# A PERRON THEOREM FOR MATRICES WITH NEGATIVE ENTRIES AND APPLICATIONS TO COXETER GROUPS 

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to Robert Labbé, in memoriam


#### Abstract

Handelman (J. Operator Theory, 1981) proved that if the spectral radius of a matrix $A$ is a simple root of the characteristic polynomial and is strictly greater than the modulus of any other root, then $A$ is conjugate to a matrix $Z$ some power of which is positive. In this article, we provide an explicit conjugate matrix $Z$, and prove that the spectral radius of $A$ is a simple and dominant eigenvalue of $A$ if and only if $Z$ is eventually positive. For $n \times n$ real matrices with each row-sum equal to 1 , this criterion can be declined into checking that each entry of some power is strictly larger than the average of the entries of the same column minus $\frac{1}{n}$. We apply the criterion to elements of irreducible infinite nonaffine Coxeter groups to provide evidences for the dominance of the spectral radius, which is still unknown.


## 1. Introduction

A primitive matrix is a real nonnegative square matrix some power of which is positive. The spectral radius of a real square matrix is the maximal modulus of its eigenvalues. Perron's theorem says that the spectral radius of a primitive matrix is a root of the characteristic polynomial (Eigenvalue) with algebraic multiplicity one (Simplicity) which is strictly greater than the modulus of any other root (Dominance) and has positive eigenvectors (Positivity). There are also matrices that are not primitive satisfying the four conclusions of Perron's theorem. For example,

$$
A=\left(\begin{array}{rr}
11 & 29 \\
14 & -1
\end{array}\right) \quad \text { is not primitive, while } \quad A^{2}=\left(\begin{array}{cc}
527 & 290 \\
140 & 407
\end{array}\right) \quad \text { is positive; }
$$

from which we deduce that $A$ shares also the four conclusions of Perron's theorem. In JT04, Nou06, they characterize exactly the matrices that satisfy the four conclusions of Perron's theorem. These are called the eventually positive matrices, which may have negative entries.

Characterizations of matrices satisfying Perron's theorem after droping one or more of the four conclusions already received attention. For instance, in EJ90, the authors considered the question of finding a converse to Perron's theorem while only keeping the Eigenvalue conclusion. Their result is expressed in terms of sign-patterns. They managed to characteristize the signpatterns that require the Eigenvalue condition, i.e., such that every matrix with that sign-pattern has its spectral radius among its eigenvalues. Nonetheless, they noticed that many sign-pattern only allow the Eigenvalue condition, i.e., some matrices with that sign-pattern have the Eigenvalue property while others do not. They were unable to characterize them all, leading to an imperfect characterization of matrices satisfying the Eigenvalue condition, see the survey COvdD09.

Handelman Han81 proved a converse to Perron's theorem while keeping the first three conclusions. He proved that the spectral radius of a matrix satisfies the Eigenvalue, Simplicity, and Dominance conditions if and only if it is conjugate to a matrix some power of which is positive. This leads to the following question: How should one find this conjugate matrix? In this article, we improve upon Handelman's result by providing a conjugate matrix to check for eventual positivity (Theorem 4.4). This provides a criterion for deciding whether the spectral radius of a matrix is a simple root of its characteristic polynomial dominating the modulus of any other root. For real $n \times n$ matrices for which each row-sum is 1 (stochastic matrices allowing negative entries), this

[^0]criterion is declined into checking that each entry of some power is strictly larger than the average of the entries of the same column minus $\frac{1}{n}$ (see Corollary 4.7).

The rest of this article is divided into four sections. In Section 2, we review some results generalizing Perron's theorem to matrices with negative entries. In Section 3, we present preliminary lemmas on the multiplicity of eigenvalues allowing to put the main result in context. In Section 4, we prove Theorem 4.4 and some corollaries giving a criterion for deciding whether the spectral radius of a matrix has the Eigenvalue, the Simplicity and the Dominance properties. In Section 5 , we apply these results to the theory of Coxeter groups. More precisely, we give many examples of elements with parabolic closure (see Section 5.1 for a definition) equal to an irreducible infinite nonaffine Coxeter group having a positive conjugate matrix.

## 2. Perron-type theorems for general matrices

Let $n \geq 1$ and $A=\left(a_{i j}\right)_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}$. The matrix $A$ is called positive if all its entries are positive, i.e. if $a_{i j}>0$ for all $i$ and $j$ with $1 \leq i, j \leq n$, and we denote it as $A>0$. Similarly, a matrix is called nonnegative if all its entries are nonnegative. A nonnegative matrix is called primitive if some of its powers is positive. The eigenvalue $\lambda$ of a matrix $A$ is called simple if it is a simple root of the characteristic polynomial of $A$, i.e. the algebraic multiplicity of $\lambda$ is one. We say that $\lambda$ is dominant if its modulus $|\lambda|$ is strictly greater than the modulus of any other eigenvalue of $A$. The spectral radius of $A$ is the maximal modulus of the eigenvalues of $A$. Therefore, a dominant eigenvalue of $A$, seen in the complex plane, lies on the circle whose radius is the spectral radius of $A$. The existence of a simple and dominant eigenvalue for primitive matrices is given by Perron's theorem.

Theorem 2.1 (Per07]). If $A \in \mathbb{R}^{n \times n}$ is a primitive matrix with spectral radius $\lambda$, then $\lambda$ is a dominant and simple eigenvalue of $A$ with positive eigenvectors.

One can say more, namely that the only positive eigenvectors are those associated with the eigenvalue $\lambda$ and this fact is used in the proof of Theorem 4.4. The exact reciprocal of Perron's theorem includes eventually positive matrices: A matrix $A$ is eventually positive if there is a positive integer $K$ such that $A^{k}>0$, for all $k>K$. As opposed to primitive matrices that integer $K$ may be arbitrarily large.

Theorem 2.2 ( JT04, Thm. 1], Nou06, Thm. 2.2]). Let $A \in \mathbb{R}^{n \times n}$. The following statements are equivalent.
(i) $A$ is eventually positive.
(ii) A has a positive, simple and dominant eigenvalue and associated positive left and right eigenvectors.
(iii) there is a positive integer $k$ such that $A^{k}>0$ and $A^{k+1}>0$.

Further, the following theorem gives a more general result about nonnegative matrices.
Theorem 2.3 (Gan59, Theorem 3, p. 66]). Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be a nonnegative matrix. Then $A$ has an eigenvalue $\lambda \geq 0$ such that $|\mu| \leq \lambda$, for all eigenvalues $\mu$ of $A$. Moreover, to this "maximal" eigenvalue $\lambda$ corresponds a nonnegative eigenvector $v: A v=\lambda v$ with $v \geq 0$ and $v \neq 0$.

As noted in [Nou06, p.136-137], the converse of the previous theorem is false as "some entries of the powers of $A$ may tend to zero from negative values". The following result of Handelman provides a converse to Perron's theorem when the Eigenvalue, Simplicity, and Dominance conditions are satisfied.

Theorem 2.4 (Han81, Theorem 2.3]). If $A \in \mathbb{R}^{n \times n}$ has a real, positive, simple and dominant eigenvalue, then $A$ is conjugate to a matrix some power of which is positive.

In Section 4 we provide a conjugate matrix to check for eventual positivity.

