Local and Global Constraints in Finite Element Modeling and the Benefits of Nodal and Edge Elements Coupling

Patrick Dular

University of Liège, Belgium

I. INTRODUCTION

Various constraints can be encountered in partial differential problems. On the one hand, there are local constraints, which locally act on fields. These are usually boundary conditions, fixing components of fields, as well as interface conditions, connecting such components. On the other hand, global behaviors of fields can be constrained, leading to define global constraints. It is the case when vector field fluxes and circulations have to be defined, again in order to be either fixed or connected.

A finite element model of a partial differential problem, through weak formulations, then leads to split up the considered constraints in two families, known as essential and natural constraints. This means that some constraints are strongly satisfied while others are only weakly satisfied.

It is the aim of this paper to make a survey of local and global constraints encountered in finite element models of electromagnetic systems. It particularly points out the benefits of using both nodal and edge finite elements to achieve their consistent discrete definitions. There are indeed properties that are worth to be kept from the continuous to the discrete level. The constraints are defined in the frame of dual formulations in order to point out their dual, or complementary, natures. Systematic explicit characterizations of constrained function spaces are shown to be quite convenient.

Detailed developments are deliberately omitted; in particular, all the studied constraints are directly expressed at the discrete level, i.e. in finite element spaces. The stress is rather laid on numerous applications benefiting from the proposed systematic approach, presented in an evolutive way, i.e. from scalar to vector fields formulations, from local to global essential and natural constraints.

II. SCALAR POTENTIAL FORMULATIONS

A. Essential constraints

Various electromagnetic problem formulations make use of scalar potentials, of which the gradient is a physical vector field (e.g. $\mathbf{e} = -\operatorname{grad} v$ where $\mathbf{e}$ is the electric field and v is the electric scalar potential, in electrostatics; also in electrokinetics and magnetostatics). These potentials define fields of local quantities and are commonly approximated with nodal finite elements [1], i.e.

$$v = \sum_{n \in N} v_n s_n \ , \quad v \in S^0(\Omega) \ , \tag{1}$$

where N is the set of nodes of the studied domain Ω, s_n is the nodal basis function associated with node n and v_n is the value of v at node n. Functions s_n, $\forall n \in N$, form a basis for the nodal finite element space $S^0(\Omega)$ without constraint. In case constraints exist, they no longer form a basis. The direct expression of these constraints has to reveal the basis functions to consider, i.e. which can serve as test functions in the finite element method.

Classical boundary conditions fixing the scalar potential on certain boundaries, i.e. Dirichlet conditions, are commonly used [1]. They simply consist in fixing values of some coefficients v_n, and then in extracting the associated functions s_n from the basis function set.

Other boundary conditions on parts of the boundary of the studied domain can imply the definition of floating values for scalar potentials [2], [3]. A floating value is an unknown constant on a region and comes from a homogeneous boundary condition for the tangential component of the associated physical vector field (e.g. $\mathbf{n} \times \mathbf{e} = -\mathbf{n} \times \operatorname{grad} v = 0$ on a surface Γ_f implies that v is a constant on Γ_f, f belonging to the set C_f of floating boundaries; $\mathbf{n}$ is the normal to Γ_f). In order to explicitly define such constraints, the nodes of Ω are classified in complementary subsets: N_v, which is the set of nodes inside Ω, and N^f, $\forall f \in C_f$, which are the sets of nodes of parts Γ_f (Fig. 1). Floating potentials being constant on each Γ_f, (1) can then be decomposed as [3]

$$v = \sum_{n \in N_v} v_n s_n + \sum_{f \in C_f} v^f s^f, v \in S_v^0(\Omega) \ , \tag{2}$$

$$\text{with} \quad s^f = \sum_{n \in N^f} s_n, \forall f \in C_f \ , \tag{3}$$

where s_n, $\forall n \in N_v$, and s^f, $\forall f \in C_f$, are basis functions for the constrained potential; $S_v^0(\Omega)$ is the constrained space. Each function s^f is associated with the group of nodes — a global geometrical entity, while nodes $n \in N_v$ are elementary entities — of boundary Γ_f (Fig. 1). The support of s^f (i.e. its domain of non-zero values) is limited to a transition layer containing all the geometrical elements having nodes on Γ_f.

