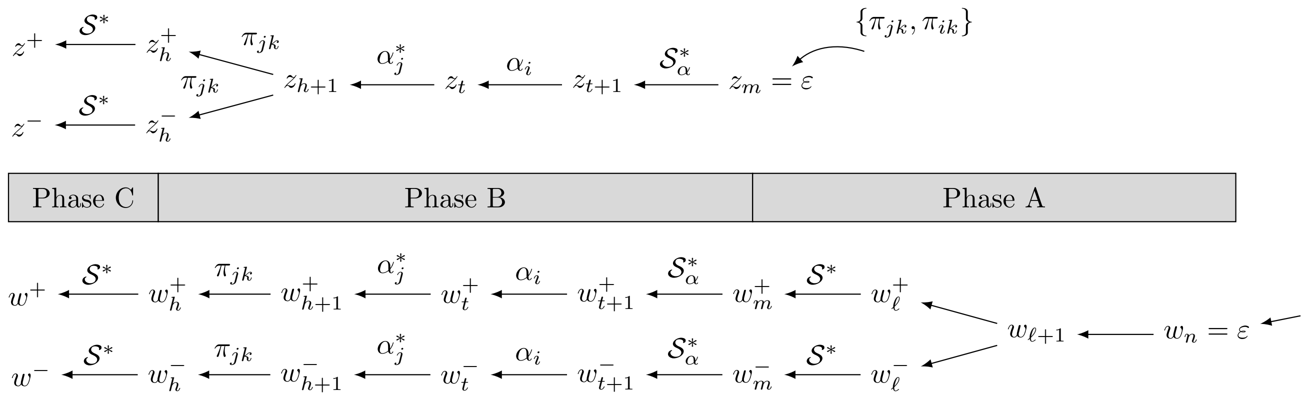

Figure 15: Three phases of the lifes of two pairs of strong and weak bispecial factors: z^+, z^- and w^+, w^-.

Proof. The proof is divided into three phases according to the lifes of the bispecial factors (see Figure 15).

(i) At the end of Phase A, we have $\overrightarrow{w_m^+} - \overrightarrow{z_m} \geq \overrightarrow{w_m^-} - \overrightarrow{w_m^+}$ is strict on two letters in $\{1,2,3\}$.

(ii) At the end of Phase B, we have $\overrightarrow{w_h^+} - \overrightarrow{z_h^-} > \overrightarrow{w_h^-} - \overrightarrow{w_h^+}$ and the words w_h^- and w_h^+ have the same left extensions.

(iii) At the end of Phase C, we have $\overrightarrow{w^+} - \overrightarrow{z^-} > \overrightarrow{w^-} - \overrightarrow{w^+}$.

Phase A. Let ℓ be the largest index such that $w_\ell^+ \neq w_\ell^-$. One has that w_ℓ^+ is strong and w_ℓ^- is weak and their antecent $w_{\ell+1}^+ = w_{\ell+1}^- = w_{\ell+1}$ are equal and neutral. The bispecial factor w_ℓ^+ contains each of the letters in $\{1,2,3\}$ because of Lemma 41 and Lemma 42. Also $\overrightarrow{w_\ell^-} - \overrightarrow{w_\ell^+} = \mathbf{e}_a$ for some $a \in \{1,2,3\}$. Thus $\overrightarrow{w_\ell^+} \geq \overrightarrow{w_\ell^-} - \overrightarrow{w_\ell^+}$ is a strict inequality on at least two coordinates. One checks that this property is preserved by each of the nine possible substitutions. This implies that $\overrightarrow{w_m^+} \geq \overrightarrow{w_m^-} - \overrightarrow{w_m^+}$ is strict on at least two letters in $\{1,2,3\}$ as well. This also proves the last part of the lemma, concerning the case where there is no younger weak bispecial factor.

Phase B. Each of the inequality below is implied by the precedent one. The substitution α_i brings the inequality (by at least two units) on the coordinate i. Then, the substitution π_{jk} spreads the strict inequality on every coordinate:

$$\begin{aligned}
&\overrightarrow{w_m^+} - \overrightarrow{z_m} \geq \overrightarrow{w_m^-} - \overrightarrow{w_m^+} \text{ is strict on at least two letters in } \{1,2,3\},\\
&\overrightarrow{w_{t+1}^+} - \overrightarrow{z_{t+1}} \geq \overrightarrow{w_{t+1}^-} - \overrightarrow{w_{t+1}^+} \text{ is strict on at least two letters in } \{1,2,3\},\\
&\overrightarrow{w_t^+} - \overrightarrow{z_t} - 2\mathbf{e}_i \geq \overrightarrow{w_t^-} - \overrightarrow{w_t^+},\\
&\overrightarrow{w_{h+1}^+} - \overrightarrow{z_{h+1}} - 2\mathbf{e}_i \geq \overrightarrow{w_{h+1}^-} - \overrightarrow{w_{h+1}^+},\\
&\overrightarrow{\pi_{jk}(w_{h+1}^+)} - \overrightarrow{\pi_{jk}(z_{h+1})} - (2,2,2) \geq \overrightarrow{\pi_{jk}(w_{h+1}^-)} - \overrightarrow{\pi_{jk}(w_{h+1}^+)},\\
&\overrightarrow{w_h^+} - \overrightarrow{z_h^-} + \mathbf{e}_j - (2,2,2) \geq \overrightarrow{w_h^-} - \overrightarrow{w_h^+},\\
&\overrightarrow{w_h^+} - \overrightarrow{z_h^-} > \overrightarrow{w_h^-} - \overrightarrow{w_h^+}.
\end{aligned}$$

The left extensions of w_m^+ and w_m^- are $\{j,k\}$ or $\{i,k\}$. Arnoux-Rauzy substitutions preserve the extensions so that $E^-(w_{h+1}^+) = E^-(w_m^+)$ and $E^-(w_{h+1}^-) = E^-(w_m^-)$. Finally, according to Table 1, π_{jk} projects those left extensions onto the same set $E^-(w_{h+1}^+) = E^-(w_{h+1}^-) = \{j,k\}$.

Phase C. We have $\overrightarrow{w_h^+}-\overrightarrow{z_h^-} > \overrightarrow{w_h^-}-\overrightarrow{w_h^+}$. Since the words w_h^- and w_h^+ have the same left extensions, then so do w_{h-1}^- and w_{h-1}^+ for all $\sigma_{h-1} \in \mathcal{S}_\pi$. But the left extensions of z_h^- can be different from the one of w_h^- and w_h^+. This can lead to $z_{h-1}^- = jk\pi_{jk}(z_h^-)$ while $w_{h-1}^- = k\pi_{jk}(w_h^-)$ and $w_{h-1}^+ = k\pi_{jk}(w_h^+)$. Thus, the proof of Phase C relies on the following recurrences on the age of bispecial factors (all other cases for left extensions are easier and follow from the same recurrences):

(i) (Recurrence **AR**) If $\overrightarrow{w^+} - \overrightarrow{z^-} > \overrightarrow{w^-} - \overrightarrow{w^+}$, then $\overrightarrow{\alpha_k(w^+)} - \overrightarrow{\alpha_k(z^-)} > \overrightarrow{\alpha_k(w^-)} - \overrightarrow{\alpha_k(w^+)}$.

(ii) (Recurrence **P**) If $\overrightarrow{w^+} - \overrightarrow{z^-} > \overrightarrow{w^-} - \overrightarrow{w^+}$, then $\overrightarrow{\pi_{jk}(w^+)} - \overrightarrow{\pi_{jk}(z^-)} - \mathbf{e}_j > \overrightarrow{\pi_{jk}(w^-)} - \overrightarrow{\pi_{jk}(w^+)}$.

Let $\overrightarrow{z^-} = (x, y, z)$, $\overrightarrow{w^+} = (a, b, c)$, $\overrightarrow{w^-} = (d, e, f)$ where the convention $\mathbf{e}_i = (1, 0, 0)$, $\mathbf{e}_j = (0, 1, 0)$, $\mathbf{e}_k = (0, 0, 1)$ is used. For the Arnoux-Rauzy recurrence, we have

$$\overrightarrow{\alpha_k(z^-)} = (x, y, x + y + z), \quad \overrightarrow{\alpha_k(w^+)} = (a, b, a + b + c), \quad \overrightarrow{\alpha_k(w^-)} = (d, e, d + e + f).$$

Then

$$\begin{aligned}
\overrightarrow{\alpha_k(w^+)} - \overrightarrow{\alpha_k(z^-)} &= \overrightarrow{w^+} - \overrightarrow{z^-} + (0, 0, a - x) + (0, 0, b - y) \\
&> \overrightarrow{w^-} - \overrightarrow{w^+} + (0, 0, a - x) + (0, 0, b - y) \\
&\geq \overrightarrow{w^-} - \overrightarrow{w^+} + (0, 0, d - a + 1) + (0, 0, e - b + 1) \\
&= \overrightarrow{w^-} - \overrightarrow{w^+} + (0, 0, d - a) + (0, 0, e - b) + (0, 0, 2) \\
&= \overrightarrow{w^-} - \overrightarrow{w^+} + (0, 0, d - a) + (0, 0, e - b) + (0, 0, 2) \\
&= (d, e, d + e + f) - (a, b, a + b + c) + (0, 0, 2) \\
&= \overrightarrow{\alpha_k(w^-)} - \overrightarrow{\alpha_k(w^+)} + 2\mathbf{e}_k
\end{aligned}$$