## 3. Preliminary lemmas

We start by setting some writing conventions. To avoid the multiple usage of transposition symbols, here and in the rest of the paper, the letter $u$ denotes a row-vector and $v$ denotes a column-vector. If $A$ is a matrix or a vector, we denote by $A^{\top}$ the transpose of $A$. Further, we denote by 1 the column vector of dimension $n$ with all entries equal to 1 , and denote the $n \times n$ identity matrix by $I$. In many of the results in this article, for instance Lemma 3.4 Lemma 4.2 or Theorem4.4, we assume that the left and right eigenvectors $u$ and $v$ associated to an eigenvalue $\lambda$ of a matrix can be chosen such that $u v=(1)$. As the next lemma shows, if it is not the case, the algebraic multiplicity of $\lambda$ is at least 2 .
Lemma 3.1. Let $A \in \mathbb{R}^{n \times n}$. Suppose that $A v=\lambda v$ and $u A=\lambda u$, for some $\lambda \in \mathbb{C}$. If $u v=(0)$, then the algebraic multiplicity of $\lambda$ is at least 2.

Proof. We suppose, without loss of generality, that both $u$ and $v$ are unit vectors. Since $u v=(0)$, there exists a orthonormal basis of column vectors of $\mathbb{R}^{n}$ containing $u^{\top}$ and $v$. From such a basis, we construct the matrix $Q=\left(v\left|u^{\top}\right| B\right)$ where $B$ is a $n \times(n-2)$ matrix made of the $n-2$ other column vectors of the basis. Notice that $v=Q e_{1}$ and $u=e_{2}^{\top} Q^{\top}$. Since $Q$ is orthogonal, we also have $Q^{-1}=Q^{\top}$. We obtain

$$
\begin{aligned}
& \left(Q^{-1} A Q\right) e_{1}=Q^{-1} A v=\lambda Q^{-1} v=\lambda e_{1} \\
& e_{2}^{\top}\left(Q^{-1} A Q\right)=e_{2}^{\top} Q^{\top} A Q=u A Q=\lambda u Q=\lambda e_{2}^{\top}
\end{aligned}
$$

Therefore

$$
Q^{-1} A Q=\left(\begin{array}{ccc}
\lambda & * & * \\
0 & \lambda & 0 \\
0 & * & *
\end{array}\right), \quad \text { which is similar to } \quad\left(\begin{array}{ccc}
\lambda & * & * \\
0 & \lambda & 0 \\
0 & 0 & *
\end{array}\right)
$$

and we conclude that the algebraic multiplicity of the eigenvalue $\lambda$ is at least 2 .
Example 3.2. Let $A=\left(\begin{array}{ccc}-1 & 2 & 0 \\ -2 & 2 & 1 \\ -2 & 3 & 0\end{array}\right)$ for which 1 is a right eigenvector and $u=(2,-1,-1)$ is a left eigenvector associated to eigenvalue $\lambda=1$. We have that $u \mathbf{1}=(0)$. From Lemma 3.1 the algebraic multiplicity of $\lambda=1$ is at least 2 . Indeed, the characteristic polynomial is $\chi_{A}(\lambda)=$ $(\lambda+1)(\lambda-1)^{2}$.

Remark 3.3. In the previous example, $\lambda$ is neither simple nor dominant eigenvalue. Therefore, the fact that entries of columns of powers of $A$ are either all positive or all negative (a property shared by elements of infinite Coxeter groups, see Section (5) is not enough to conclude the simplicity nor dominance of the spectral radius.

Now we state the following folklore lemma for real matrices with real but not necessarily positive eigenvectors for which the spectral radius is a simple and dominant eigenvalue.

Lemma 3.4. Let $A \in \mathbb{R}^{n \times n}$ with spectral radius $\rho$. Assume that $\rho$ is an eigenvalue of $A$ with left eigenvector $u$ and right eigenvector $v$, chosen such that $u v=(1)$. If $\rho$ is simple and dominant, then $\left(\frac{1}{\rho} A\right)^{k}$ converges to the matrix vu exponentially fast.

The converse of the previous lemma is false. If $\left(\frac{1}{\rho} A\right)^{k}$ converges, then one cannot conclude the simplicity of the dominant eigenvalue. Nevertheless, we may prove the semisimplicity: an eigenvalue $\lambda$ of $A \in \mathbb{R}^{n \times n}$ is called semisimple when its geometric and algebraic multiplicities are equal. The following result can be found, for instance, in Mey00, p.629-630].

Lemma 3.5. Let $A \in \mathbb{R}^{n \times n}$ with spectral radius $\rho$. Let $J=Q^{-1} A Q$ be the Jordan normal form of $A$. The following statements are equivalent.
(i) $\lim _{k \rightarrow \infty} \rho^{-k} J^{k}$ converges,
(ii) $\lim _{k \rightarrow \infty} \rho^{-k} A^{k}$ converges,
(iii) $\rho$ is a semisimple dominant eigenvalue.

Remark 3.6. The trace of the limit (when it exists) is equal to the (algebraic and geometric) multiplicity of the semisimple dominant eigenvalue.

Example 3.7. Let $A=\left(\begin{array}{ccc}-1 & 1 & 1 \\ -3 & 3 & 1 \\ -3 & 1 & 3\end{array}\right)$ of spectral radius $\rho=2$. In this case, the limit converges to a matrix of rank 2 :

$$
\lim _{k \rightarrow \infty}\left(\frac{1}{\rho} A\right)^{k}=\left(\begin{array}{lll}
-2 & 1 & 1 \\
-3 & 2 & 1 \\
-3 & 1 & 2
\end{array}\right)
$$

From Lemma 3.5. we deduce that $\rho$ is a semisimple dominant eigenvalue of $A$. Indeed, $(1,2,1)^{\top}$ and $(1,1,2)^{\top}$ form a basis of the right eigenspace associated to $\rho$.
Example 3.8. Let $A=\left(\begin{array}{ccc}-1 & 1 & 1 \\ -2 & 2 & 1 \\ -2 & 1 & 2\end{array}\right)$ of spectral radius $\rho=1$ for which

$$
A^{k}=\left(\begin{array}{rrr}
1-2 k & k & k \\
-2 k & k+1 & k \\
-2 k & k & k+1
\end{array}\right)
$$

Obviously, the limit $\lim _{k \rightarrow \infty}\left(\frac{1}{\rho} A\right)^{k}$ does not converge. We conclude from Lemma 3.5 that the spectral radius is not a semisimple eigenvalue of $A$. Indeed, $A$ has only one eigenvalue equal to 1 of algebraic degree 3 and of geometric multiplicity 2 .

We finish this section with two basic lemmas that allow to suppose that a right eigenvector of a matrix is nonnegative using similarity through signature matrices. A signature matrix is a diagonal matrix where each diagonal entry is $\pm 1$.

Lemma 3.9. Let $A \in \mathbb{R}^{n \times n}$ and $B=S A S$, where $S$ is a signature matrix. Then

$$
\begin{aligned}
& A v=\lambda v \text { if and only if } B(S v)=\lambda(S v) \text { and } \\
& u A=\lambda u \text { if and only if }(u S) B=\lambda(u S) .
\end{aligned}
$$

Proof. If $A v=\lambda v$, then

$$
B(S v)=(B S) v=(S A) v=S(A v)=S(\lambda v)=\lambda(S v)
$$

Conversely, if $B(S v)=\lambda(S v)$, then

$$
A v=A(S S) v=(A S) S v=(S B) S v=S(B S v)=S(\lambda S v)=\lambda v
$$

The proof for left eigenvectors is the same.
Lemma 3.10. Let $A \in \mathbb{R}^{n \times n}$ with spectral radius $\rho$. If there exists a signature matrix $S$ such that $S A S$ is primitive, then $\rho$ is a simple and dominant eigenvalue of $A$.