Dirichlet and floating potential constraints constitute essential constraints, being directly expressed in the scalar potential function space.
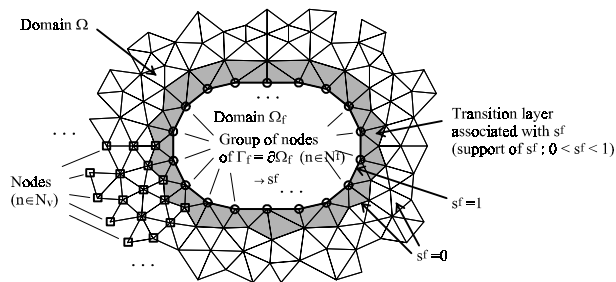

Fig. 1. Nodes and groups of nodes associated with the characterization of a scalar potential with floating values (2).

B. Natural constraints

A scalar potential v can be involved in a scalar potential formulation of the generalized problem

$$\mathbf{r} = -\operatorname{grad} v \,, \quad \operatorname{div} \mathbf{s} = \eta \,, \quad \mathbf{s} = \alpha \, \mathbf{r} \,, \qquad (4\text{-}5\text{-}6)$$

with appropriated boundary conditions. Note that (4) comes from an equation of the form curl $\mathbf{r} = 0$, and in case this original equation contains a source term $\mathbf{k}_s$, i.e. curl $\mathbf{r} = \mathbf{k}_s$, (4) becomes $\mathbf{r} = \mathbf{r}_s - \operatorname{grad} v$ where $\mathbf{r}_s$ is a source field satisfying curl $\mathbf{r}_s = \mathbf{k}_s$. Generalized fields $\mathbf{r}$, $\mathbf{s}$, v, η, $\mathbf{k}_s$ and characteristic α can be easily particularized to physical quantities involved in e.g. electrokinetics, electrostatics and magnetostatics [3].

The scalar potential weak formulation for (4-5-6) is obtained from the weak form of (5), together with (4) and (6), i.e.

$$(-\alpha \operatorname{grad} v, \operatorname{grad} v')_\Omega - <\mathbf{n}\cdot\mathbf{s}_s, v'>_{\Gamma_s} + (\eta, \operatorname{grad} v')_\Omega = 0 \,,$$

$$\forall v' \in S_v^0(\Omega) \,, \qquad (7)$$

where $\mathbf{n} \cdot \mathbf{s}_s$ is a constraint on the generalized flux density $\mathbf{s}$ associated with nonfixed potential boundaries Γ_s of domain Ω, e.g. on Neumann boundaries as well as on floating potential boundaries Γ_f, $f \in C_f$; $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ respectively denote a volume integral in Ω and a surface integral on Γ of the scalar product of their arguments. Such constraints are known as natural constraints, i.e. they only appear through integral terms in weak formulations. They are respectively of local and global types.

The key point when dealing with the considered global constraints is that each associated global test function s^f, $f \in C_f$, is equal to one on boundary Γ_f. This function therefore gives a contribution equal to $<\mathbf{n}\cdot\mathbf{s}_s, 1>_{\Gamma_f}$ and thus to the flux ψ^f of $\mathbf{s}_s$ leaving Ω through surface $f \in C_f$, leading to [3]

$$\Psi^f = (-\alpha \operatorname{grad} v, \operatorname{grad} s^f)_\Omega + (\eta, \operatorname{grad} s^f)_\Omega \,, \quad f \in C_f \,. \quad (8)$$

The flux is therefore expressed in average by the volume integral in (8) in a transition layer (support of s^f; Fig. 1). This approach is in perfect accordance with the discretized weak formulation (7) of the problem and thus with an only weakly satisfied conservation of flux. The computation of the global flux based on the explicit surface integration of $\mathbf{n} \cdot \mathbf{s}$ (i.e., $-\alpha \, \mathbf{n} \cdot \operatorname{grad} v$) would be affected by the choice of the integration surface. There would be generally no reason for the so-computed flux to be equal to the flux given by the volume integral in the transition layer, whatever the surface.

This method is useful for the computation of lumped parameter models or when both potential and flux have to be considered, e.g. in case of circuit coupling.

III. VECTOR FIELD AND POTENTIAL FORMULATIONS

The methodology presented for scalar potential formulations will now be transposed for vector field and potential formulations, i.e. from nodal to edge finite element spaces and formulations. The aim is to point out the natural and consistent character of this transposition.

A. Essential constraints

Both dual $\mathbf{h}$- and $\mathbf{b}$-conform magnetodynamic formulations are considered: a magnetic field $\mathbf{h}$ formulation and a magnetic vector potential $\mathbf{a}$ formulation [4].

