# Palindromic sequences generated from marked morphisms *

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**Abstract**

Fixed points $\mathbf{u} = \varphi(\mathbf{u})$ of marked and primitive morphisms $\varphi$ over arbitrary alphabet are considered. We show that if $\mathbf{u}$ is palindromic, i.e., its language contains infinitely many palindromes, then some power of $\varphi$ has a conjugate in class $\mathcal{P}$. This class was introduced by Hof, Knill and Simon (1995) in order to study palindromic morphic words. Our definitions of marked and well-marked morphisms are more general than the ones previously used by Frid (1999) or Tan (2007). As any morphism with an aperiodic fixed point over a binary alphabet is marked, our result generalizes the result of Tan. Labbé (2014) demonstrated that already over a ternary alphabet the property of morphisms to be marked is important for the validity of our theorem. The main tool used in our proof is the description of bispecial factors in fixed points of morphisms provided by Klouda (2012).

Keywords: palindrome, complexity, fixed point, class P, marked morphism.
2000 MSC: 68R15, 37B10

## 1 Introduction

In 1995, Hof, Knill and Simon studied the spectral properties of Schroedinger operators associated with one-dimensional structure having finite local complexity. Such structure can be coded by an infinite word over a finite alphabet. Hof, Knill and Simon showed that if the word contains infinitely many distinct palindromes, then the operator has a purely singular continuous spectrum. A prominent example of such an infinite word is the Thue-Morse word over the binary alphabet $\{0,1\}$. This word can be obtained by the rewriting rules: "0 replaced by 01" and "1 replaced by 10". If we start with the letter 0 and repeat these rules, we get

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110 \mapsto \cdots$$

The finite string we obtained in the $i^{th}$ step is a prefix of the string we obtained in the $(i+1)^{st}$ step. The Thue-Morse word is defined as the unique infinite word $\mathbf{u_{TM}}$ having the property that $i^{th}$ string in our construction is a prefix of $\mathbf{u_{TM}}$ for each positive integer $i$.

*European Journal of Combinatorics (2016) 200–214. http://dx.doi.org/10.1016/j.ejc.2015.05.006

In Combinatorics on words, the previous construction is formalized by using the notions *morphism* of a free monoid and *fixed point* of a morphism (the formal definition is provided below). In such terminology the Thue-Morse word is a fixed point of the morphism $\varphi_{TM}$ defined by the images of both letters in the alphabet, namely by $\varphi_{TM}(0) = 01$ and $\varphi_{TM}(1) = 10$. It is a well known fact that the language of $\mathbf{u_{TM}}$, i.e., the set of all finite strings occurring in $\mathbf{u_{TM}}$, contains infinitely many palindromes.

In [12], a class of morphisms is introduced: a morphism $\varphi$ over an alphabet $\mathcal{A}$ belongs to class $\mathcal{P}$ if $\varphi$ has the form $\varphi(a) = pq_a$, where $p$ is a palindrome and $q_a$ are palindromes for all letters $a \in \mathcal{A}$. Hof, Knill and Simon observed that any fixed point of a morphism from class $\mathcal{P}$ is *palindromic*, i.e., infinitely many distinct palindromes occur in the fixed point.

In Remark 3 of [12], the question "Are there (minimal) sequences containing arbitrarily long palindromes that arise from morphism none of which belongs to class $\mathcal{P}$?" is formulated. To guarantee that the generated sequence is minimal (in terminology of the symbolic dynamic), already Hof, Knill and Simon considered primitive morphisms only. Let us recall that a morphism $\varphi$ over an alphabet $\mathcal{A}$ is called primitive, if there exists a power $k$ such that each letter $b \in \mathcal{A}$ occurs in the word $\varphi^k(a)$ for any letter $a \in \mathcal{A}$. A morphism can have more fixed points. For example, $\varphi_{TM}$ has two fixed points. It is easy to see that languages of all fixed points of a primitive morphism coincide.

In 2003, Allouche et al. demonstrated that any periodic palindromic sequence is a fixed point of a morphism in class $\mathcal{P}$ [1, Theorem 13]. The Thue-Morse word is palindromic but not periodic. The morphism $\varphi_{TM}$ which generates the Thue-Morse word does not belong to class $\mathcal{P}$. Since the Thue-Morse word is a fixed point of $\varphi_{TM}$, it is a fixed point of the second iteration $\varphi_{TM}^2$ as well. But the second iteration described by $\varphi_{TM}^2(0) = 0110$ and $\varphi_{TM}^2(1) = 1001$ is in class $\mathcal{P}$. In particular, the empty word serves as the palindrome $p$ and both $q_0 = 0110$, $q_1 = 1001$ are palindromes as well.

The second famous example of an infinite palindromic word is the Fibonacci word $\mathbf{u_F}$. It is the fixed point of the morphism $\varphi_F(0) = 01$ and $\varphi_F(1) = 0$. Applying $\varphi_F$ at the starting letter 0, we get

$$0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto 01001010 \mapsto 0100101001001 \mapsto 010010100100101001010 \cdots$$

Neither the Fibonacci morphism $\varphi_F$ nor its square $\varphi_F^2$ given by $\varphi_F^2(0) = 010$ and $\varphi_F^2(1) = 01$ belongs to class $\mathcal{P}$. Images of both letters 0 and 1 under $\varphi_F^2$ start with the same letter 0. If we cyclically shift these images by one letter to the left, i.e., we move the starting 0 from the beginning to the end, we get the morphism given by $\psi_F(0) = 100$ and $\psi_F(1) = 10$. The morphism $\psi_F$ is already in class $\mathcal{P}$. If a morphism $\psi$ can be obtained from a morphism $\varphi$ by a finite number of cyclical shifts, the morphisms are *conjugate.*

Let us stress that the property "to be palindromic" is the property of the language of an infinite word and not the property of the infinite word itself. It is known (see also Lemma 14) that two primitive conjugate morphisms have the same language. Both morphisms in our examples, namely $\varphi_{TM}$ and $\varphi_F$, are obviously primitive.

In 2007, Tan proved that if a fixed point of a primitive morphism $\varphi$ over a binary alphabet is palindromic, then there exists a morphism $\psi$ in class $\mathcal{P}$ such that languages of both morphisms coincide [22, Theorem 4.1]. In fact, Tan proved a stronger result, namely, that the morphism $\varphi$ or $\varphi^2$ is conjugate to a morphism from class $\mathcal{P}$.

In view of the previous results, in 2008, Blondin Massé and Labbé [3, 16] suggested the following HKS conjecture: (Version 1) *Let* $\mathbf{u}$ *be the fixed point of a primitive morphism. Then,* $\mathbf{u}$ *is palindromic if and only if there exists a morphism* $\varphi \neq \mathrm{Id}$ *such that* $\varphi(\mathbf{u}) = \mathbf{u}$ *and* $\varphi$ *has a conjugate in class* $\mathcal{P}$.

But this statement turned out to be false already over a ternary alphabet: in 2013, Labbé found an injective primitive morphism whose fixed points contradict Version 1 of the HKS conjecture [17]. In [11], Harju, Vesti and Zamboni pointed out that after erasing the first two letters from Labbé's counterexample one gets a fixed point of a morphism from class $\mathcal{P}$. Clearly, language of this word coincides with the language of the original word. Therefore it seems that the formulation of the theorem which Tan stated for a binary alphabet is more suitable for the formulation of the HKS conjecture: (Version 2) *Let* $\mathbf{u}$ *be a fixed point of a primitive morphism* $\varphi$. *If* $\mathbf{u}$ *is palindromic then there exists a morphism* $\psi$ *in class* $\mathcal{P}$ *such that the languages of both morphisms coincide.*

The possibility of different interpretations of the phrase "arise by morphisms of class $\mathcal{P}$" in the original question of Hof, Knill and Simon, led the authors of [11] to relax the statement "to be language of a fixed point of a morphism from class $\mathcal{P}$" which both previous variants impose on the language of a palindromic word $\mathbf{u}$. Harju, Vesti and Zamboni took into their consideration not only a fixed point of a primitive morphism, but also its image under a further morphism. Such a word is called *primitive morphic*. The last modification of HKS conjecture sounds: (Version 3) *If* $\mathbf{u}$ *is a palindromic primitive morphic word then there exist morphisms* $\varphi$ *and* $\psi$ *with conjugates in class* $\mathcal{P}$ *and an infinite word* $\mathbf{v}$ *such that* $\mathbf{v} = \varphi(\mathbf{v})$ *and languages of* $\mathbf{u}$ *and* $\psi(\mathbf{v})$ *coincide.*
Inspired by Labbé's counterexample, Harju, Vesti and Zamboni concentrated on infinite words with finite defect (for definition of defect consult [7]). A corollary of their results says that for any word $\mathbf{u}$ with finite defect Version 3 is true. But there exists no example of an infinite word (with finite or infinite defect) for which Version 3 holds and the stronger Version 2 fails.

