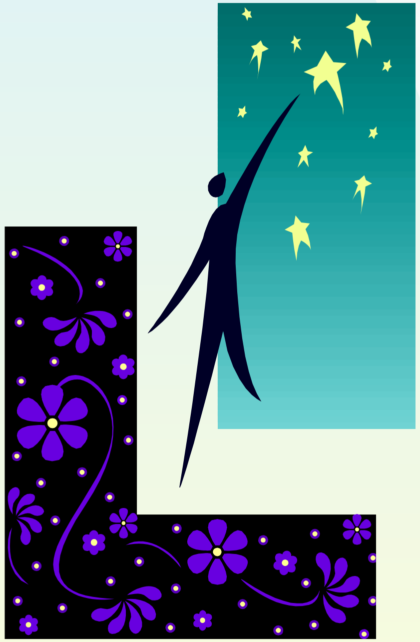
An introduction to optical/IR interferometry



$$I_q = I \; + I \; + 2 I \; |\gamma_{12}(0)| \cos\left(\beta_{12} - 2\Pi \nu \tau\right)$$

$$\gamma_{12}(\tau) = \langle V_1^*(t) V_2(t-\tau) \rangle / I$$

Fringe visibility: $\upsilon = \left(\frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \right) = \left| \gamma_{12}(0) \right|$

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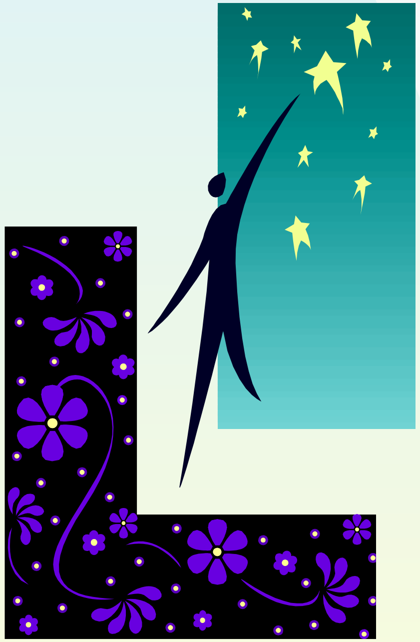

$$V = \left|\gamma_{12}(0,u,v)\right| = \left|\iint_S I'(\zeta,\eta)\exp\left\{-i2\Pi\left(u\zeta + v\eta\right)\right\} d\zeta\, d\eta\right|$$

$$I'(\zeta,\eta) = \iint \gamma_{12}(0,u,v)\exp\left\{i2\Pi\left(\zeta u + \eta v\right)\right\} d(u) d(v)$$

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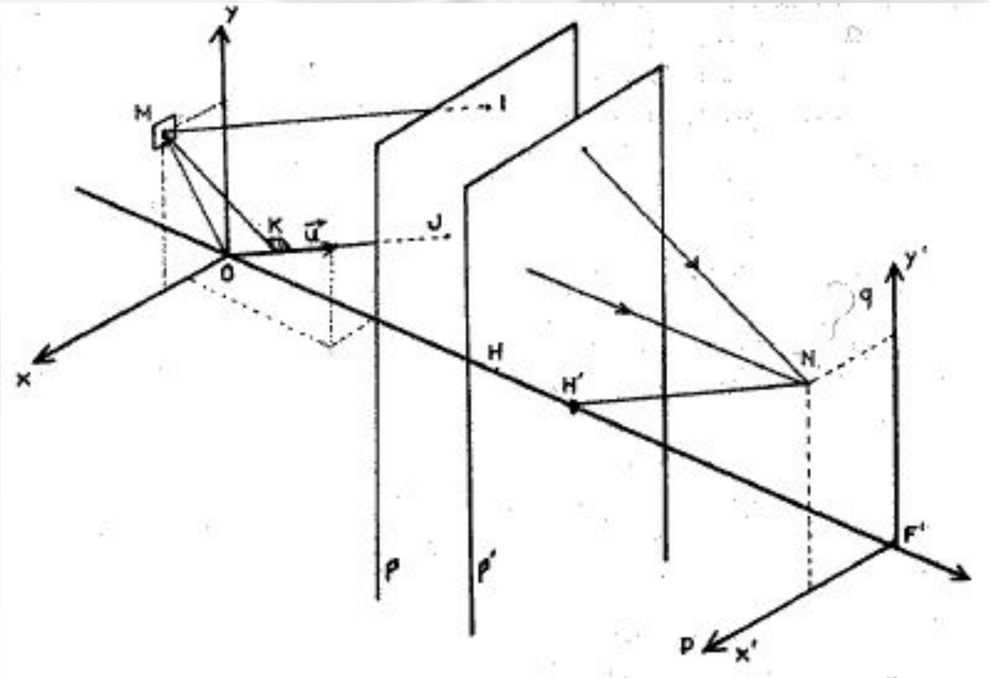
8.1 The fundamental theorem

a(p,q) = TF_(A(x,y))(p,q),

$$a(p,q) = \int_{R^2} A(x,y)\exp\left[-i2\pi\left(px+qy\right)\right]dxdy,$$



with

p = x’ /(λ f)

q = y’ /(λ f)

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8.1 The fundamental theorem

The distribution of the complex amplitude a(p,q) in the focal plane is given by the Fourier transform of the distribution of the complex amplitude A(x,y) in the entrance pupil plane.