Proof. Perron's theorem applies on the primitive matrix $S A S$ which is similar to $A$.

## 4. Eventually positive conjugate matrices

The main result of this section is Theorem 4.4 which provides a conjugate matrix to test for eventual positivity. This result is declined for a particular value of the right eigenvector in Corollary 4.7 and for more general values in Corollary 4.12 Proposition 4.10 is a result in the spirit of Theorem 2.3 for nonnegative matrices. We first prove three lemmas on conjugacy of matrices, which are used to prove the main theorem. Recall that $I$ is the $n \times n$ identity matrix.

Lemma 4.1. Let $Q=I+v\left(u^{\prime}-u\right)$, where $v$ is a column vector and $u$ and $u^{\prime}$ are row vectors such that $u v=u^{\prime} v$. Then $Q$ is invertible and $Q^{-1}=I-v\left(u^{\prime}-u\right)$.

Proof. Let $X=v\left(u^{\prime}-u\right)$. We show that $I-X$ is the inverse of $Q=I+X$. On the one hand, $Q(I-X)=(I+X)(I-X)=I-X^{2}$. On the other hand, $(I-X) Q=I-X^{2}$. Thus, we only need to verify that $X$ is nilpotent, which is the case:

$$
X^{2}=v\left(u^{\prime}-u\right) v\left(u^{\prime}-u\right)=v\left(u^{\prime} v-u v\right)\left(u^{\prime}-u\right)=(0) .
$$

In the following lemma, we conjugate $A$ in order to change some left eigenvector while preserving the right eigenvector associated to the same eigenvalue and preserving the left eigenvectors associated to other eigenvalues.

Lemma 4.2. Let $A \in \mathbb{R}^{n \times n}$ with right eigenvector $v$ and left eigenvector $u$ associated to the eigenvalue $\lambda$ chosen such that $u v=(1)$. For all row vector $u^{\prime}$ such that $u^{\prime} v=(1)$, there exists a matrix $Z$ conjugate to $A$ such that
(i) $u^{\prime} Z=\lambda u^{\prime}$,
(ii) $Z v=\lambda v$,
and
(iii) if $w A=\mu w$ and $\mu \neq \lambda$, then $w Z=\mu w$.

Proof. Let $Q=I+v\left(u^{\prime}-u\right)$. From Lemma 4.1, $Q$ is invertible and $Q^{-1}=I-v\left(u^{\prime}-u\right)$. Let $Z=Q^{-1} A Q$ be conjugate to $A$.
(i) We have

$$
u^{\prime} Q^{-1}=u^{\prime}\left(I-v\left(u^{\prime}-u\right)\right)=u^{\prime}-u^{\prime} v\left(u^{\prime}-u\right)=u^{\prime}-u^{\prime}+u=u
$$

and

$$
u Q=u\left(I+v\left(u^{\prime}-u\right)\right)=u+u v\left(u^{\prime}-u\right)=u+u^{\prime}-u=u^{\prime} .
$$

Therefore $u^{\prime} Z=u^{\prime} Q^{-1} A Q=u A Q=\lambda u Q=\lambda u^{\prime}$.
(ii) We have

$$
Q v=\left(I+v\left(u^{\prime}-u\right)\right) v=v+v\left(u^{\prime} v-u v\right)=v
$$

and $Q^{-1} v=v$. Then

$$
Z v=Q^{-1} A Q v=Q^{-1} A v=\lambda Q^{-1} v=\lambda v
$$

(iii) Suppose $w A=\mu w$ and $\mu \neq \lambda$. Then, $w$ is orthogonal to $v$, i.e., $w v=(0)$ since

$$
(\mu-\lambda) w v=\mu w v-\lambda w v=(w A) v-w(A v)=(0)
$$

Because $w v=(0)$, we have that $w Q^{-1}=w$ and $w Q=w$. Therefore

$$
w Z=w Q^{-1} A Q=w A Q=\mu w Q=\mu w
$$

Lemma 4.3. Let $A \in \mathbb{R}^{n \times n}$ with right eigenvector $v$ and left eigenvector $u$ associated to the eigenvalue $\lambda$ chosen such that $u v=(1)$. For all row vector $u^{\prime}$ such that $u^{\prime} v=(1)$, the matrix $\lambda v u^{\prime}+\left(I-v u^{\prime}\right) A$ is conjugate to $A$. Moreover, for every integer $k \geq 0$,

$$
\left(\lambda v u^{\prime}+\left(I-v u^{\prime}\right) A\right)^{k}=\lambda^{k} v u^{\prime}+\left(I-v u^{\prime}\right) A^{k} .
$$

Proof. Let $X=v\left(u^{\prime}-u\right)$ so that $Q=I+X$ and $Q^{-1}=I-X$. As in the proof of Lemma 4.2, let $Z=Q^{-1} A Q$. We have

$$
A X=A v\left(u^{\prime}-u\right)=\lambda v\left(u^{\prime}-u\right)=\lambda X
$$

and

$$
X X=v\left(u^{\prime}-u\right) v\left(u^{\prime}-u\right)=v\left(u^{\prime} v-u v\right)\left(u^{\prime}-u\right)=(0) .
$$

Then, for every integer $k \geq 0$,

$$
\begin{aligned}
Z^{k} & =Q^{-1} A^{k} Q \\
& =(I-X) A^{k}(I+X), \\
& =A^{k}+A^{k} X-X A^{k}-X A^{k} X, \\
& =A^{k}+\lambda^{k} X-X A^{k}-\lambda^{k} X X, \\
& =A^{k}+X\left(\lambda^{k} I-A^{k}\right), \\
& =A^{k}+v\left(u^{\prime}-u\right)\left(\lambda^{k} I-A^{k}\right), \\
& =A^{k}+v u^{\prime}\left(\lambda^{k} I-A^{k}\right)-v u\left(\lambda^{k} I-A^{k}\right), \\
& =\lambda^{k} v u^{\prime}+\left(I-v u^{\prime}\right) A^{k} .
\end{aligned}
$$

We are now ready to state the main theorem. In the hypothesis, we assume that the right eigenvector $v$ is positive. This assumption can be relaxed to the fact that $v$ has nonzero entries and this is done in Corollary 4.12, Recall that the column vector $(1, \ldots, 1)^{\top} \in \mathbb{R}^{n \times 1}$ is denoted by 1 .
Theorem 4.4. Let $A \in \mathbb{R}^{n \times n}$ be a matrix with positive right eigenvector $v$ and real left eigenvector $u$ associated to the eigenvalue $\lambda$ chosen such that $\mathbf{1}^{\top} v=(1)$ and $u v=(1)$. The conditions below are equivalent.
(i) $\lambda$ is a positive, simple and dominant eigenvalue of $A$.
(ii) $\lim _{k \rightarrow \infty}\left(\frac{1}{\lambda} A\right)^{k}$ converges to the matrix vu.
(iii) For all positive row vector $u^{\prime}$ with $u^{\prime} v=(1)$, the matrix $\lambda v u^{\prime}+A-v u^{\prime} A$ is eventually positive.
(iv) The matrix $\lambda v \mathbf{1}^{\top}+A-v \mathbf{1}^{\top} A$ is eventually positive.
(v) $A$ is conjugate to an eventually positive matrix $Z$ such that $Z v=\lambda v$.
(vi) There exists an integer $k \geq 1$ such that for all $i$ and $j$ with $1 \leq i, j \leq n$, the entry $a_{i j}^{(k)}$ of $A^{k}$ is strictly larger than a certain value involving the sum of the entries of the same column:

$$
a_{i j}^{(k)}>v_{i}\left(\sum_{\ell=1}^{n} a_{\ell j}^{(k)}-\lambda^{k}\right)
$$