In this paper, we show that Version 1 of the HKS conjecture is still valid for a large class of morphisms. We generalize the result of Tan for fixed point of marked morphisms on an alphabet of arbitrary size. A morphism $\varphi$ over an alphabet $\mathcal{A}$ is said to be marked if it is conjugate to morphisms $\psi$ and $\xi$ such that the first letter of $\psi(a)$ differs from the first letter of $\psi(b)$ and the last letter of $\xi(a)$ differs from the last letter of $\xi(b)$ for all $a, b \in \mathcal{A}$, $a \neq b$. Our definition of marked morphism is more general than the ones previously used by Frid [9, 10] or Tan [22]. The main result of this paper is the following.

**Theorem 1.** *Let* $\mathbf{u}$ *be a fixed point of a marked and primitive morphism* $\varphi : \mathcal{A}^* \to \mathcal{A}^*$. *If* $\mathbf{u}$ *is palindromic, then some power of* $\varphi$ *has a conjugate in class* $\mathcal{P}$.

Let us stress that any primitive morphism with an aperiodic fixed point over a binary alphabet is marked in our generalized sense. Therefore, Theorem 1 generalizes Tan's result and it is another step towards a complete characterization of the cases for which Version 1 of the HKS conjecture holds.

The article is organized as follows. In Section 2, we recall notions and results on morphisms, conjugacy, cyclic and marked morphisms. Characterization of morphisms from class $\mathcal{P}$ by the leftmost and rightmost conjugates is provided in Section 3. Section 4 is devoted to structure of bispecial factors of aperiodic palindromic words. Structure of bispecial factors of marked morphisms is described in Section 5. The crucial ingredient for our consideration in this section is the description of bispecial factors of circular D0L morphisms given by Klouda in [14]. Theorem 1 is proved in Section 6. The paper ends with some comments and open questions.

## 2 Preliminaries

### 2.1 Combinatorics on words

We borrow from Lothaire [20] the basic terminology about words. In what follows, $\mathcal{A}$ is a finite *alphabet* whose elements are called *letters*. A *word* $w$ over an alphabet $\mathcal{A}$ is a finite sequence $w = w_0w_1 \cdots w_{n-1}$ where $n \in \mathbb{N}$ and $w_i \in \mathcal{A}$ for each $i = 0, 1, \ldots, n-1$. The length of $w$ is $|w| = n$. By convention the *empty word* is denoted by $\varepsilon$ and its length is 0. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$ and the set of nonempty finite words is $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. Endowed with the concatenation, $\mathcal{A}^*$ is the *free monoid generated by* $\mathcal{A}$. The set of right infinite words is denoted by $\mathcal{A}^{\mathbb{N}}$ and the set of bi-infinite words is $\mathcal{A}^{\mathbb{Z}}$. Given a word $w \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, a *factor* $f$ of $w$ is a word $f \in \mathcal{A}^*$ satisfying $w = xfy$ for some $x \in \mathcal{A}^*$ and $y \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$. If $x = \varepsilon$ (resp. $y = \varepsilon$) then $f$ is called a *prefix* (resp. a *suffix*) of $w$. If $w = vu$ then $v^{-1}w$ and $wu^{-1}$ mean $u$ and $v$, respectively. The set of all factors of $w$, called the *language* of $w$, is denoted by $\mathcal{L}(w)$.

A factor $w$ of $\mathbf{u}$ is called *right special* if it has more than one *right extension* $x \in \mathcal{A}$ such that $wx \in \mathcal{L}(\mathbf{u})$. A factor $w$ of $\mathbf{u}$ is called *left special* if it has more than one *left extension* $x \in \mathcal{A}$ such that $xw \in \mathcal{L}(\mathbf{u})$. A factor which is both left and right special is called *bispecial*.

An infinite word $\mathbf{u}$ is *recurrent* if any factor occurring in $\mathbf{u}$ has an infinite number of occurrences. An infinite word $\mathbf{u}$ is *uniformly recurrent* if for any factor $w$ occurring in $\mathbf{u}$, there is some length $n$ such that $w$ appears in every factor of $\mathbf{u}$ of length $n$.

The *reversal* of $w = w_0w_1 \cdots w_{n-1} \in \mathcal{A}^*$ is the word $\widetilde{w} = w_{n-1}w_{n-2} \cdots w_0$. A *palindrome* is a word $w$ such that $w = \widetilde{w}$. We say that the language $\mathcal{L}(\mathbf{u})$ of an infinite word $\mathbf{u}$ is *closed under reversal* if $w \in \mathcal{L}(\mathbf{u})$ implies $\widetilde{w} \in \mathcal{L}(\mathbf{u})$ as well.

### 2.2 Periods, conjugacy and palindromes

This section gathers known results involving periods, conjugate words and palindromes that are used in this article. We recall the needed notions.

If $w = u^k$ for some $k \in \mathbb{N}$ and $u \in \mathcal{A}^*$, we say that $w$ is the $k$-th power of $u$. By $u^*$ we understand the set of all powers of $u$. A *period* of a word $v$ is an integer $p$ such that $v$ is a prefix of a power $u^k$ and $p = |u| < |v|$. An infinite word $\mathbf{w}$ is called *periodic* if $\mathbf{w} = u^\omega = uuu\cdots$ for some nonempty word $u$. Length of $u$ is called a *period* of $\mathbf{w}$. An infinite word $\mathbf{w}$ is *eventually periodic* if $\mathbf{w} = vu^\omega$ for some finite words $v$ and $u \neq \varepsilon$; their length $|v|$ and $|u|$ are called a *preperiod* and a *period*, respectively. An infinite word is *aperiodic* if it is not eventually periodic.

First we recall the result of Fine and Wilf for words having two periods.

**Lemma 2** (Fine and Wilf). *Let $u$ and $w$ be finite words over an alphabet $\mathcal{A}$. Suppose $u^h$ and $w^k$, for some integer $h$ and $k$, have a common prefix of length $|u| + |w| - \gcd(|u|, |w|)$. Then there exists $z \in \mathcal{A}^*$ of length $\gcd(|u|, |w|)$ such that $u, w \in z^*$.*

From Lothaire [18, 19] we borrow also another useful result about conjugate words, formally defined as follows.

**Definition 3.** *Let $y \in \mathcal{A}^*$ and $k \in \mathbb{N}$. The $k$-th* right conjugate *of $y$ is the word $x$ such that $xw = wy$ for some word $w$ of length $|w| = k$.*

**Lemma 4.** [18, 19] *If $z = xw = wy$ for some $x, y, w, z \in \mathcal{A}^*$, then there exist unique words $u, v \in \mathcal{A}^*$ and a unique integer $i \geq 0$ such that*

$$x = uv, w = (uv)^i u, y = vu. \tag{1}$$

*where $|u|$, $0 \leq |u| < |x|$, is the remainder and $i$ is the quotient of the division of $|w|$ by $|x|$.*

Now, we present an easy lemma about words that are product of two palindromes.

**Lemma 5.** *Let $u$, $v$ be two palindromes. If $k$ is an integer such that*

$$k = |u| + \left\lceil \frac{|v|}{2} \right\rceil + \ell \cdot |vu| \qquad \text{or} \qquad k = \left\lceil \frac{|u|}{2} \right\rceil + \ell \cdot |vu|$$

*for some integer $\ell \geq 0$, then the $k$-th conjugate of $vu$ is of the form $\alpha \cdot p$ where $\alpha \in \mathcal{A} \cup \{\varepsilon\}$ and $p$ is a palindrome. Moreover, it is a palindrome if and only if $\alpha = \varepsilon$ if and only if the ceil function is applied on an integer, that is if $|v|$ is even in the first case and if $|u|$ is even in the second case.*

*Proof.* Write $u = s\alpha\widetilde{s}$ and $v = t\beta\widetilde{t}$ for some words $s, t \in \mathcal{A}^*$ and letters (or empty word) $\alpha, \beta \in \mathcal{A} \cup \{\varepsilon\}$. The conjugates $c_1 = \beta\widetilde{t}s\alpha\widetilde{s}t$ and $c_2 = \alpha\widetilde{s}t\beta\widetilde{t}s$ of $vu$ are of the desired form. We have

$$c_1 \cdot \beta\widetilde{t}s\alpha\widetilde{s} = \beta\widetilde{t}s\alpha\widetilde{s} \cdot vu \qquad \text{and} \qquad c_2 \cdot \alpha\widetilde{s} = \alpha\widetilde{s} \cdot vu.$$

Then $c_1$ is the $k_1$-th conjugate of $vu$ for $k_1 = |\beta\widetilde{t}s\alpha\widetilde{s}| = |u| + \lceil \frac{|v|}{2} \rceil$ and $c_2$ is the $k_2$-th conjugate of $vu$ for $k_2 = |\alpha\widetilde{s}| = \lceil \frac{|u|}{2} \rceil$. Note that the $k + \ell|vu|$-th conjugate of $vu$ is equal to the $k$-th conjugate of $vu$. Finally, $c_1$ is a palindrome exactly when $|v|$ is even and $c_2$ is a palindrome exactly when $|u|$ is even. □

We finish by stating results which appeared previously in [6, Lemma 1 and 2] and [5, Proposition 6]. It was also published in a less general form in [7, Lemma 5] and as a more general form (using involutive antimorphism) as Lemma 2.10 and 2.11 of [16].