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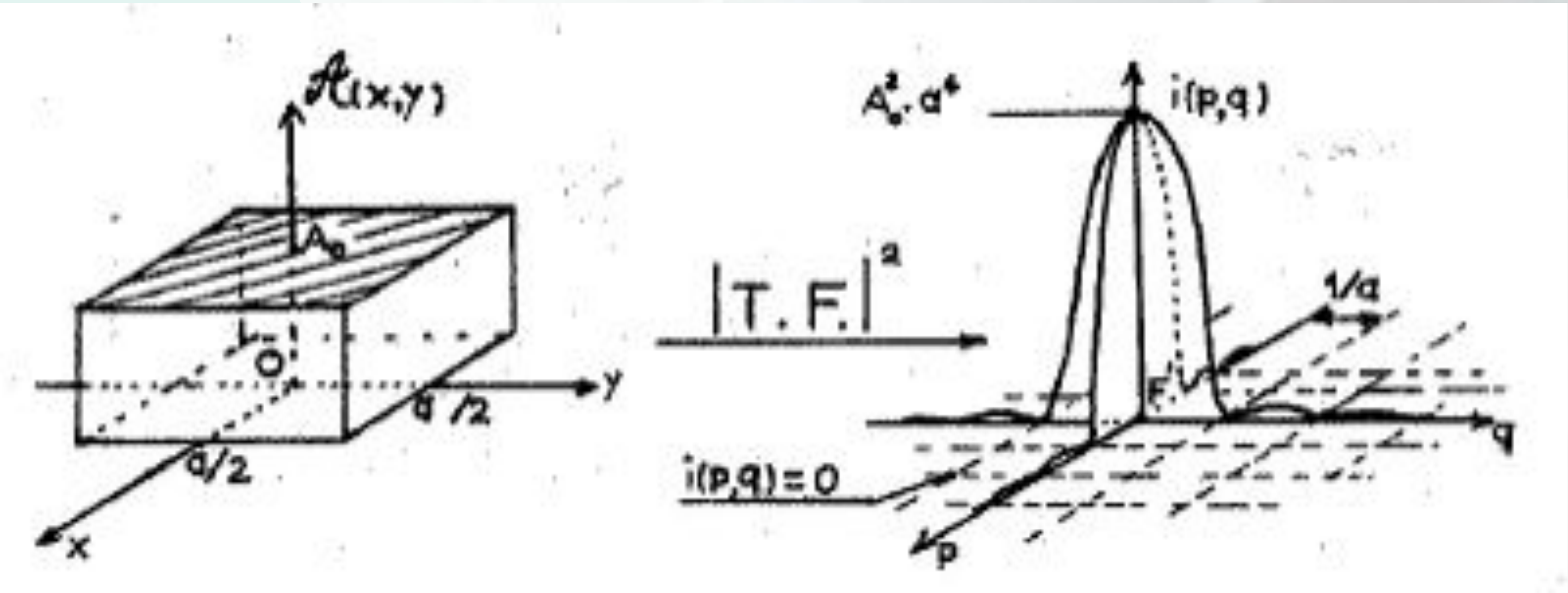
8.1 The fundamental theorem

Application: Point Spread Function determination



$\Delta p = \Delta x' /(\lambda f)$; $\Delta q = \Delta y' /(\lambda f) = 2/a \Rightarrow \Delta\phi_{x'} = \Delta\phi_{y'} = 2\lambda/a$ (8.1.7)

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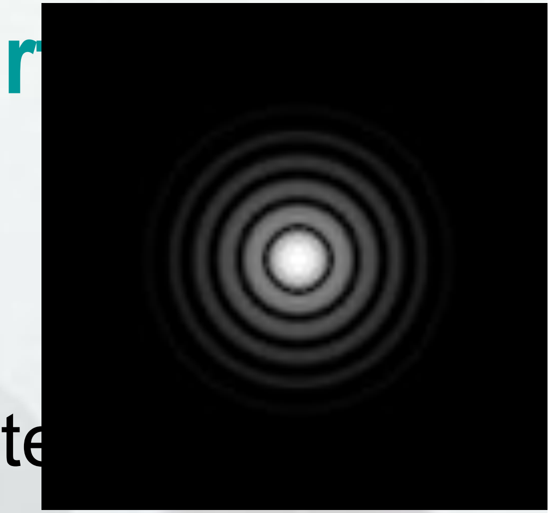
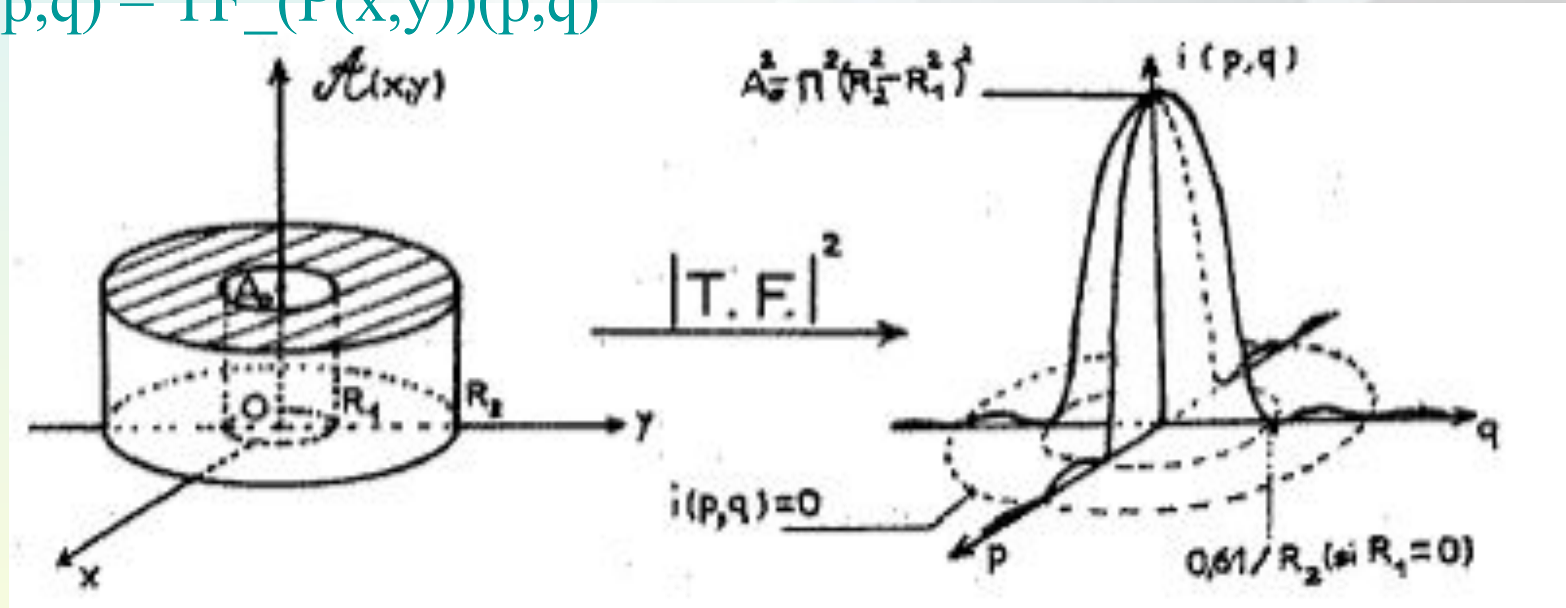