Proof. (i) $\Longrightarrow$ (ii). From Lemma [3.4, $\lambda^{-k} A^{k}$ converges to the matrix $v u$.
(ii) $\Longrightarrow$ (iii). Let $u^{\prime}$ be a positive row vector such that $u^{\prime} v=(1)$. For the purpose of the proof, we denote by $x_{i}$ the $i^{t h}$ entry of a vector $x$. By $\left|x_{i}\right|$, we denote the usual absolute value, and by $|x|_{\infty}$ the maximum norm of a vector. Let $\varepsilon$ be such that

$$
0<\varepsilon<\frac{\min \left\{\left|v_{i}\right|\right\} \min \left\{\left|u_{i}^{\prime}\right|\right\}}{1+n|v|_{\infty}\left|u^{\prime}\right|_{\infty}}
$$

Since $\lambda^{-k} A^{k}$ converges to the matrix $v u$, there exists an integer $k$ such that the entries of $E=$ $\left(E_{i j}\right)_{n \times n}=\lambda^{-k} A^{k}-v u$ are less than $\varepsilon$ in absolute value. Let $F=\left(F_{i j}\right)_{n \times n}=\left(I-v u^{\prime}\right) E$. We have $F_{i j}=E_{i j}-v_{i}\left(u_{1}^{\prime} E_{1 j}+u_{2}^{\prime} E_{2 j}+\cdots+u_{n}^{\prime} E_{n j}\right)$ so that
$\left|F_{i j}\right| \leq\left|E_{i j}\right|+\left|v_{i}\right|\left(\left|u_{1}^{\prime} E_{1 j}\right|+\left|u_{2}^{\prime} E_{2 j}\right|+\cdots+\left|u_{n}^{\prime} E_{n j}\right|\right)<\varepsilon+|v|_{\infty}\left|u^{\prime}\right|_{\infty}(n \varepsilon)<\min \left\{\left|v_{i}\right|\right\} \min \left\{\left|u_{i}^{\prime}\right|\right\}$.
Therefore $v u^{\prime}+F>0$. Note that $A^{k}=\lambda^{k}(v u+E)$. On the other hand, we have $\left(I-v u^{\prime}\right) v u=$ $v u-v u^{\prime} v u=v u-v u=0$. Therefore,

$$
\begin{aligned}
\left(\lambda v u^{\prime}+\left(I-v u^{\prime}\right) A\right)^{k} & =\lambda^{k} v u^{\prime}+\left(I-v u^{\prime}\right) A^{k} \\
& =\lambda^{k} v u^{\prime}+\left(I-v u^{\prime}\right) \lambda^{k}(v u+E) \\
& =\lambda^{k}\left(v u^{\prime}+F\right)>0
\end{aligned}
$$

(iii) $\Longrightarrow$ (iv). From the substitution $u^{\prime}=\mathbf{1}^{\top}$ and the fact that $\mathbf{1}^{\top} v=(1)$.
(iv) $\Longrightarrow(\mathrm{v})$. From Lemma4.3, we have that $A$ is conjugate to $Z=\lambda v \mathbf{1}^{\top}+A-v \mathbf{1}^{\top} A$. Moreover, $Z v=\lambda v \mathbf{1}^{\top} v+A v-v \mathbf{1}^{\top} A v=\lambda v+\lambda v-\lambda v=\lambda v$.
$(\mathrm{v}) \Longrightarrow(\mathrm{i})$. Suppose that $A$ is conjugate to $Z$ and there exists some integer $k$ such that $Z^{k}>0$. From Perron's theorem, the spectral radius of $Z^{k}$ is a simple and dominant eigenvalue of $Z^{k}$ with associated positive eigenvectors. Since $Z^{k} v=\lambda^{k} v$ and $v$ is positive, then the spectral radius of $Z^{k}$ must be $\lambda^{k}$. Therefore $\lambda^{k}$ is a simple root of the characteristic polynomial of $A^{k}$ whose modulus is strictly greater than that of any other eigenvalue. Then $\lambda$ is a positive, simple and dominant eigenvalue of $A$.
(iv) $\Longleftrightarrow\left(\right.$ vi). There exists a positive integer $k$ such that $\lambda v \mathbf{1}^{\top}+\left(I-v \mathbf{1}^{\top}\right) A$ is eventually positive if and only if $\lambda^{k} v \mathbf{1}^{\top}+\left(I-v \mathbf{1}^{\top}\right) A^{k}>0$ if and only if

$$
A^{k}>v \mathbf{1}^{\top} A^{k}-\lambda^{k} v \mathbf{1}^{\top}=v\left(\mathbf{1}^{\top} A^{k}-\lambda^{k} \mathbf{1}^{\top}\right)
$$

which holds if and only if

$$
a_{i j}^{(k)}>v_{i}\left(\sum_{\ell=1}^{n} a_{\ell j}^{(k)}-\lambda^{k}\right)
$$

for all $i$ and $j$ with $1 \leq i, j \leq n$.
Example 4.5. Let $A=\left(\begin{array}{cc}-11 & 14 \\ -26 & 29\end{array}\right)$ for which $v=\left(\frac{7}{20}, \frac{13}{20}\right)^{\top}$ is a positive right eigenvector and $u=\left(\frac{-20}{6}, \frac{20}{6}\right)$ is a left eigenvector associated to eigenvalue $\lambda=15$. We verify that $\mathbf{1}^{\top} v=(1)$ and $u v=(1)$. We compute that

$$
\lambda v \mathbf{1}^{\top}+A-v \mathbf{1}^{\top} A=15 \cdot \frac{1}{20}\left(\begin{array}{rr}
7 & 7 \\
13 & 13
\end{array}\right)+A-\frac{1}{20}\left(\begin{array}{rr}
7 & 7 \\
13 & 13
\end{array}\right) A=\frac{1}{5}\left(\begin{array}{rr}
36 & 21 \\
39 & 54
\end{array}\right)
$$

is positive. Using Theorem 4.4, we conclude 15 is a simple and dominant eigenvalue of $A$.
The next example illustrates that the entries of $u$ can be zero.
Example 4.6. Let $A=\left(\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right)$ for which $v=\left(\frac{1}{2}, \frac{1}{2}\right)^{\top}$ is a positive right eigenvector and $u=(0,2)$ is a left eigenvector associated to eigenvalue $\lambda=1$. We verify that $\mathbf{1}^{\top} v=(1)$ and $u v=(1)$. We compute that

$$
\lambda v \mathbf{1}^{\top}+A-v \mathbf{1}^{\top} A=\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)+A-\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) A=\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)
$$

is positive. Using Theorem 4.4, we conclude that 1 is a simple and dominant eigenvalue of $A$.
In the case where $\mathbf{1}$ is a right eigenvector associated to eigenvalue 1 , the criteria can be declined in terms of the average of the entries of the same column.