For the Poincaré recurrence, we have

$$\overrightarrow{\pi_{jk}(z^-)} = (x, x + y, x + y + z), \quad \overrightarrow{\pi_{jk}(w^+)} = (a, a + b, a + b + c), \quad \overrightarrow{\pi_{jk}(w^-)} = (d, d + e, d + e + f).$$

Then

$$\begin{aligned}
\overrightarrow{\pi_{jk}(w^+)} - \overrightarrow{\pi_{jk}(z^-)} - \mathbf{e}_j &= \overrightarrow{w^+} - \overrightarrow{z^-} + (0, a - x, a - x) + (0, 0, b - y) - (0, 1, 0) \\
&> \overrightarrow{w^-} - \overrightarrow{w^+} + (0, a - x, a - x) + (0, 0, b - y) - (0, 1, 0) \\
&\geq \overrightarrow{w^-} - \overrightarrow{w^+} + (0, d - a + 1, d - a + 1) + (0, 0, e - b + 1) - (0, 1, 0) \\
&= \overrightarrow{w^-} - \overrightarrow{w^+} + (0, d - a, d - a) + (0, 0, e - b) + (0, 0, 2) \\
&= \overrightarrow{w^-} - \overrightarrow{w^+} + (0, d - a, d - a) + (0, 0, e - b) + (0, 0, 2) \\
&= (d, d + e, d + e + f) - (a, a + b, a + b + c) + (0, 0, 2) \\
&= \overrightarrow{\pi_{jk}(w^-)} - \overrightarrow{\pi_{jk}(w^+)} + 2\mathbf{e}_k. \quad \square
\end{aligned}$$

5.3 Linear growth for the factor complexity

We now have gathered all the elements for proving Theorem 1.

Proof of Theorem 1. Since $\mathbf{x}$ is totally irrational, Proposition 12 certifies that lemmas of the previous two sections can be applied since the $\mathcal{S}$-adic words $\mathbf{u}^{(m)}$ are proper and uniformly recurrent for all m. The set of bispecial factors of length n contains at most one weak or strong bispecial factor. Indeed, suppose on the contrary that it contains two of them: w and z. They cannot have the same age according to Lemma 43 since this would otherwise imply $|w| \neq |z|$. Also, if one is older, e.g. $\mathrm{age}(w) > \mathrm{age}(z)$, then $|w| > |z|$ from Lemma 44. Then $b(n) \in \{-1, 0, +1\}$ according to Equation (7) of Theorem 20. Finally, it remains to prove that the assumptions of Lemma 19 are satisfied. The first non-zero value of $b(n)$ is $+1$ because strong and weak bispecial factors come in pairs and the strong one is smaller than the weak one from Lemma 43. Moreover, non-zero values are alternating. Indeed, let z^+ and w^+ be two strong bispecial factors such that $\mathrm{age}(w^+) > \mathrm{age}(z^+)$. Let z^- be the weak bispecial factor such that $\mathrm{age}(z^-) = \mathrm{age}(z^+)$. From Lemma 43 and Lemma 44, $|z^+| < |z^-| < |w^+|$. Hence, there is always a -1 between two $+1$ in the sequence $(b(n))_{n\geq 0}$. This alternance of non-zero values in the sequence $(b(n))_n$ shows that $p(n+1)-p(n) \in \{2,3\}$ (Lemma 19), so that $2n+1 \leq p(n) \leq 3n+1$ for $n \geq 0$.