8.1 The fundamental theorem

Application: Point Spread Function dete

h(p,q) = TF_(P(x,y))(p,q)



$$i(\rho') = |a(\rho')|^2 = (A_0\,\pi)^2\,[R_2^2\,2\,J_1(Z_2)/Z_2\; - R_1^2\,2\,J_1(Z_1)/Z_1]^2, \qquad (8.1.8)$$

$$\text{with } Z_2 = 2\pi\,R_2\,\rho' / (\lambda f) \text{ and } Z_1 = 2\pi\,R_1\,\rho' / (\lambda f). \qquad (8.1.9)$$

From the previous result, i.e. for the case of a circular aperture with a radius R, the distribution of the complex amplitude in the focal plane is given by the expression:

$a(\rho') = (A_0 \pi)\, [R^2\, 2\, J_1(Z) / Z]$,

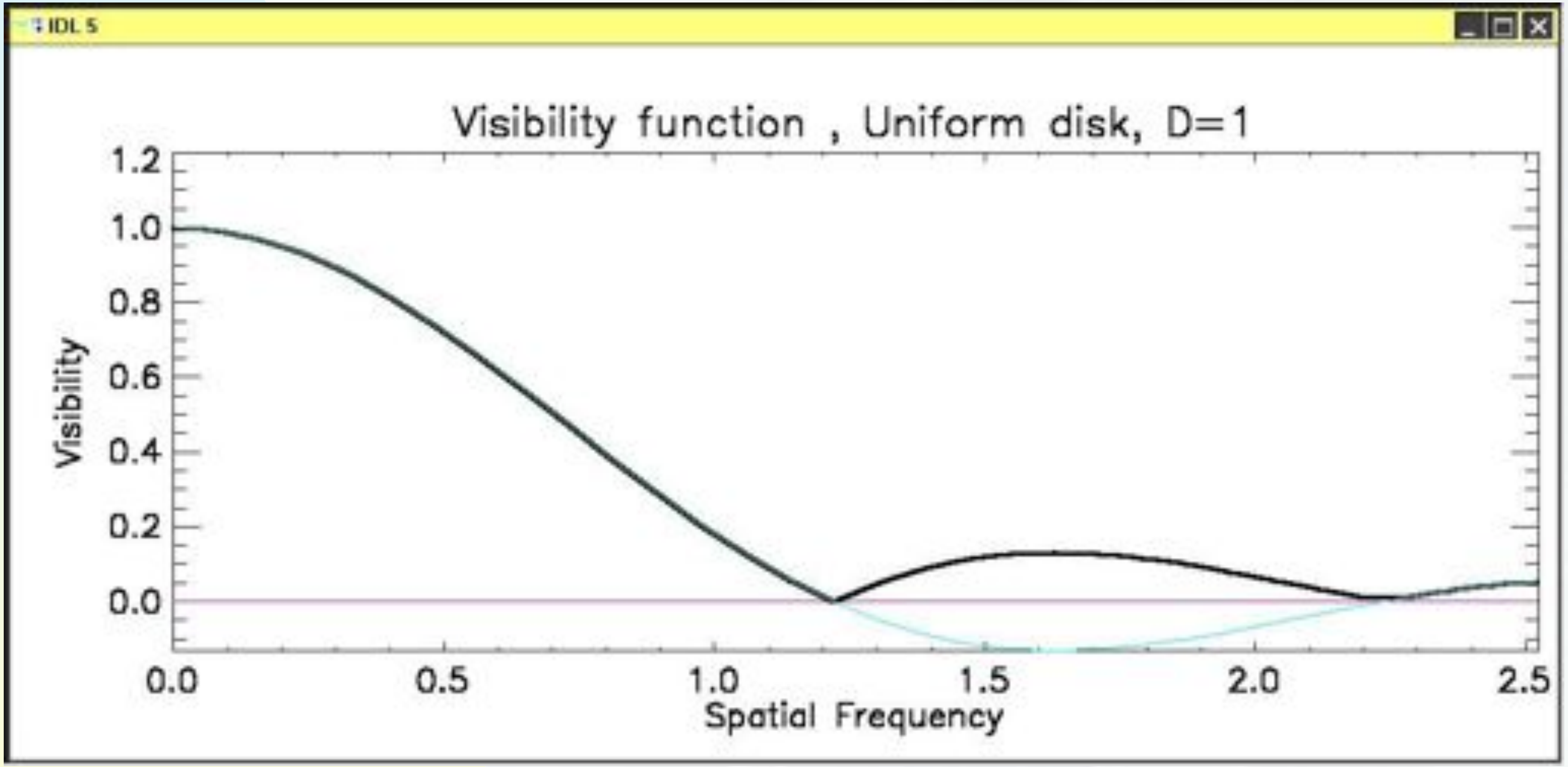
where $Z = 2\pi\, R\, \rho' / (\lambda f)$

one should be able to demonstrate the next result, i.e., the visibility V of the fringes observed for the case of a uniformly bright circular disk source with an angular diameter θ_{UD} by means of an interferometer with a baseline B is given by:

$$\upsilon = \left(\frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \right) = \left| \gamma_{12}(0) \right| = TF(I') = \frac{2J_1(\pi\theta_{UD} B / \lambda)}{\pi\theta_{UD} B / \lambda}$$

If the source is characterized by a uniform disk light distribution, the corresponding visibility function is given by

$$\upsilon = \left(\frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \right) = \left| \gamma_{12}(0) \right| = TF(I') = \frac{2J_1(\pi\theta_{UD}B/\lambda)}{\pi\theta_{UD}B/\lambda}$$