Corollary 4.7. Let $A \in \mathbb{R}^{n \times n}$ with positive right eigenvector 1 and real left eigenvector $u$ associated to the eigenvalue 1 chosen such that $u \mathbf{1}=(1)$. The following conditions are equivalent.
(i) 1 is a simple and dominant eigenvalue of $A$.
(ii) $\lim _{k \rightarrow \infty} A^{k}$ converges to the matrix $\frac{1}{n} \mathbf{1} u$.
(iii) For all positive row vector $u^{\prime}$ with $u^{\prime} \frac{1}{n} \mathbf{1}=(1)$, the matrix $\frac{1}{n} \mathbf{1} u^{\prime}+A-\frac{1}{n} \mathbf{1} u^{\prime} A$ is eventually positive.
(iv) The matrix $\frac{1}{n} \mathbf{1 1}^{\top}+A-\frac{1}{n} \mathbf{1 1}^{\top} A$ is eventually positive.
(v) $A$ is conjugate to an eventually positive matrix $Z$ such that $Z \mathbf{1}=\mathbf{1}$.
(vi) There exists an integer $k \geq 1$ such that for all $i$ and $j$ with $1 \leq i, j \leq n$, the entry $a_{i j}^{(k)}$ of $A^{k}$ is strictly larger than the average of the entries of the same column minus $\frac{1}{n}$ :

$$
a_{i j}^{(k)}>\frac{1}{n} \sum_{\ell=1}^{n} a_{\ell j}^{(k)}-\frac{1}{n} .
$$

Proof. Substituting $v=\frac{1}{n} \mathbf{1}$ and $\lambda=1$ in Theorem 4.4.
Example 4.8. Let $A=\frac{1}{15}\left(\begin{array}{cc}-11 & 26 \\ -14 & 29\end{array}\right)$ for which 1 is a right eigenvector and $u=\frac{1}{6}(-7,13)$ is a left eigenvector associated to eigenvalue $\lambda=1$. We verify that $u \mathbf{1}=(1)$. We compute that

$$
\frac{1}{2} \mathbf{1 1} \mathbf{1}^{\top}+A-\frac{1}{2} \mathbf{1 1} \mathbf{1}^{\top} A=\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)+A-\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) A=\frac{1}{5}\left(\begin{array}{ll}
3 & 2 \\
2 & 3
\end{array}\right)
$$

is positive. Using Corollary 4.7 we conclude that 1 is a simple and dominant eigenvalue of $A$.
We illustrate when the corollary is not satisfied with a nonexample.
Example 4.9. Let $A=\frac{1}{3}\left(\begin{array}{cc}-11 & 14 \\ -26 & 29\end{array}\right)$ for which 1 is a right eigenvector and $u=\frac{1}{6}(13,-7)$ is a left eigenvector associated to eigenvalue $\lambda=1$. We verify that $u \mathbf{1}=(1)$. The criteria is not satisfied when $k=1$ since $d_{21}=\frac{-26}{3} \ngtr \frac{-20}{3}=\frac{1}{2}\left(\frac{-11}{3}+\frac{-26}{3}\right)-\frac{1}{2}=\frac{1}{2}\left(d_{11}+d_{21}\right)-\frac{1}{2}$. Also the matrix

$$
\frac{1}{2} \mathbf{1 1} \mathbf{1}^{\top}+A-\frac{1}{2} \mathbf{1 1}^{\top} A=\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)+A-\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) A=\left(\begin{array}{rr}
3 & -2 \\
-2 & 3
\end{array}\right)
$$

is not positive and neither of its powers. From Corollary 4.7, we conclude that 1 is not a simple and dominant eigenvalue of $A$. Indeed, 5 is another eigenvalue of $A$.

Now we adapt the theorem to the nonnegative case. As it is the case for Theorem 2.3, there is no equivalence possible.
Proposition 4.10. Let $A \in \mathbb{R}^{n \times n}$ with positive right eigenvector $v$ and real left eigenvector $u$ associated to the eigenvalue $\lambda$ chosen such that $\mathbf{1}^{\top} v=(1)$ and $u v=(1)$. If the matrix $\lambda v \mathbf{1}^{\top}+A-$ $v \mathbf{1}^{\top} A$ is nonnegative, then the spectral radius of $A$ is a nonnegative eigenvalue which is greater than or equal to the modulus of any other eigenvalue.
Proof. From Lemma 4.3, $A$ is similar to $Z=\lambda v \mathbf{1}^{\top}+A-v \mathbf{1}^{\top} A$. If $Z$ is nonnegative, then from Theorem 2.3. $Z$ has a nonnegative eigenvalue $\lambda$ such that the moduli of all other eigenvalues do not exceed $\lambda$.

We illustrate the proposition on an example.
Example 4.11. Let $A=\left(\begin{array}{cc}-4 & 5 \\ -3 & 4\end{array}\right)$ for which 1 is a right eigenvector and $u=\frac{1}{2}(-3,5)$ is a left eigenvector associated to eigenvalue $\lambda=1$. We verify that $u \mathbf{1}=(1)$. We compute that

$$
\frac{1}{2} \mathbf{1 1}{ }^{\top}+A-\frac{1}{2} \mathbf{1 1} \mathbf{1}^{\top} A=\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)+A-\frac{1}{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) A=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

is nonnegative. Using Proposition 4.10 we conclude that 1 is an eigenvalue of $A$ greater than or equal to the modulus of any other eigenvalue. Indeed, 1 and -1 are eigenvalues of $A$ as the characteristic polynomial of $A$ is $\chi_{A}(\lambda)=(\lambda-1)(\lambda+1)$.

Using a signature matrix, we can adapt the theorem to the case where the right eigenvector $v$ is not positive with nonzero entries.
Corollary 4.12. Let $A \in \mathbb{R}^{n \times n}$ with right eigenvector $v$ and left eigenvector $u$ associated to the eigenvalue $\lambda$ chosen such that $\mathbf{1}^{\top} S v=(1)$ and $u v=(1)$ where $S$ is a signature matrix such that $S v$ is positive. The following conditions are equivalent.
(i) $\lambda$ is a positive, simple, dominant eigenvalue of $A$.
(ii) $\lim _{k \rightarrow \infty}\left(\frac{1}{\lambda} A\right)^{k}$ converges to the matrix vu.
(iii) For all positive row vector $u^{\prime}$ with $u^{\prime} S v=(1)$, the matrix $\lambda S v u^{\prime}+S A S-S v u^{\prime} S A S$ is eventually positive.
(iv) The matrix $\lambda S v \mathbf{1}^{\top}+S A S-S v \mathbf{1}^{\top} S A S$ is eventually positive.
(v) $A$ is conjugate to an eventually positive matrix $Z$ such that $Z S v=\lambda S v$.

Proof. From Lemma 3.9 $S v$ is a right eigenvector and $u S$ is a left eigenvector of the matrix $B=S A S$ associated to eigenvalue $\lambda$. Since $(u S)(S v)=u v=(1)$ and $S v$ is positive, the hypotheses of Theorem4.4 are satisfied. Then (i) $\lambda$ is a positive, simple and dominant eigenvalue of $B$ therefore also of $A$. Then (ii) $\lim _{k \rightarrow \infty}\left(\frac{1}{\lambda} B\right)^{k}=S\left(\lim _{k \rightarrow \infty}\left(\frac{1}{\lambda} A\right)^{k}\right) S$ converges to the matrix $S v u S$. Then (iii) for all positive row vector $u^{\prime}$ with $u^{\prime} S v=(1)$, the matrix $\lambda S v u^{\prime}+S A S-S v u^{\prime} S A S$ is eventually positive. Then (iv) the matrix $\lambda S v \mathbf{1}^{\top}+S A S-S v \mathbf{1}^{\top} S A S$ is eventually positive. Then (v) $S A S$ is conjugate to an eventually positive matrix $Z$ such that $Z S v=\lambda S v$.