Now we show that $p(n) \leq \frac{5}{2}n + 1$. We prove by recurrence that $p(q+1) \leq \frac{5}{2}(q+1)+1$ for each q such that $b(q) = -1$. By assuming that $b(-1) = -1$, we remark that the statement is valid for $q = -1$ because $p(0) = 1 \leq 1$. Suppose q and t are two consecutive occurrences of -1 in the sequence $(b(\ell))_\ell$, that is, $b(q) = b(t) = -1$ and $b(\ell) \neq -1$ for all ℓ such that $q < \ell < t$. We show that if $p(q+1) \leq \frac{5}{2}(q+1)+1$ then $p(n+1) \leq \frac{5}{2}(n+1)+1$ for each n such that $q < n \leq t$. From the alternance of non-zero values $+1$ and -1 in the sequence $(b(\ell))_\ell$, there exists an integer r with $q < r < t$ such that $b(r) = +1$ and such that for all integers $r' \neq r$ with $q < r' < t$ then $b(r') = 0$. Since the first non-zero value of $(b(\ell))_{\ell\geq 0}$ is $+1$, then $\sum_{\ell=0}^{q} b(\ell) = 0$. The consequence of Lemma 45 is that $r - q > t - r$ which is true if and only if $\frac{t-q}{2} > t - r$. Note that if n is such that $q < n \leq r$, then $s(n) = s(0) + \sum_{\ell=0}^{n-1} b(\ell) = 2 + \sum_{\ell=0}^{q} b(\ell) = 2$. Also, if n is such that $r < n \leq t$, then $s(n) = s(0) + \sum_{\ell=0}^{n-1} b(\ell) = 2 + \sum_{\ell=0}^{q} b(\ell) + b(r) = 2 + 1 = 3$. Therefore, for each n such that $r < n \leq t$ we have

$$\begin{aligned}
p(n+1) - p(q+1) &= \sum_{\ell=q+1}^{r} s(\ell) + \sum_{\ell=r+1}^{n} s(\ell) = 2(r-q) + 3(n-r) \\
&= 2(n-q) + (n-r) < 2(n-q) + \frac{n-q}{2} = \frac{5}{2}(n-q).
\end{aligned}$$

But since $p(q+1) \leq \frac{5}{2}(q+1)+1$ we conclude that $p(n+1) \leq \frac{5}{2}(n+1)+1$. We get the same conclusion for each n such that $q < n \leq r$. From this we conclude that $p(n) \leq \frac{5}{2}n+1$ and $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2}$. □

We in fact prove the more general result.

Theorem 46. *Let* $\mathbf{u}$ *be a word of the Arnoux-Rauzy-Poincaré system.*

- *If* $\mathbf{u}$ *is of Type* 1, *then it has a bounded factor complexity.*
- *If* $\mathbf{u}$ *is of Type* 2, *then its factor complexity satisfies ultimately* $p(n) = n+k$ *for some constant* k.
- *If* $\mathbf{u}$ *is of Type* 3, *then* $p(n+1) - p(n) \in \{2,3\}$ *and* $2n+1 \leq p(n) \leq \frac{5}{2}n+1$ *for all* $n \geq 0$.

Proof. Words $\mathbf{u}$ of Type 1 are periodic and thus have a bounded factor complexity. A word $\mathbf{u}$ of Type 2 is an image by a substitution of a Sturmian sequences. Then, according to [Cas97b], its factor complexity satisfies ultimately $p(n) = n + k$ for some constant k. A word $\mathbf{u}$ of Type 3 is weakly primitive, $\mathbf{u}^{(m)}$ is recurrent and proper for all m, and its factor complexity was proven to satisfy the desired bounds in Theorem 1. □

6 Convergence and unique ergodicity

We start with some terminology. Let $\mathbf{u}$ be an infinite word in $\mathcal{A}^{\mathbb{N}}$. Let $X_{\mathbf{u}}$ be the orbit closure of the infinite word $\mathbf{u}$ under the action of the shift S, that is, $X_{\mathbf{u}}$ is the closure in $\mathcal{A}^{\mathbb{N}}$ of the set $\{S^n(\mathbf{u}) \mid n \in \mathbb{N}\} = \{(u_k)_{k\geq n} \mid k \in \mathbb{N}\}$, where the shift S satisfies $S((u_n)_n) = (u_{n+1})$. The set $X_{\mathbf{u}}$ coincides with the set of infinite words whose language is contained in $\mathcal{L}(\mathbf{u})$, and is called the *symbolic dynamical system* generated by $\mathbf{u}$. The topological dynamical system $(X_{\mathbf{u}}, S)$ can be endowed with a structure of a measure-theoretic dynamical system $(X_u, T, \mu, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra, by taking any probability measure μ preserved by T, that is, for all $B \in \mathcal{B}$, $\mu(S^{-1}(B)) = \mu(B)$. The system $X_{\mathbf{u}}$ is said to be *uniquely ergodic* if there exists a unique shift-invariant probability measure on X.

One natural way for getting an S-invariant measure is to consider factor frequencies (for more details, see [FM10]). The *frequency* of a letter i in $\mathbf{u}$ is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0u_1 \cdots u_{n-1}$ divided by n. The infinite word $\mathbf{u}$ has *uniform letter frequencies* if, for every letter i of u, the number of occurrences of i in $u_k \cdots u_{k+n-1}$ divided by n has a limit when n tends to infinity, uniformly in k. Similarly, we can define the frequency and the uniform frequency of a factor, and we say that u has *uniform frequencies* if all its factors have uniform frequency. The property of having uniform factor frequencies for a shift X is actually equivalent to unique ergodicity (see e.g. [FM10]).