**Lemma 6.** [5–7, 16] *Assume that $z = xw = wy$. Let $u, v$ and $i \geq 0$ be such that* (1) *holds. The following conditions are equivalent:*

(i) $x = \widetilde{y}$*;*

(ii) *$u$ and $v$ are palindromes, i.e., $\widetilde{u} = u$ and $\widetilde{v} = v$;*

(iii) *$z$ is a palindrome, i.e., $\widetilde{z} = z$;*

(iv) *$xwy$ is a palindrome, i.e., $\widetilde{xwy} = xwy$.*

*Moreover, if one of the equivalent conditions above holds then*

(v) *$w$ is a palindrome, i.e., $\widetilde{w} = w$.*

**Lemma 7.** [6, 16] *Assume that $z = xw = wy$ with $|w| \geq |x|$. Then, conditions* (i)-(v) *in Lemma 6 are equivalent.*

## 2.3 Morphisms

A *morphism* is a function $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ compatible with concatenation, that is, such that $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \mathcal{A}^*$. The *identity morphism* on $\mathcal{A}$ is denoted by $\mathrm{Id}_{\mathcal{A}}$ or simply Id when the context is clear. A matrix $M$ with elements $M_{ab}$ = number of occurrences of the letter $a$ in $\varphi(b)$ for each $a, b \in \mathcal{A}$ is called *incidence matrix of* $\varphi$. Obviously, $M$ is a $(d \times d)$-matrix, where $d = \mathrm{Card}\,\mathcal{A}$. It is easy to see that a morphism $\varphi$ is *primitive* if and only if some power of its incidence matrix has only positive elements. A morphism can be naturally extended to a map over $\mathcal{A}^{\mathbb{N}}$ by

$$\varphi(u_0 u_1 u_2 \cdots) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$$

A morphism is *erasing* if the image of one of the letters is the empty word. If $\varphi$ is a nonerasing morphism, we define $\textsc{Fst}(\varphi) : \mathcal{A} \to \mathcal{A}$ to be the function such that $\textsc{Fst}(\varphi)(a)$ is the first letter of $\varphi(a)$. Similarly, let $\textsc{Lst}(\varphi) : \mathcal{A} \to \mathcal{A}$ be the function such that $\textsc{Lst}(\varphi)(a)$ is the last letter of $\varphi(a)$.

A morphism $\varphi$ is *prolongable at the letter* $a$ if $\varphi(a) = aw$, where $w$ is a nonempty word $w$. If $\varphi$ is prolongable at $a$, then

$$\mathbf{w} = aw\varphi(w)\varphi(\varphi(w))\cdots\varphi^n(w)\cdots$$

is a *fixed point* of $\varphi$, i.e., $\varphi(\mathbf{w}) = \mathbf{w}$. A morphism may be prolongable at more letters, in other words a morphism may have more fixed points. The fixed point starting with letter $a$ will be denoted $\varphi^\infty(a)$. As we have already mentioned, all fixed points of a primitive morphism $\varphi$ have the same language. Therefore, notation $\mathcal{L}(\varphi)$ is used instead of $\mathcal{L}(\mathbf{u})$, where $\mathbf{u}$ is a fixed point of $\varphi$. The *reversal* of a morphism $\varphi$, denoted by $\widetilde{\varphi}$, is the morphism such that $\widetilde{\varphi}(\alpha) = \widetilde{\varphi(\alpha)}$ for all $\alpha \in \mathcal{A}$. We now define formally class $\mathcal{P}$ morphisms.

**Definition 8.** *A morphism $\varphi$ is in* class $\mathcal{P}$ *if there exists a palindrome $p$ such that $p$ is a prefix of $\varphi(\alpha)$ and $\varphi(\alpha)p$ is a palindrome for every $\alpha \in \mathcal{A}$.*

A straightforward consequence of the above definition is that the mapping $\Psi : v \mapsto \varphi(v)p$ assigns to any palindrome $v$ a new palindrome $\varphi(v)p$. If $\varphi$ is primitive then $\Psi^n(\alpha)$ is a palindrome for any letter $\alpha \in \mathcal{A}$ and any $n \in \mathbb{N}$ and clearly, $\Psi^n(\alpha)$ belongs to $\mathcal{L}(\varphi)$. Therefore, the language of any fixed point of a primitive morphism in class $\mathcal{P}$ is palindromic.

**Remark 9.** *The primitivity of $\varphi \in \mathcal{P}$ is not necessary for palindromicity of a fixed point of $\varphi$. The following example was provided by Starosta (personal communication, spring, 2014). Consider the morphism $\varphi$ on a binary alphabet defined by $0 \mapsto 000$ and $1 \mapsto 10110100$. Clearly, the morphism belongs to class $\mathcal{P}$. The fixed point $\varphi^\infty(1)$ is not uniformly recurrent as it contains arbitrarily long blocks of zeros. It can be shown that $\varphi^\infty(1)$ has defect $0$ and thus contains infinitely many palindromes.*

## 2.4 Conjugacy of acyclic morphisms

Recall from Lothaire [20] (Section 2.3.4) that $\varphi$ is *right conjugate* of $\psi$, or that $\psi$ is *left conjugate* of $\varphi$, noted $\psi \triangleright \varphi$, if there exists $w \in \mathcal{A}^*$ such that

$$w\psi(x) = \varphi(x)w, \quad \text{for all words } x \in \mathcal{A}^*, \tag{2}$$

or equivalently that $w\psi(\alpha) = \varphi(\alpha)w$, for all letters $\alpha \in \mathcal{A}$. We say that the word $w$ is the *conjugate word of the relation* $\psi \triangleright \varphi$.

A morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is *cyclic* [19] if there exists a word $w \in \mathcal{A}^*$ such that $\varphi(\alpha) \in w^*$ for all $\alpha \in \mathcal{A}$. Otherwise, we say that $\varphi$ is *acyclic*. If $\varphi$ is cyclic and $|w| > 1$, then the fixed point of $\varphi$ is $wwww\cdots$ and is periodic. Observe that the converse does not hold. For example, $a \mapsto aba, b \mapsto bab$ is acyclic but both its fixed points are periodic. We have the following statement.

**Lemma 10.** *A morphism is cyclic if and only if it is conjugate to itself with a nonempty conjugate word.*

*Proof.* If $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is cyclic, then there exists a word $w \in \mathcal{A}^*$ such that for all $\alpha \in \mathcal{A}$ there exists an integer $n$ such that $\varphi(\alpha) = w^n$. If $w$ is empty, then $\varphi$ is conjugate to itself with any word of $\mathcal{A}^+$ as conjugate word. Suppose that $w$ is not empty. Therefore $\varphi(\alpha)w = w^n \cdot w = w \cdot w^n = w\varphi(\alpha)$ for all $\alpha \in \mathcal{A}$ and $\varphi$ is conjugate to itself with a nonempty conjugate word $w$.

For the reciprocal, recall that $xy = yx$ if and only if $x$ and $y$ are powers of the same word [19, Prop. 1.3.2]. Suppose there exists $w \in \mathcal{A}^+$ such that $\varphi(\alpha)w = w\varphi(\alpha)$ for all $\alpha \in \mathcal{A}$. Then, there exists a nonempty word $z_\alpha$ such that $\varphi(\alpha)$ and $w$ are powers of $z_\alpha$ for all $\alpha \in \mathcal{A}$. If $w = z_\alpha$ for all $\alpha \in \mathcal{A}$, then $\varphi(\alpha) \in w^*$ for all $\alpha \in \mathcal{A}$ and $\varphi$ is cyclic. If there is only one letter $\beta \in \mathcal{A}$ such that $w = (z_\beta)^n$ with $n > 1$ and $w = z_\alpha$ for all $\alpha \in \mathcal{A} \setminus \{\beta\}$, then $\varphi(\alpha) \in (z_\beta)^*$ for all $\alpha \in \mathcal{A}$ and $\varphi$ is cyclic. If there are more than one letter $\beta \in \mathcal{A}$ such that $w = (z_\beta)^{n_\beta}$ with $n_\beta > 1$, then from Lemma 2 there exists a word $z \in \mathcal{A}^+$ such that $z_\beta \in z^*$ for all those letters $\beta$ and $\varphi(\alpha) \in z^*$ for all $\alpha \in \mathcal{A}$ and $\varphi$ is cyclic. $\square$

**Definition 11.** *Let $\varphi$ be a morphism. The* rightmost conjugate of $\varphi$ *is a morphism $\varphi_R$ such that the following two conditions hold:*

(i) *$\varphi_R$ is right conjugate of $\varphi$;*

(ii) *if $\psi$ is right conjugate to $\varphi_R$, then $\psi = \varphi_R$.*

*The* leftmost conjugate of $\varphi$ *is defined analogously and denoted by $\varphi_L$.*

In other words, $\varphi_R$ is the rightmost conjugate of $\varphi$ if $\varphi_R$ is a right conjugate of $\varphi$ and $\textsc{Lst}(\varphi_R)$ is not constant. Also, $\varphi_L$ is the leftmost conjugate of $\varphi$ if $\varphi_L$ is a left conjugate of $\varphi$ and $\textsc{Fst}(\varphi_L)$ is not constant. Some morphisms do not have a leftmost or rightmost conjugate. For example let $f : a \mapsto abab, b \mapsto ab$ and $g : a \mapsto baba, b \mapsto ba$. We have that $f \triangleright g \triangleright f$.