## 5. Application to Coxeter groups

5.1. Geometric representations of Coxeter systems. A Coxeter system consists of a pair $(W, S)$, where $W$ is a group (with identity denoted by $e$ ) generated by a set $S=\left\{s_{1}, s_{2}, \ldots, s_{n}\right\}$ of letters, with $n \in \mathbb{N}$ satisfying the following conditions.
(i) $s_{i}^{2}=e$ for all $i$ such that $1 \leq i \leq n$.
(ii) $\left(s_{i} s_{j}\right)^{m_{i, j}}=e$, with $m_{i, j} \geq 2$ or $m_{i, j}=\infty$.

Among others, the following books give a general background on Coxeter groups and their related structures: Bou68, Hum92, BB05]. An element $w \in W$ is represented as a word in the alphabet $S$. A word representing an element $w$ is called reduced if it is shortest with this property.

Given a Coxeter system, we represent it as a group of linear transformations of a vector space stabilizing a bilinear form as follows.

Let $V$ be a real vector space with basis $\Delta=\left\{\alpha_{s} \mid s \in S\right\}$. Define a symmetric bilinear form $\mathcal{B}$ on the basis $\Delta$ by the matrix

$$
B=\left[b_{i, j}=\left\{\begin{array}{l}
-\cos \left(\pi / m_{i, j}\right), \text { if } m_{i, j}<\infty \\
-c_{i, j}, \text { if } m_{i, j}=\infty
\end{array}\right]_{1 \leq i, j \leq n}\right.
$$

where $c_{i, j} \geq 1$. If all $c_{i, j}$ are equal to 1 , we recover the "classical" bilinear form which is canonical. The signature of $B$ is $(p, q, r)$, where $p$ is the number of positive eigenvalues of $B, q$ the number of negative eigenvalues of $B$ and $r$ the dimension of the kernel of $B$. If $B$ is positive-definite, we say that ( $W, S$ ) is of finite type (notice that there is a unique bilinear form in this case). If $B$ is positive-semidefinite, we say that $(W, S)$ with the associated bilinear form is of affine type. If the associate space $V$ cannot be partitionned into 2 proper subspaces orthogonal with respect to $\mathcal{B}$, then $(W, S)$ is said to be irreducible. This bilinear form $\mathcal{B}$ allows to define the reflection $\sigma_{\alpha}$ of $V$ along a nonisotropic vector $\alpha \in V$ using the formula

$$
\begin{aligned}
\sigma_{\alpha}: V & \rightarrow V \\
\lambda & \mapsto \lambda-\frac{\mathcal{B}(\lambda, \alpha)}{\mathcal{B}(\alpha, \alpha)} \alpha .
\end{aligned}
$$

Finally, the morphism $\phi: W \rightarrow G L(V)$ sends the generators in $S$ to the reflections $\sigma_{\alpha_{s}}$. This morphism is well-defined, injective and its image preserves the bilinear form $\mathcal{B}$, i.e. $\phi(W)$ is a subgroup of $O_{\mathcal{B}}(V)=\{\tau \in G L(V) \mid \mathcal{B}(\tau(v), \tau(v))=\mathcal{B}(v, v)$, for all $v \in V\}$. A subgroup of $W$ generated by elements in a subset $I$ of $S$ is called standard parabolic subgroup. A subgroup of $W$ conjugate to a standard parabolic subgroup is a parabolic subgroup. The parabolic closure of an element $w \in W$ is the smallest parabolic subgroup that contains $w$. Similarly, the standard parabolic closure of an element $w \in W$ is the smallest standard parabolic subgroup $W_{I}$ with $I \subseteq S$ that contains $w$. It follows from the properties of the geometric representation that the columns of a matrix representing an element of the group are either nonnegative or nonpositive. In the case of the classical bilinear form, we refer to the representation as the classical geometric representation. For more details about the classical geometric representation of Coxeter groups, we refer the reader to [Hum92, Chapter 5]. For more details on general geometric representations and precise references to proofs, we refer the reader to HLR14 Section 1] and the references therein.
5.2. Spectrum of matrices in geometric representations of Coxeter systems. Investigations on the eigenvalues have been done before in the classical representation for Coxeter elements, i.e. the products of the generators in $S$ taken in some order: If the Coxeter graph is a forest, A'Campo showed that the spectrum of the Coxeter element is contained in the union of the unit circle and the positive real line $A^{\prime} \mathrm{C} 76$. This allows to show that $W$ is infinite if and only if a Coxeter element has infinite order. This was later generalized to all graphs by Howlett How82 which studied Coxeter elements using $M$-matrices. The following result gives a general description of the spectral radius.

Theorem 5.1 (McMullen McM02, Theorem 1.1]). Let $(W, S)$ be a Coxeter system, $\phi$ the classical geometric representation and $w \in W$. The spectral radius $\rho_{w}$ of $\phi(w)$ is either 1 or $\rho_{w} \geq \lambda_{\text {Lehmer }}$, where $\lambda_{\text {Lehmer }} \approx 1.1762808$ is Lehmer's number.

In McM02, McMullen also gives a proof that the elements of irreducible, infinite, and nonaffine Coxeter systems have the Eigenvalue property, which derives from results of Vinberg Vin71, see also Dye13, Lemma 7.3].

In trying to generalize Perron's theorem to more general matrices containing negative entries, there is a framework of cone-preserving maps which generalize the notion of $K$-primitivity with respect to a pointed convex closed full-dimensional cone $K$, see for instance the survey article Tam04. The Tits cone and imaginary cone which are left invariant by the action of the group seem like good candidates, but it turns out that the eigenvectors considered are on the boundary
of the cones which cannot be obtained from $K$-primitive matrices as they would have to be in the interior of the cone. Nevertheless, it is possible to say more about the spectrum as the following theorem shows.

Theorem 5.2 (Krammer Kra09, Section 6.5]). Let (W,S) be infinite, nonaffine, irreducible and $w \in W$ whose parabolic closure is $W$. The spectral radius of $\phi(w)$ is an eigenvalue, which is simple and strictly greater than 1.

Remark 5.3. This theorem does not show the dominance of the spectral radius, that is to say, there are no other eigenvalues with the same modulus. Krammer mentionned in his 1992 thesis (republished in 2009) that he tried to find a stable cone to apply Perron-Frobenius techniques without success and left it as a conjecture, Kra09, Section 6.5]. The proof of this theorem rather relies on the structure of root systems. An original motivation of the current work was to show the dominance of the spectral radius, based on Perron-Frobenius theory.
5.3. A conjecture on elements with Perron's properties. In this section, we describe and motivate a closer study of the spectral radius of matrices in the geometric representation of Coxeter groups.