The general expression of the magnetic field $\mathbf{h}$ in Ω in a $\mathbf{h}$ formulation is [5]

$$\mathbf{h} = \mathbf{h}_s + \mathbf{h}_r - \operatorname{grad} \phi \,. \qquad (9)$$

Field $\mathbf{h}_s$ is a source magnetic field [5], [6] associated with the imposed current density $\mathbf{j}_s$ in stranded conductors Ω_s, i.e. a global constraint, through equation

$$\operatorname{curl} \mathbf{h}_s = \mathbf{j}_s \,. \qquad (10)$$

Field $\mathbf{h}_r$ is the associated reaction field in conducting regions Ω_c while ϕ is the associated reaction magnetic scalar potential in nonconducting regions Ω_c^C.

Classical essential boundary conditions concern tangential components of $\mathbf{h}$ and can be transposed to constraints on ϕ. The source field constraint is favorably approximated with edge finite elements to reduce the cancellation error in magnetic materials [7]. Indeed, both $\mathbf{h}_r$ and the gradient of a nodal finite element approximated ϕ are included as well in this space [8], [9]. An exact complementarity between these three fields can therefore be obtained.

Two kinds of essential constraints can be associated as well with the magnetic scalar potential ϕ. Potential ϕ defined in Ω_c^C is also defined on boundary $\partial\Omega_c$ of Ω_c, which leads to interface constraints between $\mathbf{h}$ and ϕ [10]. Also, potential ϕ can be multivalued when Ω_c^C is multiply connected, in which case cuts must be defined to make this domain simply connected [10], [11].

A discrete characterization of the magnetic field using edge finite elements enables to satisfy the considered constraints, i.e. [5], [10]

$$\mathbf{h} = \mathbf{h}_s + \sum_{k \in E_c} h_k \, \mathbf{s}_k + \sum_{n \in N_c^C} \phi_n \, \mathbf{v}_n + \sum_{i \in C} I_i \, \mathbf{c}_i \,, \quad (11)$$

where E_c is the set of inner edges of Ω_c, N_c^C is the set of nodes inside Ω_c^C and on its boundary $\partial\Omega_c^C$, and C is a set of well defined cuts which make Ω_c^C simply connected. The so-defined constrained function space is noted $S_h^1(\Omega)$.

Characterization (11) explicitly defines a coupling between field $\mathbf{h}$ (in Ω_c; given by all three sums) and a scalar potential ϕ (the gradient of ϕ is given by the second and third sums; the third sum enables multivalued potentials to be considered). Actually, potential ϕ in Ω_c^C is decomposed in continuous and discontinuous parts, of which the gradients are respectively given by the second and third sums in (11). Coefficients I_i represent circulations of $\mathbf{h}$ along well defined paths (equal to the fluxes of their curl and thus to the currents through associated surfaces) and functions $\mathbf{c}_i$ are global vector basis functions associated with cuts C_i. Note that such a characterization enables function $\mathbf{v}_n$, and thus the associated scalar potential, to be fully continuous in a multiply connected domain, the discontinuity being taken into account by functions $\mathbf{c}_i$.

A similar treatment can be made for the $\mathbf{t}$-ω formulation [12], with $\mathbf{h} = \mathbf{h}_s + \mathbf{t} - \operatorname{grad} \omega$ in Ω.

With the magnetic vector potential formulation, the general expression of the electric field $\mathbf{e}$ via a magnetic vector potential $\mathbf{a}$ involves the gradient of an electric scalar potential v in the conducting regions, i.e. [13], [14]

$$\mathbf{e} = -\partial_t \mathbf{a} - \operatorname{grad} v \text{ in } \Omega_c, \quad \text{with} \quad \mathbf{b} = \operatorname{curl} \mathbf{a} \text{ in } \Omega, \quad (12\text{-}13)$$

so that the Faraday equation $\operatorname{curl}\mathbf{e} = -\partial_t \mathbf{b}$ is satisfied.

Classical essential boundary conditions concern tangential components of $\mathbf{a}$, i.e. normal component of $\mathbf{b}$. The tree-cotree gauge condition [15] can be used as an additional essential constraint, which consists in fixing the circulations of $\mathbf{a}$ to zero along the edges of a tree built in the gauged domain. Another kind of gauge condition can be defined as a natural constraint, i.e. as a constraint only satisfied weakly through an additional penalty term in the weak formulation [13], as it will be explained hereafter.