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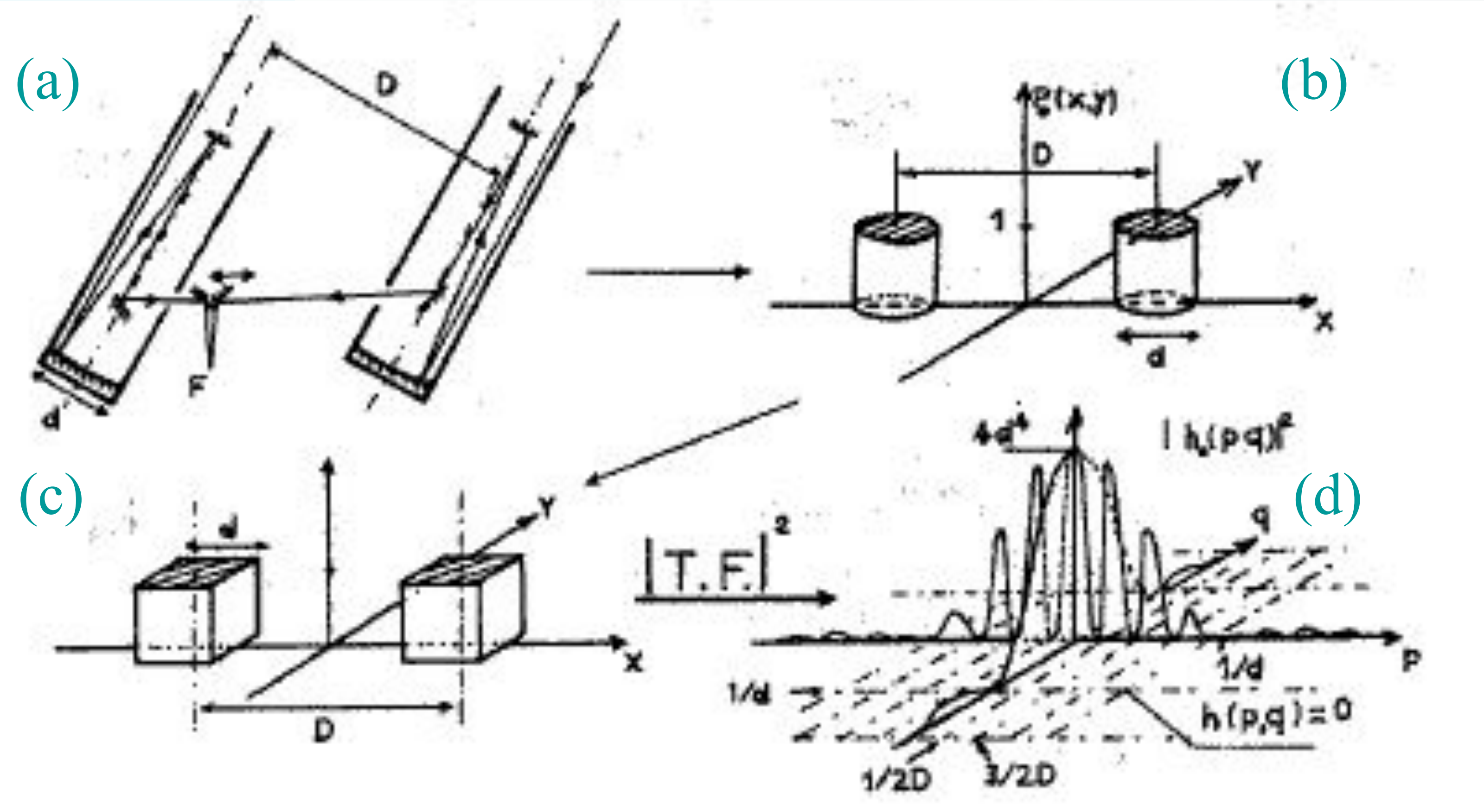
8.1 The fundamental theorem: 2 telescope interferometer



Two coupled optical telescopes: simplified optical scheme (a). Distribution of the complex amplitude for the case of two circular (b) or square (c) apertures and corresponding impulse response (d).

An introduction to optical/IR interferometry

8.1 The fundamental theorem: 2 telescope interferometer

$$h\,(p,q) = TF(P(x,y)\big(p,q\big) = \int_{R^2} P\,(x,y)\exp\big[-i2\pi\big(px+qy\big)\big]dxdy \qquad (8.1.10)$$

$$\begin{aligned}
&h\,(p,q) = TF(P_0(x+D/2)+P_0(x-D/2))(p,q) = \\
&TF(P_0(x+D/2))(p,q) + TF(P_0(x-D/2))(p,q) = \\
&\exp(i\pi D)\,TF(P_0(x))(p,q) + \exp(-i\pi D)\,TF(P_0(x))(p,q) = \\
&(\exp(i\pi D)+\exp(-i\pi D))\,TF(P_0(x))(p,q) = \\
&2\cos(\pi D)\,TF(P_0(x))(p,q)
\end{aligned} \qquad (8.1.11)$$

For the particular case of two square apertures:

$$i(p,q) = \left|h\,(p,q)\right|^2 = 4\cos^2(\pi pD)\,d^4\left(\frac{\sin(\pi qd)}{\pi qd}\right)^2\left(\frac{\sin(\pi pd)}{\pi pd}\right)^2 \qquad (8.1.12)$$

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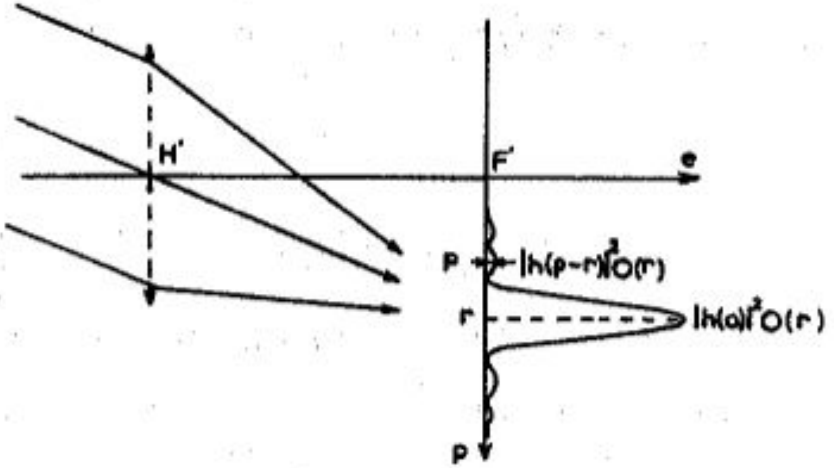
8.2 The convolution theorem