The conjecture below is motivated by the study of infinite reduced words and their associated inversion set, see HL15. An infinite reduced word is an infinite sequence of generators in $S$ where every prefix is reduced. Its inversion set is a special set of vectors called roots which characterizes geometrically the infinite reduced word. It is conjectured that the inversion set of an infinite reduced word, seen in projective space, has a unique accumulation point. This conjecture is known to hold when the Coxeter system is Lorentzian CL14. This conjecture calls for a better knowledge of the spectral properties of elements in geometric representations of Coxeter system.

The following conjecture characterizes those elements in a geometric representation of a Coxeter system which possess the first three conclusions of Perron's theorem along with a positive right eigenvector. It is closely related to Krammer's conjecture [Kra09, Conjecture 6.5.16], however it also includes elements whose parabolic closure is not necessarily the whole group. In McM02, elements whose parabolic closure is the whole group are called essential and otherwise are called peripherical.

Conjecture 5.4. Let $(W, S)$ be Coxeter system with $|S|>2, \phi$ be a geometric representation and $w \in W$. If the parabolic closure of $w$ is irreducible, infinite and nonaffine, then the spectral radius of $\phi(w)$ is a simple and dominant eigenvalue of $\phi(w)$. Moreover, it is strictly greater than 1 with positive right eigenvectors.

To support this conjecture, we present below three examples respectively of rank 3,4 and 5 . We consider elements $w$ such that the parabolic closure is infinite, irreducible and nonaffine. We apply the computational criterion of Theorem4.4 to conclude that $\phi(w)$ has an eventually positive conjugate matrix and its spectral radius is a simple and dominant eigenvalue.

Example 5.5. Let $S=\left\{s_{1}, s_{2}, s_{3}\right\}$ and $W$ be the free Coxeter group on $S$, i.e. the product of any two generators has infinite order. Consider the bilinear form where the values of $c_{i, j}$ are all equal to 2 . The bilinear form then has signature $(2,1,0)$. In this case, it is known that elements of the group are elliptic, parabolic or hyperbolic, see [L14] for a detailed description. Hyperbolic elements have a unique real simple and dominant eigenvalue greater than 1 with a corresponding positive right eigenvector. The generating set $S$ is

$$
\left\{\left(\begin{array}{rrr}
-1 & 4 & 4 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{rrr}
1 & 0 & 0 \\
4 & -1 & 4 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{rrr}
1 & 0 & 0 \\
0 & 1 & 0 \\
4 & 4 & -1
\end{array}\right)\right\}
$$

The matrix corresponding to $s_{1} s_{2} s_{3} s_{2}$ is

$$
H=\phi\left(s_{1} s_{2} s_{3} s_{2}\right)=\left(\begin{array}{rrr}
399 & -76 & 284 \\
80 & -15 & 56 \\
20 & -4 & 15
\end{array}\right)
$$

for which $\lambda \approx 397.9974$ is an eigenvalue with right eigenvector $v \approx(0.7995,0.1603,0.04008)^{\top}$. Using Theorem 4.4, we compute that

$$
\lambda v \mathbf{1}^{\top}+H-v \mathbf{1}^{\top} H \approx\left(\begin{array}{lll}
318.23857 & 318.19990 & 318.38071 \\
63.807131 & 64.038071 & 62.893420 \\
15.951783 & 15.759518 & 16.723355
\end{array}\right)>0
$$

As expected, $\lambda$ is a simple and dominant eigenvalue of $H$.
The previous example is a Lorentzian Coxeter system covered by the work in [CL14 while the conjecture on the uniqueness of the accumulation point for infinite reduced words is still open for the next two examples of rank 4 and 5 .

Example 5.6. Let $S=\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}$ and $W$ be the free Coxeter group on $S$, i.e. the product of any two generators has infinite order. Consider the bilinear form where the value of $c_{i, j}$ is 1 , except for $c_{1,2}=2$ and $c_{3,4}=6$. The bilinear form then has signature $(2,2,0)$. The generating set $S$ is

$$
\left\{\left(\begin{array}{rrrr}
-1 & 4 & 2 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right),\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
4 & -1 & 2 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right),\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 2 & -1 & 12 \\
0 & 0 & 0 & 1
\end{array}\right),\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2 & 2 & 12 & -1
\end{array}\right)\right\} .
$$

The matrix corresponding to $s_{1} s_{3} s_{2} s_{4} s_{2} s_{3}$ is

$$
H=\phi\left(s_{1} s_{3} s_{2} s_{4} s_{2} s_{3}\right)=\left(\begin{array}{rrrr}
1763 & 1264 & -670 & 8150 \\
84 & 61 & -32 & 388 \\
672 & 480 & -255 & 3104 \\
42 & 30 & -16 & 195
\end{array}\right),
$$

for which $\lambda \approx 1761.9994$ is an eigenvalue with right eigenvector

$$
v \approx(0.6884,0.03279,0.2623,0.01639)^{\top}
$$

Using Theorem 4.4 we compute that

$$
\lambda v \mathbf{1}^{\top}+H-v \mathbf{1}^{\top} H \approx\left(\begin{array}{llll}
1212.9657 & 1213.7462 & 1212.7815 & 1214.3404 \\
57.793025 & 58.605604 & 57.707147 & 57.543055 \\
462.34420 & 460.84483 & 462.65718 & 460.34444 \\
28.896513 & 28.802802 & 28.853573 & 29.771528
\end{array}\right)>0
$$

Therefore, $\lambda$ is a simple and dominant eigenvalue of $H$.
Example 5.7. Let $S=\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}\right\}$ and $W$ be the Coxeter group given by the relations $\left(s_{1} s_{2}\right)^{\infty}=\left(s_{2} s_{3}\right)^{\infty}=\left(s_{3} s_{4}\right)^{\infty}=\left(s_{4} s_{5}\right)^{\infty}=\left(s_{1} s_{5}\right)^{\infty}=e$ and all other pairs of generators commute. Further, choose the parameters $c_{i j}$ in the bilinear form for the 5 infinite labels to be equal to 2 . Then, the signature of the bilinear form is $(2,3,0)$. The generating set $S$ is
$\left\{\left(\begin{array}{rrrrr}-1 & 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{array}\right),\left(\begin{array}{rrrrr}1 & 0 & 0 & 0 & 0 \\ 4 & -1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{array}\right),\left(\begin{array}{rrrrr}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & -1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{array}\right),\left(\begin{array}{rrrrr}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -1 & 4 \\ 0 & 0 & 0 & 0 & 1\end{array}\right),\left(\begin{array}{rrrrr}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 4 & -1\end{array}\right)\right\}$.
The matrix corresponding to $s_{1} s_{2} s_{3} s_{4} s_{5} s_{1} s_{2}$ is

$$
H=\phi\left(s_{1} s_{2} s_{3} s_{4} s_{5} s_{1} s_{2}\right)=\left(\begin{array}{rrrrr}
16065 & -4280 & 17360 & 976 & 3960 \\
3960 & -1055 & 4280 & 240 & 976 \\
976 & -260 & 1055 & 60 & 240 \\
240 & -64 & 260 & 15 & 60 \\
60 & -16 & 64 & 4 & 15
\end{array}\right)
$$

for which $\lambda \approx 16094.04766330161$ is an eigenvalue with right eigenvector

$$
v \approx(0.7541,0.1859,0.04582,0.01126,0.002814)^{\top}
$$

Using Theorem 4.4, we compute that

$$
\lambda v \mathbf{1}^{\top}+H-v \mathbf{1}^{\top} H \approx\left(\begin{array}{rrrrr}
12137.980 & 12137.949 & 12137.286 & 12137.261 & 12137.694 \\
2991.9849 & 2992.0443 & 2992.5946 & 2991.2642 & 2991.8114 \\
737.41275 & 737.47739 & 737.69244 & 738.10571 & 736.83822 \\
181.32465 & 181.30789 & 181.96510 & 181.76536 & 182.18656 \\
45.345076 & 45.268805 & 44.509779 & 45.651798 & 45.517668
\end{array}\right)>0
$$

Therefore, $\lambda$ is a simple and dominant eigenvalue of $H$.
As the examples show, it is possible to obtain information about the simplicity and the dominance of a real eigenvalue of a matrix representing an element of an infinite Coxeter group using Perron-Frobenius theory. This is an achievement that was left open in the work of Krammer (see Remark (5.3). Further, this criterion is a characterization: if the obtained matrix is not eventually positive, then either the eigenvalue is not simple or there is another eigenvalue with the same modulus. More research has to be done to see whether Theorem 4.4 and this approach can lead to a proof of Conjecture 5.4.