**Lemma 12.** *If a morphism is acyclic, it has a leftmost and a rightmost conjugate.*

*Proof.* Suppose that a morphism $\varphi$ does not have a rightmost conjugate. Then there exists arbitrarily long words $w$ satisfying $\psi(\alpha)w = w\varphi(\alpha)$ for each $\alpha \in \mathcal{A}$. Take a word $w$ such that $|\varphi(\alpha)|$ divides the length of $w$ for all letters $\alpha \in \mathcal{A}$. According to Lemma 4, $\psi(\alpha) = \varphi(\alpha)$ for each $\alpha \in \mathcal{A}$. This means that $\varphi$ is conjugate to itself with a nonempty conjugate word. From Lemma 10, $\varphi$ is cyclic. The argument for leftmost conjugates is the same. $\square$

**Example 13.** *Consider the following morphisms:*

$$\begin{array}{ll}
\varphi_1 : a \mapsto babba, b \mapsto bab, & \varphi_5 : a \mapsto ababb, b \mapsto abb, \\
\varphi_2 : a \mapsto abbab, b \mapsto abb, & \varphi_6 : a \mapsto babba, b \mapsto bba, \\
\varphi_3 : a \mapsto bbaba, b \mapsto bba, & \varphi_7 : a \mapsto abbab, b \mapsto bab, \\
\varphi_4 : a \mapsto babab, b \mapsto bab. &
\end{array}$$

*The morphism $\varphi_1$ is right conjugate to $\varphi_2$ with conjugate word $b$ because $\varphi_1(a)\cdot b = babbab = b\cdot\varphi_2(a)$ and $\varphi_1(b)\cdot b = babb = b\cdot\varphi_2(b)$. In general, the following relations are satisfied:*

$$\varphi_L = \varphi_7 \triangleright \varphi_6 \triangleright \varphi_5 \triangleright \varphi_4 \triangleright \varphi_3 \triangleright \varphi_2 \triangleright \varphi_1 = \varphi_R.$$

*The morphism $\varphi_7$ is leftmost conjugate and $\varphi_1$ is rightmost conjugate with conjugate word $babbab$.*

Recall some simple properties of conjugate morphisms.

**Lemma 14.** *Let a morphism $\varphi$ be right conjugate of a morphism $\psi$ and $w \in \mathcal{A}^*$ be the conjugate word of the relation $\psi \triangleright \varphi$.*

1. *For any $k \in \mathbb{N}$, the morphism $\varphi^k$ is right conjugate of $\psi^k$.*
2. *$\varphi$ is injective if and only if $\psi$ is injective.*
3. *$\varphi$ is primitive if and only if $\psi$ is primitive.*
4. *If $\varphi$ is primitive, then $\mathcal{L}(\varphi) = \mathcal{L}(\psi)$.*
5. *$\widetilde{\psi}$ is right conjugate of $\widetilde{\varphi}$ and the corresponding conjugate word is $\widetilde{w}$.*

*Proof.* 1. Let us define recursively $w_{(1)} = w$ and $w_{(k+1)} = \varphi(w_{(k)})w = w\psi(w_{(k)})$ for any $k \in \mathbb{N}$. We show that $\varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u)$ for any $u \in \mathcal{A}^*$. We proceed by induction on $k$.

Assume that $\varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u)$. As $\varphi(v)w = w\psi(v)$ for any word $v$, we can apply this relation to $v = \varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u)$. We get $\varphi^{k+1}(u)\varphi(w_{(k)})w = w\psi(w_{(k)})\psi^{k+1}(u)$, or equivalently $\varphi^{k+1}(u)w_{(k+1)} = w_{(k+1)}\psi^{k+1}(u)$.

2. Let $u, v \in \mathcal{A}^*$. Then

$$\psi(u) = \psi(v) \;\Leftrightarrow\; \psi(u)w = \psi(v)w \;\Leftrightarrow\; w\varphi(u) = w\varphi(v) \;\Leftrightarrow\; \varphi(u) = \varphi(v)\,.$$

3. Let us recall that a morphism is primitive if and only if there exists an integer $k$ such that all elements of $k^{th}$ powers of its incidence matrix are positive. Two mutually conjugate morphisms have the same incidence matrix.

4. Let us fix an arbitrary $n \in \mathbb{N}$. We will show that $\mathcal{L}_n(\varphi) = \mathcal{L}_n(\psi)$, where $\mathcal{L}_n(\varphi)$ denotes $\{w \in \mathcal{L}(\varphi) : |w| = n\}$ and $\mathcal{L}_n(\psi)$ is defined analogously. According to point 3, $\psi$ is primitive as well, and thus there exists a number $R(n)$ such that any factor of $\mathcal{L}(\varphi)$ longer than $R(n)$ contains any factor of $\mathcal{L}_n(\varphi)$ and any factor of $\mathcal{L}(\psi)$ longer than $R(n)$ contains any factor of $\mathcal{L}_n(\psi)$. Let us find $k$ such that for some letter $a$ both words $\varphi^k(a)$ and $\psi^k(a)$ are longer than $2R(n)$.

As $\varphi^k$ and $\psi^k$ are conjugate, there exists a word, say $y$, such that $\varphi^k(a)y = y\psi^k(a)$. Using the fact that the equation $xy = yz$ implies $x = uv$ and $z = vu$ for some $u, v$, we can write $\varphi^k(a) = uv$ and $\psi^k(a) = vu$, in particular $u, v \in \mathcal{L}(\varphi)$ and $u, v \in \mathcal{L}(\psi)$. Since $|\varphi^k(a)| \geq 2R(n)$, either $u$ or $v$ are longer than $R(n)$ and thus all elements from $\mathcal{L}_n(\varphi)$ occur in $u$ or $v$ and consequently in $\mathcal{L}(\psi)$ as well.

5. Applying reversal mapping to relation $\varphi(a)w = w\psi(a)$ for any letter $a \in \mathcal{A}$, we get $\widetilde{w}\widetilde{\varphi(a)} = \widetilde{\psi(a)}\widetilde{w}$ as desired. □

**Lemma 15.** *Let $\varphi$ be a primitive acyclic morphism. Denote $\varphi_L$ and $\varphi_R$ the leftmost and the rightmost conjugate of $\varphi$, respectively. Let $w \in \mathcal{A}^*$ be the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. If $u \in \mathcal{L}(\varphi)$ then $\varphi_R(u)w = w\varphi_L(u) \in \mathcal{L}(\varphi)$.*

*Proof.* Let $\mathbf{u}$ be a fixed point of $\varphi$. Since $\varphi$ is primitive, $u$ occurs in $\mathbf{u}$ infinitely many times. Hence there exists arbitrarily long word $v$ such that $vu \in \mathcal{L}(\varphi)$. Clearly, $\varphi_L(vu) \in \mathcal{L}(\varphi)$. Consequently, $w^{-1}w\varphi_L(vu) = w^{-1}\varphi_R(vu)w = w^{-1}\varphi_R(v)\varphi_R(u)w \in \mathcal{L}(\varphi)$. Since the factor $v$ is long enough, $|w^{-1}\varphi_R(v)| > 0$ and thus $\varphi_R(u)w \in \mathcal{L}(\varphi)$. □

## 2.5 Marked morphisms

Frid [9, 10] defined a morphism $\varphi$ to be marked if both $\textsc{Fst}(\varphi)$ and $\textsc{Lst}(\varphi)$ are injective. On a binary alphabet, Tan [22] defined a morphism $\varphi$ to be marked if $\textsc{Fst}(\varphi)$ is injective and well-marked if $\textsc{Fst}(\varphi)$ is the identity. It is convenient to extend the definition of Frid to morphisms such that the cardinality of their conjugacy class is larger than one. Also, we do not need that $\textsc{Fst}(\varphi)$ be the identity in the proof of Lemma 27. Thus, we introduce the following definitions that will be useful in the sequel.