$e(p,q) = O(p,q) * |h(p,q)|^2,$



$$e(p,q) = \int_{R^2} O(r,s)\left|h(p-r,q-s)\right|^2 drds$$

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8.2 The convolution theorem

More generally, since
$TF\_(f * g) = TF\_(f)\ TF\_(g).$ (8.2.4)

We find, because
$e(p,q) = O(p,q) * |h(p,q)|^2$ (8.2.5)

that:
$TF\_(e(p,q)) = TF\_(O(p,q))\ TF\_(|h(p,q)|^2),$ (8.2.6)

and, finally,
$O(p,q) = TF^{-1}\,[TF\_(e(p,q)) / TF\_(|h(p,q)|^2)].$ (8.2.7)

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8.2 The convolution theorem

$$O(p,q) = (\lambda^2 E / \phi^2)\, \Pi(p\lambda/\phi)\, \Pi(q\lambda/\phi). \qquad (8.2.8)$$

$$e(p,q) = O(p,q) * |h_0(p,q)|^2.$$

$$e(p) = O(p) * |h_0(p)|^2, \qquad (8.2.9)$$

$$e(p) = 2d^2(\lambda/\phi)\sqrt{E}\int_{p-\phi/2\lambda}^{p+\phi/2\lambda}\left(\frac{\sin(\pi rd)}{\pi rd}\right)^2 \cos^2(\pi rD)\,dr \qquad (8.2.10)$$

$\left(\frac{\sin(\pi rd)}{\pi rd}\right)^2 \approx$ Cte sur $[p-\phi/2\lambda,\ p+\phi/2\lambda]$, and (8.2.11)

$$e(p) = 2d^2(\lambda/\phi)\sqrt{E}\left(\frac{\sin(\pi pd)}{\pi pd}\right)^2 \int_{p-\phi/2\lambda}^{p+\phi/2\lambda} \cos^2(\pi rD)\,dr\,. \qquad (8.2.12)$$

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8.2 The convolution theorem

$$e(p) = 2d^2\left(\frac{\sin(\pi pd)}{\pi pd}\right)^2\left[O(p) * \cos^2(\pi pD)\right], \qquad (8.2.13)$$

$$e(p) = 2d^2\left(\frac{\sin(\pi pd)}{\pi pd}\right)^2\left[\frac{1}{2}\int_R O(p)dp + \frac{1}{2}O(p) * \cos(2\pi pD)\right] \qquad (8.2.14)$$

$$e(p) = A\left[B + \frac{1}{2}\mathrm{Re}\big(O(p) * \exp(i2\pi pD)\big)\right], \qquad (8.2.15)$$

$$A = 2d^2\left(\frac{\sin(\pi pd)}{\pi pd}\right)^2 \quad et \quad B = \frac{1}{2}\int_R O(p)dp, \qquad (8.2.16)$$

An introduction to optical/IR interferometry

8.2 The convolution theorem

$$e(p) = A\left[B + \frac{1}{2}\mathrm{Re}\left(\int_R O(r)\exp\left(i2\pi(p-r)D\right)dr\right)\right], \qquad (8.2.17)$$

$$e(p) = A\left[B + \frac{1}{2}\cos(2\pi pD)\,TF\_(O(r))(D)\right], \qquad (8.2.18)$$

$$\gamma(\mathrm{D}) = (\mathrm{e}_{\mathrm{max}} - \mathrm{e}_{\mathrm{min}}) / (\mathrm{e}_{\mathrm{max}} + \mathrm{e}_{\mathrm{min}}), \qquad (8.2.19)$$

$$\gamma(\mathrm{D}) = \mathrm{TF\_(O(r))(D)} / (2\mathrm{B}) = \mathrm{TF\_(O(r))(D)} / \int \mathrm{O(p)\,dp}. \qquad (8.2.20)$$

Two possible types of beam combination:

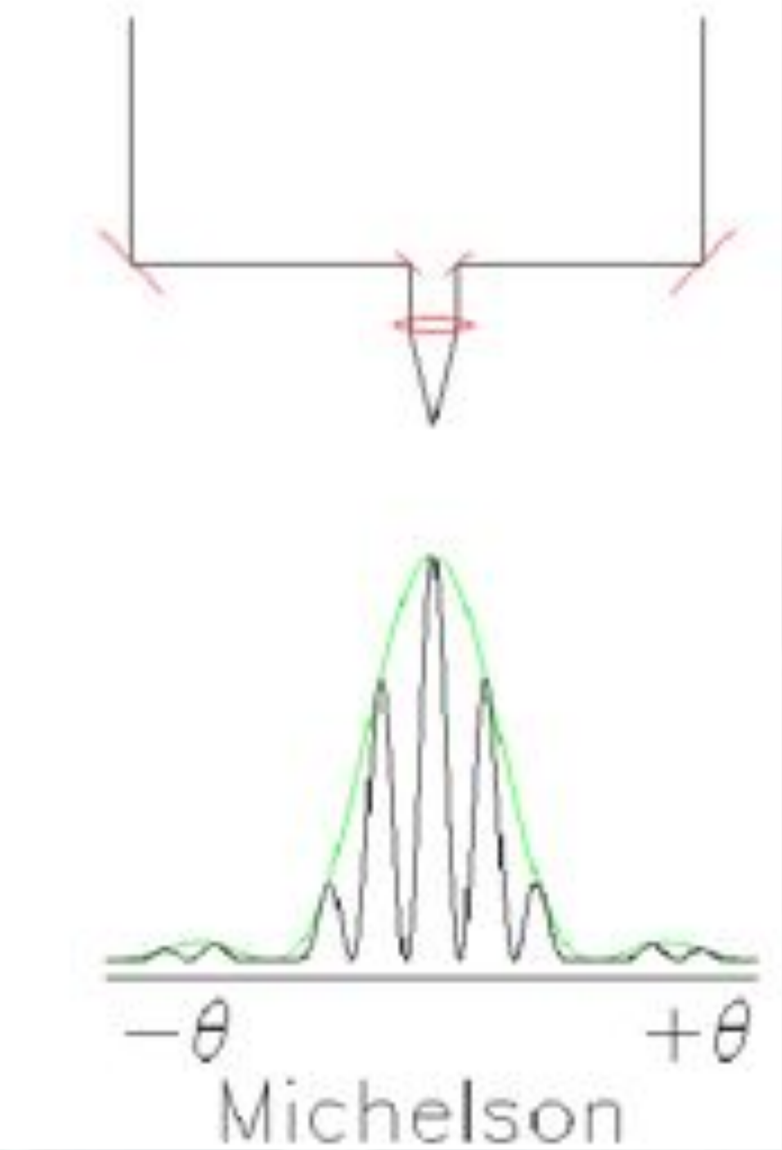
1) Image plane combination

- Mix the signals in the focal plane as in Young’s slit experiment:

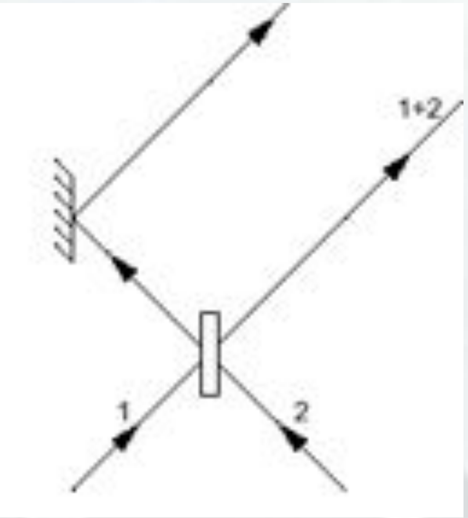
– In the focused image the transverse coordinate measures the delay between the beams.

– Fringes encoded by use of a non redundant input pupil.

– Possible to use dispersion prior to detection in the direction perpendicular to the fringes. Allows measurement of the visibility function at multiple λ



2) Pupil plane combination



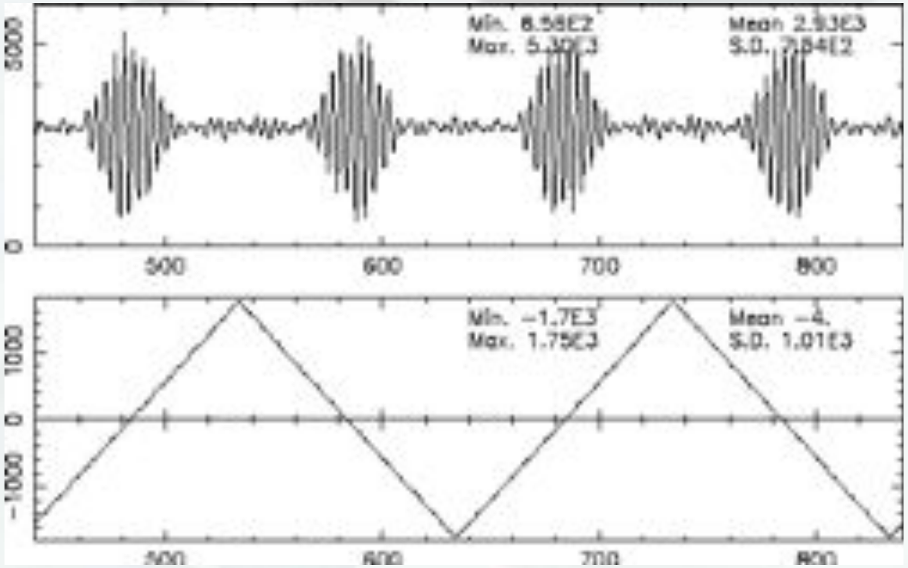
- Mix the signals by superposing afocal (collimated) beams at a beam splitter plate:

Then focus superposed beams onto a single element detector to measure I.

– Fringes are visualised by measuring intensity versus time.

– Fringes encoded by use of a non-Redundant modulation of delay of each beam.

– Can use spectral dispersion prior to detection to measure at multiple wavelengths.



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8.3 The Wiener-Khintchin theorem

In our case, this theorem merely states that the Fourier tranform of the PSF (see Eq. (8.2.7)) is the auto-correlation function of the distribution of the complex amplitude in the pupil plane:

$$TF(\left|h(a,b)\right|^2) = \iint P^*(x,y)P(x-a,y-b)\,dx\,dy$$

Démonstration (1/2):

Let us evaluate: $\iint P^*(x,y)P\,(x-a,y-b)\,dx\,dy$

Let us also remind:

$$h(p,q)=\iint P\,(x,y)\exp(-i2\pi(px+qy))\,dx\,dy$$

And thus (Fourier inverse transform),

$$P(x,y)=\iint h(p,q)\exp(i2\pi(px+qy))\,dp\,dq$$, and also

$$P^*(x,y)=\iint h^*(p',q')\exp(-i2\pi(p'x+q'y))\,dp'\,dq'$$

We then find that

$$\iint P^*(x,y)P\,(x-a,y-b)\,dx\,dy=$$

$$\iint\ \iint\ h^*(p',q')h\,(p,q)\exp(-i2\pi(ap+bq))dpdq$$

$$\iint\ \exp(-i2\pi((p'-p)x+(q'-q)y))dxdydp'dq'$$

Démonstration (2/2):