Factor complexity is a priori a topological notion. However it may yield (in particular when it has a linear growth order) measure-theoretical information on the the symbolic dynamical system $X_{\mathbf{u}}$. Indeed, according to [Bos85], if $\mathbf{u}$ is assumed to be uniformly recurrent, and if $\limsup p(n)/n < 3$, then $(X_{\mathbf{u}}, S)$ is uniquely ergodic.

Proof of Theorem 2. We now have gathered all the elements for observing that Theorem 2 is a direct consequence of Theorem 1 together with the above mentioned result of [Bos85] and Proposition 12. □

7 Conclusion

Given a totally irrational vector of frequencies $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ (with $\sum x_i = 1$), we thus have shown how to construct an infinite word $\mathbf{u}$ over the alphabet $\mathcal{A} = \{1, 2, 3\}$ such that the frequency of each letter $i \in \mathcal{A}$ exists and is equal to x_i, with this word $\mathbf{u}$ having a linear factor complexity. This word is contructed by translating symbolically within the S-adic formalism a multidimensional continued fraction algorithm, namely the Arnoux-Rauzy-Poincaré algorithm.

Observe that usual proofs of convergence for multidimensional continued fraction algorithms rely on linear algebra and on the use of the Hilbert projective metric (see e.g. [Sch00]). Let us stress the fact that we provide here a purely combinatorial proof of convergence for a two-dimensional continued fraction algorithm based on the unique ergodicity.

The restriction to the regular language $\mathcal{L}(\mathcal{G})$ is clearly important; there exist examples of $\mathcal{S}$-adic words constructed with the alphabet of substitutions $\mathcal{S}$ for which the upper bound $\frac{5}{2}n + 1$ does not hold. Moreover, a quadratic complexity is even also achievable (see Section 4.5). Hence, the present study gives some more insight on a statement of the S-adic conjecture (it rather should be qualified of problem) which is to find conditions for which S-adic sequences have a linear complexity (see e.g. [DLR13, Ler12]). Note that any uniformly recurrent word $\mathbf{u}$ whose complexity function $p(n)$ satisfies $p(n+1) - p_u(n) \leq k$, for all n, is S_k-adic, with a set S_k of substitutions that depends on k ([Fer96]).

The upper bound $\limsup_{n\to\infty} \frac{p(n)}{n} \leq \frac{5}{2}$ is not sharp. Numerical experimentations tend to indicate that the worst case in the language $\mathcal{L}(\mathcal{G})$ of the Arnoux-Rauzy-Poincaré algorithm is obtained with the fixed point of $\pi_{23}\alpha_1$ for which the value is approximately $\limsup_{n\to\infty} \frac{p(n)}{n} \approx 2.26079201$.

Factor complexity of Poincaré and Arnoux-Rauzy substitutions can be described exactly by considering left and right extensions of length one. It is not always the case, and the study of Brun substitutions (provided by the Brun multidimensional continued fraction algorithm) seems to be an example for which extensions of length longer than one are necessary to describe bispecial factors. Recently, Klouda [Klo12] described bispecial factors in fixed points of morphisms where extensions of length longer than one were considered. Extending this work to S-adic words deserves further research.

Balance properties of the Poincaré and Arnoux-Rauzy S-adic system have also nice properties and their study should be done more deeply. An infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is said to be C-balanced if for any pair v, w of factors of the same length of $\mathbf{u}$, and for any letter $i \in \mathcal{A}$, one has $||v|_i - |w|_i| \leq C$. It is said balanced if there exists $C > 0$ such that it is C-balanced. For example, it was proven in [DHS13] that words generated by Brun algorithm gives almost everywhere balanced sequences. Balance properties are intimately connected with Diophantine properties of the algorithm. Indeed, an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is balanced if and only if it has uniform letter frequencies and there exists a constant B such that for any factor w of u, we have $||w|_i - f_i|w|| \leq B$ for all letter i in $\mathcal{A}$, where f_i is the frequency of i.

8 Appendix

Proof of Proposition 6. We define

$$\widetilde{\mathcal{P}} = \{A_j H_{jk} : \{i,j,k\} = \{1,2,3\}\} \cup \{P_{jk} H_{jk} : \{i,j,k\} = \{1,2,3\}\}$$

which describes another partition of Δ into 12 triangles shown in Figure 16.