Using the software Sage $\mathbf{S}^{+} 15$, we tested this criterion on several different irreducible infinite nonaffine Coxeter groups (of rank $\leq 8$ ), representations, and several elements (in particular elements different from Coxeter elements). Interestingly, it seems that the matrix obtained using Theorem 4.4 is already positive when the parabolic closure is irreducible, infinite and nonaffine. Thus, the eventual positivity of the conjugate matrix follows immediately.
5.4. Towards an equivalence. In this section, we consider the reverse of the conjecture. We remark in the next example that the reverse is false: the parabolic closure of an element whose spectral radius is a simple and dominant eigenvalue may be reducible. Then, we propose an adaptation of the conjecture that is stated as an equivalence for the case where the parabolic closure is irreducible. We support our conjecture with some examples and a proof in the case when the spectral radius is 1 (Proposition 5.12).
Example 5.8. Let $S=\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}\right\}$ and $W$ be the Coxeter group on $S$ with the relations $c_{1,2}=3,\left(s_{2} s_{3}\right)^{3}=\left(s_{3} s_{4}\right)^{3}=e, c_{4,5}=2$, and the other pairs commute. The parabolic closure of the element $w=s_{1} s_{2} s_{4} s_{5}$ is infinite, reducible, and nonaffine. Computing the eigenvalues of $\phi(w)$, one gets

$$
\{\approx 0.029437, \approx 0.071796,1, \approx 13.928203, \approx 33.970562\}
$$

Thus, the spectral radius is a simple and dominant eigenvalue.
Therefore, if the parabolic closure of $w$ is reducible, the spectral radius of $\phi(w)$ may be a dominant and simple eigenvalue. We thus propose the equivalence in the next conjecture which is restricted to the case where the parabolic closure of $w$ is irreducible.
Conjecture 5.9. Let $(W, S)$ be Coxeter system with $|S|>2$, $\phi$ be a geometric representation and $w \in W$. Assume that the parabolic closure of $w$ is irreducible. The following statements are equivalent.
(i) The parabolic closure of $w$ is infinite and nonaffine.
(ii) The spectral radius of $\phi(w)$ is a simple and dominant eigenvalue.

Moreover, if one of the above condition holds, then the spectral radius of $\phi(w)$ is strictly greater than 1 with positive right eigenvectors.

The next example consists of an element whose parabolic closure is an infinite, irreducible, and affine Coxeter group on which we cannot apply Theorem 4.4 nevertheless, Lemma 3.1 allows to conclude that it still satisfies Conjecture 5.9 .

Example 5.10. Let $S=\left\{s_{1}, s_{2}\right\}$ and $W$ be the free Coxeter group on $S$, i.e. the product of any two generators has infinite order. Consider the bilinear form where the value of $c_{1,2}$ is 1 . The bilinear form then has signature $(1,0,1)$ and the group with this represenation is of affine type. The generating set $S$ is

$$
\left\{\left(\begin{array}{rr}
-1 & 2 \\
0 & 1
\end{array}\right),\left(\begin{array}{rr}
1 & 0 \\
2 & -1
\end{array}\right)\right\} .
$$

The matrix corresponding to $s_{1} s_{2}$ is

$$
\phi\left(s_{1} s_{2}\right)=\left(\begin{array}{ll}
3 & -2 \\
2 & -1
\end{array}\right)
$$

for which $\lambda=1$ is an eigenvalue with right eigenvector $v=(1,1)^{\top}$ and left eigenvector $u=(1,-1)$. Since $u v=(0)$, by Lemma 3.1 we conclude that $\lambda$ has algebraic multiplicity 2 .

The next example illustrates Conjecture 5.9 and the distinction between elements for which the infinite and irreducible parabolic closure is affine or not.

Example 5.11. Let $S=\left\{s_{1}, s_{2}, s_{3}\right\}$ and $W$ be the free Coxeter group on $S$, i.e. the product of any two generators has infinite order. Consider the bilinear form where the values of $c_{i, j}$ are all equal to 1 . The element $s_{3} s_{1} s_{2} s_{3}$ is conjugated to an element in the standard parabolic subgroup generated by $\left\{s_{1}, s_{2}\right\}$, which is infinite, irreducible, and affine. In this case, the spectral radius of $s_{3} s_{1} s_{2} s_{3}$ is dominant but not simple since 1 is the only eigenvalue with algebraic multiplicity 3. Thus, it verifies Conjecture 5.9. One can verify that the parabolic closure of the element $s_{1} s_{2} s_{1} s_{3} s_{2} s_{3}$ is the whole group $W$ which is infinite, irreducible and nonaffine. Moreover, the eigenvalues are $1, \approx 0.005154$, and $\approx 193.994845$ so that its spectral radius is simple and dominant. Again, it verifies the Conjecture 5.9,

We give here the proof of Conjecture 5.9 for elements $w$ for which $\phi(w)$ has spectral radius 1 . In particular, the conjecture holds for elements of finite order.

Proposition 5.12. Let $(W, S)$ be Coxeter system with $|S|>2$, $\phi$ be a geometric representation and $w \in W$. Assume that the spectral radius of $w$ is 1. Both of the following statements are false.
(i) The parabolic closure of $w$ is infinite, irreducible and nonaffine.
(ii) The spectral radius of $\phi(w)$ is a simple and dominant eigenvalue.

Proof. We prove that the matrix $\phi(w)$ does not satisfy condition (i). If the element $w$ is of finite order, its parabolic closure is finite, see [Kra09, Proposition 3.2.1] and (i) is not satisfied in this case. Assume $w$ has infinite order. By contradiction, assume that the parabolic closure of $\phi(w)$ is infinite, irreducible and nonaffine. Then, by Theorem 5.2, the spectral radius is strictly greater than 1; a contradiction. Thus, the parabolic closure of $w$ has to be reducible or affine.

Now we prove that $\phi(w)$ does not satisfy condition (ii). The determinant, i.e. the product of the eigenvalues, of $\phi(w)$ is either 1 or -1 . Then all eigenvalues should have modulus 1 to have a product equal to 1 or -1 . Therefore if an element has at least two different eigenvalues, the Dominance property does not hold. If there is only one eigenvalue and the dimension of the representation is strictly greater than one, the algebraic multiplicity of the eigenvalue has to be strictly greater than 1.

It remains to show that elements of spectral radius strictly greater than 1 also satisfy Conjecture 5.9 which would imply Conjecture 5.4

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