Since

$$\iint \exp(-i2\pi((p'-p)x+(q'-q)y))dxdy =$$

$\delta(p-p')\delta(q-q')$, we then find

$$\iint P^*(x,y)P(x-a,y-b)dx\,dy =$$

$$\iint \iint h^*(p',q')h(p,q)\delta(p-p')\delta(q-q')$$

$$\exp(-i2\pi(ap+bq))dpdqdp'dq'$$

and finally

$$\iint P^*(x,y)P(x-a,y-b)dx\,dy =$$

$$\iint \left|h(p,q)\right|^2 \exp(-i2\pi(ap+bq))dpdq =$$

$$= TF(\left|h(p,q)\right|^2)(a,b).$$

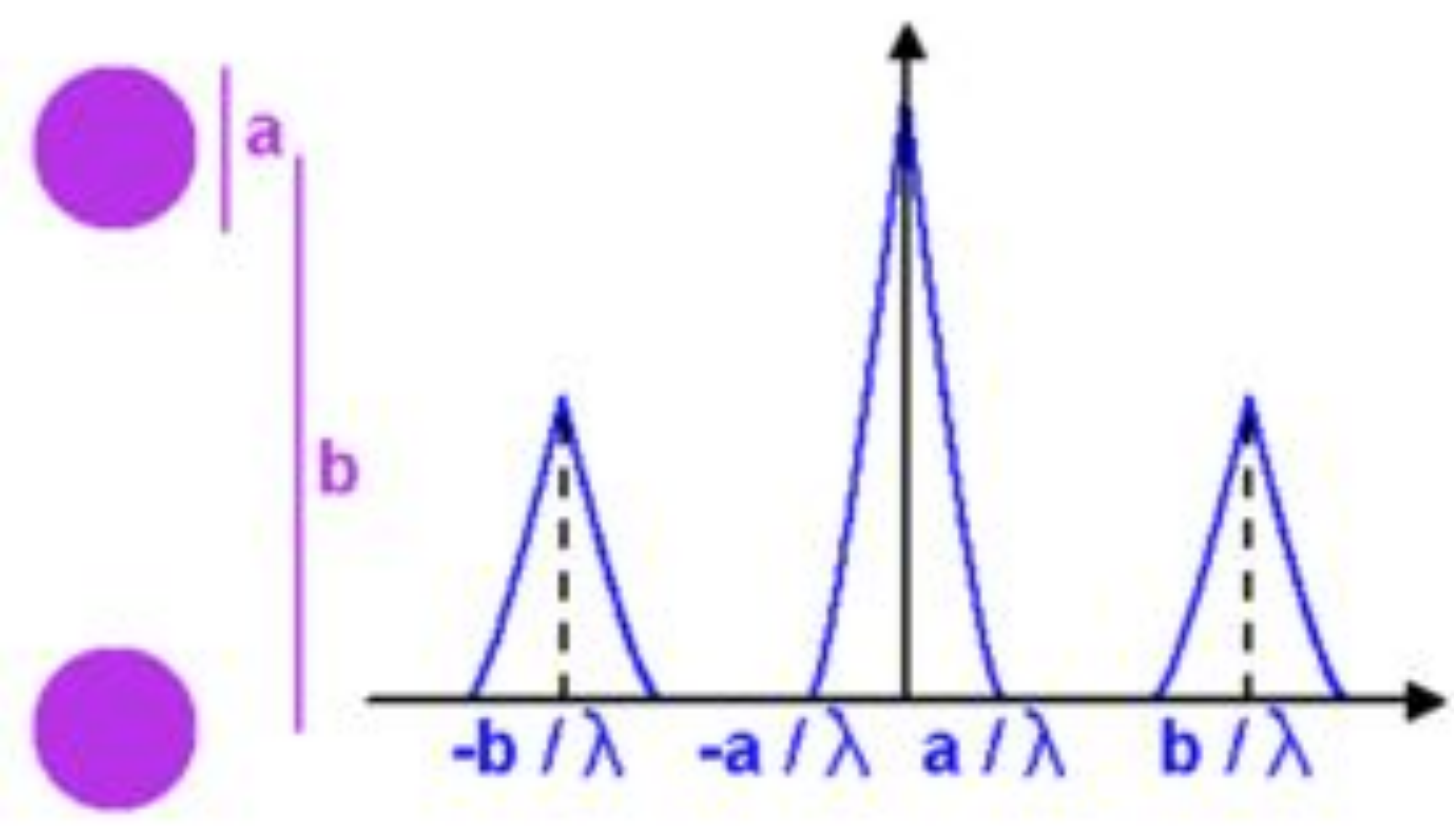
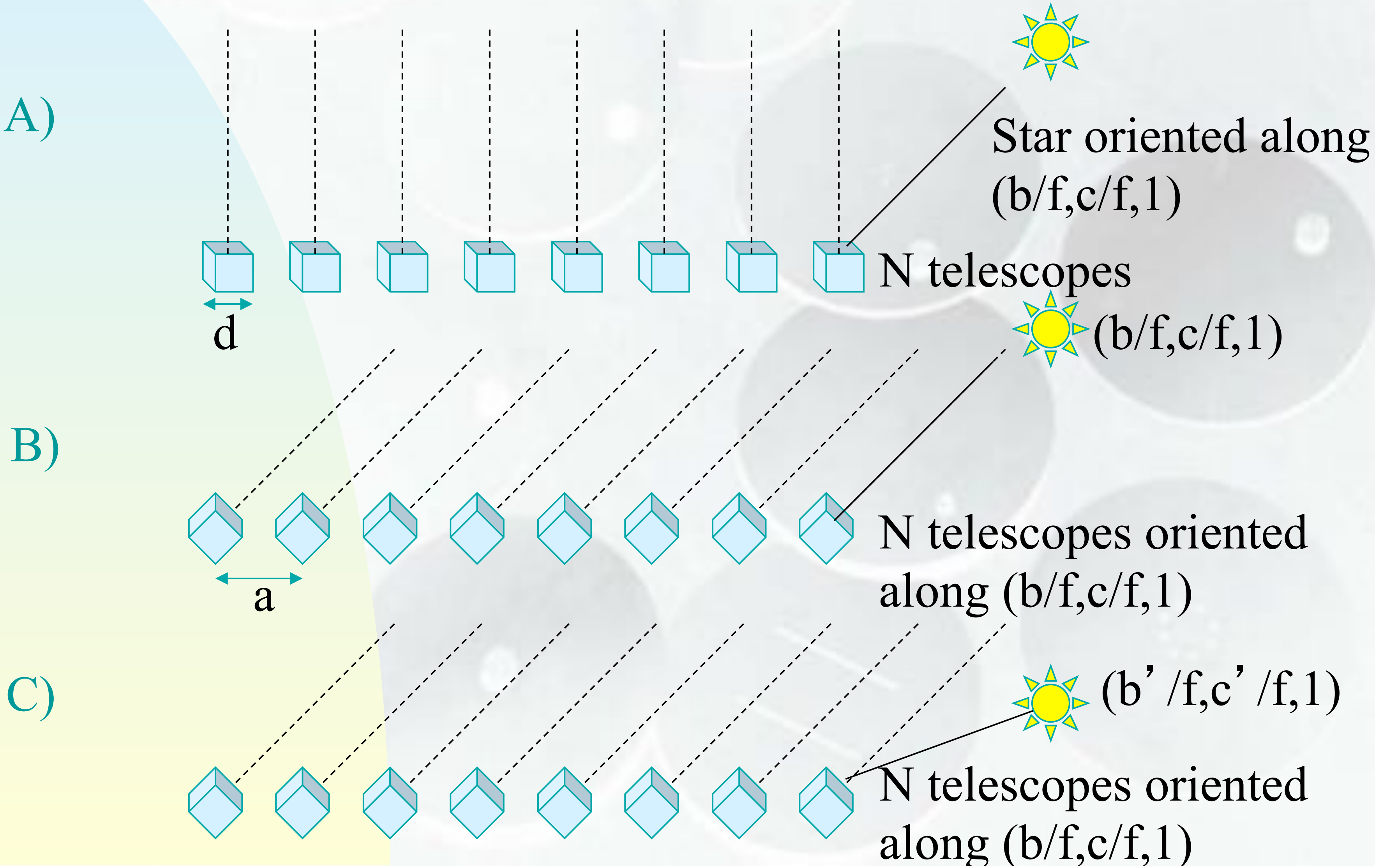