First, we show that the transformation T is a Markov transformation for the partition $\widetilde{\mathcal{P}}$. Let

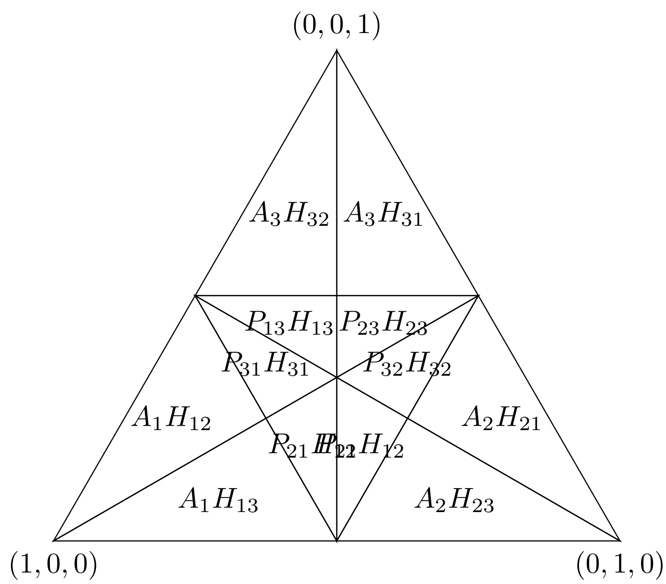

Figure 16: The Markov partition $\widetilde{\mathcal{P}}$ of Arnoux-Rauzy-Poincaré algorithm.

$\{i, j, k\} = \{1, 2, 3\}$. The image of $A_j H_{jk}$ and of $P_{jk} H_{jk}$ under T are the same and are equal to the half triangle H_{jk}:

$$T(A_j H_{jk}) = T(P_{jk} H_{jk}) = H_{jk}.$$

But the half triangle H_{jk} is a union of elements of $\widetilde{\mathcal{P}}$:

$$H_{jk} = A_i H_{ik} \cup A_i H_{ij} \cup A_j H_{jk} \cup P_{ij} H_{ij} \cup P_{ji} H_{ji} \cup P_{ki} H_{ki}.$$

Thus, the transformation T is a Markov transformation for the partition $\widetilde{\mathcal{P}}$. This defines an automaton $\widetilde{G} = (\widetilde{\mathcal{P}}, \widetilde{\Sigma}, \widetilde{\delta}, \widetilde{I}, \widetilde{F})$ where the alphabet is

$$\widetilde{\Sigma} = \{A_1^{-1}, A_2^{-1}, A_3^{-1}, P_{31}^{-1}, P_{13}^{-1}, P_{23}^{-1}, P_{32}^{-1}, P_{12}^{-1}, P_{21}^{-1}\},$$

the transitions are

$$\widetilde{\delta} = \{(p, M, q) \in \widetilde{\mathcal{P}} \times \widetilde{\Sigma} \times \widetilde{\mathcal{P}} : q \subseteq M \cdot p = T(p)\},$$

or, more precisely,

$$\widetilde{\delta} = \left\{ \begin{array}{ll} A_j H_{jk}, A_j^{-1} \to A_i H_{ik}, & P_{jk} H_{jk}, P_{jk}^{-1} \to A_i H_{ik}, \\ A_j H_{jk}, A_j^{-1} \to A_i H_{ij}, & P_{jk} H_{jk}, P_{jk}^{-1} \to A_i H_{ij}, \\ A_j H_{jk}, A_j^{-1} \to A_j H_{jk}, & P_{jk} H_{jk}, P_{jk}^{-1} \to A_j H_{jk}, \\ A_j H_{jk}, A_j^{-1} \to P_{ij} H_{ij}, & P_{jk} H_{jk}, P_{jk}^{-1} \to P_{ij} H_{ij}, \\ A_j H_{jk}, A_j^{-1} \to P_{ji} H_{ji}, & P_{jk} H_{jk}, P_{jk}^{-1} \to P_{ji} H_{ji}, \\ A_j H_{jk}, A_j^{-1} \to P_{ki} H_{ki}, & P_{jk} H_{jk}, P_{jk}^{-1} \to P_{ki} H_{ki} \end{array} \text{for each } \{i, j, k\} = \{1, 2, 3\} \right\},$$

the initial states and final states are all of the twelve states, i.e., $\widetilde{I} = \widetilde{F} = \widetilde{\mathcal{P}}$. The automaton $\widetilde{G}$ recognize all the expansions of the Arnoux-Rauzy-Poincaré continued fraction algorithm. It is clearly not deterministic. A minimized and deterministic version of it is the automaton $\mathcal{G}$ shown in Figure 2 where the alphabet considered is $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \pi_{31}, \pi_{13}, \pi_{23}, \pi_{32}, \pi_{12}, \pi_{21}\}$ instead of $\widetilde{\Sigma}$. In fact, amongst all the elements of $2^{\widetilde{\mathcal{P}}}$ considered in the determinization process, only the states in the set $Q = \{\Delta, H_{12}, H_{13}, H_{21}, H_{23}, H_{31}, H_{32}\}$ survive the minimization. □