**Definition 16.** *Let $\varphi$ be an acyclic morphism. We say that $\varphi$ is* marked *if*

$$\textsc{Fst}(\varphi_L) \text{ and } \textsc{Lst}(\varphi_R) \text{ are injective}$$

*and that $\varphi$ is* well-marked *if*

$$\text{it is marked and if } \textsc{Fst}(\varphi_L) = \textsc{Lst}(\varphi_R)$$

*where $\varphi_L$ ($\varphi_R$ resp.) is the leftmost (rightmost resp.) conjugate of $\varphi$.*

**Example 17.** *The morphism $\varphi_3 : a \mapsto bbaba, b \mapsto bba$ is acyclic. It has a leftmost conjugate $\varphi_L = \varphi_7 : a \mapsto abbab, b \mapsto bab$ and a rightmost conjugate $\varphi_R = \varphi_1 : a \mapsto babba, b \mapsto bab$. The morphism $\varphi_3$ is marked since $\textsc{Fst}(\varphi_L)$ and $\textsc{Lst}(\varphi_R)$ are injective. It is also well-marked since $\textsc{Fst}(\varphi_L) = \textsc{Lst}(\varphi_R)$.*

**Lemma 18.** *A marked morphism has a well-marked power.*

*Proof.* Let us realize that $\textsc{Fst}(\varphi \circ \psi) = \textsc{Fst}(\varphi) \circ \textsc{Fst}(\psi)$ for each morphisms $\varphi, \psi : \mathcal{A}^* \to \mathcal{A}^*$. If $\varphi$ is marked, then $\textsc{Fst}(\varphi_L)$ and $\textsc{Lst}(\varphi_R)$ are permutations of the alphabet $\mathcal{A}$. Let $d = \operatorname{Card} \mathcal{A}$ Then for $k = d!$, the permutations $\big(\textsc{Fst}(\varphi_L)\big)^k$ and $\big(\textsc{Lst}(\varphi_R)\big)^k$ are the identity. Note that $\varphi_L^k$ is the leftmost conjugate and $\varphi_R^k$ is the rightmost conjugate of $\varphi^k$. Since $\textsc{Fst}(\varphi_L^k) = \big(\textsc{Fst}(\varphi_L)\big)^k = \mathrm{Id} = \big(\textsc{Lst}(\varphi_R)\big)^k = \textsc{Lst}(\varphi_R^k)$ the power $\varphi^k$ is well-marked. $\square$

The power need not be $d!$. In fact, the least positive integer $a(d)$ for which $p^{a(d)} = 1$ for all permutations $p$ in $S_d$ is sufficient. This sequence $(a(d))_d$ is well-known and indexed by A003418 in the OEIS [21]. Of course, there might be an integer $N < a(d)$ such that $\textsc{Fst}(\varphi_L)^N = \textsc{Lst}(\varphi_R)^N$. In particular, this happens for the binary alphabet.

**Lemma 19.** *A marked morphism is injective.*

*Proof.* Let $\varphi_L$ be the leftmost conjugate of a marked morphism $\varphi$. By definition $\textsc{Fst}(\varphi_L)$ is injective so that $\varphi_L$ is injective. From Lemma 14, injectivity is preserved by conjugacy. Therefore $\varphi$ is injective. $\square$

# 3 Equivalent conditions for a morphism to be in class $\mathcal{P}$

Let $\varphi$ be an acyclic and primitive morphism. Let $\varphi_R$ ($\varphi_L$ resp.) be the rightmost (leftmost resp.) conjugate of $\varphi$ (their existence follows from the acyclic hypothesis, see Lemma 12). Let $w$ be the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. Then, Equation (2) is satisfied:

$$\varphi_R(x)w = w\varphi_L(x), \quad \text{for all words } x \in \mathcal{A}^*.$$

Note that we have $|\varphi_R(x)| = |\varphi_L(x)|$ for all words $x \in \mathcal{A}^*$. Obviously, if a word $x \in \mathcal{A}^*$ is such that $|\varphi_L(x)| \geq |w|$, then $w$ is a suffix of $\varphi_L(x)$ and $w$ is a prefix of $\varphi_R(x)$.

The next lemma holds whatever is the size of the alphabet. An important consequence is that HKS conjecture is satisfied for all morphisms satisfying one of the equivalent conditions.

**Lemma 20.** *Let $\varphi$ be a morphism and $\varphi_R$ ($\varphi_L$ resp.) be the rightmost (leftmost resp.) conjugate of $\varphi$. Let $w$ be the conjugate word such that $w\varphi_L(u) = \varphi_R(u)w$ for all words $u \in \mathcal{A}^*$. Let $\mathcal{B} \subseteq \mathcal{A}$ be the set of letters for which the image under $\varphi$ is larger than the conjugate word $w$, i.e.,*

$$\mathcal{B} = \{b \in \mathcal{A} : |\varphi(b)| > |w|\}.$$

*For all $b \in \mathcal{B}$, let $p_b$ be the nonempty word such that $\varphi_L(b) = p_b w$. Then, the following conditions are equivalent:*

(1) *the $\lfloor \frac{|w|+1}{2} \rfloor$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$;*

(2) *$\varphi$ has a conjugate in class $\mathcal{P}$;*

(3) *$\varphi$ and $\widetilde{\varphi}$ are conjugates;*

(4) *$\varphi_L = \widetilde{\varphi_R}$;*

(5) *$w$ is a palindrome and $p_b$ is a palindrome for all $b \in \mathcal{B}$.*

*Proof.* (1) $\implies$ (2): This is clear.

(2) $\implies$ (3): Let $\varphi' : \alpha \mapsto pq_\alpha$ be the conjugate of $\varphi$ in class $\mathcal{P}$. Then $\widetilde{\varphi}$ is conjugate to $\widetilde{\varphi}' : \alpha \mapsto q_\alpha p$. But $\varphi'$ and $\widetilde{\varphi}'$ are conjugates. We conclude by transitivity of the conjugacy of morphisms.

(3) $\implies$ (4): The leftmost conjugate of $\widetilde{\varphi}$ is $\widetilde{\varphi_R}$. If $\varphi$ and $\widetilde{\varphi}$ are conjugates, they must share the same leftmost conjugate. Therefore $\varphi_L = \widetilde{\varphi_R}$.

(4) $\implies$ (5): Let $u$ be a word long enough so that $|\varphi_L(u)| \geq |w|$. Then $w$ is a suffix of $\varphi_L(u)$ because $w\varphi_L(u) = \varphi_R(u)w$. But $\varphi_L(u) = \widetilde{\varphi_R}(u) = \widetilde{\varphi_R(\widetilde{u})}$. Therefore, $\widetilde{w}$ is a prefix of $\varphi_R(\widetilde{u})$. But again $w$ is a prefix of $\varphi_R(\widetilde{u})$. We conclude that $w$ is a palindrome. If $b \in \mathcal{B}$, then $\widetilde{w}\widetilde{p_b}w = \widetilde{\varphi_L}(b)w = \varphi_R(b)w = w\varphi_L(b) = wp_bw$ and $p_b$ is a palindrome.

(5) $\implies$ (1): Let $k = \lfloor \frac{|w|+1}{2} \rfloor$ and $x_\alpha^{(k)}$ be $k$-th conjugate of the word $\varphi_L(\alpha)$.

First, suppose $b \in \mathcal{B}$. Let $z$ be the word such that $w = \widetilde{z}cz$ for some letter (or empty word) $c \in \mathcal{A} \cup \{\varepsilon\}$.

$$x_b^{(k)} = czp_b\widetilde{z}$$

is a palindrome if $|w|$ is even and the product of the letter $c$ and a palindrome if $|w|$ is odd.

Now suppose $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Then $|w| \geq |\varphi_L(\alpha)|$. The equation $\varphi_R(\alpha)w = w\varphi_L(\alpha)$ is depicted in Figure 1. Hypothesis of Lemma 7 are satisfied and we conclude the existence of palindromes

<table>
<tr><td colspan="2">$w$</td><td>$\varphi_L(\alpha)$</td></tr>
<tr><td>$\varphi_R(\alpha)$</td><td colspan="2">$w$</td></tr>
</table>

Figure 1: When $w$ is longer than $\varphi_L(\alpha)$.

$u_\alpha, v_\alpha \in \mathcal{A}^*$ and an integer $i \geq 1$ such that $\varphi_L(\alpha) = v_\alpha u_\alpha$, $w = (u_\alpha v_\alpha)^i u_\alpha$, $\varphi_R(\alpha) = \widetilde{\varphi_L(\alpha)} = u_\alpha v_\alpha$ and

$$|w| = i \cdot n_\alpha + |u_\alpha|, \quad 0 \leq |u_\alpha| < n_\alpha.$$

where $n_\alpha = |\varphi_L(\alpha)|$. We show that the $\lfloor \frac{|w|+1}{2} \rfloor$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$. Observe that $k \leq |w|$ so that the $k$-th conjugate of $\varphi_L$ exists. We have

$$k = \left\lfloor \frac{|w|+1}{2} \right\rfloor = \left\lceil \frac{|w|}{2} \right\rceil = \left\lceil \frac{i|u_\alpha v_\alpha| + |u_\alpha|}{2} \right\rceil = \begin{cases} \left\lceil \frac{|u_\alpha|}{2} \right\rceil + \frac{i}{2}|u_\alpha v_\alpha| & \text{if } i \text{ is even,} \\ \left\lceil \frac{|v_\alpha|}{2} \right\rceil + |u_\alpha| + \frac{i-1}{2}|u_\alpha v_\alpha| & \text{if } i \text{ is odd.} \end{cases}$$

If $|w|$ is even, then $i$ is odd and $|v_\alpha|$ is even or $i$ and $|u_\alpha|$ are both even. In other words, the ceil function is applied on an integer in the above formulas for $k$. From Lemma 5, the $k$-th conjugate of the word $\varphi_L(\alpha)$ is a palindrome. We conclude that the $k$-th conjugate of $\varphi_L$ is mapping each letter $\alpha \mapsto x_\alpha^{(k)}$ on a palindrome, that is, the $k$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$.