As for the global constraint treatment, it consists in defining, for each massive conductor i, a unit source electric scalar potential v_0^i (also noted v_0) associated with a unit voltage [14], leading to

$$v = \sum_{i \in \Gamma_j} V_i \, v_0^i \,, \qquad (14)$$

each function v_0^i being then a basis function for the associated voltage V_i. By this specific way, voltage essential global constraints can be defined.

Such a determination can be done using an electrokinetic finite element formulation for each conductor in Ω_c and with appropriate boundary conditions, in particular v_0 equal to 0 and 1 on its two electrodes. A generalized source potential can be used as well, with a support limited to the finite elements located on one side of a cross-section for each conductor (Fig. 2) [14]. The source potential v_0 can then be simply the sum of the nodal basis functions s_n of all the nodes located on that cross-section, noted s^i for section Γ_j^i, i.e.

$$v_0 = s^i = \sum_{n \in \Gamma_j^i} s_n \,, \qquad (15)$$

with a support limited to a transition layer.

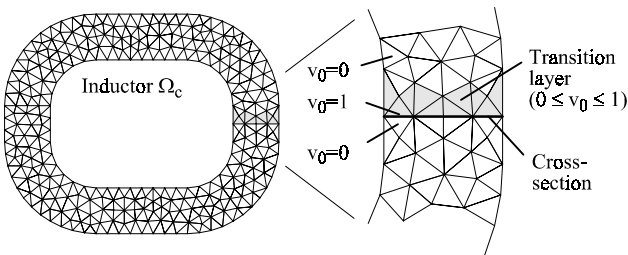
Fig. 2. Cross-section and associated transition layer in a conductor.

B. Natural constraints

The $\mathbf{h}$-ϕ magnetodynamic formulation is obtained from the weak form of the Faraday equation $\operatorname{curl}\mathbf{e} = -\partial_t \mathbf{b}$, with $\mathbf{b} = \mu\mathbf{h}$, and $\mathbf{j} = \sigma\mathbf{e}$, i.e. [4], [16]

$$\partial_t(\mu\mathbf{h},\mathbf{h}')_\Omega + (\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{h}')_{\Omega_c} + (\sigma^{-1}\mathbf{j}_s,\operatorname{curl}\mathbf{h}')_{\Omega_s},$$

$$+ <\mathbf{n}\times\mathbf{e}_s,\mathbf{h}'>_{\Gamma_e} = 0, \quad \forall\mathbf{h}'\in S_h^1(\Omega), \qquad (16)$$

where $S_h^1(\Omega)$ is the constrained function space defined on Ω and containing the basis functions for $\mathbf{h}$ (coupled to ϕ) as well as for the test function $\mathbf{h}'$. Surface electric field $\mathbf{n}\times\mathbf{e}_s$ is a natural boundary condition on boundaries Γ_e of two kinds. Either, it can be a locally specified field, i.e. a classical natural boundary condition, or a field for which only associated global quantities are known (functionals of $\mathbf{e}_s$), i.e. voltages. Indeed, each global test function c_i from (11) gives a contribution to the surface integral term equal to the voltage V_i, i.e.

$$\partial_t(\mu\mathbf{h},\mathbf{c}_i)_\Omega + (\sigma^{-1}\operatorname{curl}\mathbf{h},\operatorname{curl}\mathbf{c}_i)_{\Omega_c} = -V_i \,, \qquad (17)$$

which is the natural weak circuit relation for massive conductor i [5], [16]. It is the natural global constraint for the voltage.

For stranded conductors, the basis functions of $\mathbf{h}_s$ in (11), i.e. the source magnetic fields $\mathbf{h}_{s,i}$ due to unit currents, lead to, when used as test functions $\mathbf{h}'$,

$$\partial_t(\mu\mathbf{h},\mathbf{h}_{s,i})_\Omega + I_{s,i}(\sigma^{-1}\mathbf{j}_{s,i},\operatorname{curl}\mathbf{h}_{s,i})_{\Omega_{s,i}} = -V_i, \qquad (18)$$

This relation allows a natural computation of the total magnetic flux through all the wires of the conductor in perfect accordance with the weak formulation (16), without the need of any supplementary integral formula [5], [16].

Voltages therefore appear as the weak global quantities in $\mathbf{h}$-formulations whereas currents are the associated strong quantities. These quantities have to be fixed respectively through essential constraints in (11) and natural constraints (17-18).