Proof of Lemma 38. (i) Let $\overrightarrow{z} = (z_1, z_2, z_3) = \overrightarrow{\alpha_k(v)}$. Let $\overrightarrow{z'} = (z'_1, z'_2, z'_3) = \overrightarrow{\alpha_k(v')}$. We have

$$\left\{ \begin{array}{l} z_i = v_i \\ z_j = v_j \\ z_k = v_i + v_j + v_k \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} z'_i = v'_i \\ z'_j = v'_j \\ z'_k = v'_i + v'_j + v'_k. \end{array} \right.$$

Then

$$z_k + 2 = v_i + v_j + v_k + 2 = (v_i + 1) + (v_j + 1) + (v_k + 1) - 1 \leq v'_i + v'_j + v'_k - 1 < z'_k$$

and $\overrightarrow{z} + 2e_k < \overrightarrow{z'}$.

(ii) Let $\overrightarrow{z} = (z_1, z_2, z_3) = \overrightarrow{\pi_{jk}(v)}$. Let $\overrightarrow{z'} = (z'_1, z'_2, z'_3) = \overrightarrow{\pi_{jk}(v')}$. We have

$$\left\{ \begin{array}{l} z_i = v_i \\ z_j = v_i + v_j \\ z_k = v_i + v_j + v_k \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} z'_i = v'_i \\ z'_j = v'_i + v'_j \\ z'_k = v'_i + v'_j + v'_k \end{array} \right.$$

As above we have $z_k + 2 < z'_k$. Moreover,

$$z_j + 1 = v_i + v_j + 1 = (v_i + 1) + (v_j + 1) - 1 \leq v'_i + v'_j - 1 < z'_j.$$

Then $\overrightarrow{z} + e_j + 2e_k < \overrightarrow{z'}$. □

Proof of Lemma 39. (i) Under Arnoux-Rauzy substitution, the extended image of v and v' are uniquely determined: $w = k\alpha_k(v)$ and $w' = k\alpha_k(v')$. From Lemma 38, $\overrightarrow{\alpha_k(v)} < \overrightarrow{\alpha_k(v')}$. Then

$$\overrightarrow{w} = \overrightarrow{\alpha_k(v)} + e_k < \overrightarrow{\alpha_k(v')} + e_k = \overrightarrow{w'}.$$

(ii) The proof is divided into four disjoint cases depending on the values of $w \in \{jk\pi_{jk}(v), k\pi_{jk}(v)\}$ and $w' \in \{jk\pi_{jk}(v'), k\pi_{jk}(v')\}$. The proof relies on the fact that $\overrightarrow{\pi_{jk}(v)} < \overrightarrow{\pi_{jk}(v')}$ but only the fourth case makes a stronger use of Lemma 38, i.e., $\overrightarrow{\pi_{jk}(v)} + e_j < \overrightarrow{\pi_{jk}(v')}$.

(ii.i) If $w = jk\pi_{jk}(v)$ and $w' = jk\pi_{jk}(v')$, then

$$\overrightarrow{w} = \overrightarrow{\pi_{jk}(v)} + e_j + e_k < \overrightarrow{\pi_{jk}(v')} + e_j + e_k = \overrightarrow{w'}.$$

(ii.ii) If $w = k\pi_{jk}(v)$ and $w' = k\pi_{jk}(v')$, then

$$\overrightarrow{w} = \overrightarrow{\pi_{jk}(v)} + e_k < \overrightarrow{\pi_{jk}(v')} + e_k = \overrightarrow{w'}.$$

(ii.iii) If $w = k\pi_{jk}(v)$ and $w' = jk\pi_{jk}(v')$, then

$$\overrightarrow{w} = \overrightarrow{\pi_{jk}(v)} + e_k < \overrightarrow{\pi_{jk}(v')} + e_j + e_k = \overrightarrow{w'}.$$

(ii.iv) If $w = jk\pi_{jk}(v)$ and $w' = k\pi_{jk}(v')$, then

$$\overrightarrow{w} = \overrightarrow{\pi_{jk}(v)} + e_j + e_k < \overrightarrow{\pi_{jk}(v')} + e_k = \overrightarrow{w'}.$$

□

References

[Ada03] B. Adamczewski. Balances for fixed points of primitive substitutions. *Theoret. Comput. Sci.*, 307(1):47–75, 2003.