If $|w|$ is odd, then $i$ is odd and $|v_\alpha|$ is odd or $i$ is even and $|u_\alpha|$ is odd. In other words, the ceil function is applied on a half-integer in the above formulas for $k$. From Lemma 5, the $k$-th conjugate of the word $\varphi_L(\alpha)$ is of the form $c_\alpha p$ where $c_\alpha \in \mathcal{A}$ is a letter and $p$ is a palindrome. Since $k \geq 1$ and because of the definition of conjugacy, the first letter of each word $x_\alpha^{(k)}$ is the same, i.e., $c_\alpha = c$. We conclude that the $k$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$. □

**Corollary 21.** *Let $w$ be a conjugate word of the relation $\triangleright$ under which a morphism $\varphi$ is right conjugate of some morphism. If $w$ is a palindrome and $|w| \geq |\varphi(\alpha)|$ for all $\alpha \in \mathcal{A}$, then the $\lfloor \frac{|w|+1}{2} \rfloor$-th conjugate of $\varphi$ is in class $\mathcal{P}$.*

The morphisms $\varphi_R$ and $\varphi_L$ from Example 13 satisfy the assumptions of the previous corollary. Therefore a conjugate of them belongs to class $\mathcal{P}$.

# 4 Properties of palindromic words

This section contains results on properties of palindromic words. It ends with a lemma giving sufficient conditions for the presence of infinitely many palindromic bispecial factors. First we recall the relationship between two properties of language: "to be palindromic" and "to be closed under reversal".

**Lemma 22.** *Let $\mathbf{u}$ be a uniformly recurrent palindromic word. Then its language is closed under reversal.*

*Proof.* Let $v \in \mathcal{L}(\mathbf{u})$ and $|v| = n$. Since $\mathbf{u}$ is uniformly recurrent, there exists an integer $R(n)$ such that any factor $u \in \mathcal{L}(\mathbf{u})$ with $|u| \geq R(n)$ contains all factors from $\mathcal{L}_n(\mathbf{u})$. In particular, any palindrome $p \in \mathcal{L}(\mathbf{u})$ with $|p| \geq R(n)$ contains all factors of length $n$, i.e., $v$ occurs in $p$. And clearly $\widetilde{v}$ occurs in $p$ as well. □

Berstel, Boasson, Carton and Fagnot provided in [2] a nice example of uniformly recurrent word whose language is closed under reversal, but the language is not palindromic. It means that the opposite implication in the previous lemma does not hold. If we restrict ourselves to eventually periodic words, the opposite implication is valid.

The next result is well-known and already present in [1,7].

**Lemma 23.** *Let* $\mathbf{u}$ *be an eventually periodic word with language closed under reversal. Then* $\mathbf{u}$ *is periodic and there exist two palindromes* $p$ *and* $q$ *such that* $\mathbf{u} = (pq)^\omega$*.*

*Proof.* Let $\mathbf{u} = zw^\omega$, where the length of $z$ is minimal and $w$ has the shortest length among all possible periods. First we show the fact that $w$ occurs in $ww$ only twice: once as a prefix and once as a suffix. Indeed, otherwise there exist nonempty words $x$ and $y$ such that $w = xy = yx$. Then according to Lemma 4 the word $w$ is a power of a shorter word and it contradicts our choice of $w$.

Since $\mathcal{L}(\mathbf{u})$ is closed under reversal, any prefix of $\mathbf{u}$ has infinitely many occurrences in $\mathbf{u}$. In particular, the prefix $zw$ occurs in the factor $w^n$ for some sufficiently large $n \in \mathbb{N}$. Due to the previous fact, $z$ is a suffix of $w^{n-1}$. The minimality of the length of $z$ implies that $z$ is empty.

Then $\mathbf{u}$ is purely periodic, i.e., $\mathbf{u} = w^\omega$. Closedness under reversal gives that $\widetilde{w}$ is a factor of $\mathbf{u}$. Any factor of length $|w|$, in particular $\widetilde{w}$, occurs in $ww$. Therefore there exist factors $p, q$ of $w$ such that $p\widetilde{w}q = ww$. Consequently $w = pq$ and $\widetilde{w} = qp$. It implies that $\widetilde{p} = p$ and $\widetilde{q} = q$ and thus both $p$ and $q$ are palindromes. □

**Lemma 24.** *Let* $\mathbf{u} \in \mathcal{A}^\mathbb{N}$ *be a palindromic word. Then there exists a bi-infinite word* $\mathbf{p} := \cdots p_3p_2p_1p_0p_1p_2p_3\cdots$*, where* $p_0 \in \mathcal{A} \cup \{\varepsilon\}$ *and* $p_i \in \mathcal{A}$ *for* $i \geq 1$*, such that for any nonnegative integer* $m$ *the string* $p_mp_{m-1}\cdots p_0\cdots p_{m-1}p_m$ *is a palindrome occurring in* $\mathcal{L}(\mathbf{u})$*. The word* $\mathbf{p}$ *is called infinite palindromic branch of* $\mathbf{u}$*.*

*Proof.* Let us construct a directed infinite graph $G$: vertices of $G$ are palindromes occurring in $\mathbf{u}$. A pair of palindromes $p$ and $q$ are connected with an edge starting in $p$ and ending in $q$ if there exists a letter $a$ such that $q = apa$. It is readily seen that

– for any palindrome $q$ with length at least 2 there exists exactly one edge ending in $q$;
– for any palindrome $q$ there exists at most $\operatorname{Card}\mathcal{A}$ edges starting in $q$;
– no edge ends in a vertex from $\mathcal{A} \cup \{\varepsilon\}$;
– $G$ contains no directed cycle because any edge starts in a shorter palindrome than it ends;
– any palindrome $p$ is reachable by an oriented path from one of vertices from $\mathcal{A} \cup \{\varepsilon\}$.

It means that $G$ is a forest with $1 + \operatorname{Card}\mathcal{A}$ components. Since the language of $\mathbf{u}$ contains infinitely many palindromes, at least one of these components is an infinite tree. According to the famous Kőnig's lemma [15], this component contains an infinite directed path. Denote its starting vertex $p_0 \in \mathcal{A} \cup \{\varepsilon\}$. The $m^{th}$-vertex $P_m$ along the path is a palindrome $P_m := p_mp_{m-1}\cdots p_0\cdots p_{m-1}p_m$.

It implies that there exists a bi-infinite word $\mathbf{p} := \cdots p_3p_2p_1p_0p_1p_2p_3\cdots$ as desired. □

**Lemma 25.** *Let* $\mathbf{u}$ *be a uniformly recurrent palindromic word. If* $\mathbf{u}$ *is not eventually periodic, then its language contains infinitely many palindromic bispecial factors.*

*Proof.* Let $w$ be a factor of $\mathbf{u}$. Recall that $u$ is a *complete return word* of $w$ in $\mathbf{u}$ if $u$ is a factor of $\mathbf{u}$, $w$ is a prefix and a suffix of $u$ and $u$ contains no other occurrences of $w$.

First we show the existence of a bispecial word containing any factor $w$. Since $\mathbf{u}$ is uniformly recurrent, the gaps between consecutive occurrences of $w$ are bounded, or in other words, the set of complete return words to $w$ is finite. Let $v$ be the longest common prefix of all complete return words to $w$. Clearly, it has the form $v = wV$ for some (possibly empty) factor $V$ of $\mathbf{u}$. If $wV$ contains two occurrences of $w$, then $wV$ is the unique complete return word to $w$ and $\mathbf{u}$ is periodic which is a contradiction. Thus $wV$ contains only one occurrence of $w$ and $wV$ must be shorter than the shortest complete return word to $w$. Therefore, $wV$ is right special because it is the longest common prefix of two longer complete return words to $w$. Moreover, $wV$ is the unique right prolongation of the factor $w$ with length $|wV|$. Analogously, the longest common suffix of all complete return words to $w$ has the form $Uw$, it is left special and $Uw$ is the unique left prolongation of the factor $w$ with length $|Uw|$. As $Uw$ is left special and $wV$ is the unique right prolongation of the factor $w$ with length $|wV|$, the factor $UwV$ is left special as well. For the same reason, $UwV$ is right special and thus $UwV$ is a bispecial factor containing the factor $w$.

Let us show that if $w$ is a palindromic factor of $\mathbf{u}$, then the bispecial factor $UwV$ is palindromic as well. Indeed, as $\mathbf{u}$ is closed under reversal (Lemma 22), the set of complete return words to the palindrome $w$ is closed under reversal as well. Therefore the longest common suffix of all complete return words to $w$ is just the reversal of the longest common prefix of all complete return words to $w$, in other words $\widetilde{U} = V$, i.e., the bispecial factor $UwV$ is a palindrome.

We have shown that for any palindrome $w$ there exists a palindromic bispecial factor with length at least $|w|$. Since $\mathbf{u}$ contains infinitely many palindromes, necessarily $\mathbf{u}$ contains infinitely many palindromic bispecial factors. $\square$

**Remark 26.** *In the proof of the previous lemma, we have shown that any palindrome $w$ can be extended to a bispecial factor of the form $Uw\widetilde{U}$, where $Uw\widetilde{U}$ is the unique palindromic extension of $w$ with length $|Uw\widetilde{U}|$. It implies that any palindromic branch $\mathbf{p} := \cdots p_3p_2p_1p_0p_1p_2p_3\cdots$ contains infinitely many bispecial palindromes $P_m := p_mp_{m-1}\cdots p_0\cdots p_{m-1}p_m$.*

# 5 Properties of well-marked morphisms

**Lemma 27.** *Let $\varphi$ be a well-marked morphism. Denote $\varphi_L$ and $\varphi_R$ the leftmost and the rightmost conjugate of $\varphi$, respectively. Let $w \in \mathcal{A}^*$ be the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. If there exist $u, v \in \mathcal{A}^*$ such that*

$$\widetilde{\varphi_R(u)}w = \varphi_R(v)w\,, \tag{3}$$

*then $w$ is a palindrome, $\widetilde{u} = v$ and $\varphi_L(a) = \widetilde{\varphi_R}(a)$ for any letter $a$ occurring in $u$.*

The hypothesis of this lemma can be made more general (injectivity of $\varphi_L$ and $\varphi_R$ instead of well-marked) but we use it only for well-marked morphisms.