Diagram illustrating the autocorrelation as a function of the space frequency, for the case of an interferometer composed of 2 telescopes, with a diameter a, separated by the baseline b. The autocorrelation of the pupil gives access to high space frequencies.

- **Exercices:** Calculate the response functions (PSFs) for the cases described in the figures below!



Star oriented along (b/f,c/f,1)

N telescopes

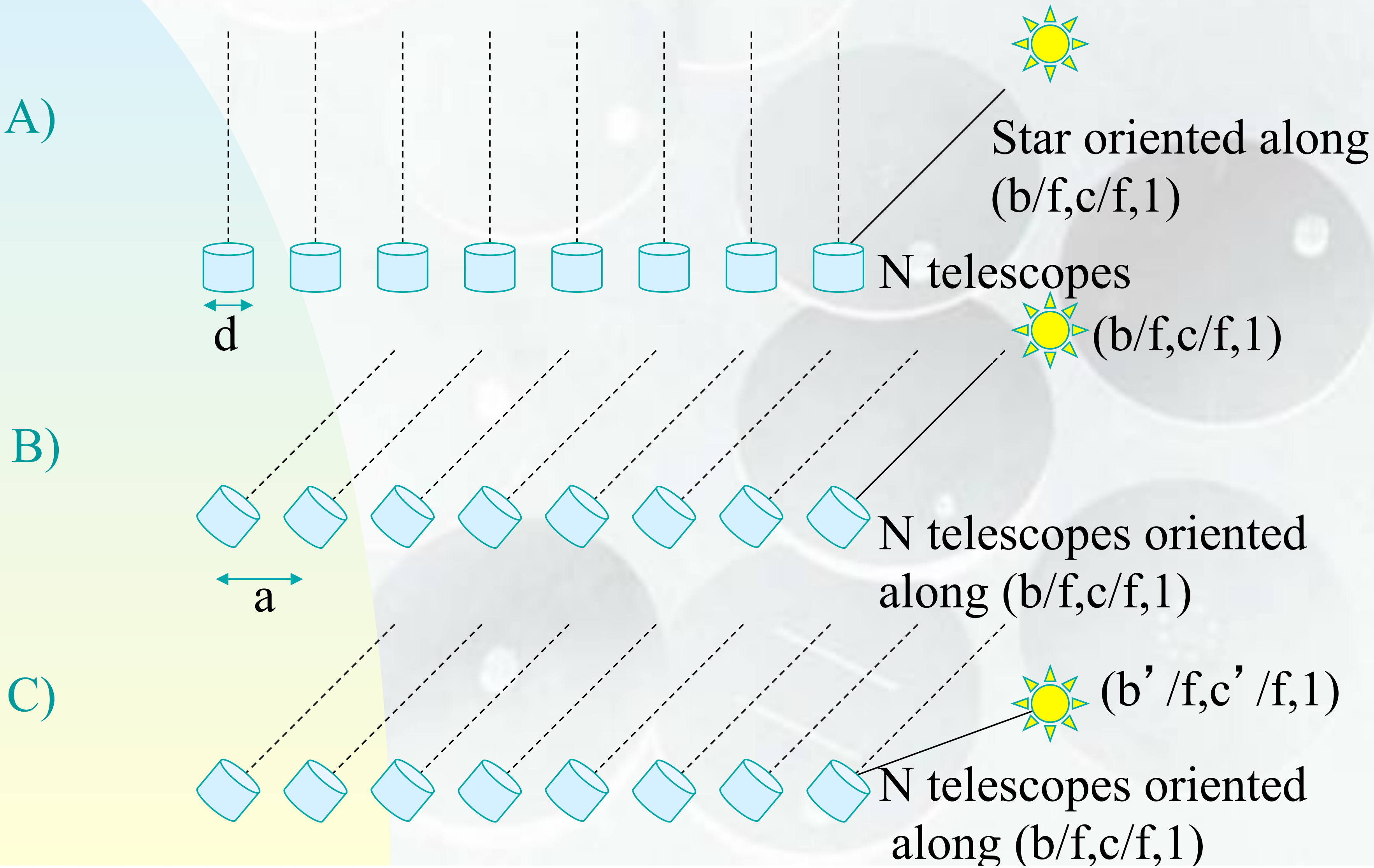
(b/f,c/f,1)

N telescopes oriented along (b/f,c/f,1)

(b’/f,c’/f,1)

N telescopes oriented along (b/f,c/f,1)

- **Exercices:** Calculate the response functions (PSFs) for the cases described in the figures below!



A)

Star oriented along $(b/f,c/f,1)$

N telescopes

d

B)

$(b/f,c/f,1)$

N telescopes oriented along $(b/f,c/f,1)$

a

C)

$(b'/f,c'/f,1)$

N telescopes oriented along $(b/f,c/f,1)$