[And03] E. Andres. Discrete linear objects in dimension n: the standard model. *Graphical Models*, 65:92–111, 2003.

[AR91] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité 2n+1. *Bull. Soc. Math. France*, 119(2):199–215, 1991.

[Bar95] Y. Baryshnikov. Complexity of trajectories in rectangular billiards. *Comm. Math. Phys.*, 174(1):43–56, 1995.

[BD14] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: S-adic expansions. *To appear in RIMS Lecture note 'Kokyuroku Bessatu'*, 2014.

[Bed03] N. Bedaride. Billiard complexity in rational polyhedra. *Regul. Chaotic Dyn.*, 8(1):97–104, 2003.

[BL11] V. Berthé and S. Labbé. Uniformly balanced words with linear complexity and prescribed letter frequencies. In P. Ambrož, Š. Holub, and Z. Masáková, editors, *Proc. 8th Int. Conf. on Words,* Prague, Czech Republic, September 12-16th, 2011, volume 63 of *Electronic Proceedings in Theoretical Computer Science*, pages 44–52. Open Publishing Association, 2011.

[BL13] V. Berthé and S. Labbé. Convergence and factor complexity for the Arnoux-Rauzy-Poincaré algorithm. In *WORDS*, volume 8079 of *Lecture Notes in Computer Science*, pages 71–82. Springer, 2013.

[Bos85] M. Boshernitzan. A unique ergodicity of minimal symbolic flows with linear block growth. *J. Analyse Math.*, 44:77–96, 1984/85.

[Bre81] A. J. Brentjes. *Multidimensional continued fraction algorithms*. Mathematisch Centrum, Amsterdam, 1981.

[Cas97a] J. Cassaigne. Complexité et facteurs spéciaux. *Bull. Belg. Math. Soc. Simon Stevin*, 4(1):67–88, 1997. Journées Montoises (Mons, 1994).

[Cas97b] Julien Cassaigne. Sequences with grouped factors. In Symeon Bozapalidis, editor, *Developments in Language Theory*, pages 211–222. Aristotle University of Thessaloniki, 1997.

[CN10] J. Cassaigne and F. Nicolas. Factor complexity. In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 163–247. Cambridge Univ. Press, Cambridge, 2010.

[DHS13] V. Delecroix, T. Hejda, and W. Steiner. Balancedness of Arnoux-Rauzy and Brun words. In J. Karhumäki, A. Lepistö, and L. Q. Zamboni, editors, *WORDS*, volume 8079 of *Lecture Notes in Computer Science*, pages 119–131. Springer, 2013.

[DLR13] F. Durand, J. Leroy, and G. Richomme. Do the Properties of an S-adic Representation Determine Factor Complexity? *Journal of Integer Sequences*, 16(Article 13.2.6), 2013.

[Dur03] F. Durand. Corrigendum and addendum to: "Linearly recurrent subshifts have a finite number of non-periodic subshift factors" [Ergodic Theory Dynam. Systems **20** (2000), no. 4, 1061–1078; MR1779393 (2001m:37022)]. *Ergodic Theory Dynam. Systems*, 23(2):663–669, 2003.

[Fer96] S. Ferenczi. Rank and symbolic complexity. *Ergodic Theory Dynam. Systems*, 16(4):663–682, 1996.

[FM10] S. Ferenczi and T. Monteil. Infinite words with uniform frequencies, and invariant measures. In *Combinatorics, automata and number theory*, volume 135 of *Encycl. Math. Appl.*, pages 373–409. Cambridge Univ. Press, 2010.

[Fog02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[Klo12] K. Klouda. Bispecial factors in circular non-pushy D0L languages. *Theoret. Comput. Sci.*, 445:63–74, 2012.

[Lab12] S. Labbé. *Structure des pavages, droites discrètes 3D et combinatoire des mots*. PhD thesis, Université du Québec à Montréal, May 2012.

[Ler12] J. Leroy. Some improvements of the S-adic conjecture. *Adv. in Appl. Math.*, 48(1):79–98, 2012.

[Lot02] M. Lothaire. *Algebraic Combinatorics on Words*. Cambridge University Press, Cambridge, 2002.

[Nog95] A. Nogueira. The three-dimensional Poincaré continued fraction algorithm. *Israel J. Math.*, 90(1-3):373–401, 1995.

[Sch00] F. Schweiger. *Multidimensional continued fractions*. Oxford Science Publications. Oxford University Press, Oxford, 2000.

[Zor97] A. Zorich. Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems*, 17(6):1477–1499, 1997.