*Proof.* Suppose $u = u_0u_1\cdots u_n$ and $v = v_0v_1\cdots v_m$. Due to (3) we have $\widetilde{w}\widetilde{\varphi_R(u)} = w\varphi_L(v)$. It immediately gives $\widetilde{w} = w$ and moreover, $\widetilde{\varphi_R(u_n)}\cdots\widetilde{\varphi_R(u_0)} = \varphi_L(v_0)\cdots\varphi_L(v_m)$. Hence,

$$\textsc{Fst}(\varphi_L(v_0)) = \textsc{Fst}(\widetilde{\varphi_R(u_n)}) = \textsc{Lst}(\varphi_R(u_n)).$$

Since $\varphi$ is well-marked, then

$$\textsc{Lst}(\varphi_R(u_n)) = \textsc{Fst}(\varphi_L(u_n)).$$

We get $\textsc{Fst}(\varphi_L)(v_0) = \textsc{Fst}(\varphi_L)(u_n)$. But $\textsc{Fst}(\varphi_L)$ is injective since $\varphi$ is well-marked. We conclude that $u_n = v_0$. As $|\varphi_L(a)| = |\varphi_R(a)| = |\widetilde{\varphi_R}(a)|$ for any letter $a \in \mathcal{A}$, we also deduce $\varphi_L(v_0) = \widetilde{\varphi_R}(v_0)$. It implies $\varphi_L(v_1)\cdots\varphi_L(v_m) = \widetilde{\varphi_R}(u_{n-1})\cdots\widetilde{\varphi_R}(u_0)$. For the same reason as above, we have $v_1 = u_{n-1}$ and $\varphi_L(v_1) = \widetilde{\varphi_R}(v_1)$ and $m = n$. $\square$

**Proposition 28.** *Let $\varphi$ be a primitive marked morphism. Denote $\varphi_L$ and $\varphi_R$ the leftmost and the rightmost conjugate of $\varphi$, respectively. Let $w \in \mathcal{A}^*$ be the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. Define $\Phi : \mathcal{L}(\varphi) \mapsto \mathcal{L}(\varphi)$ by*

$$\Phi(u) = \varphi_R(u)w\,.$$

1. *If $u \in \mathcal{L}(\varphi)$ is a left (resp. right) special factor, then $\Phi(u)$ is a left (resp. right) special factor, too.*

2. *There exist a finite number of bispecial factors, say $u^{(1)}, u^{(2)}, \ldots, u^{(N)} \in \mathcal{L}(\varphi)$, such that any bispecial factor of $\mathcal{L}(\varphi)$ equals $\Phi^n(u^{(j)})$ for some $j = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, where $\Phi^n$ denotes the $n^{th}$ iteration of $\Phi$.*

The proof of the proposition is based on results of Klouda from [14]. They concern a very broad class of circular non-pushy $D0L$-systems. Any primitive injective morphism belongs to this class. For two words $x, y \in \mathcal{A}^*$, we define $\text{LCP}\{x, y\}$ to be the longest common prefix of $x$ and $y$ and $\text{LCS}\{x, y\}$ to be the longest common suffix of $x$ and $y$. Let us summarize the relevant consequences of Theorems 22 and 36 from [14] in the case of injective primitive morphism.

**Theorem 29** (Klouda, [14])**.** *Let $\varphi$ be an injective primitive morphism. Then, there exist a finite set $\mathcal{I} \subset \mathcal{L}(\varphi)$ of bispecial factors and two finite sets $\mathcal{B}_L$ and $\mathcal{B}_R$ satisfying*

$$\mathcal{B}_L \subseteq \left\{\text{LCS}\{\varphi(x), \varphi(y)\} : x, y \in \mathcal{A}^+, \text{LST}(x) \neq \text{LST}(y)\right\},$$
$$\mathcal{B}_R \subseteq \left\{\text{LCP}\{\varphi(x), \varphi(y)\} : x, y \in \mathcal{A}^+, \text{FST}(x) \neq \text{FST}(y)\right\}$$

*such that any bispecial factor $u \in \mathcal{L}(\varphi) \setminus \mathcal{I}$ has the form $u = f_L\,\varphi(u')\,f_R$, where $u' \in \mathcal{L}(\varphi)$ is a bispecial factor, $f_L \in \mathcal{B}_L$ and $f_R \in \mathcal{B}_R$.*

*Proof of Proposition 28.* According to Lemma 15, the definition of the mapping $\Phi$ is correct. There exist $a, b \in \mathcal{A}$, $a \neq b$ such that $au, bu \in \mathcal{L}(\varphi)$. According to Lemma 15, words $\varphi_R(a)\varphi_R(u)w$ and $\varphi_R(b)\varphi_R(u)w$ belong to $\mathcal{L}(\varphi)$ too. Since $\varphi$ is marked, the last letters of $\varphi_R(a)$ and $\varphi_R(b)$ differ and thus $\varphi_R(u)w$ is left special.

Analogously, there exist $c, d \in \mathcal{A}$, $c \neq d$ such that $uc, ud \in \mathcal{L}(\varphi)$. According to Lemma 15, words $\varphi_R(u)\varphi_R(c)w = \varphi_R(u)w\varphi_L(c)$ and $\varphi_R(u)\varphi_R(d)w = \varphi_R(u)w\varphi_L(d)$ belong to $\mathcal{L}(\varphi)$ too. Since $\varphi$ is marked, the first letters of $\varphi_L(c)$ and $\varphi_L(d)$ differ and thus $\varphi_R(u)w$ is right special.

Let us apply Theorem 29 to the morphism $\varphi_R$. Since $\text{LST}(\varphi_R)$ is injective, $\text{LCS}\{\varphi_R(x), \varphi_R(y)\}$ is empty for any pair of nonempty words $x, y$ with distinct last letters. Thus the set $\mathcal{B}_L$ contains only the empty word. On the other hand, since $\varphi_R(x)w = w\varphi_L(x)$ and $\varphi_R(y)w = w\varphi_L(y)$ and $\text{FST}(\varphi_L)$ is injective, we have $\text{LCP}\{\varphi_R(x), \varphi_R(y)\} = w$ for any pair of nonempty words $x, y$ with distinct first letters. Therefore, the set $\mathcal{B}_R$ contains only the word $w$. It means that any bispecial factor $u$ from $\mathcal{L}(\varphi_R) \setminus \mathcal{I}$ where $\mathcal{I}$ is finite equals to $\varphi_R(u')w = \Phi(u')$ for some bispecial factor $u' \in \mathcal{L}(\varphi_R)$. $\square$

The fixed point of a cyclic primitive morphism is periodic. As we have already illustrated on the example

$$\xi(a) = aba \quad \text{and} \quad \xi(b) = bab$$

the converse does not hold. This morphisms has two periodic fixed points $(ab)^\omega$ and $(ba)^\omega$. Let us point out that the shortest period is 2 and both letters of the binary alphabet occur in any factor of length 2. Moreover, the morphism $\xi$ is primitive and well-marked, i.e., it satisfies the assumption of the previous propositions. The following corollary describes morphisms of this type.

**Corollary 30.** *Let $\varphi$ be a primitive marked morphism over an alphabet $\mathcal{A}$ and $\mathbf{u}$ be a fixed point of $\varphi$. If $\mathbf{u}$ is eventually periodic, then there exists a word $w \in \mathcal{A}^*$ such that $\mathbf{u} = w^\omega$, $|w| = \text{Card}\,\mathcal{A}$, every letter of $\mathcal{A}$ occurs exactly once in $w$ and $\varphi(w) = w^k$ for some $k \in \mathbb{N}$.*

*Proof.* According to Point 1 of Proposition 28, if $\mathcal{L}(\varphi)$ contains one left or right special factor of length at least 1, then it contains infinitely many left and right special factors. Since language of an eventually periodic word has only finitely many left and right special factors, necessarily there are no nonempty special factors in $\mathcal{L}(\varphi)$ at all. Let $\mathbf{u} = vw^\omega$, where $|w|$ is the shortest period and $|v|$ is the shortest preperiod of $\mathbf{u}$. Obviously, $v$ is empty, otherwise the first letter of $w$ is left special. For the same reason, any letter of the alphabet occurs in $w$ exactly once.