As for the $\mathbf{a}$-v magnetodynamic formulation, it is obtained from the weak form of the Ampere equation $\operatorname{curl}\mathbf{h} = \mathbf{j}$, with $\mathbf{b} = \mu\mathbf{h}$, and $\mathbf{j} = \sigma\mathbf{e}$, i.e. [13]

$$(\mu^{-1}\operatorname{curl}\mathbf{a},\operatorname{curl}\mathbf{a}')_\Omega + (\sigma\partial_t\mathbf{a},\mathbf{a}')_{\Omega_c} + (\sigma\operatorname{grad} v,\mathbf{a}')_{\Omega_c} = 0,$$

$$\forall\mathbf{a}'\in S_a^1(\Omega), \qquad (19)$$

where $S_a^1(\Omega)$ is the constrained function space (with boundary and gauge conditions) defined on Ω and containing the basis functions for $\mathbf{a}$ as well as for the test function $\mathbf{a}'$.

In the same way as $\operatorname{curl}\mathbf{h} = \mathbf{j}$ implies $\operatorname{div}\mathbf{j} = 0$, weak formulation (19) implies, by taking $\mathbf{a}' = \operatorname{grad} v'$ as a test function, that

$$(\sigma\partial_t\mathbf{a},\operatorname{grad} v')_{\Omega_c} + (\sigma\operatorname{grad} v,\operatorname{grad} v')_{\Omega_c} = <\mathbf{n}\cdot\mathbf{j},v'>_{\Gamma_j},$$

$$\forall v'\in F_v(\Omega_c), \qquad (20)$$

where Γ_j is the part of the boundary of Ω_c which is crossed by a current. Formulation (19) is actually also the weak form of $\operatorname{div}\mathbf{j} = 0$ in Ω_c. At the discrete level, this implication is only true when the gradient of v' is included in the space of $\mathbf{a}'$, which is the case when edge and nodal elements are used respectively for $\mathbf{a}'$ and v', and thus for $\mathbf{a}$ and v [8], [9]. Otherwise, e.g. using nodal elements for $\mathbf{a}$ and v, both (19) and (20) must be taken into account, with an additional penalty term in (19) to ensure a gauge condition in Ω [13].

This justifies the definition of $S_a^1(\Omega)$ as an edge finite element function space, and $S_v^0(\Omega_c)$ as the associated nodal fi-

nite element space, with the relation grad $S_v^0(\Omega_c) \subset S_a^1(\Omega)$ [8], [9].

Each global test function $s^i = v_0^i$ from (14), equal to one on boundary Γ_i, gives a contribution equal to $<\mathbf{n} \cdot \mathbf{j}, 1>_{\Gamma_i}$ and thus to the current I_i through that surface, i.e.

$$I_i = (\sigma \partial_t \mathbf{a}, \mathrm{grad}\, s^i)_{\Omega_c} + V_i\, (\sigma\, \mathrm{grad}\, v_0^i, \mathrm{grad}\, s^i)_{\Omega_c}. \quad (21)$$

Equation (21) is the circuit relation associated with massive conductor i, i.e. a relation between its voltage V_i and its current I_i [14]. It is the natural global constraint for the current. Again, the form of this relation is coherent with the way the problem is approximated, i.e. with (19) and (20), and thus with an only weakly satisfied conservation of current. The current is obtained rather from a volume integration in a transition layer located on one side of the cross-section (Fig. 2; because the support of s^i is reduced to this layer in Ω_c) than from a numerical surface integration of $\mathbf{n} \cdot \mathbf{j} = \mathbf{n} \cdot \sigma \mathbf{e}$ on this section. This explicit surface integration would be affected by the choice of the integration surface.

IV. Conclusions

The systematic explicit characterization of constrained function spaces has been revealed quite useful for expressing both local and global constraints in various electromagnetic problems. It usually enables direct interpretations of the degrees of freedom. It has been particularly pointed out that edge finite elements are ideal complementary components for nodal elements in the sense that they enable consistent discrete forms of both essential and natural constraints. On the one hand, they enable to transpose scalar field treatments to vector field treatments, i.e. local boundary and interface conditions for tangential vector fields. On the other hand, they can be strongly coupled with nodal elements when both scalar and vector fields are considered, either in common or complementary domains, i.e. respectively for source fields and interface conditions.

This systematic approach can be efficiently considered as well at the software level to contribute to a software environment open to various coupling aspects encountered in numerical modeling.

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