Let us denote by $c$ the first letter of $w$. As $\mathbf{u}$ is a fixed point of $\varphi$ the first letter of $\varphi(c)$ is $c$. Since any letter occurs in $w$ exactly once, any prefix $u$ of $\mathbf{u}$ which starts and ends by the letter $c$ has the form $u = w^k c$ for some $k \in \mathbb{N}$. As $\varphi(w)c$ is a prefix of $\varphi(ww) = \varphi(w)\varphi(w)$ which is a prefix $\mathbf{u}$, the word $\varphi(w)c = w^k c$ for some $k \in \mathbb{N}$. □

**Example 31.** *The morphism $\varphi$ defined by $a \mapsto abcabcab$, $b \mapsto cabca$ and $c \mapsto bc$ is a primitive marked morphism over the alphabet $\{a, b, c\}$. Its fixed point is $(abc)^\omega$ and $\varphi(abc) = (abc)^5$. But the language of the fixed point $(abc)^\omega$ unlike the fixed point $(ab)^\omega$ of the morphisms $\xi$ is not palindromic.*

**Corollary 32.** *Let $\varphi$ be a primitive marked morphism over an alphabet $\mathcal{A}$ and $\mathbf{u}$ be an eventually periodic fixed point of $\varphi$. If $\mathbf{u}$ is palindromic, then $\mathcal{A}$ is a binary alphabet and $\varphi$ belongs to class $\mathcal{P}$.*

*Proof.* According to Corollary 30 $\mathbf{u} = w^\omega$ and the period $w$ contains each letter of alphabet exactly once. Due to Lemma 23 the period $w = pq$, where $p$ and $q$ are palindromes. The only possibility is that $|p| = |q| = 1$ and the period has the form $w = ab$, where $a, b \in \mathcal{A}$, $a \neq b$. Thus, the cardinality of the alphabet is $|w| = 2$.

Corollary 30 moreover says that $\varphi(ab) = (ab)^k$. If $\varphi(a) = (ab)^\ell$ for some $\ell < k$ then $\varphi(b) = (ab)^{k-\ell}$ and the morphism $\varphi$ is not marked. Therefore, $\varphi(a) = (ab)^\ell a$ and $\varphi(b) = b(ab)^{k-\ell-1}$ and obviously belongs to class $\mathcal{P}$. □

# 6 Proof of Theorem 1

**Proposition 33.** *Let $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ be a primitive and well-marked morphism. If the language $\mathcal{L}(\varphi)$ is palindromic, then $\varphi$ has a conjugate in class $\mathcal{P}$.*

*Proof.* Because of Corollary 32, we can focus on $\varphi$ with aperiodic fixed points. Since $\varphi$ is primitive, the language $\mathcal{L}(\varphi)$ is uniformly recurrent and thus there exists a constant $K$ such that any factor longer than $K$ contains all letters from $\mathcal{A}$. From Lemma 25, $\mathcal{L}(\varphi)$ contains an infinite number of bispecial palindromes. If a bispecial palindrome $u$ is long enough, then according to Proposition 28 there exists $u'$ such that $u = \varphi_R(u')w = w\varphi_L(u')$. Without loss of generality, we can assume that $u'$ is longer than $K$ and thus all letters of $\mathcal{A}$ occur in $u'$. Due to Lemma 27, $\varphi_L = \widetilde{\varphi_R}$. This together with Lemma 20 implies the statement. □

We may now prove the main result, namely that Version 1 of HKS Conjecture holds for general alphabet for the case of marked morphisms.

*Proof of Theorem 1.* From Lemma 18, $\varphi^k$ is well-marked for some integer $k$. From Proposition 33, $\varphi^k$ has a conjugate in class $\mathcal{P}$. □

The result of Tan thus becomes a corollary of this theorem.

**Corollary 34.** *Let $\mathcal{A}$ be a binary alphabet and $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ be an acyclic and primitive morphism. If the language $\mathcal{L}(\varphi)$ is palindromic, then $\varphi$ or $\varphi^2$ has a conjugate in class $\mathcal{P}$.*

*Proof.* Any acyclic binary morphism $\varphi$ is marked. If $\varphi$ is not well-marked then so is $\varphi^2$. □

It turns out that the square $\varphi^2$ is necessary only in some particular cases for a binary alphabet $\mathcal{A} = \{a, b\}$, namely when $|w| < |\varphi(a)|$ and $|w| < |\varphi(b)|$.

As we have already mentioned, for a uniformly recurrent word $\mathbf{u}$, the closedness of languages under reversal does not imply that $\mathbf{u}$ is palindromic. The following proposition is an analogy of Theorem 3.13 from [22] for larger alphabet, which is stated for binary alphabet.

**Proposition 35.** *Let $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ be a primitive and marked morphism. If $\mathcal{L}(\varphi)$ is closed under reversal, then $\mathcal{L}(\varphi)$ is palindromic.*

*Proof.* Since $\mathcal{L}(\varphi) = \mathcal{L}(\varphi^k)$, we can without loss of generality assume that $\varphi$ is well-marked. We exploit Proposition 28. Consider a bispecial factor $u$ which contains all letters from the alphabet and which is longer than any initial bispecial factor on the list $u^{(1)}, \ldots, u^{(N)}$. The same proposition says that the factor $\Phi(u) = \varphi_R(u)w$ is bispecial. Since $\mathcal{L}(\varphi)$ is closed under reversal, the factor $\widetilde{\varphi_R(u)w}$ is bispecial, too. By Proposition 28, there exists a bispecial factor $v$ such that $\widetilde{\varphi_R(u)w} = \Phi(v) = \varphi_R(v)w$. Lemma 27 forces $u = \widetilde{v}$ and $\widetilde{\varphi_R} = \varphi_L$. Thus, $\varphi$ has a conjugate in class $\mathcal{P}$. ∎

# 7 Comments and open questions

In our article we focused exclusively on infinite words generated by marked morphisms. Moreover, we studied only Version 1 of HKS conjecture. Let us comment some problems we did not touch.

- We believe that Version 1 of HKS conjecture can be proved for a larger class of primitive morphisms and not only for morphisms having a well-marked power. For example, consider a word coding a three interval exchange transformation $T$ under permutation (321). Such word is palindromic. If the transformation $T$ satisfies the so-called infinite distinct orbit condition (i.d.o.c.) introduced by Keane in [13], then a primitive morphism fixing a coding of $T$ cannot be marked.

  Is some power of this morphism conjugate to a morphism from class $\mathcal{P}$?

  The counterexample to Version 1 of HKS conjecture constructed by the first author is an infinite word over a ternary alphabet. In fact this word - say $\mathbf{u}$ - is the coding of a three interval exchange transformation $T$ with permutation (321). But it does not satisfy i.d.o.c. It means that the factor complexity $\mathcal{C}$ of $\mathbf{u}$ is bounded by $\mathcal{C}(n) \leq n + K$ for some constant $K$. Such word is usually referred to as a degenerate 3iet word and it is just a morphic image of a sturmian word.

- On the binary alphabet one may consider besides reversal mapping also the mapping $E$ defined by $E(w_1 w_2 \cdots w_n) = (1 - w_n)(1 - w_{n-1}) \cdots (1 - w_1)$. Obviously, the mapping $E$ is an involutive antimorphism, i.e., $E^2 = \mathrm{Id}$ and $E(uv) = E(v)E(u)$ for all words $u, v \in \{0, 1\}^*$.

  Let us look at the Thue-Morse morphism $\varphi_{TM} : 0 \mapsto 01,\ 1 \mapsto 10$. The second iteration $\varphi_{TM}^2 : 0 \mapsto 0110,\ 1 \mapsto 1001$ belongs to class $\mathcal{P}$ and thus the language $\mathcal{L}(\varphi_{TM})$ contains infinitely many palindromes. On the other hand, the language $\mathcal{L}(\varphi_{TM})$ is closed under involutive antimorphism $E$ as well and it contains infinitely many $E$-palindromes, i.e., factors $w$ satisfying $E(w) = w$. In fact, images of letters $\varphi_{TM}(0) = 01$ and $\varphi_{TM}(1) = 10$ are both $E$-palindromes. The question is: What is an $E$-analogy of class $\mathcal{P}$ and HKS conjecture?

- As we have already mentioned, Harju, Vesti and Zamboni verified Version 3 of HKS conjecture for words with finite defect. Words coding symmetric exchange transformation and also Arnoux-Rauzy words belong to most prominent examples of words with defect zero. Of the same interest is the opposite question: Which morphisms from class $\mathcal{P}$ have a fixed point with finite defect?

  In this context we have to mention the conjecture stated in the last chapter of the article [4].

  **Conjecture**: *Let* $\mathbf{u}$ *be a fixed point of a primitive morphism* $\varphi$*. If the defect of* $\mathbf{u}$ *is finite but nonzero, then* $\mathbf{u}$ *is periodic.*

  In [8], Bucci and Vaslet provided a morphism $\varphi$ over a ternary alphabet which contradicts this conjecture. But their morphism $\varphi$ is not injective. Therefore, the validity of Conjecture is still open for morphisms over a binary alphabet or injective morphisms. It can be deduced from [4], that the conjecture is true if $\varphi$ is a uniform marked morphism.

- The main open problem remains to prove validity of Version 2 of HKS conjecture or at least validity of its relaxed Version 3.

# Acknowledgements

We are grateful to the anonymous referees for their many valuable comments. The first author is supported by NSERC postdoctoral fellowship (Canada). The second author acknowledges support of GAČR 13-03538S (Czech Republic).

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